symmetry

Article

New Oscillation Criteria for Neutral Delay Differential Equations of Fourth-Order

Saeed Althubiti [1], Omar Bazighifan [2,3], Hammad Alotaibi [1] and Jan Awrejcewicz [4,*]

[1] Department of Mathematics and Statistics, College of Science, Taif University, P.O. Box 11099, Taif 21944, Saudi Arabia; sathubiti@tu.edu.sa (S.A.); hm.alotaibi@tu.edu.sa (H.A.)
[2] Section of Mathematics, International Telematic University Uninettuno, Corso Vittorio Emanuele II, 39, 00186 Roma, Italy; o.bazighifan@hu.edu.ye
[3] Department of Mathematics, Faculty of Science, Hadhramout University, Hadhramout 50512, Yemen
[4] Department of Automation, Biomechanics and Mechatronics, Lodz University of Technology, 1/15 Stefanowski St., 90-924 Lodz, Poland
* Correspondence: jan.awrejcewicz@p.lodz.pl

Abstract: New oscillatory properties for the oscillation of solutions to a class of fourth-order delay differential equations with several deviating arguments are established, which extend and generalize related results in previous studies. Some oscillation results are established by using the Riccati technique under the case of canonical coefficients. The symmetry plays an important and fundamental role in the study of the oscillation of solutions of the equations. Examples are given to prove the significance of the new theorems.

Keywords: fourth-order; delay differential equations; oscillation

Citation: Althubiti, S.; Bazighifan, O.; Alotaibi, A.; Awrejcewicz, J. New Oscillation Criteria for Neutral Delay Differential Equations of Fourth-Order. *Symmetry* **2021**, *13*, 1277. https://doi.org/10.3390/sym13071277

Academic Editor: José Carlos R. Alcantud

Received: 1 July 2021
Accepted: 14 July 2021
Published: 16 July 2021

1. Introduction

In this article, we present some oscillatory properties of the equation

$$\left(j(y)\left(\xi'''(y)\right)^{r_1}\right)' + \sum_{i=1}^{n} \varsigma_i(y)\xi^{r_2}(z_i(y)) = 0,\ y \geq y_0. \tag{1}$$

Throughout this article, we suppose that

$$\begin{cases} j \in C^1([y_0,\infty)),\ \varsigma_i \in C([y_0,\infty)), j(y) > 0, \varsigma_i(y) > 0, j'(y) \geq 0, \\ z_i(y) \in C([y_0,\infty),\mathbb{R}), z_i(y) \leq y, \lim_{y\to\infty} z_i(y) = \infty, i = 1,2,\ldots,n, \\ r_1 \text{ and } r_2 \text{ are quotients of odd positive integers.} \end{cases}$$

Definition 1. *A solution of (1) is said to be non-oscillatory if it is positive or negative, ultimately; otherwise, it is said to be oscillatory.*

Definition 2. *The equation (1) is said to be oscillatory if every solution of it is oscillatory.*

Delayed differential equations contribute to many real-life applications and real-world problems, as they play an important role in physics, chemistry, medicine, biology, engineering and aviation. In addition, for networks containing lossless transmission lines, see [1–3].

On the other hand, a study of the oscillation of solutions to fourth-order differential equations in the non-canonical case has interested some researchers due to its utmost importance in many applications (see [4–6]).

In addition, there are some papers and books dealing with the oscillation of the solutions of delay differential equations with/without deviating arguments (see [7–13]).

The motivation for this article is to complement the results reported in [14,15]; therefore, we discuss their findings and results below.

The authors in [14] presented some oscillatory properties for the equation

$$\left(j(y)\left(\xi'''(y)\right)^{r_1}\right)' + \varsigma(y)\xi^{r_2}(z(y)) = 0,\ y \geq y_0, \tag{2}$$

They also used the comparison technique.

Agarwal et al. [12] investigated the oscillation of equation:

$$\left(j(y)\left(\xi'''(y)\right)^{r_1}\right)' + \varsigma(y)f(\xi^{r_1}(z(y))) = 0,$$

The authors used the integral averaging technique to obtain oscillation results for this equation.

Zhang et al. [16] presented criteria for the oscillation of Equation (2), under the assumption that $\int_{y_0}^{\infty} \frac{1}{j^{1/r_1}(s)} \mathrm{d}s < \infty$. Moreover, the authors used the Riccati method to find the oscillation criteria for this equation.

