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Existence of Solutions for a Singular Fractional q -Differential Equations under Riemann–Liouville Integral Boundary Condition

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Abstract: We investigate the existence of solutions for a system of m -singular sum fractional q -differential equations in this work under some integral boundary conditions in the sense of Caputo fractional q -derivatives. By means of a fixed point Arzelá–Ascoli theorem, the existence of positive solutions is obtained. By providing examples involving graphs, tables, and algorithms, our fundamental result about the endpoint is illustrated with some given computational results. In general, symmetry and q -difference equations have a common correlation between each other. In Lie algebra, q -deformations can be constructed with the help of the symmetry concept.

Keywords: Caputo q -derivative; singular sum fractional q -differential; fixed point; equations; Riemann–Liouville q -integral

MSC: 34A08; 34B16; 39A13

1. Introduction

There are many definitions of fractional derivatives that have been formulated according to two basic conceptions: one of a global (classical) nature and the other of a local nature. Under the first formulation, the fractional derivative is defined as an integral, Fourier, or Mellin transformation, which provides its non-local property with memory. The second conception is based on a local definition through certain incremental ratios. This global conception is associated with the appearance of the fractional calculus itself and dates back to the pioneering works of important mathematicians, such as Euler, Laplace, Lacroix, Fourier, Abel, and Liouville, until the establishment of the classical definitions of Riemann–Liouville and Caputo.

Until relatively recently, the study of these fractional integrals and derivatives was limited to a purely mathematical context; however, in recent decades, their applications in various fields of natural Sciences and technology, such as fluid mechanics, biology, physics, image processing, or entropy theory, have revealed the great potential of these fractional integrals and derivatives [1–9]. Furthermore, the study from the theoretical and practical point of view of the elements of fractional differential equations has become a focus for interested researchers [10–15].

The q -difference equations (qDifEqs) were first proposed by Jackson in 1910 [16]. After that, qDifEqs were investigated in various studies [17–24]. On the contrary, integro-differential equations (InDifEqs) have been recently studied via various fractional derivatives and formulations based on the original idea of qDifEqs (see [25–32]). The concept of symmetry and q -difference equations are connected to each other while theoretically investigating the differential equation symmetries.

The solution existence and uniqueness for the fractional qDifEqs were investigated in 2012 by Ahmad et al. as: ${}^c\mathcal{D}_q^\alpha[u](t) = T(t, u(t))$ with boundary conditions (B.Cs):

$$\alpha_1 u(0) - \beta_1 \mathcal{D}_q[u](0) = \gamma_1 u(\eta_1), \quad \alpha_2 u(1) - \beta_2 \mathcal{D}_q[u](1) = \gamma_2 u(\eta_2),$$

where $\alpha \in (1, 2]$, $\alpha_i, \beta_i, \gamma_i, \eta_i$ are real numbers, for $i = 1, 2$ and $T \in C(J \times \mathbb{R}, \mathbb{R})$ [20]. The q -integral problem was studied in in 2013 by Zhao et al. as:

$$\mathcal{D}_q^\alpha[u](t) + f(t, u(t)) = 0,$$

with B.Cs: $u(1) = \mu \mathcal{I}_q^\beta[u](\eta)$ and $u(0) = 0$ almost $\forall t \in (0, 1)$, where $q \in (0, 1)$, $\alpha \in (1, 2]$, $\beta \in (0, 2]$, $\eta \in (0, 1)$, μ is positive real number, and \mathcal{D}_q^α is the q -derivative of Riemann–Liouville (RL) and the real values continuous map u defined on $I \times [0, \infty)$ [24]. The problem:

$${}^c\mathcal{D}_q^\beta({}^c\mathcal{D}_q^\gamma + \lambda)[u](t) = pf(t, u(t)) + k\mathcal{I}_q^\xi[g](t, u(t))$$

was investigated in 2014 by Ahmad et al. with B.Cs:

$$\alpha_1 u(0) - \beta_1 (t^{(1-\gamma)} \mathcal{D}_q[u](0))|_{t=0} = \sigma_1 u(\eta_1)$$

and

$$\alpha_2 u(1) + \beta_2 \mathcal{D}_q[u](1) = \sigma_2 u(\eta_2),$$

where $t, q \in [0, 1]$, ${}^c\mathcal{D}_q^\beta$ is the Caputo fractional q -derivative (CpFqDr), $0 < \beta, \gamma \leq 1$, $\mathcal{I}_q^\xi(\cdot)$ represents the RL integral with $\xi \in (0, 1)$, f and g are given continuous functions, λ and p, k are real constants, $\alpha_i, \beta_i, \sigma_i \in \mathbb{R}$ and $\eta_i \in (0, 1)$ for $i = 1, 2$ [19]. The solutions' existence was studied in 2019 by Samei et al. for some multi-term q -integro-differential equations with non-separated and initial B.Cs ([23]).

Inspired by all previous works, we investigate in this work the positive solutions for the singular fractional q -differential equation (SFqDEqs) as follows:

$${}^c\mathcal{D}_q^\alpha[u](t) + h(t, u(t)) = 0, \tag{1}$$

with the B.Cs: $u(0) = 0$, $cu(1) = \mathcal{I}_q^\gamma[u](1)$ and $u''(0) = \dots = u^{(n-1)}(0) = 0$, where $t \in J = (0, 1)$, $\mathcal{I}_q^\gamma[u]$ is the RL q -integral of order γ for the given function: u , here $q \in J$, $c \geq 1$, $n = [\alpha] + 1$, $\alpha \geq 3$, $\gamma \in [1, \infty)$, $2\Gamma_q(\gamma) \geq \Gamma_q(\alpha)$, $h : (0, 1] \times [0, \infty) \rightarrow [0, \infty)$ is continuous, $\lim_{t \rightarrow 0^+} h(t, \cdot) = +\infty$ that is, h is singular at $t = 0$, and ${}^c\mathcal{D}_q^\alpha$ represents the CpFqDr of order α , $q \in J$.

This work is divided into the following: some essential notions and basic results of q -calculus are reviewed in Section 2. Our original important results are stated in Section 3. In Section 4, illustrative numerical examples are provided to validate the applicability of our main results.

2. Essential Preliminaries

Assume that $q \in (0, 1)$ and $a \in \mathbb{R}$. Define $[a]_q = \frac{1-q^a}{1-q}$ [16]. The power function: $(x - y)_q^n$ with $n \in \mathbb{N}_0$ is written as:

$$(x - y)_q^{(n)} = \prod_{k=0}^{n-1} (x - yq^k)$$

for $n \geq 1$ and $(x - y)_q^{(0)} = 1$, where x and y are real numbers and $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$ ([17]). In addition, for $\sigma \in \mathbb{R}$ and $a \neq 0$, we obtain:

$$(x - y)_q^{(\sigma)} = x^\sigma \prod_{k=0}^{\infty} \frac{x - yq^k}{x - yq^{\sigma+k}}.$$

