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Identifying the Positive Definiteness of Even-Order Weakly Symmetric Tensors via Z-Eigenvalue Inclusion Sets

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Abstract: The positive definiteness of even-order weakly symmetric tensors plays important roles in asymptotic stability of time-invariant polynomial systems. In this paper, we establish two Brauer-type Z-eigenvalue inclusion sets with parameters by Z-identity tensors, and show that these inclusion sets are sharper than existing results. Based on the new Z-eigenvalue inclusion sets, we propose some sufficient conditions for testing the positive definiteness of even-order weakly symmetric tensors, as well as the asymptotic stability of time-invariant polynomial systems. The given numerical experiments are reported to show the efficiency of our results.

Keywords: Z-eigenvalue inclusion set; positive definiteness; asymptotic stability; weakly symmetric tensor

MSC: 15A18; 15A69

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1. Introduction

Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in (\mathbb{R}^n)^{\otimes m}$ be an m-th order n dimensional real square tensor and x be a real n-vector and $N = \{1, 2, \cdots, n\}$. Consider the following real n-vector:

$$(\mathcal{A}x^{m-1})_i = \sum_{i_2, \ldots, i_m \in N} a_{i i_2 \cdots i_m} x_{i_2} \ldots x_{i_m}.$$

If there exists a unimodular vector $x \in \mathbb{R}^n$ and a real number λ such that

$$\mathcal{A}x^{m-1} = \lambda x,$$

then λ is called a Z-eigenvalue of $\mathcal{A}$ and x is called a Z-eigenvector of $\mathcal{A}$ associated with λ [1,2].

Z-eigenvalue problems of tensors were constantly emerging due to their wide applications in medical resonance [3,4], spectral hypergraph theory [5,6], automatic control [7,8] and machine learning [9]. Some effective algorithms for finding Z-eigenvalues and the corresponding eigenvectors have been proposed [5,10–15]. However, it is difficult to compute all Z-eigenvalues, even the smallest Z-eigenvalue when m and n are large [16,17]. Thus, many researchers turned to investigating Z-eigenvalue inclusion sets [10,18–22]. Later, Qi et al. [13] investigated Z-eigenvalues to identify the positive definiteness of a degree m with n variables homogeneous polynomials with unit constraint:

$$\begin{aligned} f_{\mathcal{A}}(x) = \mathcal{A}x^m = & \sum_{i_1, i_2, \ldots, i_m \in N} a_{i_1 i_2 \ldots i_m} x_{i_1} x_{i_2} \ldots x_{i_m} \\ s.t. \quad & x^\top x = 1. \end{aligned} \tag{1}$$

We say $f_{\mathcal{A}}(x)$ is positive definite if $f_{\mathcal{A}}(x) > 0$ for all $x^\top x = 1$. Note that $f_{\mathcal{A}}(x)$ is positive definite if and only if the even-order symmetric tensor $\mathcal{A}$ is positive definite, and $\mathcal{A}$ is positive definite if and only if its Z-eigenvalues are positive [13]. Unfortunately, the mentioned inclusion sets always include zero and could not be used to identify the positive definiteness of even-order tensors. Recently, several significant results have arisen to solve the problem of deciding positive-definiteness of an even-order symmetric tensor based on their special structure [23–25]. For more general cases, such as even-order real symmetric tensors, Li et al. [26] proposed Gershgorin-type Z-eigenvalue inclusion sets with parameters, and identified the positive-definiteness. It is worth noting that the symmetry of a tensor is a relatively strict condition, and Brauer-type inclusion sets are tighter than Gershgorin-type inclusion sets [20]. Therefore, it is necessary to establish new Brauer-type Z-eigenvalue inclusion sets to exactly characterize Z-eigenvalues and test the positive definiteness of even-order real weakly symmetric tensors.

As we know in [7,8,26–28], Z-eigenvalue problems also play a fundamental role in the time-invariant polynomial systems with unit constraint:

$$\sum: \begin{array}{l} \dot{x} = \mathcal{A}^{(2)}x + \mathcal{A}^{(4)}x^3 + \ldots + \mathcal{A}^{(2k)}x^{2k-1}, \\ s.t. \quad x^\top x = 1, \end{array} \tag{2}$$

where $\mathcal{A}^{(t)} = (a_{i_1 i_2 \ldots i_t}) \in (\mathbb{R}^n)^{\otimes t} (t = 2, 4, \ldots, 2k)$. Deng et al. [27] investigated asymptotical stability of time-invariant polynomial systems without constraint via Lyapunov's method. Further, Li et al. [26] established asymptotical stability of systems (2) based on Gershgorin-type Z-eigenvalue inclusion sets. So, we want to exactly verify the asymptotical stability of time-invariant polynomial systems use of Brauer-type Z-eigenvalue inclusion sets, which constitutes the second motivation of the paper.

This paper is organized as follows. In Section 2, important definitions and preliminary results are recalled. In Section 3, two Brauer-type Z-eigenvalue inclusion sets with parameters are established. In Section 4, some sufficient conditions are proposed for identifying positive definiteness of even-order weakly symmetric tensors and asymptotic stability of time-invariant polynomial systems.

2. Preliminaries

In this section, we introduce some definitions and important properties related to Z-eigenvalues of a tensor [12,13,26].

Definition 1. *Let $\mathcal{A}$ be m-order n-dimensional tensors.*

(i) We define $\sigma_Z(\mathcal{A})$ as the set of all Z-eigenvalues of $\mathcal{A}$. Assume $\sigma_Z(\mathcal{A}) \neq \emptyset$. Then, the Z-spectral radius of $\mathcal{A}$ is denoted by

$$\rho(\mathcal{A}) = \max\{|\lambda| : \lambda \in \sigma(\mathcal{A})\}.$$

(ii) We say that $\mathcal{A}$ is symmetric if

$$a_{i_{p(1)} \ldots i_{p(m)}} = a_{i_1 \ldots i_m}, \ for \ all \ i_1, \ldots, i_m \in N, \forall p \in II_m,$$

where II_m is the permutation group of m indices.

(iii) We say that $\mathcal{A}$ is weakly symmetric if the associated homogeneous polynomial $\mathcal{A}x^m$ satisfies

$$\nabla \mathcal{A}x^m = m\mathcal{A}x^{m-1},$$

where ∇ is the differential operator.