Moaaz et al. [15] presented conditions for oscillation of equation

$$\left(j(y)\left(\xi'''(y)\right)^{r_1}\right)' + \varsigma(y)f(\xi^{r_2}(z(y))) = 0,$$

under the condition

$$\int_{y_0}^{\infty} \epsilon_1^{r_2 - r_1} \varsigma(s) \frac{z^{3r_1}(s)}{s^{3r_1}} \mathrm{d}s = \infty,$$

where ϵ_1 is a positive constant. Additionally, the authors used the comparison technique.

In [17,18], the authors studied the equation

$$\varsigma^{(n)}(y) + \varsigma(y)\xi(z(y)) = 0, \tag{3}$$

By using the comparison technique, they proved that this equation is oscillatory if

$$\liminf_{y\to\infty} \int_{z(y)}^{y} z^{n-1}(s)\mathrm{d}s > \frac{(n-1)2^{(n-1)(n-2)}}{\mathrm{e}} \tag{4}$$

and

$$\liminf_{y\to\infty} \int_{z(y)}^{y} z^{n-1}(s)\mathrm{d}s > \frac{(n-1)!}{\mathrm{e}}, \tag{5}$$

where $n \geq 4$ is an even natural number.

Our main goal in this article is to obtain some oscillatory properties of (1) under the hypothesis

$$\int_{y_0}^{\infty} \frac{1}{j^{1/r_1}(s)} \mathrm{d}s = \infty, \tag{6}$$

which complement some properties that have been studied in the literature, where we use a different technique based on using the Riccati method. The benefit gained using this approach is to get more effective oscillation conditions.

2. Oscillation Criteria

We present some lemmas, which are required for our theorem proofs.

Lemma 1 ([19]). *Let $h \in C^n([y_0,\infty))$ and $h(y) > 0$. Suppose that $h^{(n)}(y)$ is of a fixed sign, on $[y_0,\infty)$, $h^{(n)}(y)$ not identically zero and that there exists a $y_1 \geq y_0$, such that for all $y \geq y_1$,*

$$h^{(n-1)}(y)h^{(n)}(y) \leq 0.$$

If we have $\lim_{y\to\infty} h(y) \neq 0$, *then there exists* $y_\lambda \geq y_0$, *such that*

$$h(y) \geq \frac{\lambda}{(n-1)!} y^{n-1} \left| h^{(n-1)}(y) \right|,$$

for every $\lambda \in (0,1)$ *and* $y \geq y_\lambda$.

Lemma 2 ([20])**.** *If* $\xi^{(i)}(y) > 0$, $i = 0,1,\ldots,n$, *and* $\xi^{(n+1)}(y) < 0$, *then*

$$\xi(y)\frac{n!}{y^n} \geq \xi'(y)\frac{(n-1)!}{y^{n-1}}.$$

Lemma 3 ([21])**.** *Let*

$$\xi(y) \text{ be an eventually positive solution of (1).} \tag{7}$$

Then, there exist two possible cases: either

$$(N_1)\ \xi^{(\kappa)}(y) > 0 \ \text{ for } \kappa = 0,1,2,3;$$

or

$$(N_2)\ \xi^{(\kappa)}(y) > 0 \ \text{ for } \kappa = 0,1,3, \text{ and } \xi''(y) < 0,$$

holds.

Lemma 4. *Suppose that (7) holds.*

$(\mathbf{i}_1)$ *If* ξ *satisfies* (N_1), *then*

$$\delta_1'(y) + \epsilon_1^{r_2-r_1}\sum_{i=1}^{n} \varsigma_i(y)\frac{z_i^{3r_1}(y)}{y^{3r_1}} + \frac{r_1\kappa}{2}\frac{y^2}{j^{1/r_1}(y)}\delta_1^{1+1/r_1}(y) \leq 0; \tag{8}$$