If $y = 0$, then it is obvious that $x^{(\sigma)} = x^\sigma$. The q -Gamma function is expressed by

$$\Gamma_q(z) = \frac{(1 - q)^{(z-1)}}{(1 - q)^{z-1}},$$

where $z \in \mathbb{R} \setminus \{0, -1, -2, \dots\}$ ([16]). We know that $\Gamma_q(z + 1) = [z]_q \Gamma_q(z)$. The value of the q -Gamma function, $\Gamma_q(z)$, for input values q and z with counting the sentences' number n in summation by simplification analysis. A pseudo-code is constructed for estimating q -Gamma function of order n . The q -derivative of function w , is expressed as:

$$\mathcal{D}_q[w](x) = \left(\frac{d}{dx} \right)_q w(x) = \frac{w(x) - w(qx)}{(1 - q)x}$$

and $\mathcal{D}_q[w](0) = \lim_{x \rightarrow 0} \mathcal{D}_q[w](x)$ ([17]). In addition, the higher order q -derivative of a function w is defined by $\mathcal{D}_q^n[w](x) = \mathcal{D}_q \mathcal{D}_q^{n-1}[w](x)$ for all $n \geq 1$, where $\mathcal{D}_q^0[w](x) = w(x)$ ([17,18]). The q -integral of a function f defined on $[0, b]$ is expressed as:

$$\mathcal{I}_q[w](x) = \int_0^x w(s) d_qs = x(1 - q) \sum_{k=0}^{\infty} q^k w(xq^k),$$

for $0 \leq x \leq b$, provided that the series is absolutely convergent ([17,18]). If a in $[0, b]$, then we have:

$$\int_a^b w(u) d_qu = \mathcal{I}_q[w](b) - \mathcal{I}_q[w](a) = (1 - q) \sum_{k=0}^{\infty} q^k [bw(bq^k) - aw(aq^k)],$$

if the series exists. The operator \mathcal{I}_q^n is given by $\mathcal{I}_q^0[w](x) = w(x)$ and $\mathcal{I}_q^n[w](x) = \mathcal{I}_q \mathcal{I}_q^{n-1}[w](x)$ for $n \geq 1$ and $g \in C([0, b])$ ([17,18]). It is proven that $\mathcal{D}_q \mathcal{I}_q[w](x) = w(x)$ and $\mathcal{I}_q \mathcal{D}_q[w](x) = w(x) - w(0)$ whenever w is continuous at $x = 0$ ([17,18]). The fractional RL type q -integral of the function w on J for $\sigma \geq 0$ is defined by $\mathcal{I}_q^\sigma[w](t) = w(t)$, and

$$\begin{aligned} \mathcal{I}_q^\alpha[w](t) &= \frac{1}{\Gamma_q(\sigma)} \int_0^t (t - qs)^{(\sigma-1)} w(s) d_qs \\ &= t^\sigma (1 - q)^\sigma \sum_{k=0}^{\infty} q^k \frac{\prod_{i=1}^{k-1} (1 - q^{\sigma+i})}{\prod_{i=1}^{k-1} (1 - q^{i+1})} w(tq^k), \end{aligned}$$

for $t \in J$ and $\sigma > 0$ ([22,33]). In addition, the CpFqDr of a function w is expressed as:

$$\begin{aligned} {}^c\mathcal{D}_q^\sigma[w](t) &= \mathcal{I}_q^{[\sigma]-\sigma} [{}^c\mathcal{D}_q^{[\sigma]}[w]](t) \\ &= \frac{1}{\Gamma_q([\sigma] - \alpha)} \int_0^t (t - qs)^{([\sigma]-\sigma-1)} {}^c\mathcal{D}_q^{[\sigma]}[w](s) d_qs \\ &= \frac{1}{t^\sigma (1 - q)^\sigma} \sum_{k=0}^{\infty} q^k \frac{\prod_{i=1}^{k-1} (1 - q^{i-\sigma})}{\prod_{i=1}^{k-1} (1 - q^{i+1})} w(tq^k), \end{aligned} \tag{2}$$

where $t \in J$ and $\sigma > 0$ ([22]). It is proven that

$$\mathcal{I}_q^\beta [\mathcal{I}_q^\sigma [w]](x) = \mathcal{I}_q^{\sigma+\beta} [w](x) \text{ and } {}^c\mathcal{D}_q^\sigma [\mathcal{I}_q^\sigma [w]](x) = w(x),$$

where $\sigma, \beta \geq 0$ ([22]).

Some essential notions and lemmas are now presented as follows: In our work, $L^1(\bar{J})$ and $C_{\mathbb{R}}(\bar{J})$ are denoted by $\bar{\mathcal{L}}$ and $\bar{\mathcal{B}}$, respectively, where $\bar{J} = [0, 1]$.

Lemma 1 ([34]). *If $x \in \bar{\mathcal{B}} \cap \bar{\mathcal{L}}$ with $\mathcal{D}_q^\alpha x \in \mathcal{B} \cap \mathcal{L}$, then*

$$\mathcal{I}_q^\alpha \mathcal{D}_q^\alpha x(t) = x(t) + \sum_{i=1}^n c_i t^{\alpha-i},$$

where n is the smallest integer $\geq \alpha$, and c_i is some real number.

Here, we restate the well-known Arzelá–Ascoli theorem. Assume that $S = \{s_n\}_{n \geq 1}$ is a sequence of bounded and equicontinuous real valued functions on $[a, b]$. Then, S has a uniformly convergent subsequence. We need the following fixed point theorem in our main result:

Lemma 2 ([35]). *Assume that \mathcal{A} is a Banach space, $P \subseteq \mathcal{A}$ is a cone, and $\mathcal{O}_1, \mathcal{O}_2$ are two bounded open balls of \mathcal{A} centered at the origin with $\bar{\mathcal{O}}_1 \subset \mathcal{O}_2$. Assume that $\Omega : P \cap (\bar{\mathcal{O}}_2 \setminus \mathcal{O}_1) \rightarrow P$ is a completely continuous operator such that either $\|\Omega(a)\| \leq \|a\|$ for all $a \in P \cap \partial \mathcal{O}_1$ and $\|\Omega(a)\| \geq \|a\|$ for all $a \in P \cap \partial \mathcal{O}_2$, or $\|\Omega(a)\| \geq \|a\|$ for each $a \in P \cap \partial \mathcal{O}_1$ and $\|\Omega a\| \leq \|a\|$ for $a \in P \cap \partial \mathcal{O}_2$. Then, Ω has a fixed point in $P \cap (\mathcal{O}_2 \setminus \mathcal{O}_1)$.*

3. Main Results

Differential Equation

Let us now present our fundamental lemma as follows:

Lemma 3. *The u_0 is a solution for the q -differential equation $\mathcal{D}_q^\alpha[u](t) + g(t) = 0$ with the B.Cs: $u(0) = 0, cu(1) = \mathcal{I}_q^\gamma u(1)$ and $u''(0) = \dots = u^{(n-1)}(0) = 0$ if u_0 is a solution for the q -integral equation*

$$u(t) = \int_0^1 G_q(t, s) f(s) d_q s,$$

where

$$G_q(t, s) = \begin{cases} \frac{-(t - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} & s \leq t, \\ + t^2 \frac{\Gamma_q(\gamma + 3) [a\Gamma_q(\alpha + \gamma)(1 - qs)^{(\alpha-1)} - \Gamma_q(\alpha)(1 - qs)^{(c+\gamma-1)}]}{\Gamma_q(\alpha)\Gamma_q(\alpha + \gamma) [c\Gamma_q(\gamma + 3) - 2\Gamma_q(\gamma)]}, & \\ t^2 \frac{\Gamma_q(\gamma + 3) [c\Gamma_q(\alpha + \gamma)(1 - qs)^{(\alpha-1)} - \Gamma_q(\alpha)(1 - qs)^{(c+\gamma-1)}]}{\Gamma_q(\alpha)\Gamma_q(\alpha + \gamma) [c\Gamma_q(\gamma + 3) - 2\Gamma_q(\gamma)]}, & t \leq s, \end{cases} \tag{3}$$

for $s, t \in \bar{J}, n = [\alpha] + 1$, the function $g \in \bar{\mathcal{B}}, \alpha \geq 3$ and $\gamma \in [1, \infty)$ with $2\Gamma_q(\gamma) \geq \Gamma_q(\alpha)$.