Obviously, if tensor $\mathcal{A}$ is symmetric, then $\mathcal{A}$ is weakly symmetric. However, the converse result may not hold.

Based on variational property of weakly symmetric tensors given in [10], we establish the following result.

Lemma 1. *Let $\mathcal{A} \in (\mathbb{R}^n)^{\otimes m}$ be a weakly symmetric tensor. Then, $f_{\mathcal{A}}(x) = \mathcal{A}x^m$ is positive definite if and only if its Z-eigenvalues are positive.*

Proof. Since $\mathcal{A}$ is a weakly symmetric tensor, we have

$$f_{\mathcal{A}}(x) = \mathcal{A}x^m = \frac{1}{m}\langle \nabla f_{\mathcal{A}}(x), x\rangle = \langle \mathcal{A}x^{m-1}, x\rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product. Thus, we observe that $\lambda \in \sigma_Z(\mathcal{A})$ if and only if λ is a critical value of $\mathcal{A}x^m$ on the standard unit sphere. Letting (λ, x) be a Z-eigenpair of $\mathcal{A}$, one has

$$f_{\mathcal{A}}(x) = \langle \mathcal{A}x^{m-1}, x\rangle = \langle \lambda x, x\rangle = \lambda.$$

Hence, $f_{\mathcal{A}}(x) > 0$ if and only if $\lambda > 0$. The conclusion follows. □

A Z-identity tensor was introduced by [2,12] to propose a shifted power method for computing tensor Z-eigenpairs and investigate a generalization of the characteristic polynomial for symmetric even-order tensors, respectively.

Definition 2. *Assume that m is even. We call I_Z a Z-identity tensor if*

$$I_Z x^{m-1} = x, \text{for all unimodular vector } x \in \mathbb{R}^n.$$

Note that there is no Z-identity tensor for m odd [12]. Meanwhile, Z-identity tensor is not unique in general. For instance, each even tensor in the following is a Z-identity tensor:
Case I: $(I_Z)_{iii_2i_2\ldots i_ki_k} = 1, \forall k \in N$ and $m = 2k$;
Case II: (Property 2.4 of [12]): $(I_Z)_{i_1i_2\ldots i_m} = \frac{1}{m!}\sum\limits_{p\in II_m} \delta_{i_{p(1)}i_{p(2)}}\ldots\delta_{i_{p(m-1)}i_{p(m)}}$, where δ is the standard Kronecker, i.e.,

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases}$$

To end this section, we introduce the results given in [26].

Lemma 2 (Theorem 2 of [26])**.** *Let $\mathcal{A} = (a_{i_1i_2\ldots i_m}) \in (\mathbb{R}^n)^{\otimes m}$ and $I_Z \in (\mathbb{R}^n)^{\otimes m}$ be a Z-identity tensor with m being even. For any real vector $\alpha = (\alpha_1, \ldots, \alpha_n)^\top \in \mathbb{R}^n$, then*

$$\sigma_Z(\mathcal{A}) \subseteq \mathcal{G}(\mathcal{A}, \alpha) = \bigcup_{i\in N} \mathcal{G}_i(\mathcal{A}, \alpha) := \{z \in \mathbb{R} : |z - \alpha_i| \le R_i(\mathcal{A}, \alpha_i)\},$$

where $R_i(\mathcal{A}, \alpha_i) = \sum\limits_{i_2,\ldots,i_m\in N} |a_{ii_2\ldots i_m} - \alpha_i(I_Z)_{ii_2\ldots i_m}|$. Further, $\sigma_Z(\mathcal{A}) \subseteq \bigcap\limits_{\alpha\in\mathbb{R}^n} \mathcal{G}(\mathcal{A}, \alpha)$.

3. Sharp Z-Eigenvalue Inclusion Sets for Even Tensors

In this section, we establish Brauer-type Z-eigenvalue inclusion sets and give comparisons among different Z-eigenvalue inclusion sets for even-order tensors.

Theorem 1. *Let $\mathcal{A} = (a_{i_1i_2\ldots i_m}) \in (\mathbb{R}^n)^{\otimes m}$ and $I_Z \in (\mathbb{R}^n)^{\otimes m}$ be a Z-identity tensor. For any real vector $\alpha = (\alpha_1, \ldots, \alpha_n)^\top \in \mathbb{R}^n$, then*

$$\sigma_Z(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{A}, \alpha) = \bigcup_{i\in N} \bigcap_{j\in N, j\neq i} \mathcal{L}_{i,j}(\mathcal{A}, \alpha),$$

where $\mathcal{L}_{i,j}(\mathcal{A},\alpha) = \{z \in \mathbb{R} : (|z-\alpha_i| - R_i^j(\mathcal{A},\alpha_i))|z-\alpha_j| \leq |a_{ij\ldots j} - \alpha_i(I_Z)_{ij\ldots j}|R_j(\mathcal{A},\alpha_j)\}$ *and* $R_i^j(\mathcal{A},\alpha_i) = R_i(\mathcal{A},\alpha_i) - |a_{ij\ldots j} - \alpha_i(I_Z)_{ij\ldots j}|$. *Further,* $\sigma_Z(\mathcal{A}) \subseteq \bigcap\limits_{\alpha\in\mathbb{R}^n} \mathcal{L}(\mathcal{A},\alpha)$.

Proof. Let (λ, x) be a Z-eigenpair of $\mathcal{A}$ and $I_Z \in (\mathbb{R}^n)^{\otimes m}$ be a Z-identity tensor, i.e.,

$$\mathcal{A}x^{m-1} = \lambda x = \lambda I_Z x^{m-1} \ \ and \ \ x^\top x = 1. \tag{3}$$

Assume that $|x_t| = \max\limits_{i\in N}|x_i|$, then $0 < |x_t|^{m-1} \leq |x_t| \leq 1$. From (3), we have

$$\sum_{i_2,\ldots,i_m\in N} \lambda(I_Z)_{ti_2\ldots i_m}x_{i_2}\ldots x_{i_m} = \sum_{i_2,\ldots,i_m\in N} a_{ti_2\ldots i_m}x_{i_2}\ldots x_{i_m}.$$