$(\mathbf{i}_2)$ *If* ξ *satisfies* (N_2), *then*

$$\delta_2'(y) + \delta_2^2(y) + \epsilon_1^{r_2-r_1}G(y) \leq 0, \tag{9}$$

where

$$G(y) := \lambda^{r_2/r_1}\epsilon_2^{r_2/r_1}\int_y^\infty \left(\frac{1}{j(u)}\int_u^\infty \sum_{i=1}^{n}\varsigma_i(s)\left(\frac{z_i(s)}{s}\right)^{r_2} \mathrm{d}s\right)^{1/r_1} \mathrm{d}u,$$

for every $\kappa \in (0,1)$ *and* ϵ_1, ϵ_2 *are positive constants.*

Proof. Suppose that (7) holds. By Lemma 3, we see that cases (N_1) and (N_2) hold Suppose that (N_1) holds. From Lemma 1, we find

$$\xi'(y) \geq \frac{\kappa}{2}y^2\xi'''(y) \tag{10}$$

and by using Lemma 2, we obtain $\xi(y) \geq \frac{1}{3}y\xi'(y)$. Hence,

$$\xi(z_i(y)) \geq \frac{z_i^3(y)}{y^3}\xi(y). \tag{11}$$

Define

$$\delta_1(y) := \frac{j(y)(\xi'''(y))^{r_1}}{\xi^{r_1}(y)}.$$

Differentiating δ_1 and using (1), (10) and (11), we obtain

$$\delta_1'(y) \leq -\sum_{i=1}^{n} \varsigma_i(y) \frac{z_i^{3r_1}(y)}{y^{3r_1}} \xi^{r_2-r_1}(z_i(y)) - \frac{r_1\kappa}{2} \frac{y^2}{j^{1/r_1}(y)} \delta_1^{1+1/r_1}(y).$$

Since $\xi'(y) > 0$, there exist a $y_2 \geq y_1$ and a constant $\epsilon_1 > 0$, such that $\xi(y) > \epsilon_1$, for all $y \geq y_2$. Thus, we see that

$$\delta_1'(y) \leq -\sum_{i=1}^{n} \varsigma_i(y) \frac{z_i^{3r_1}(y)}{y^{3r_1}} \epsilon_1^{r_2-r_1}(z_i(y)) - \frac{r_1\kappa}{2} \frac{y^2}{j^{1/r_1}(y)} \delta_1^{1+1/r_1}(y),$$

Thus, (8) is satisfied.

Suppose that (N_2) holds. Integrating (1) from y to l, we see that

$$j(l)\left(\xi'''(l)\right)^{r_1} = j(y)\left(\xi'''(y)\right)^{r_1} - \int_y^l \sum_{i=1}^{n} \varsigma_i(s) \xi^{r_2}(z_i(s)) \mathrm{d}s. \tag{12}$$

By Lemma 2, we find

$$\xi(y) \geq y\xi'(y). \tag{13}$$

Thus, $\xi(z_i(y)) \geq (z_i(y)/y)\xi(y)$, from (12) and $\xi'(y) > 0$, we obtain

$$j(l)\left(\xi'''(l)\right)^{r_1} - j(y)\left(\xi'''(y)\right)^{r_1} + \xi^{r_2}(y) \int_y^l \sum_{i=1}^{n} \varsigma_i(s) \left(\frac{z_i(s)}{s}\right)^{r_2} \mathrm{d}s \leq 0.$$

Letting $l \to \infty$, we obtain

$$\xi'''(y) \geq \frac{\lambda^{r_2/r_1}}{j^{1/r_1}(y)} \xi^{r_2/r_1}(y) \left(\int_y^\infty \sum_{i=1}^{n} \varsigma_i(s) \left(\frac{z_i(s)}{s}\right)^{r_2} \mathrm{d}s\right)^{1/r_1}.$$

Integrating the above inequality from y to ∞, we obtain

$$\begin{aligned} \xi''(y) &\leq -\lambda^{r_2/r_1} \xi^{r_2/r_1}(y) \int_y^\infty \left(\frac{1}{j(u)} \int_u^\infty \sum_{i=1}^{n} \varsigma_i(s) \left(\frac{z_i(s)}{s}\right)^{r_2} \mathrm{d}s\right)^{1/r_1} \mathrm{d}u \\ &\leq -G(y)\xi^{r_2/r_1}(y). \end{aligned} \tag{14}$$