Proof. Let us first assume that u_0 is a solution for the equation $\mathcal{D}_q^\alpha u(t) + g(t) = 0$ with the B.Cs. By using Lemma 1, we obtain:

$$u_0(t) = -\mathcal{I}_q^\alpha [g](t) + c_0 + c_1 t + c_2 t^2 + \dots c_{n-1} t^{n-1}$$

and by using the condition $u_0(0) = u_0''(0) = \dots = u_0^{(n-1)}(0) = 0$, we have

$$u_0(t) = -\mathcal{I}_q^\alpha [g](t) + c_2 t^2.$$

Indeed,

$$\mathcal{I}_q^\gamma [u_0](t) = -\mathcal{I}_q^{\alpha+\gamma} [g](t) + c_2 \frac{2\Gamma_q(\gamma)}{\Gamma_q(\gamma+3)} t^{\gamma+2},$$

and thus

$$\mathcal{I}_q^\gamma [u_0](1) = -\mathcal{I}_q^{\alpha+\gamma} [g](1) + c_2 \frac{2\Gamma_q(\gamma)}{\Gamma_q(\gamma+3)}.$$

Note that $cu_0(1) = -c\mathcal{I}_q^\alpha [g](1) + cc_2$ and

$$\begin{aligned} c_2 \left(c - \frac{2\Gamma_q(\gamma)}{\Gamma_q(\gamma+3)} \right) &= c\mathcal{I}_q^\alpha g(1) - \mathcal{I}_q^{\alpha+\gamma} g(1) \\ &= \frac{c\Gamma_q(\alpha+\gamma)}{\Gamma_q(\alpha+\gamma)} \mathcal{I}_q^\alpha [g](1) - \frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha)} \mathcal{I}_q^{\alpha+\gamma} [g](1) \\ &= \int_0^1 \frac{c\Gamma_q(\alpha+\gamma)(1-qs)^{(\alpha-1)} - \Gamma_q(\alpha)(1-qs)^{(\alpha+\gamma-1)}}{\Gamma_q(\alpha)\Gamma_q(\alpha+\gamma)} g(s) \, d_qs. \end{aligned}$$

On the other hand,

$$c - \frac{2\Gamma_q(\gamma)}{\Gamma_q(\gamma+3)} = \frac{c\Gamma_q(\gamma+3) - 2\Gamma_q(\gamma)}{\Gamma_q(\gamma+3)}.$$

Hence,

$$c_2 = \int_0^1 \frac{\Gamma_q(\gamma+3) [c\Gamma_q(\alpha+\gamma)(1-qs)^{(\alpha-1)} - \Gamma_q(\alpha)(1-qs)^{(\alpha+\gamma-1)}]}{\Gamma_q(\alpha)\Gamma_q(\alpha+\gamma) [c\Gamma_q(\gamma+3) - 2\Gamma_q(\gamma)]} g(s) \, d_qs.$$

Therefore, we have

$$\begin{aligned} u_0(t) &= -\mathcal{I}_q^\alpha [g](t) \\ &\quad + t^2 \int_0^1 \frac{\Gamma_q(\gamma+3) [c\Gamma_q(\alpha+\gamma)(1-qs)^{(\alpha-1)} - \Gamma_q(\alpha)(1-qs)^{(\alpha+\gamma-1)}]}{\Gamma_q(\alpha)\Gamma_q(\alpha+\gamma) [c\Gamma_q(\gamma+3) - 2\Gamma_q(\gamma)]} g(s) \, d_qs \\ &= \int_0^1 G_q(s,t) g(s) \, d_qs, \end{aligned}$$

where

$$\begin{aligned} G_q(t,s) &= \frac{-(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \\ &\quad + t^2 \frac{\Gamma_q(\gamma+3) [c\Gamma_q(\alpha+\gamma)(1-qs)^{(\alpha-1)} - \Gamma_q(\alpha)(1-qs)^{c+\gamma-1}]}{\Gamma_q(\alpha)\Gamma_q(\alpha+\gamma) [c\Gamma_q(\gamma+3) - 2\Gamma_q(\gamma)]}, \end{aligned}$$

whenever $0 \leq s \leq t \leq 1$ and

$$t^2 \frac{\Gamma_q(\gamma+3) [c\Gamma_q(\alpha+\gamma)(1-qs)^{(\alpha-1)} - \Gamma_q(\alpha)(1-qs)^{c+\gamma-1}]}{\Gamma_q(\alpha)\Gamma_q(\alpha+\gamma) [c\Gamma_q(\gamma+3) - 2\Gamma_q(\gamma)]}$$

whenever $0 \leq t \leq s \leq 1$. Hence, u_0 is an integral equation's solution. By simple review, we can see that u_0 is a solution for the equation $\mathcal{D}_q^\alpha u(t) + g(t) = 0$ with the B.Cs whenever u_0 is an integral equation's solution. \square

Remark 1. By applying some simple calculations, one can show that $G_q(t, s) \geq 0$ for each $s, t \in \bar{J}$. Now, let us define the operator Ω on the Banach space \bar{B} by

$$\Omega(u(t)) = \int_0^1 G_q(t, s)h(s, u(s)) \, d_qs.$$

It is easy to check that u_0 is a fixed point of the operator Ω if u_0 is a solution for Equation (1).

Consider \bar{B} together the supremum norm and cone, P is the set of all $u \in \bar{B}$ such that $u(t) \geq 0 \forall t \in \bar{J}$. Suppose that $h : (0, 1) \times [0, \infty) \rightarrow [0, \infty)$ is the singular function at $t = 0$ in the Equation (1) and $G_q(t, s)$ is the q -Green function in Lemma 3. Now, define the self operator Ω on P by

$$\Omega(u(t)) = \int_0^1 G_q(t, s)h(s, u(s)) \, d_qs,$$

for all $t \in \bar{J}$. At present, we can provide our first main result on the solution’s existence for problem (1) under some assumptions.

Theorem 1. Problem (1) has a unique solution if the following conditions hold.

I. There exists a continuous function $h : (0, 1] \times [0, \infty) \rightarrow [0, \infty)$ such that

$$\lim_{t \rightarrow 0^+} h(t, s) = \infty,$$

for $s \in [0, \infty)$.

II. There exists $L > 0, \beta \in J$ and positive constant k such that

$$kc\Gamma_q(\gamma + 3) < (c\Gamma_q(\gamma + 3) - 2\Gamma_q(\gamma)),$$

$$|t^\beta h(t, 0)| \leq L \text{ for each } t \in \bar{J} \text{ and}$$

$$|t^\beta h(t, u(t)) - t^\beta h(t, v(t))| \leq k\|u - v\|,$$

for each u, v belong to P .