Hence, for any real number α_t, it holds that

$$\begin{aligned}(\lambda-\alpha_t)x_t &= \sum_{i_2,\ldots,i_m\in N}(\lambda-\alpha_t)(I_Z)_{ti_2\ldots i_m}x_{i_2}\ldots x_{i_m}\\ &= \sum_{i_2,\ldots,i_m\in N}\big(a_{ti_2\ldots i_m} - \alpha_t(I_Z)_{ti_2\ldots i_m}\big)x_{i_2}\ldots x_{i_m}\\ &= \sum_{\delta_{ji_2,\ldots,i_m}=0}\big(a_{ti_2\ldots i_m} - \alpha_t(I_Z)_{ti_2\ldots i_m}\big)x_{i_2}\ldots x_{i_m}\\ &+ \big(a_{tj\ldots j} - \alpha_t(I_Z)_{tj\ldots j}\big)x_j^{m-1}.\end{aligned} \tag{4}$$

Taking modulus in (4) and using the triangle inequality give

$$\begin{aligned}|\lambda-\alpha_t||x_t| &\leq \sum_{\delta_{ji_2,\ldots,i_m}=0}|a_{ti_2\ldots i_m} - \alpha_t(I_Z)_{ti_2\ldots i_m}||x_{i_2}|\ldots|x_{i_m}|\\ &+ |a_{tj\ldots j} - \alpha_t(I_Z)_{tj\ldots j}||x_j|^{m-1}\\ &\leq R_t^j(\mathcal{A},\alpha_t)|x_t| + |a_{tj\ldots j} - \alpha_t(I_Z)_{tj\ldots j}||x_j|,\end{aligned}$$

i.e.,

$$(|\lambda-\alpha_t| - R_t^j(\mathcal{A},\alpha_t))|x_t| \leq |a_{tj\ldots j} - \alpha_t(I_Z)_{tj\ldots j}||x_j|. \tag{5}$$

If $|x_j| = 0$, by (5), we deduce $|\lambda-\alpha_t| - R_t^j(\mathcal{A},\alpha_t) \leq 0$. Thus, $\lambda \in \mathcal{L}_{t,j}(\mathcal{A},\alpha) \subseteq \mathcal{L}(\mathcal{A},\alpha)$. Otherwise, $|x_j| > 0$. For any $j \in N, j \neq t$ and any real number α_j, we obtain

$$(\lambda-\alpha_j)x_j = \sum_{i_2,\ldots,i_m\in N}\big(a_{ji_2\ldots i_m} - \alpha_j(I_Z)_{ji_2\ldots i_m}\big)x_{i_2}\ldots x_{i_m},$$

which implies

$$|\lambda-\alpha_j||x_j| \leq R_j(\mathcal{A},\alpha_j)|x_t|. \tag{6}$$

Multiplying (5) with (6) yields

$$(|\lambda-\alpha_t| - R_t^j(\mathcal{A},\alpha_t))|\lambda-\alpha_j||x_j||x_t| \leq |a_{tj\ldots j} - \alpha_t(I_Z)_{tj\ldots j}|R_j(\mathcal{A},\alpha_j)|x_j||x_t|,$$

i.e.,

$$(|\lambda-\alpha_t| - R_t^j(\mathcal{A},\alpha_t))|\lambda-\alpha_j| \leq |a_{tj\ldots j} - \alpha_t(I_Z)_{tj\ldots j}|R_j(\mathcal{A},\alpha_j),$$

which implies $\lambda \in \mathcal{L}_{t,j}(\mathcal{A},\alpha)$. Because of $\lambda \in \mathcal{L}_{t,j}(\mathcal{A},\alpha)$ for all j, it follows that λ lives in the intersection and hence $\lambda \in \bigcap\limits_{j\in N, j\neq t}\mathcal{L}_{t,j}(\mathcal{A},\alpha)$. Further, $\lambda \in \bigcup\limits_{i\in N}\bigcap\limits_{j\in N, j\neq i}\mathcal{L}_{i,j}(\mathcal{A},\alpha)$ and $\sigma_Z(\mathcal{A}) \subseteq \bigcap\limits_{\alpha\in\mathbb{R}^n}\mathcal{L}(\mathcal{A},\alpha)$. □

Next, we show $\mathcal{L}(\mathcal{A},\alpha) \subseteq \mathcal{G}(\mathcal{A},\alpha)$.

Corollary 1. *Let $\mathcal{A} = (a_{i_1 i_2 \ldots i_m}) \in (\mathbb{R}^n)^{\otimes m}$ and $I_Z \in (\mathbb{R}^n)^{\otimes m}$ be a Z-identity tensor. For any real vector $\alpha = (\alpha_1, \ldots, \alpha_n)^\top \in \mathbb{R}^n$, then*

$$\sigma_Z(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{A}, \alpha) \subseteq \mathcal{G}(\mathcal{A}, \alpha),$$

where $\mathcal{G}(\mathcal{A}, \alpha)$ is defined in Lemma 2.

Proof. For any $\lambda \in \mathcal{L}(\mathcal{A}, \alpha)$, without loss of generality, there exists $t \in N$ with any $j \neq t$ such that $\lambda \in \mathcal{L}_{t,j}(\mathcal{A}, \alpha)$, that is,

$$(|\lambda - \alpha_t| - R_t^j(\mathcal{A}, \alpha_t))|\lambda - \alpha_j| \leq |a_{tj\ldots j} - \alpha_t (I_Z)_{tj\ldots j}| R_j(\mathcal{A}, \alpha_j), \forall j \neq t. \tag{7}$$

If $|a_{tj\ldots j} - \alpha_t (I_Z)_{tj\ldots j}| R_j(\mathcal{A}, \alpha_j) = 0$, then $\lambda = \alpha_j$ or $|\lambda - \alpha_t| - R_t^j(\mathcal{A}, \alpha_t) \leq 0$. Hence, $\lambda \in \mathcal{G}_t(\mathcal{A}, \alpha) \bigcup \mathcal{G}_j(\mathcal{A}, \alpha)$. Otherwise, it follows from (7) that

$$\frac{|\lambda - \alpha_t| - R_t^j(\mathcal{A}, \alpha_t)}{|a_{tj\ldots j} - \alpha_t (I_Z)_{tj\ldots j}|} \times \frac{|\lambda - \alpha_j|}{R_j(\mathcal{A}, \alpha_j)} \leq 1.$$