Define

$$\delta_2(y) := \frac{\xi'(y)}{\xi(y)}.$$

Differentiating δ_2 and using (14), we obtain

$$\delta_2'(y) + \delta_2^2(y) + \epsilon_1^{r_2-r_1} G(y) \leq 0.$$

Lemma 4 is proved. □

Theorem 1. *Let*

$$\int_{y_0}^\infty \epsilon_1^{r_2-r_1} \sum_{i=1}^{n} \varsigma_i(s) \frac{z_i^{3r_1}(s)}{s^{3r_1}} \mathrm{d}s = \infty \tag{15}$$

and

$$\int_{y_0}^\infty \lambda^{r_2/r_1} \epsilon_2^{r_2/r_1} \int_y^\infty \left(\frac{1}{j(u)} \int_u^\infty \sum_{i=1}^{n} \varsigma_i(s) \left(\frac{z_i(s)}{s}\right)^{r_2} \mathrm{d}s\right)^{1/r_1} \mathrm{d}u\mathrm{d}s = \infty, \tag{16}$$

then (1) is oscillatory.

Proof. Suppose that $\xi(y) > 0$. By Lemma 3, there exist two possible cases for $y \geq y_1$, where $y_1 \geq y_0$ is sufficiently large.

For case (N_1), by Lemma 4, we find (8) holds, which yields

$$\delta_1'(y) + \epsilon_1^{r_2-r_1} \sum_{i=1}^{n} \varsigma_i(y) \frac{z_i^{3r_1}(y)}{y^{3r_1}} \leq 0. \tag{17}$$

Integrating (17) from y_2 to y and using (15), we obtain

$$\delta_1(y) \leq \delta_1(y_2) - \int_{y_2}^{y} \epsilon_1^{r_2-r_1} \sum_{i=1}^{n} \varsigma_i(s) \frac{z_i^{3r_1}(s)}{s^{3r_1}} ds \to -\infty \quad \text{as} \quad y \to \infty.$$

This contradicts that $\delta_1(y) > 0$.

Similarly, suppose that case (N_2) holds, we obtain a contradiction with (16), which is omitted here for convenience. Theorem 1 is proved. □

Definition 3. *Let sequence $\{\phi_n(y)\}_{n=0}^{\infty}$ and $\{\varphi_n(y)\}_{n=0}^{\infty}$ be defined as*

$$\phi_n(y) = \phi_0(y) + \int_y^{\infty} \frac{r_1 \kappa}{2} \frac{s^2}{j^{1/r_1}(s)} \phi_{n-1}^{\frac{r_1+1}{r_1}}(s) \mathrm{d}s \tag{18}$$

and

$$\varphi_n(y) = \varphi_0(y) + \int_y^{\infty} \varphi_{n-1}^2(s) \mathrm{d}s, \tag{19}$$

where

$$\phi_0(y) = \int_y^{\infty} \epsilon_1^{r_2-r_1} \sum_{i=1}^{n} \varsigma_i(s) \frac{z_i^{3r_1}(s)}{s^{3r_1}} \mathrm{d}s$$

and

$$\varphi_0(y) = \int_y^{\infty} \lambda^{r_2/r_1} \epsilon_2^{r_2/r_1} \int_y^{\infty} \left(\frac{1}{j(u)} \int_u^{\infty} \sum_{i=1}^{n} \varsigma_i(s) \left(\frac{z_i(s)}{s} \right)^{r_2} \mathrm{d}s \right)^{1/r_1} \mathrm{d}u \mathrm{d}s.$$

Theorem 2. *Assume that*

$$\liminf_{y \to \infty} \frac{1}{\phi_0(y)} \int_y^{\infty} \frac{r_1 \kappa}{2} \frac{s^2}{j^{1/r_1}(s)} \phi_0^{\frac{r_1+1}{r_1}}(s) \mathrm{d}s > \frac{r_1}{(r_1+1)^{\frac{r_1+1}{r_1}}} \tag{20}$$

and

$$\liminf_{y \to \infty} \frac{1}{\varphi_0(y)} \int_y^{\infty} \varphi_0^2(s) \mathrm{d}s > \frac{1}{4}. \tag{21}$$

Then, (1) is oscillatory.