Proof. Note that,

$$|\Omega(u(t))| \leq t^2 \frac{c\Gamma_q(\gamma + 3)}{c\Gamma_q(\gamma + 3) - 2\Gamma_q(\gamma)} \mathcal{I}_q^\alpha [h](1, u(1))$$

for all $t \in \bar{J}$. Now, put

$$\ell = L \frac{c\Gamma_q(\gamma + 3)\Gamma_q(1 - \beta)}{c\Gamma_q(\gamma + 3) - 2\Gamma_q(\gamma)}$$

and define $B = \{u \in P : \|u\| \leq \ell\}$. Clearly, B is a bounded and closed subset of \mathcal{A} , and thus B is complete. If $u \in B$, then we obtain:

$$|\Omega(u(t))| \leq \frac{c\Gamma_q(\gamma + 3)}{\Gamma_q(\alpha)[c\Gamma_q(\gamma + 3) - 2\Gamma_q(\gamma)]} \int_0^1 (1 - qs)^{(\alpha-1)} s^{-\beta} s^\beta h(s, u(s)) \, d_qs$$

$\forall t \in \bar{J}$ and thus

$$\begin{aligned} |F(x(t))| &\leq \frac{c\Gamma_q(\gamma + 3)}{\Gamma_q(\alpha)[c\Gamma_q(\gamma + 3) - 2\Gamma_q(\gamma)]} \\ &\times \int_0^1 (1 - qs)^{(\alpha-1)} s^{-\beta} s^\beta (|h(s, u(s)) - h(s, 0)| + |h(s, 0)|) \, d_qs \\ &\leq (k\ell + L) \frac{c\Gamma_q(\gamma + 3)}{\Gamma_q(\alpha)[c\Gamma_q(\gamma + 3) - 2\Gamma_q(\gamma)]} B_q(1 - \beta, \alpha) \end{aligned}$$

$$\begin{aligned}
 &= (k\ell + L) \frac{c\Gamma(\gamma + 3)\Gamma_q(1 - \beta)}{[c\Gamma_q(\gamma + 3) - 2\Gamma_q(\gamma)]\Gamma_q(\alpha - \beta + 1)} \\
 &\leq \frac{[c\Gamma_q(\gamma + 3) - 2\Gamma_q(\gamma)]\ell}{c\Gamma_q(\gamma + 3)\Gamma_q(1 - \beta)} \left[\frac{c\Gamma_q(\gamma + 3)\Gamma_q(1 - \beta)}{(c\Gamma_q(\gamma + 3) - 2\Gamma_q(\gamma))\Gamma_q(\alpha - \beta + 1)} \right] \\
 &\quad + L \frac{c\Gamma_q(\gamma + 3)\Gamma_q(1 - \beta)}{[c\Gamma_q(\gamma + 3) - 2\Gamma_q(\gamma)]\Gamma_q(\alpha - \beta + 1)} \\
 &= \frac{\ell}{\Gamma_q(\alpha - \beta + 1)} + \frac{\ell}{\Gamma_q(\alpha - \beta + 1)} \\
 &< \frac{\ell}{\Gamma_q(\alpha)} + \frac{\ell}{\Gamma_q(\alpha)} \leq \frac{\ell}{2} + \frac{\ell}{2} = \ell.
 \end{aligned}$$

Indeed, $\Omega(B) \subseteq B$, and therefore a restriction of Ω on B is an operator on B . Let $u, v \in B$. Then, we obtain

$$\begin{aligned}
 \|\Omega(u(t)) - \Omega(v(t))\| &\leq \frac{1}{\Gamma_q(\alpha)} \int_0^1 (t - qs)^{(\alpha-1)} |h(s, u(s)) - h(s, v(s))| \, d_qs \\
 &\quad + \frac{ct^2\Gamma_q(\gamma + 3)}{\Gamma_q(\alpha)[c\Gamma_q(\gamma + 3) - 2\Gamma_q(\gamma)]} \\
 &\quad \times \int_0^1 (1 - qs)^{(\alpha-1)} s^{-\beta} s^\beta \|h(s, u(s)) - h(s, v(s))\| \, d_qs \\
 &\leq k\|u - v\| \\
 &\quad \times \left[\frac{\Gamma_q(1 - \beta)}{\Gamma_q(\alpha - \beta + 1)} + \frac{c\Gamma_q(\gamma + 3)\Gamma_q(1 - \beta)}{[c\Gamma_q(\gamma + 3) - 2\Gamma_q(\gamma)]\Gamma_q(\alpha - \beta + 1)} \right] \\
 &\leq \left[\frac{c\Gamma_q(\gamma + 3) - 2\Gamma_q(\gamma)}{c\Gamma_q(\gamma + 3)\Gamma_q(\alpha - \beta + 1)} + \frac{1}{\Gamma_q(\alpha - \beta + 1)} \right] \|u - v\| \\
 &< \left[\frac{c\Gamma_q(\gamma + 3) - 2\Gamma_q(\gamma)}{c\Gamma_q(\gamma + 3)\Gamma_q(\alpha)} + \frac{1}{\Gamma_q(\alpha)} \right] \|u - v\|
 \end{aligned}$$

for all $t \in \bar{J}$. Take

$$\lambda = \frac{c\Gamma_q(\omega + 3) - 2\Gamma_q(\omega)}{c\Gamma_q(\omega + 3)\Gamma_q(\alpha)} + \frac{1}{\Gamma_q(\alpha)}.$$

Since $\alpha \geq 3$, we obtain $\lambda \in J$, and therefore $\Omega : B \rightarrow B$ is a contraction. Thus, Ω has a unique fixed point in B . By employing Lemma 3, the problem (1) has a unique solution in B . \square

Lemma 4. Suppose that there exists $\beta \in J$ such that the map $t^\beta g(t)$ is a continuous map on J . If $G_q(t, s)$ is the q -Green function (3) in Lemma 3, then

$$\Omega(t) = \int_0^1 G_q(t, s)g(s) \, d_qs,$$

is also a continuous map on J . The self-operator Ω is completely continuous whenever there exists $\beta \in J$ such that the map $t^\beta g(t)$ is a continuous map on \bar{J} .

Proof. Since the map $t^\beta g(t)$ is continuous and $\Omega(t) = \int_0^t G_q(t, s)s^{-\beta} s^\beta g(s) \, d_qs$, we obtain

$$|\Omega(t)| \leq \sup_{s \in \delta} |G_q(t, s)s^\beta g(s)| \int_0^t s^{-\beta} \, ds = \frac{mt^{1-\beta}}{1 - \beta},$$

where $\delta = [0, t]$,

$$m = \sup_{s \in \delta} |G_q(t, s)s^\beta g(s)| < \infty.$$

Indeed, $\Omega(0) = 0$. Note that, $G_q(t, s)$ is continuous in \bar{J}^2 . First, suppose that $t_1 = 0$ and $t_2 \in (0, 1]$. By continuity $t^\beta g(t)$, there exists $L > 0$ such that

$$\sup_{t \in \bar{J}} |t^\beta g(t)| \leq L.$$

Thus, we have:

$$\begin{aligned} |\Omega(t_2) - \Omega(t_1)| &= |\Omega(t_2)| \leq \int_0^{t_2} \frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} s^{-\beta} s^\beta g(s) \, d_qs \\ &\quad + t_2^2 \int_0^1 \frac{\Gamma_q(\gamma+3) [c\Gamma_q(\alpha+\gamma) + \Gamma_q(\alpha)]}{\Gamma_q(\alpha) [c\Gamma_q(\gamma+3) - 2\Gamma_q(\gamma)]} (1-qs)^{(\alpha-1)} s^{-\beta} s^\beta g(s) \, d_qs \\ &\leq \frac{L}{\Gamma_q(\alpha)} B_q(1-\beta, \alpha) t_2^{\alpha-\beta} \\ &\quad + L t_2^2 \frac{\Gamma_q(\gamma+3) [c\Gamma_q(\alpha+\gamma) + \Gamma_q(\alpha)]}{\Gamma_q(\alpha) [c\Gamma_q(\gamma+3) - 2\Gamma_q(\gamma)]} B_q(1-\beta, \alpha) \\ &= \frac{L\Gamma_q(1-\beta)}{\Gamma_q(\alpha-\beta+1)} t_2^{\alpha-\beta} \\ &\quad + L \frac{\Gamma_q(\gamma+3)\Gamma_q(1-\beta) [c\Gamma_q(\alpha+\gamma) + \Gamma_q(\alpha)]}{[c\Gamma_q(\gamma+3) - 2\Gamma_q(\gamma)]\Gamma_q(\alpha-\beta+1)} t_2^2. \end{aligned}$$