Further,

$$\frac{|\lambda - \alpha_t| - R_t^j(\mathcal{A}, \alpha_t)}{|a_{tj\ldots j} - \alpha_t (I_Z)_{tj\ldots j}|} \leq 1$$

or

$$\frac{|\lambda - \alpha_j|}{R_j(\mathcal{A}, \alpha_j)} \leq 1,$$

which shows $\lambda \in \mathcal{G}_t(\mathcal{A}, \alpha) \bigcup \mathcal{G}_j(\mathcal{A}, \alpha) \subseteq \mathcal{G}(\mathcal{A}, \alpha)$. Thus, $\mathcal{L}(\mathcal{A}, \alpha) \subseteq \mathcal{G}(\mathcal{A}, \alpha)$. □

By classifying the index set, we can get an accurate characterization for $\sigma(\mathcal{A})$. Define

$$\Theta_j = \{(i_2, i_3, \ldots, i_m) : i_k = j \text{ for some } k \in \{2, \ldots, m\}, \text{where } j, i_2, \ldots, i_m \in N\},$$

$$\overline{\Theta}_j = \{(i_2, i_3, \ldots, i_m) : i_k \neq j \text{ all any } k \in \{2, \ldots, m\}, \text{where } j, i_2, \ldots, i_m \in N\},$$

$$r_i^{\Theta_j}(\mathcal{A}, \alpha_i) = \sum_{\{i_2, \ldots, i_m\} \in \Theta_j} |a_{i i_2 \ldots i_m} - \alpha_i (I_Z)_{i i_2 \ldots i_m}|,$$

$$r_i^{\overline{\Theta}_j}(\mathcal{A}, \alpha_i) = \sum_{\{i_2, \ldots, i_m\} \in \overline{\Theta}_j} |a_{i i_2 \ldots i_m} - \alpha_i (I_Z)_{i i_2 \ldots i_m}|.$$

Obviously, $R_i(\mathcal{A}, \alpha_i) = r_i^{\Theta_j}(\mathcal{A}, \alpha_i) + r_i^{\overline{\Theta}_j}(\mathcal{A}, \alpha_i)$.

Theorem 2. *Let $\mathcal{A} = (a_{i_1 i_2 \ldots i_m}) \in (\mathbb{R}^n)^{\otimes m}$ and $I_Z \in (\mathbb{R}^n)^{\otimes m}$ be a Z-identity tensor. For any real vector $\alpha = (\alpha_1, \ldots, \alpha_n)^\top \in \mathbb{R}^n$, then*

$$\sigma_Z(\mathcal{A}) \subseteq \mathcal{N}(\mathcal{A}, \alpha) = \bigcup_{i,j \in N, j \neq i} (\mathcal{N}_{i,j}(\mathcal{A}, \alpha) \bigcup \mathcal{H}_{i,j}(\mathcal{A}, \alpha)),$$

where $\mathcal{N}_{i,j}(\mathcal{A}, \alpha) = \{z \in \mathbb{R} : (|z - \alpha_i| - R_i^j(\mathcal{A}, \alpha_i))(|z - \alpha_j| - r_j^{\overline{\Theta}_i}(\mathcal{A}, \alpha_j)) \leq |a_{ij\ldots j} - \alpha_i (I_Z)_{ij\ldots j}| r_j^{\Theta_i}(\mathcal{A}, \alpha_j)\}$, and $\mathcal{H}_{i,j}(\mathcal{A}, \alpha) = \{z \in \mathbb{R} : |z - \alpha_i| - R_i^j(\mathcal{A}, \alpha_i) < 0$ and $|z - \alpha_j| < r_j^{\overline{\Theta}_i}(\mathcal{A}, \alpha_j)\}$. Further, $\sigma_Z(\mathcal{A}) \subseteq \bigcap_{\alpha \in \mathbb{R}^n} \mathcal{N}(\mathcal{A}, \alpha)$.

Proof. Let (λ, x) be a Z-eigenpair of $\mathcal{A}$ and $I_Z \in (\mathbb{R}^n)^{\otimes m}$ be a Z-identity tensor, i.e.,

$$\mathcal{A} x^{m-1} = \lambda x = \lambda I_Z x^{m-1} \text{ with } x^\top x = 1.$$

Assume that $|x_t| \geq |x_s| \geq \max_{k\in N}|x_k|$, then $|x_t| > 0$. For $t, s \in N, s \neq t$, following the characterization of (5), we have

$$(|\lambda - \alpha_t| - R_t^s(\mathcal{A}, \alpha_t))|x_t| \leq |a_{ts\ldots s} - \alpha_t(I_Z)_{ts\ldots s}||x_s|. \tag{8}$$

If $|x_s| = 0$, then $|\lambda - \alpha_t| - R_t^s(\mathcal{A}, \alpha_t) \leq 0$ and $\lambda \in \mathcal{N}_{t,s}(\mathcal{A}, \alpha)$.
Otherwise, $|x_s| > 0$. In virtue of $0 \leq |x_i| \leq 1$ for $i \in N$, it holds that

$$\begin{aligned}
|\lambda - \alpha_s||x_s| &= |\sum_{i_2,\ldots,i_m\in N}(\lambda - \alpha_s)(I_Z)_{si_2\ldots i_m}x_{i_2}\ldots x_{i_m}| \\
&\leq \sum_{i_2,\ldots,i_m\in\Theta_t}|a_{si_2\ldots i_m} - \alpha_s(I_Z)_{si_2\ldots i_m}||x_{i_2}|\ldots|x_{i_m}| \\
&+ \sum_{i_2,\ldots,i_m\in\overline{\Theta}_t}|a_{si_2\ldots i_m} - \alpha_s(I_Z)_{si_2\ldots i_m}||x_{i_2}|\ldots|x_{i_m}| \\
&\leq \sum_{i_2,\ldots,i_m\in\Theta_t}|a_{si_2\ldots i_m} - \alpha_s(I_Z)_{si_2\ldots i_m}||x_t| + \sum_{i_2,\ldots,i_m\in\overline{\Theta}_t}|a_{si_2\ldots i_m} - \alpha_s(I_Z)_{si_2\ldots i_m}||x_s|.
\end{aligned} \tag{9}$$

Thus,

$$(|\lambda - \alpha_s| - r_s^{\overline{\Theta}_t}(\mathcal{A}, \alpha_s))|x_s| \leq r_s^{\Theta_t}(\mathcal{A}, \alpha_s)||x_t|. \tag{10}$$

The following argument is divided into two cases.