Proof. Suppose that $\xi(y) > 0$. By Lemma 3, there exist two possible cases, (N_1) and (N_2). Let case (N_1) hold. In Lemma 4, integrating (8) from y to l, we obtain

$$\delta_1(l) - \delta_1(y) + \int_y^{l} \epsilon_1^{r_2-r_1} \sum_{i=1}^{n} \varsigma_i(s) \frac{z_i^{3r_1}(s)}{s^{3r_1}} \mathrm{d}s + \int_y^{l} \frac{r_1 \kappa}{2} \frac{s^2}{j^{1/r_1}(s)} \delta_2^{\frac{r_1+1}{r_1}}(s) \mathrm{d}s \leq 0. \tag{22}$$

From (22), it is obvious that

$$\delta_1(l) - \delta_1(y) + \int_y^{l} \frac{r_1 \kappa}{2} \frac{s^2}{j^{1/r_1}(s)} \delta_1(s) \mathrm{d}s \leq 0. \tag{23}$$

Then, we conclude from (23) that either

$$\int_y^\infty \frac{r_1\kappa}{2}\frac{s^2}{j^{1/r_1}(s)}\delta_1(s)\mathrm{d}s < \infty, \text{ for } y \geq Y, \tag{24}$$

or, otherwise,

$$\delta_1(l) \leq \delta_1(y) - \int_y^l \frac{r_1\kappa}{2}\frac{s^2}{j^{1/r_1}(s)}\delta_1(s)\mathrm{d}s \to -\infty \text{ as } l \to \infty,$$

which contradicts that $\delta_1(y) > 0$. Since $\delta_1(y)$ is positive and decreasing, $\lim_{y\to\infty}\delta_1(y) = k \geq 0$. By (24), we see $k = 0$. So, from (22), we find

$$\delta_1(y) \geq \widetilde{Q}(y) + \int_y^\infty \frac{r_1\kappa}{2}\frac{s^2}{j^{1/r_1}(s)}\delta_1(s)\mathrm{d}s = \phi_0(y) + \int_y^\infty \frac{r_1\kappa}{2}\frac{s^2}{j^{1/r_1}(s)}\delta_1(s)\mathrm{d}s. \tag{25}$$

From (25), we have

$$\frac{\delta_1(y)}{\phi_0(y)} \geq 1 + \frac{1}{\phi_0(y)}\int_y^\infty \frac{r_1\kappa}{2}\frac{s^2}{j^{1/r_1}(s)}\phi_0^{\frac{r_1+1}{r_1}}(s)\left(\frac{\delta_1(s)}{\phi_0(s)}\right)^{\frac{r_1+1}{r_1}}\mathrm{d}s,\ y \geq Y. \tag{26}$$

If we set $a = \inf_{y\geq Y}\delta_1(y)/\phi_0(y)$, then obviously $a \geq 1$. Hence, from (20) and (26), we see that

$$a \geq 1 + r_1\left(\frac{a}{r_1+1}\right)^{(r_1+1)/r_1}$$

or

$$\frac{a}{r_1+1} \geq \frac{1}{r_1+1} + \frac{r_1}{r_1+1}\left(\frac{a}{r_1+1}\right)^{(r_1+1)/r_1}$$

which contradicts the admissible value of r_1 and a. Similarly, in the case (N_2), if we set $a_1 = \inf_{y\geq Y_1}\delta_2(y)/\varphi_0(y)$ and taking 21 into account, then we arrive at a contradiction with the admissible value of a_1. Therefore, Theorem 2 is proved. □

Theorem 3. *Let*

$$\limsup_{y\to\infty}\phi_n(y)\left(\frac{\kappa}{2}y^2\int_{y_0}^y j^{-1/r_1}(s)\mathrm{d}s\right)^{r_1} > 1 \tag{27}$$

and

$$\limsup_{y\to\infty} y\varphi_n(y) > 1, \tag{28}$$

hold. Then (1) is oscillatory.

Proof. Suppose that $\xi(y) > 0$ and case (N_1) holds. By Lemma 1, we obtain

$$\xi(y) \geq \frac{\kappa}{6}y^3\xi'''(y). \tag{29}$$

From the definition of δ_1 and (29), we have

$$\begin{aligned}\frac{1}{\delta_1(y)} &= \frac{1}{j(y)}\left(\frac{\xi(y)}{\xi'''(y)}\right)^{r_1}\\ &\geq \frac{1}{j(y)}\left(\frac{\kappa}{6}y^3\right)^{r_1}\end{aligned}$$

Thus,

$$\delta_1(y)\frac{1}{j(y)}\left(\frac{\kappa}{6}y^3\right)^{r_1} \leq 1 \tag{30}$$

and

$$\limsup_{y\to\infty} \delta_1(y)\left(\frac{\kappa y^3}{6 j^{1/r_1}(y)}\right)^{r_1} \leq 1,$$

and this contradicts (27).