This implies that $\lim_{t_2 \rightarrow t_1} |\Omega(t_2) - \Omega(t_1)| = 0$. At present, in the next case, we assume that $t_1 \in J$ and $t_2 \in (t_1, 1]$. Thus, we obtain:

$$\begin{aligned} |\Omega(t_2) - \Omega(t_1)| &\leq \left| \frac{1}{\Gamma_q(\alpha)} \right| - \int_0^{t_2} (t_2 - qs)^{(\alpha-1)} s^{-\beta} s^\beta g(s) \, d_qs \\ &\quad + \left| \int_0^{t_1} (t_1 - qs)^{(\alpha-1)} s^{-\beta} s^\beta g(s) \, d_qs \right| \\ &\quad + |t_2^2 - t_1^2| \frac{\Gamma_q(\gamma+3) [c\Gamma_q(\gamma+3) + \Gamma_q(\alpha)]}{\Gamma_q(\alpha) [c\Gamma_q(\gamma+3) - 2\Gamma_q(\gamma)]} \\ &\quad \times \int_0^1 (1-qs)^{(\alpha-1)} s^{-\beta} s^\beta g(s) \, d_qs. \end{aligned}$$

On the other hand,

$$\begin{aligned} &\left| \frac{1}{\Gamma_q(\alpha)} \right| - \int_0^{t_2} (t_2 - qs)^{(\alpha-1)} s^{-\beta} s^\beta g(s) \, d_qs + \int_0^{t_1} (t_1 - qs)^{(\alpha-1)} s^{-\beta} s^\beta g(s) \, d_qs \\ &\leq \left| \frac{1}{\Gamma_q(\alpha)} \right| \int_0^{t_1} (t_2 - qs)^{\alpha-1} s^{-\beta} s^\beta g(s) \, d_qs \\ &\quad - \left| \int_0^{t_2} (t_2 - qs)^{(\alpha-1)} s^{-\beta} s^\beta g(s) \, d_qs \right| \\ &= \left| \frac{1}{\Gamma_q(\alpha)} \right| \int_{t_2}^{t_1} (t_2 - qs)^{(\alpha-1)} s^{-\beta} s^\beta g(s) \, d_qs \\ &\leq \frac{L}{\Gamma_q(\alpha)} \int_{t_1}^{t_2} (t_2 - qs)^{(\alpha-1)} s^{-\beta} \, d_qs \\ &\leq \frac{L}{\Gamma_q(\alpha)} \sup_{s \in [t_2, t_1]} (t_2 - qs)^{(\alpha-1)} \int_{t_1}^{t_2} s^{-\beta} \, d_qs \\ &= \frac{L}{\Gamma_q(\alpha)} (t_2 - t_1)^{\alpha-1} \frac{t_2^{1-\beta} - t_1^{1-\beta}}{1-\beta} \end{aligned}$$

and therefore $\lim_{t_2 \rightarrow t_1} |\Omega(t_2) - \Omega(t_1)| = 0$. By applying in a similar way, we conclude that

$$\lim_{t_2 \rightarrow t_1} |\Omega(t_2) - \Omega(t_1)| = 0,$$

whenever $t_1 \in \bar{J}$ and $t_2 \in [0, t_1)$. Now, we prove that the self-operator Ω is completely continuous. Assume that $\varepsilon > 0$. Since the function $t^\beta h(t, u(t))$ is continuous, there exist $\delta > 0$ such that

$$|t^\beta h(t, u(t)) - t^\beta h(t, v(t))| < \varepsilon,$$

for each $u, v \in P$ with $\|u - v\| < \delta$. Thus, we obtain

$$\begin{aligned} \|\Omega(u) - \Omega(v)\| &= \sup_{t \in \bar{J}} |\Omega(u(t)) - \Omega(v(t))| \\ &= \sup_{t \in \bar{J}} \left| \int_0^t \frac{(t - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} s^{-\beta} (s^\beta h(s, u(s)) - s^\beta h(s, v(s))) \, d_qs \right. \\ &\quad + t^2 \int_0^1 \frac{\Gamma_q(\gamma + 3) [c\Gamma_q(\gamma + \alpha)(1 - qs)^{(\alpha-1)} - \Gamma_q(\alpha)(1 - qs)^{(\alpha+\gamma-1)}]}{\Gamma_q(\alpha)\Gamma_q(\alpha + \gamma) [c\Gamma_q(\gamma + 3) - 2\Gamma_q(\gamma)]} \\ &\quad \left. \times s^{-\beta} [s^\beta h(s, u(s)) - s^\beta h(s, v(s))] \, d_qs \right| \\ &\leq \sup_{t \in \bar{J}} \left[\varepsilon \int_0^t \frac{(t - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \, d_qs \right. \\ &\quad \left. + \varepsilon t^2 \int_0^1 \frac{\Gamma_q(\gamma + 3) [c\Gamma_q(\gamma + \alpha) + \Gamma_q(\alpha)]}{\Gamma_q(\alpha)\Gamma_q(\alpha + \gamma) [c\Gamma_q(\gamma + 3) - 2\Gamma_q(\gamma)]} (1 - qs)^{(\alpha-1)} s^{-\beta} \, d_qs \right] \\ &\leq \sup_{t \in \bar{J}} \varepsilon t^{\alpha-\beta} \frac{\Gamma_q(1 - \beta)}{\Gamma_q(\alpha - \beta + 1)} \\ &\quad + \sup_{t \in \bar{J}} \varepsilon t^2 \frac{\Gamma_q(\gamma + 3)\Gamma_q(1 - \beta) [c\Gamma_q(\gamma + \alpha) + \Gamma_q(\alpha)]}{\Gamma_q(\alpha + \gamma)\Gamma_q(\alpha - \beta + 1) [c\Gamma_q(\gamma + 3) - 2\Gamma_q(\gamma)]} \\ &= \left[\frac{\Gamma_q(1 - \beta)}{\Gamma_q(\alpha - \beta + 1)} + \frac{\Gamma_q(\gamma + 3)\Gamma_q(1 - \beta) [c\Gamma_q(\alpha + \gamma) + \Gamma_q(\alpha)]}{\Gamma_q(\alpha + \gamma)\Gamma_q(\alpha - \beta + 1) [c\Gamma_q(\gamma + 3) - 2\Gamma_q(\gamma)]} \right] \varepsilon. \end{aligned}$$

Therefore, Ω is continuous. Let $Q \subset P$ be bounded. Choose $k > 0$ such that $\|u\| \leq k$ for each $u \in Q$. Since the function $t^\beta h(t, u)$ is continuous on $\bar{J} \times [0, \infty)$, the function: $t^\beta h(t, u)$ is also continuous on $\bar{J} \times [0, k]$. Select $r \geq 0$ such that $|t^\beta h(t, u)| \leq r$ for all $u \in Q$, and t belongs to \bar{J} . Thus,

$$\begin{aligned} |\Omega(u(t))| &\leq \int_0^1 G_q(t, s) s^{-\beta} |s^\beta h(s, u(s))| \, d_qs \\ &\leq r \left[\int_0^t \frac{(t - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} s^{-\beta} \, d_qs \right. \\ &\quad \left. + t^2 \frac{\Gamma_q(\gamma + 3) [c\Gamma_q(\alpha + \gamma) + \Gamma_q(\alpha)]}{\Gamma_q(\alpha + \gamma) [c\Gamma_q(\gamma + 3) - 2\Gamma_q(\gamma)]} \int_0^1 (1 - qs)^{(\alpha-1)} s^{-\beta} \, d_qs \right], \end{aligned}$$

for each $t \in \bar{J}$, and thus

$$\begin{aligned} \|\Omega(x(t))\| &= \sup_{t \in \bar{J}} |\Omega(x(t))| \\ &\leq \frac{\Gamma_q(1 - \beta)}{\Gamma_q(\alpha - \beta + 1)} + \frac{\Gamma_q(\gamma + 3)\Gamma_q(1 - \beta) [c\Gamma_q(\alpha + \gamma) + \Gamma_q(\alpha)]}{\Gamma_q(\alpha + \gamma)\Gamma_q(\alpha - \beta + 1) [c\Gamma_q(\gamma + 3) - 2\Gamma_q(\gamma)]} \\ &< \infty. \end{aligned}$$