Case 1. Either $|\lambda - \alpha_t| - R_t^s(\mathcal{A}, \alpha_t) \geq 0$ or $|\lambda - \alpha_s| - r_s^{\overline{\Theta}_t}(\mathcal{A}, \alpha_s) \geq 0$. Multiplying inequalities (8) with (10), we obtain

$$(|\lambda - \alpha_t| - R_t^s(\mathcal{A}, \alpha_t))(|\lambda - \alpha_s| - r_s^{\overline{\Theta}_t}(\mathcal{A}, \alpha_s)) \leq |a_{ts\ldots s} - \alpha_t(I_Z)_{ts\ldots s}|r_s^{\Theta_t}(\mathcal{A}, \alpha_s),$$

which shows $\lambda \in \mathcal{N}_{t,s}(\mathcal{A}, \alpha) \subseteq \mathcal{N}(\mathcal{A}, \alpha)$.

Case 2. $|\lambda - \alpha_t| - R_t^s(\mathcal{A}, \alpha_t) < 0$ and $|\lambda - \alpha_s| - r_s^{\overline{\Theta}_t}(\mathcal{A}, \alpha_s) < 0$. We obtain $\lambda \in \mathcal{H}_{t,s}(\mathcal{A}, \alpha) \subseteq \mathcal{N}(\mathcal{A}, \alpha)$.

Summing up the above two situations, we draw the conclusion. □

Now, we show the set $\mathcal{N}(\mathcal{A}, \alpha)$ is tighter than $\mathcal{G}(\mathcal{A}, \alpha)$.

Corollary 2. *Let $\mathcal{A} = (a_{i_1i_2\ldots i_m}) \in (\mathbb{R}^n)^{\otimes m}$ and $I_Z \in (\mathbb{R}^n)^{\otimes m}$ be a Z-identity tensor. For any real vector $\alpha = (\alpha_1, \ldots, \alpha_n)^\top \in \mathbb{R}^n$, then*

$$\sigma_z(\mathcal{A}) \in \mathcal{N}(\mathcal{A}, \alpha) \subseteq \mathcal{G}(\mathcal{A}, \alpha).$$

Proof. For any $\lambda \in \mathcal{N}(\mathcal{A}, \alpha)$, we break the proof into two parts.

Case 1. There exist $t, s \in N$ with $s \neq t$ such that $\lambda \in \mathcal{N}_{t,s}(\mathcal{A}, \alpha)$, that is,

$$(|\lambda - \alpha_t| - R_t^s(\mathcal{A}, \alpha_t))(|\lambda - \alpha_s| - r_s^{\overline{\Theta}_t}(\mathcal{A}, \alpha_s)) \leq |a_{ts\ldots s} - \alpha_t(I_Z)_{ts\ldots s}|r_s^{\Theta_t}(\mathcal{A}, \alpha_s). \tag{11}$$

If $|a_{ts\ldots s} - \alpha_t(I_Z)_{ts\ldots s}|r_s^{\Theta_t}(\mathcal{A}, \alpha_s) = 0$, then

$$|\lambda - \alpha_s| \leq r_s^{\overline{\Theta}_t}(\mathcal{A}, \alpha_s) \leq R_s(\mathcal{A}, \alpha_s)$$

or

$$|\lambda - \alpha_t| \leq R_t^s(\mathcal{A}, \alpha_t) \leq R_t(\mathcal{A}, \alpha_t),$$

which shows $\lambda \in \mathcal{G}_t(\mathcal{A}, \alpha) \bigcup \mathcal{G}_s(\mathcal{A}, \alpha)$.

Otherwise, $|a_{ts\ldots s} - \alpha_t(I_Z)_{ts\ldots s}|r_s^{\Theta_t}(\mathcal{A}, \alpha_s) > 0$. Then, (11) entails

$$\frac{(|\lambda - \alpha_t| - R_t^s(\mathcal{A}, \alpha_t))}{|a_{ts\ldots s} - \alpha_t(I_Z)_{ts\ldots s}|} \times \frac{|\lambda - \alpha_s| - r_s^{\overline{\Theta}_t}(\mathcal{A}, \alpha_s)}{r_s^{\Theta_t}(\mathcal{A}, \alpha_s)} \leq 1.$$

Further,

$$\frac{|\lambda - \alpha_t| - R_t^s(\mathcal{A}, \alpha_t)}{|a_{ts\ldots s} - \alpha_t (I_Z)_{ts\ldots s}|} \leq 1$$

or

$$\frac{|\lambda - \alpha_s| - r_s^{\overline{\Theta}_t}(\mathcal{A}, \alpha_s)}{r_s^{\Theta_t}(\mathcal{A}, \alpha_s)} \leq 1,$$

which implies $\lambda \in \mathcal{G}_t(\mathcal{A}, \alpha) \bigcup \mathcal{G}_s(\mathcal{A}, \alpha) \subseteq \mathcal{G}(\mathcal{A}, \alpha)$.

Case 2. There exist $t, s \in N$ with $s \neq t$ such that $\lambda \in \mathcal{H}_{t,s}(\mathcal{A}, \alpha)$, that is

$$|\lambda - \alpha_t| - R_t^s(\mathcal{A}, \alpha_t) < 0 \text{ and } |\lambda - \alpha_s| - r_s^{\overline{\Theta}_t}(\mathcal{A}, \alpha_s) < 0.$$

Obviously, $\lambda \in \mathcal{G}_t(\mathcal{A}, \alpha) \bigcap \mathcal{G}_j(\mathcal{A}, \alpha) \subseteq \mathcal{G}(\mathcal{A}, \alpha)$. □

Next, we give a numerical comparison among Theorems 1 and 2 and existing results.

Example 1. *Consider a* 4 *order* 2 *dimensional tensor* $\mathcal{A} = (a_{ijkl})$ *defined by*

$$a_{ijkl} = \begin{cases} a_{1111} = 10; a_{1122} = 9; a_{1121} = a_{1222} = -1; \\ a_{2222} = 5; a_{2211} = 6; a_{2122} = a_{2111} = -1; \\ a_{ijkl} = 0, \ \textit{otherwise}. \end{cases}$$

By simple computation, all Z-eigenvalues of $\mathcal{A}$ are 5.0000 and 10.0000. Taking positive vector $\alpha = [10, 7]^\top$ and I_Z as follows:

$$I_{ijkl} = \begin{cases} I_{1111} = I_{1122} = I_{2211} = I_{2222} = 1; \\ I_{ijkl} = 0, \ \textit{otherwise}, \end{cases}$$

We compute Table 1 to show the comparisons different methods with our results.