Similarly, when (N_2) holds, we find a contradiction with (28). Theorem 3 is proved. □

Corollary 1. *If there exist ϕ_n and φ_n such that*

$$\int_y^y \epsilon_1^{r_2-r_1} \sum_{i=1}^n \varsigma_i(s) \frac{z_i^{3r_1}(s)}{s^{3r_1}} \exp\left(\int_y^s \frac{r_1\kappa}{2} \frac{u^2}{j^{1/r_1}(u)} \phi_n^{1/r_1}(u)\mathrm{d}u\right)\mathrm{d}s = \infty \tag{31}$$

and

$$\int_y^y G(s) \exp\left(\int_y^s \varphi_n(u)\mathrm{d}u\right)\mathrm{d}s = \infty, \tag{32}$$

where

$$G(s) := \lambda^{r_2/r_1} \epsilon_2^{r_2/r_1} \int_y^\infty \left(\frac{1}{j(u)} \int_u^\infty \sum_{i=1}^n \varsigma_i(s) \left(\frac{z_i(s)}{s}\right)^{r_2} \mathrm{d}s\right)^{1/r_1} \mathrm{d}u,$$

then (1) is oscillatory.

Proof. Let case (N_1) hold. From (25), we find

$$\delta_1(y) \geq \phi_0(y).$$

Moreover, by using Lebesgue monotone convergence theorem, we find

$$\phi(y) = \phi_0(y) + \int_y^\infty \frac{r_1\kappa}{2} \frac{s^2}{j^{1/r_1}(s)} \phi^{\frac{r_1+1}{r_1}}(s)\mathrm{d}s. \tag{33}$$

From (33), we have that

$$\phi'(y) = -\frac{r_1\kappa}{2} \frac{y^2}{j^{1/r_1}(y)} \phi^{\frac{r_1+1}{r_1}}(y) - \epsilon_1^{r_2-r_1} \sum_{i=1}^n \varsigma_i(y) \frac{z_i^{3r_1}(y)}{y^{3r_1}}. \tag{34}$$

Since $\phi_n(y) \leq \phi(y)$, it follows from (34) that

$$\phi'(y) \leq -\frac{r_1\kappa}{2} \frac{y^2}{j^{1/r_1}(y)} \phi_n^{1/r_1}(y)\phi(y) - \epsilon_1^{r_2-r_1} \sum_{i=1}^n \varsigma_i(y) \frac{z_i^{3r_1}(y)}{y^{3r_1}}.$$

Hence, we obtain

$$\begin{aligned}\phi(y) \quad \leq \quad & \exp\left(-\int_y^y \frac{r_1\kappa}{2} \frac{s^2}{j^{1/r_1}(s)} \phi_n^{1/r_1}(s)\mathrm{d}s\right) \\ & \left(\phi(y) - \int_y^y \epsilon_1^{r_2-r_1} \sum_{i=1}^n \varsigma_i(s) \frac{z_i^{3r_1}(s)}{s^{3r_1}} \exp\left(\int_y^s \frac{r_1\kappa}{2} \frac{u^2}{j^{1/r_1}(u)} \phi_n^{1/r_1}(u)\mathrm{d}u\right)\mathrm{d}s\right).\end{aligned}$$

The above inequality follows

$$\int_y^y \epsilon_1^{r_2-r_1} \sum_{i=1}^n \varsigma_i(s) \frac{z_i^{3r_1}(s)}{s^{3r_1}} \exp\left(\int_y^s \frac{r_1\kappa}{2} \frac{u^2}{j^{1/r_1}(u)} \phi_n^{1/r_1}(u)\mathrm{d}u\right)\mathrm{d}s \leq \phi(y) < \infty,$$

and this contradicts (31).

Similarly, when (N_2) holds, we find a contradiction with (32). Corollary 1 is proved. □

3. Example

This section presents some interesting examples to examine the applicability of the theoretical outcomes.