This implies that $\Omega(Q)$ is bounded. Assume that $u \in Q$ and $t_1, t_2 \in \bar{J}$ with $t_1 < t_2$. Then, we obtain

$$\begin{aligned} |\Omega(u(t_2)) - \Omega(u(t_1))| &\leq \left| \int_0^{t_2} \frac{(t_2 - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} h(s, u(s)) \, d_qs \right. \\ &\quad \left. - \int_0^{t_1} \frac{(t_1 - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} h(s, u(s)) \, d_qs \right| \\ &\quad + |t_2^2 - t_1^2| \frac{\Gamma_q(\gamma + 3) [c\Gamma_q(\alpha + \gamma) + \Gamma_q(\alpha)]}{\Gamma_q(\alpha) [c\Gamma_q(\gamma + 3) - 2\Gamma_q(\gamma)]} \int_0^1 h(s, u(s)) \, d_qs \\ &\leq r \int_{t_1}^{t_2} \frac{(t_2 - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} s^{-\beta} \, d_qs \\ &\quad + r |t_2^2 - t_1^2| \int_0^1 \frac{\Gamma_q(\gamma + 3) [c\Gamma_q(\alpha + \gamma) + \Gamma_q(\alpha)]}{\Gamma_q(\alpha) [c\Gamma_q(\gamma + 3) - 2\Gamma_q(\gamma)]} s^{-\beta} \, d_qs \\ &\leq \frac{r}{\Gamma_q(\alpha)} \sup_{s \in [t_1, t_2]} (t_2 - qs)^{(\alpha-1)} \frac{t_2^{1-\beta} - t_1^{1-\beta}}{1 - \beta} \\ &\quad + r (t_2^2 - t_1^2) \frac{\Gamma_q(\gamma + 3) [c\Gamma_q(\alpha + \gamma) + \Gamma_q(\alpha)] \Gamma_q(1 - \beta)}{\Gamma_q(\alpha) \Gamma_q(\alpha - \gamma + 1) [c\Gamma_q(\gamma + 3) - 2\Gamma_q(\gamma)]}. \end{aligned}$$

Thus,

$$\lim_{t_2 \rightarrow t_1} |\Omega(u(t_2)) - \Omega(u(t_1))| = 0.$$

In other cases, one can prove a similar result. Hence, $\Omega(Q)$ is equicontinuous. Now, by applying the Arzelà–Ascoli theorem, $\overline{\Omega(Q)}$ is compact, and therefore Ω is completely continuous. \square

Theorem 2. The problem (1) has at least one positive solution whenever the hypothesis as follows holds:

- I. There exists $\beta \in J$ such that the map $t^\beta g(t)$ is a continuous map on J .
- II. There exists $r'_1 > 0$ and $r'_2 > 0$ with $r'_2 < r'_1$ such that $t^\beta h(t, u) \leq r'_1$ and $t^\beta h(t, u) \leq r'_2$ for each $(t, u) \in \bar{J} \times [0, r_1]$ and $(t, u) \in \bar{J} \times [0, r_2]$, respectively, where

$$\begin{aligned} r_1 &> \frac{\Gamma_q(\gamma + 3) \Gamma_q(1 - \beta) [c\Gamma_q(\alpha + \gamma) + \Gamma_q(\alpha)]}{\Gamma_q(\alpha + \gamma) \Gamma(\alpha - \sigma + 1) [c\Gamma_q(\gamma + 3) - 2\Gamma_q(\gamma)]} r'_1 \\ &> r_2 \\ &> \frac{[2\Gamma_q(\gamma) \Gamma_q(\alpha + \gamma) - \Gamma_q(\gamma + 3) \Gamma_q(\alpha)] \Gamma_q(1 - \beta)}{\Gamma_q(\alpha + \gamma) [c\Gamma_q(\gamma + 3) - 2\Gamma_q(\gamma)] \Gamma_q(\alpha - \gamma + 1)} r'_2. \end{aligned}$$

Proof. We take the set \mathcal{X}_1 and \mathcal{X}_2 of all $u \in P$ such that

$$\|u\| < \frac{[2\Gamma_q(\gamma) \Gamma_q(\alpha + \gamma) - \Gamma_q(\gamma + 3) \Gamma_q(\alpha)] \Gamma_q(1 - \beta)}{\Gamma_q(\alpha + \gamma) [c\Gamma_q(\gamma + 3) - 2\Gamma_q(\gamma)] \Gamma_q(\alpha - \beta + 1)} r'_2$$

and

$$\|u\| < \frac{\Gamma_q(\gamma + 3) \Gamma_q(1 - \beta) [c\Gamma_q(\alpha + \gamma) + \Gamma_q(\alpha)]}{\Gamma_q(\alpha + \gamma) \Gamma_q(\alpha - \beta + 1) [c\Gamma_q(\gamma + 3) - 2\Gamma_q(\gamma)]} r'_1,$$

respectively. Since $2\Gamma_q(\gamma) > \Gamma_q(\alpha)$ and $\Gamma_q(\alpha + \gamma) > \Gamma_q(\gamma + 3)$, we have:

$$\frac{2\Gamma_q(\gamma) \Gamma_q(\alpha + \gamma) - \Gamma_q(\gamma + 3) \Gamma_q(\alpha)}{\Gamma_q(\alpha + \gamma) [c\Gamma_q(\gamma + 3) - 2\Gamma_q(\gamma)]} > 0.$$

Since $\gamma \in [1, \infty)$ and $r'_1 > r'_2$, $2\Gamma_q(\gamma) < \Gamma_q(\gamma + 3)$ and

$$\frac{\Gamma_q(\gamma + 3) [c\Gamma_q(\alpha + \gamma) + \Gamma_q(\alpha)] r'_1}{\Gamma_q(\alpha + \gamma) [c\Gamma_q(\gamma + 3) - 2\Gamma_q(\gamma)]} > \frac{2\Gamma_q(\gamma)\Gamma_q(\alpha + \gamma) - \Gamma_q(\gamma + 3)\Gamma_q(\alpha) r'_2}{\Gamma_q(c + \gamma) [c\Gamma_q(\gamma + 3) - 2\Gamma_q(\gamma)]},$$

therefore, $\mathcal{X}_1 \subset \overline{\mathcal{X}_2}$. If $u \in P \cap \overline{\partial\mathcal{X}_1}$, then

$$0 \leq u(t) \leq \frac{[2\Gamma_q(\gamma)\Gamma_q(\alpha + \gamma) - \Gamma_q(\gamma + 3)\Gamma_q(\alpha)] \Gamma_q(1 - \beta)}{\Gamma_q(\alpha + \gamma)\Gamma_q(\alpha - \beta + 1) [c\Gamma_q(\gamma + 3) - 2\Gamma_q(\gamma)]} r'_2$$