Table 1. Comparisons among different methods.

References	Inclusion Set
Theorem 3.1 of [20]	$[-21, 21]$
Theorems 3.2–3.3 of [20]	$[-20.6301, 20.6301]$
Theorem 7 of [18]	$[-20.6301, 20.6301]$
Theorem 2 of [26]	$[2, 13]$
Theorem 1	$[2.595, 12.851]$
Theorem 2	$[2.595, 12.791]$

Numerical results show that Theorems 1 and 2 are tighter than existing results.

In the following, setting $\alpha_1 = [10, 7]^\top, \alpha_2 = [9, 5]^\top$ and $\alpha_3 = [9, 5.5]^\top$, we obtain inclusion sets by different theorems. Consequently, the parameter α has a great influence on the numerical effects. The parameter α has a great influence on the numerical effects from Table 2.

Table 2. The effect of parameters on the inclusion set.

	Inclusion Set	Inclusion Set	Inclusion Set
	Theorem 2 of [26]	**Theorem 1**	**Theorem 2**
$\alpha = [10, 7]^\top$	$[2, 13]$	$[2.595, 12.851]$	$[2.595, 12.791]$
$\alpha = [9, 5]^\top$	$[2, 12]$	$[2.522, 11.462]$	$[2.618, 11.462]$
$\alpha = [9, 5.5]^\top$	$[2.5, 12]$	$[3, 11.5]$	$[3.541, 11.5]$

4. Positive Definiteness of even Order Weakly Symmetric Tensors and Asymptotic Stability of Polynomial Systems

In this section, based on the inclusion sets $\mathcal{L}(\mathcal{A},\alpha)$ and $\mathcal{N}(\mathcal{A},\alpha)$ in Theorems 1 and 2, we propose some sufficient conditions for the positive definiteness of weakly symmetric tensors, as well as the asymptotic stability of time-invariant polynomial systems.

4.1. Positive Definiteness of even Order Weakly Symmetric Tensors

Li et al. [26] proposed the following theorem to test the positive definiteness of polynomial systems via Gershgorin-type Z-eigenvalue inclusion sets.

Lemma 3 (Theorem 3.2 of [26]). *Let $\mathcal{A}=(a_{i_1i_2\ldots i_m})\in(\mathbb{R}^n)^{\otimes m}$ and λ be a Z-eigenvalue of $\mathcal{A}$. If there exists a positive real vector $\alpha=(\alpha_1,\ldots,\alpha_n)^\top$ such that*

$$\alpha_i > R_i(\mathcal{A},\alpha_i), \forall i\in N,$$

then $\lambda>0$. Further, if $\mathcal{A}$ is symmetric, then $\mathcal{A}$ is positive definite and $f_{\mathcal{A}}(x)$ defined in (1) is positive definite.

Theorem 3. *Let λ be a Z-eigenvalue of $\mathcal{A}=(a_{i_1i_2\ldots i_m})\in(\mathbb{R}^n)^{\otimes m}$ and $I_Z\in(\mathbb{R}^n)^{\otimes m}$ be a Z-identity tensor. If there exists a positive real vector $\alpha=(\alpha_1,\ldots,\alpha_n)^\top$ and $i,j\in N$ with $j\neq i$ such that*

$$(\alpha_i - R_i^j(\mathcal{A},\alpha_i))\alpha_j > |a_{ij\ldots j}-\alpha_i(I_Z)_{ij\ldots j}|R_j(\mathcal{A},\alpha_j), \tag{12}$$

then $\lambda>0$. Further, if $\mathcal{A}$ is weakly symmetric, then $\mathcal{A}$ is positive definite and $f_{\mathcal{A}}(x)$ defined in (1) is positive definite.

Proof. Suppose on the contrary that $\lambda\leq 0$. From Theorem 1, there exists $t\in N$ with any $j\neq t$ such that $\lambda\in\mathcal{L}_{t,j}(\mathcal{A},\alpha)$, i.e.,

$$(|\lambda-\alpha_t|-R_t^j(\mathcal{A},\alpha_t))|\lambda-\alpha_j|\leq|a_{tj\ldots j}-\alpha_t(I_Z)_{tj\ldots j}|R_j(\mathcal{A},\alpha_j),\forall j\neq t.$$

Further, it follows from $\alpha_t>0$ and $\lambda\leq 0$ that

$$(\alpha_t - R_t^j(\mathcal{A},\alpha_t))\alpha_j\leq|a_{ij\ldots j}-\alpha_t(I_Z)_{tj\ldots j}|R_j(\mathcal{A},\alpha_j),\forall j\neq t,$$

which contradicts (12). Thus, $\lambda>0$. When $\mathcal{A}$ is a weakly symmetric tensor and all Z-eigenvalues are positive, we obtain that $\mathcal{A}$ is positive definite and $f_{\mathcal{A}}(x)$ defined in (1) is positive definite by Lemma 1. □

Theorem 4. *Let λ be a Z-eigenvalue of $\mathcal{A}=(a_{i_1i_2\ldots i_m})\in(\mathbb{R}^n)^{\otimes m}$ and $I_Z\in(\mathbb{R}^n)^{\otimes m}$ be a Z-identity tensor. If there exist positive real vector $\alpha=(\alpha_1,\ldots,\alpha_n)^\top$ such that*

$$(\alpha_i - R_i^j(\mathcal{A},\alpha_i))(\alpha_j - r_j^{\overline{\Theta}_i}(\mathcal{A},\alpha_j)) > |a_{ij\ldots j}-\alpha_i(I_Z)_{ij\ldots j}|r_j^{\Theta_i}(\mathcal{A},\alpha_j),\forall i,j\in N, j\neq i \tag{13}$$

and

$$\alpha_i > R_i^j(\mathcal{A},\alpha_i)\ \text{ and }\ \alpha_j > r_j^{\overline{\Theta}_i}(\mathcal{A},\alpha_j),\forall i,j\in N, j\neq i, \tag{14}$$

then $\lambda>0$. Further, if $\mathcal{A}$ is weakly symmetric, then $\mathcal{A}$ is positive definite and $f_{\mathcal{A}}(x)$ defined in (1) is positive definite.

Proof. Suppose on the contrary that $\lambda\leq 0$. The following argument is divided into two cases.