Example 1. *Consider the equation*

$$\xi^{(4)}(y) + \frac{\varsigma_0}{y^4}\xi\left(\frac{9}{10}y\right) = 0, \ y \geq 1. \tag{35}$$

We note that $r_1 = r_2 = 1, \ n = 4, j(y) = 1, z_i(y) = 9y/10$ *and* $\varsigma(y) = \varsigma_0/y^4$. *Applying the conditions (4) and (5) to Equation (35), we obtain*

The condition	*(4)*	*(5)*
The criterion	$\varsigma_0 > 1839.2$	$\varsigma_0 > 59.5$

Using Theorem 2, Equation (35) is oscillatory if $\varsigma_0 > 57.5$.

Observe that, as shown in the table, the value of the condition $\varsigma_0 > 57.5$ *is smaller than the other values for the other conditions. Hence, the condition* $\varsigma_0 > 57.5$ *provides a better result than the results obtained by conditions (4) and (5) in [17,18]. However, these conditions for oscillation cannot be applied to examples where there is no delay term.*

Example 2. *Let the equation be*

$$\left(y\left(\xi'''(y)\right)\right)' + y\xi(ay) = 0, \ y \geq 1, \tag{36}$$

Let $r_1 = r_2 = 1, j(y) = y, \ z(y) = ay, \ \varsigma(y) = y$ and $a \in (0,1)$. Moreover, we see

$$\int_{y_0}^{\infty} \frac{1}{j^{1/r_1}(s)} \mathrm{d}s = \int_{y_0}^{\infty} \frac{ds}{s} = \infty.$$

It is easy to see that all conditions of Theorem 1 are satisfied. Hence, every solution of Equation (36) is oscillatory.

Example 3. *Consider the equation*

$$\xi^{(4)}(y) + \frac{\varsigma_0}{y^4}\xi\left(\frac{1}{2}y\right) = 0, \tag{37}$$

where $\varsigma_0 > 0$. *We note that* $r_1 = r_2 = 1, j(y) = 1, z_i(y) = y/2$ *and* $\varsigma(y) = \varsigma_0/y^4$. *Hence, it is easy to see that*

$$\phi_0 = \frac{\varsigma_0}{24y}$$

and

$$\varphi_0(y) = \frac{\varsigma_0}{2y}.$$

Using Theorem 2, Equation (37) is oscillatory if $\varsigma_0 > 36$.

Furthermore, we see that

$$\int^{\infty} \epsilon_1^{r_2 - r_1} \sum_{i=1}^{n} \varsigma_i(s) \frac{z_i^{3r_1}(s)}{s^{3r_1}} \mathrm{d}s \neq \infty,$$

and hence Theorem 1 fails.

4. Conclusions

In this manuscript, we are interested in studying the oscillation conditions of Equation (1). By the Riccati method, some new oscillation results are established, which extend and generalize related results in the literature. Two examples are given to clarify our results.

Additionally, in future work, we will contribute by providing more effective conditions for the oscillation of the equation

$$\left(j(y)\left(\xi'''(y)\right)^{r_1}\right)' + a(y)\left(\xi'''(y)\right)^{r_1} + \varsigma(y)f\left(\xi^{r_2}(z(y))\right) = 0,\ y \geq y_0.$$

under the condition $\int_{y_0}^{\infty} \frac{1}{j^{1/r_1}(s)} \mathrm{d}s < \infty$.

Author Contributions: Conceptualization, S.A., O.B., H.A. and J.A.; methodology, S.A., O.B., H.A. and J.A.; investigation, S.A., O.B., H.A. and J.A.; resources, S.A., O.B., H.A. and J.A.; data curation, S.A., O.B., H.A. and J.A.; writing—original draft preparation, S.A., O.B., H.A. and J.A.; writing—review and editing, S.A., O.B., H.A. and J.A.; supervision, S.A., O.B., H.A. and J.A.; project administration, S.A., O.B., H.A. and J.A.; funding acquisition, S.A., O.B., H.A. and J.A. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the Deanship of Scientific Research, Taif University, KSA. Research Project Number 1-441-139. This work has been supported by the Polish National Science Centre under the Grant OPUS 14 No. 2017/27/B/ST8/01330. This work has been supported by the Polish National Science Centre under the Grant OPUS 18 No. 2019/35/B/ST8/00980.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

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