$\forall t \in \bar{J}$, and also

$$\begin{aligned} \Omega(u(1)) &= - \int_0^1 \frac{(1 - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} h(s, u(s)) \, d_qs \\ &\quad + \int_0^1 \frac{\Gamma_q(\gamma + 3) [c\Gamma_q(\alpha + \gamma)(1 - qs)^{(\alpha-1)} - \Gamma_q(\alpha)(1 - qs)^{(\alpha+\gamma-1)}]}{\Gamma_q(\alpha)\Gamma_q(\alpha + \gamma) [c\Gamma_q(\gamma + 3) - 2\Gamma_q(\gamma)]} \\ &\quad \times h(s, u(s)) \, d_qs \\ &\geq \int_0^1 \frac{\Gamma_q(\gamma + 3) [c\Gamma_q(\alpha + \gamma) - \Gamma_q(\alpha)] - \Gamma_q(\alpha + \gamma) [c\Gamma_q(\gamma + 3) - 2\Gamma_q(\gamma)]}{\Gamma_q(\alpha)\Gamma_q(\alpha + \gamma) [c\Gamma_q(\gamma + 3) - 2\Gamma_q(\gamma)]} \\ &\quad \times (1 - qs)^{(\alpha-1)} s^{-\beta} h(s, u(s)) \, d_qs \\ &\geq r'_2 \int_0^1 \frac{2\Gamma_q(\gamma)\Gamma_q(\alpha + \gamma) - \Gamma_q(\gamma + 3)\Gamma_q(\alpha)}{\Gamma_q(\alpha)\Gamma_q(\alpha + \gamma) [c\Gamma_q(\gamma + 3) - 2\Gamma_q(\gamma)]} (1 - qs)^{(\alpha-1)} s^{-\beta} \, d_qs \\ &= A_2 \frac{[2\Gamma_q(\gamma)\Gamma_q(\alpha + \gamma) - \Gamma_q(\gamma + 3)\Gamma_q(\alpha)] \Gamma_q(1 - \beta)}{\Gamma_q(\alpha + \gamma) [c\Gamma_q(\gamma + 3) - 2\Gamma_q(\gamma)] \Gamma_q(\alpha - \beta + 1)} = \|u\|. \end{aligned}$$

Hence, $\|\Omega(u)\| \geq \|u\|$ on $P \cap \partial\mathcal{X}_1$. If $u \in P \cap \partial\mathcal{X}_2$, then

$$\begin{aligned} \Omega(u(t)) &= \int_0^t \frac{(t - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} h(s, u(s)) \, d_qs \\ &\quad + t^2 \int_0^1 \frac{\Gamma_q(\gamma + 3) [c\Gamma_q(\alpha + \gamma)(1 - qs)^{(\alpha-1)} - \Gamma_q(\alpha)(1 - qs)^{(\alpha+\gamma-1)}]}{\Gamma_q(\alpha)\Gamma_q(\alpha + \gamma) [c\Gamma_q(\gamma + 3) - 2\Gamma_q(\gamma)]} \\ &\quad \times h(s, u(s)) \, d_qs \\ &\leq \int_0^1 \frac{\Gamma_q(\gamma + 3) [c\Gamma_q(\alpha + \gamma) + \Gamma_q(\alpha)] (1 - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)\Gamma_q(\alpha + \gamma) [c\Gamma_q(\gamma + 3) - 2\Gamma_q(\gamma)]} s^{-\beta} h(s, u(s)) \, d_qs \\ &\leq r'_1 \frac{\Gamma_q(\gamma + 3) [c\Gamma_q(\alpha + \gamma) + \Gamma_q(\alpha)]}{\Gamma_q(\alpha)\Gamma_q(\alpha + \gamma) [c\Gamma_q(\gamma + 3) - 2\Gamma_q(\gamma)]} \int_0^1 (1 - qs)^{(\alpha-1)} s^{-\beta} \, d_qs \\ &= r'_1 \frac{\Gamma_q(\gamma + 3)\Gamma_q(1 - \beta) [c\Gamma_q(\alpha + \gamma) + \Gamma_q(\alpha)]}{\Gamma_q(\alpha + \gamma)\Gamma_q(\alpha - \sigma + 1) [c\Gamma_q(\gamma + 3) - 2\Gamma_q(\gamma)]} = \|u\| \end{aligned}$$

for $t \in \bar{J}$. Thus, $\|\Omega(u)\| \leq \|u\|$ on $P \cap \partial\mathcal{X}_2$. Since the self-operator Ω defined on P is completely continuous and $P \cap (\overline{\mathcal{X}_2} \setminus \mathcal{X}_1)$ is a closed subset of P , the restriction $\Omega : P \cap (\overline{\mathcal{X}_2} \setminus \mathcal{X}_1) \rightarrow P$ is completely continuous. At present, by employing Lemma 2, Ω has a fixed point in $P \cap (\overline{\mathcal{X}_2} \setminus \mathcal{X}_1)$. By simple review, we can see that the fixed point of Ω is a positive solution for problem (1). \square

4. Illustrative Examples with Application

Some illustrative examples are provided in this section to validate our original results. At the same time, a computational technique is constructed for testing the problem (1) and (2). A simplified analysis is also studied for executing the q -Gamma function's values. As

a result, a pseudo-code that describes our simplified method is presented for calculating the q -Gamma function of order n in Algorithm A1 (for more details, see the following online resources: https://en.wikipedia.org/wiki/Q-gamma_function and <https://www.dm.uniba.it/members/garrappa/software>, accessed on 10 March 2021).

When the analytical solution is impossible to find for certain problems, we need to find the numerical approximation with a tiny step h via the implicit trapezoidal PI rule, which usually shows excellent accuracy [36]. Our numerical experiments were performed with the help of MATLAB software. Some additional supporting information are provided in Appendix A of this paper including some algorithms of the proposed method (see Algorithms A1–A5), and Tables A1–A3 present various numerical experiments to provide additional support to the validity of our results in this work.

Example 1. Consider the SFqDEq with the B.C:

$$\begin{cases} {}^c\mathcal{D}_q^{\frac{17}{5}}[u](t) + \frac{|\cos t|}{t^2} [1 + (u(t))^3] = 0, \\ \frac{15}{7}u(1) = \mathcal{I}_q^{\frac{29}{7}}[u](1), \\ u(0) = u''(0) = u'''(0) = 0 = 0, \end{cases} \tag{4}$$

for all $t \in J = (0, 1)$ and $q \in J$.

In Problem (1), define

$$\alpha = \frac{17}{5} \geq 3, \quad n = \left[\frac{17}{5}\right] + 1 = 4, \quad c = \frac{15}{7} \geq 1, \quad \gamma = \frac{29}{7} \in [1, \infty).$$

Define the continuous map:

$$h(t, u(t)) = \frac{|\cos t|}{t^2} [1 + (u(t))^3],$$

such that

$$\lim_{t \rightarrow 0^+} h(t, \cdot) = +\infty,$$

that is, h is singular at $t = 0$. In addition to, Table 1 shows that

$$2\Gamma_q(\gamma) \geq \Gamma_q(\alpha),$$

holds for each q .