Case 1. There exist $t,s\in N$ with $s\neq t$ such that $\lambda\in\mathcal{N}_{t,s}(\mathcal{A},\alpha)$, i.e.,

$$(|\lambda-\alpha_t|-R_t^s(\mathcal{A},\alpha_t))(|\lambda-\alpha_s|-r_s^{\overline{\Theta}_t}(\mathcal{A},\alpha_s))\leq|a_{ts\ldots s}-\alpha_t(I_Z)_{ts\ldots s}|r_s^{\Theta_t}(\mathcal{A},\alpha_s),\forall s\neq t.$$

Further, it follows from $\alpha_s > 0, \alpha_t > 0$ and $\lambda \leq 0$ that

$$(\alpha_t - R_t^s(\mathcal{A}, \alpha_t))(\alpha_s - r_s^{\overline{\Theta}_t}(\mathcal{A}, \alpha_s)) \leq |a_{ts\ldots s} - \alpha_t (I_Z)_{ts\ldots s}| r_s^{\Theta_t}(\mathcal{A}, \alpha_s),$$

which contradicts (13). Thus, $\lambda > 0$.

Case 2. There exist $t, s \in N$ with $s \neq t$ such that $\lambda \in \mathcal{H}_{t,s}(\mathcal{A}, \alpha)$, i.e.,

$$|\lambda - \alpha_t| < R_t^s(\mathcal{A}, \alpha_t) \ \ and \ \ |\lambda - \alpha_s| < r_s^{\overline{\Theta}_t}(\mathcal{A}, \alpha_s).$$

Further, it follows from $\alpha_s > 0, \alpha_t > 0$ and $\lambda \leq 0$ that

$$\alpha_t < R_t^s(\mathcal{A}, \alpha_t) \ \ and \ \ \alpha_s < r_s^{\overline{\Theta}_t}(\mathcal{A}, \alpha_s),$$

which contradicts (14). Thus, $\lambda > 0$.

From the above two cases, when $\mathcal{A}$ is a weakly symmetric tensor and all Z-eigenvalues are positive, we obtain that $\mathcal{A}$ is positive definite and $f_{\mathcal{A}}(x)$ defined in (1) is positive definite from Lemma 1. □

Remark 1. *Compared with Theorem 3.2 of [26], our conclusions can more accurately determine the positive definiteness for even order weakly symmetric tensors.*

The following example reveals that Theorems 3 and 4 can judge the positive definiteness of weakly symmetric tensors.

Example 2. *Let $\mathcal{A} = (a_{ijkl}) \in (\mathbb{R}^2)^{\otimes 4}$ be a tensor with elements defined as follows:*

$$\begin{aligned}
&a_{1111} = 6, a_{1112} = -0.3, a_{1121} = -0.1, a_{1122} = 3;\\
&a_{1211} = -0.2, a_{1212} = a_{1221} = 0, a_{1222} = -0.4;\\
&a_{2111} = -0.2, a_{2112} = a_{2121} = 0, a_{1222} = -0.4;\\
&a_{2211} = 3, a_{2212} = -0.6, a_{2221} = -0.2, a_{2222} = 1.55.
\end{aligned}$$

Firstly, we can verify that $\mathcal{A}$ is not a symmetric tensor, but a weakly symmetric tensor. By computations, we obtain that the minimum Z-eigenvalue and corresponding with the Z-eigenvector are $(\bar{\lambda}, \bar{x}) = (1.3543, (0.2300, 0.9732))$. Hence, $\mathcal{A}$ is positive definite.

Taking the Z-identity tensor I_Z as Case I or Case II in Definition 2, we cannot find positive real number α_2 such that

$$\alpha_2 > R_2(\mathcal{A}, \alpha_2),$$

which shows that Theorem 3.2 of [26] fails to check the positive definiteness of weakly symmetric tensor $\mathcal{A}$.

Setting $\alpha = (6, 3)^\top$, from Theorem 3, we verify

$$(\alpha_1 - R_1^2(\mathcal{A}, \alpha_1))\alpha_2 = 7.2 > 1.22 = |a_{1222} - \alpha_1 (I_Z)_{1222}| R_2(\mathcal{A}, \alpha_2)$$

and

$$(\alpha_2 - R_2^1(\mathcal{A}, \alpha_2))\alpha_1 = 0.9 > 0.8 = |a_{2111} - \alpha_1 (I_Z)_{2111}| R_1(\mathcal{A}, \alpha_1),$$

which implies that $\mathcal{A}$ is positive definite.

The verification method of Theorem 4 is similar to Theorem 3, hence omitted.

4.2. Asymptotic Stability of Time-Invariant Polynomial Systems

Li et al. [26] investigated asymptotical stability of time-invariant polynomial systems via Lyapunov's method in automatic control.

Lemma 4 (Corollary 2 of [26]). *For the nonlinear system $\sum$ in (2), if $-\mathcal{A}^{(t)}(t = 2, 4, \ldots, 2k)$ satisfies all conditions of Theorem 2, then the equilibrium point of $\sum$ is asymptotically stable.*

Theorem 5. *For the nonlinear system $\sum$ in (2), if $-\mathcal{A}^{(t)}(t = 2, 4, \ldots, 2k)$ is weakly symmetric and positive definite, then the equilibrium point of $\sum$ in (2) is asymptotically stable.*

Proof. Following the proof of Theorem 3.3 of [27], we find that the equilibrium point of $\sum$ in (2) is asymptotically stable via Lyapunov's method. □

The following example shows the validity of Theorem 5.

Example 3. *Consider the following polynomial systems:*

$$\begin{aligned}\sum:\ \dot{x}_1 &= -4x_1 + 1.5x_2 + 1.5x_3 - 1.4x_1^3 + 0.3x_1^2x_2 - 2.55x_1x_2^2 - 3.15x_1x_3^2\\ &\quad + 0.1x_2^3 + 0.3x_2x_3^2 - 0.1x_3^3;\\ \dot{x}_2 &= 1.5x_1 - 5x_2 + 2x_3 + 0.1x_1^3 - 2.55x_1^2x_2 + 0.3x_1x_2^2 + 0.3x_1x_3^2\\ &\quad - 3.2x_2^3 - 0.3x_2^2x_3 - 3.0x_2x_3^2;\\ \dot{x}_3 &= 1.5x_1 + 2x_2 - 4x_3 - 3.15x_1^2x_3 + 0.6x_1x_2x_3 - 0.3x_1x_3^2 - 0.1x_2^3\\ &\quad - 3.0x_2^2x_3 - 2.6x_3^3.\end{aligned}$$

Thus, $\sum$ can be written as $\dot{x} = \mathcal{A}^{(2)}x + \mathcal{A}^{(4)}x^3$, where $x = (x_1, x_2, x_3)^T$,

$$\mathcal{A}^{(2)} = \begin{bmatrix} -4 & 1.5 & 1.5 \\ 1.5 & -5 & 2 \\ 1.5 & 2 & -4 \end{bmatrix}$$

and $\mathcal{A}^{(4)} = (a_{ijkl}) \in (\mathbb{R}^3)^{\otimes 4}$ with

$$\begin{aligned}&a_{1111} = -1.4; a_{2222} = -3.2; a_{3333} = -2.6;\\ &a_{1112} = a_{1121} = a_{1211} = a_{1222} = a_{2111} = a_{2122} = a_{2212} = a_{2221} = 0.1;\\ &a_{1122} = a_{1212} = a_{1221} = a_{2112} = a_{2121} = a_{2211} = -0.85;\\ &a_{1133} = a_{1313} = a_{1331} = a_{3113} = a_{3131} = a_{3311} = -1.05;\\ &a_{1233} = a_{1323} = a_{1332} = a_{2133} = a_{2313} = a_{2331} = 0.1;\\ &a_{3123} = a_{3132} = a_{3213} = a_{3231} = a_{3312} = a_{3321} = 0.1;\\ &a_{2223} = a_{2232} = a_{2322} = a_{3222} = a_{1333} = a_{3133} = a_{3313} = a_{3331} = -0.1;\\ &a_{2233} = a_{2323} = a_{2332} = a_{3223} = a_{3232} = a_{3322} = -1.0; a_{ijkl} = 0, \textit{otherwise}.\end{aligned}$$

Obviously, $-\mathcal{A}^{(2)}$ is symmetric and positive definite and $-\mathcal{A}^{(4)}$ is symmetric. Taking the Z-identity tensor I_Z as Case I or Case II in Definition 2, we cannot find positive real number α_1 such that

$$\alpha_1 > R_1(-\mathcal{A}^{(4)}, \alpha_1),$$

which shows that Theorem 2 of [26] fails to check the positive definiteness of $-\mathcal{A}^{(4)}$ and asymptotical stability of polynomial systems $\sum$.

Taking $\alpha = (2.85, 3.0, 2.7)^\top$ and Z-identity tensor I_Z as Case II in Definition 2, we have

$$I_{ijkl} = \begin{cases} I_{1111} = I_{2222} = I_{3333} = 1;\\ I_{1122} = I_{1212} = I_{1221} = I_{1133} = I_{1313} = I_{1331} = \frac{1}{3};\\ I_{2112} = I_{2121} = I_{2221} = I_{2233} = I_{2323} = I_{2332} = \frac{1}{3};\\ I_{3113} = I_{3131} = I_{3311} = I_{3223} = I_{3232} = I_{3322} = \frac{1}{3};\\ 0, \ \textit{otherwise}, \end{cases}$$

By Theorem 3, we may compute Table 3 as follows:

Table 3. Verification of positive definiteness by Theorem 3.

	$(\alpha_i - R_i^j(-\mathcal{A}^{(4)}, \alpha_i))\alpha_j$	$\lvert a_{ij\ldots j} - \alpha_i(I_Z)_{ij\ldots j}\rvert R_j(-\mathcal{A}^{(4)}, \alpha_j)$
$i = 1, j = 2$	0.3	0.165
$i = 1, j = 3$	0.27	0.185
$i = 2, j = 1$	4.1325	0.285
$i = 2, j = 3$	3.645	0
$i = 3, j = 1$	2.4225	0
$i = 3, j = 2$	2.85	0.165

From table 3, we verify

$$(\alpha_i - R_i^j(-\mathcal{A}^{(4)}, \alpha_i))\alpha_j > |a_{ij\ldots j} - \alpha_i(I_Z)_{ij\ldots j}|R_j(-\mathcal{A}^{(4)}, \alpha_j), \forall i, j \in N, i \neq j,$$

which implies that $-\mathcal{A}^{(4)}$ is positive definite.

By Theorem 4, we may propose Table 4 as follows:

Table 4. Verification of positive definiteness by Theorem 4.

	$(\alpha_i - R_i^j(-\mathcal{A}^{(4)}, \alpha_i))(\alpha_j - r_j^{\overline{\Theta}_i}(-\mathcal{A}^{(4)}, \alpha_j))$	$\lvert a_{ij\ldots j} - \alpha_i(I_Z)_{ij\ldots j}\rvert r_j^{\Theta_i}(-\mathcal{A}^{(4)}, \alpha_j)$
$i = 1, j = 2$	0.25	0.115
$i = 1, j = 3$	0.22	0.135
$i = 2, j = 1$	1.45	0.1
$i = 2, j = 3$	2.4975	0
$i = 3, j = 1$	0.595	0
$i = 3, j = 2$	1.8525	0.06

Hence, we have

$$(\alpha_i - R_i^j(-\mathcal{A}^{(4)}, \alpha_i))(\alpha_j - r_j^{\overline{\Theta}_i}(-\mathcal{A}^{(4)}, \alpha_j)) > |a_{ij\ldots j} - \alpha_i(I_Z)_{ij\ldots j}|r_j^{\Theta_i}(-\mathcal{A}^{(4)}, \alpha_j), \forall j \neq i \in N,$$

$$\alpha_i > R_i^j(-\mathcal{A}^{(4)}, \alpha_i) \ \ and \ \ \alpha_j > r_j^{\overline{\Theta}_i}(-\mathcal{A}^{(4)}, \alpha_j), \forall i, j \in N, i \neq j.$$

Hence, $-\mathcal{A}^{(4)}$ is positive definite.

Further, it follows from Theorem 5 that the equilibrium point of $\sum$ is asymptotically stable.

5. Conclusions

In this paper, we established new Brauer-type Z-eigenvalue inclusion sets for even-order tensors by Z-identity tensor and proposed some sufficient conditions for the positive definiteness of weakly symmetric tensors, as well as the asymptotic stability of time-invariant polynomial systems. The given numerical experiments showed its validity. It is remarkable that suitable parameter α has a great influence on the numerical effect and positive definiteness. Therefore, how to select the suitable parameter α is our further research.

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