Table 1. Numerical experiment for calculating $\Gamma_q(\alpha), \Gamma_q(\gamma)$ in Example 1 for $q = \frac{1}{10}, \frac{1}{2}, \frac{8}{9}$.

n	$q = \frac{1}{10}$		$q = \frac{1}{2}$		$q = \frac{8}{9}$	
	$\Gamma_q(\alpha)$	$2\Gamma_q(\gamma)$	$\Gamma_q(\alpha)$	$2\Gamma_q(\gamma)$	$\Gamma_q(\alpha)$	$2\Gamma_q(\gamma)$
1	1.1479	2.4817	2.2951	7.2266	34.0843	265.2795
2	1.1467	2.4792	2.0569	6.414	21.5589	153.3424
3	<u>1.1466</u>	<u>2.479</u>	1.9515	6.056	15.299	101.2765
4	1.1466	2.479	1.9018	5.8876	11.7053	73.0841
⋮	⋮	⋮	⋮	⋮	⋮	⋮
17	1.1466	2.479	1.8539	5.7258	3.4748	16.2557
18	1.1466	2.479	1.8539	5.7258	3.3755	15.6765
19	1.1466	2.479	<u>1.8539</u>	<u>5.7257</u>	3.2907	15.1843
20	1.1466	2.479	1.8539	5.7257	3.2177	14.7638
⋮	⋮	⋮	⋮	⋮	⋮	⋮
106	1.1466	2.479	1.8539	5.7257	2.709	11.8963
107	1.1466	2.479	1.8539	5.7257	2.709	11.8963
108	1.1466	2.479	1.8539	5.7257	2.709	11.8963
109	1.1466	2.479	1.8539	5.7257	<u>2.709</u>	<u>11.8962</u>
110	1.1466	2.479	1.8539	5.7257	2.709	11.8962

To numerically show our results, we consider the problem (2) as follows:

$$\begin{aligned}
 & \mathcal{D}_q^{\frac{10}{3}} [u](t) + \Gamma_q(5)t^{-\frac{1}{9}}|u|^{\frac{1}{3}} + \Gamma_q(4)t^{-\frac{1}{9}}|u'|^{\frac{2}{5}} \\
 & \quad + \Gamma_q(6)t^{-\frac{1}{9}}|\mathcal{D}_q^{\frac{4}{15}} [u](t)|^{\frac{3}{4}} + \Gamma_q(3)t^{-\frac{1}{9}}|v_u|^{\frac{7}{9}} \\
 & \quad + \frac{1}{1 + u^2(t)} + \frac{1}{1 + (u')^2} + \frac{1}{1 + (\mathcal{D}_q^{\frac{4}{15}} [u])^2} + \frac{1}{1 + (v_u)^2} \\
 & \leq \mathcal{D}_q^{\frac{10}{3}} [u](t) + \Gamma_q(5)t^{-\frac{1}{9}}|u|^{\frac{1}{3}} + \Gamma_q(4)t^{-\frac{1}{9}}|u'|^{\frac{2}{5}} \\
 & \quad + \Gamma_q(6)t^{-\frac{1}{9}}|\mathcal{D}_q^{\frac{4}{15}} [u](t)|^{\frac{3}{4}} + \Gamma_q(3)t^{-\frac{1}{9}}|v_u|^{\frac{7}{9}} \\
 & \quad + (u(t))^{-2} + (u')^{-2} + (\mathcal{D}_q^{\frac{4}{15}} [u])^{-2} + (v_u)^{-2} = 0.
 \end{aligned}$$

Thus,

$$\begin{aligned}
 & \mathcal{D}_q^{\frac{10}{3}} [u](t) + \Gamma_q(5)t^{-\frac{1}{9}}|u|^{\frac{1}{3}} + \Gamma_q(4)t^{-\frac{1}{9}}|u'|^{\frac{2}{5}} \\
 & \quad + \Gamma_q(6)t^{-\frac{1}{9}}|\mathcal{D}_q^{\frac{4}{15}} [u](t)|^{\frac{3}{4}} + \Gamma_q(3)t^{-\frac{1}{9}}|v_u|^{\frac{7}{9}} \\
 & \quad + (u(t))^{-2} + (u')^{-2} + (\mathcal{D}_q^{\frac{4}{15}} [u])^{-2} + (v_u)^{-2} = 0. \tag{5}
 \end{aligned}$$

Table 2 shows numerically the values of $x(t)$ in Equation (5). In addition, the curve of $x(t)$ w.r.t t in Figures 1–3 for $q = \frac{1}{10}, \frac{1}{2},$ and $\frac{6}{7},$ respectively (Algorithm A1).

Table 2. Numerical experiment of Equation (5) in Example 1 for $q \in \left\{ \frac{1}{10}, \frac{1}{2}, \frac{6}{7} \right\}$ and $n = 1, \dots, 20$ (Algorithm A1).

n	$q = \frac{1}{10}$		$q = \frac{1}{2}$		$q = \frac{6}{7}$	
	t	u(t)	t	u(t)	t	u(t)
1	$n = 1$					
1	0	0	0	0	0	0
1	0.25	0.00172	0.25	0.00806	0.25	0.38812
1	0.5	0.01733	0.5	0.08187	0.5	4.1244
1	0.75	0.06744	0.75	0.32299	0.75	17.97576
1	1	0.17909	1	0.87607	1	56.89764
2	$n = 2$					
2	0	0	0	0	0	0
2	0.25	0.00171	0.25	0.0071	0.25	0.21494
2	0.5	0.01731	0.5	0.07216	0.5	2.26527
2	0.75	0.06737	0.75	0.2846	0.75	9.69401
2	1	0.17891	1	0.77148	1	29.82949
⋮						
20	$n = 20$					
	0	0	0	0	0	0
	0.25	Inf	0.25	Inf	0.25	Inf
	0.5	Inf	0.5	Inf	0.5	Inf
	0.75	Inf	0.75	Inf	0.75	Inf
	1	Inf	1	Inf	1	Inf
	1.25	Inf	1.25	Inf	1.25	Inf
	1.5	Inf	1.5	Inf	1.5	Inf
	1.75	Inf	1.75	Inf	1.75	Inf
	⋮	⋮	⋮	⋮	⋮	⋮

We can see that all conditions of Theorem 2 hold. Thus, the fixed point of Ω is a positive solution for problem (4).

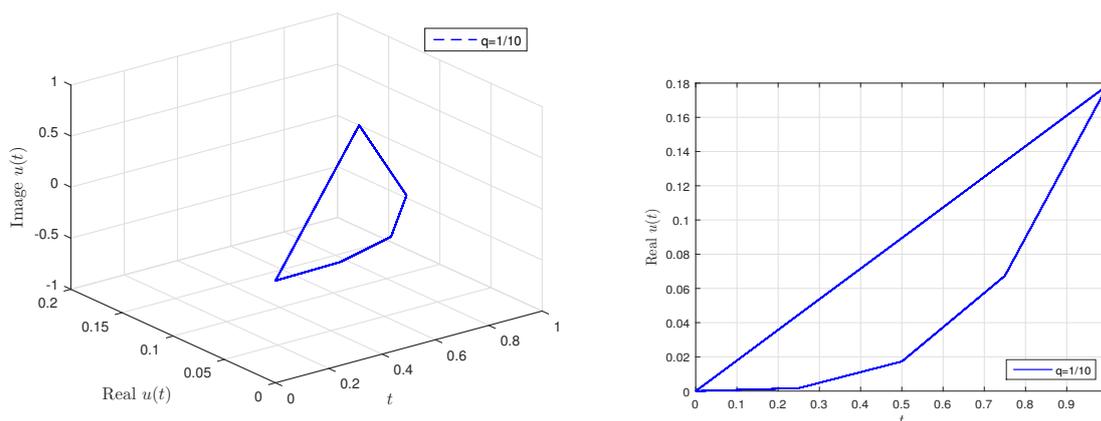


Figure 1. $u(t)$ with respect to t in Equation (5) in Example 1 for $q = \frac{1}{10}$ according to Table 2.

Linear motion is the most basic of all motion. According to Newton’s first law of motion, objects that do not experience any net force will continue to move in a straight line with a constant velocity until they are subjected to a net force. In the next example, we consider an application to examine the validity of our theoretical results on the fractional order representation of the motion of a particle along a straight line.

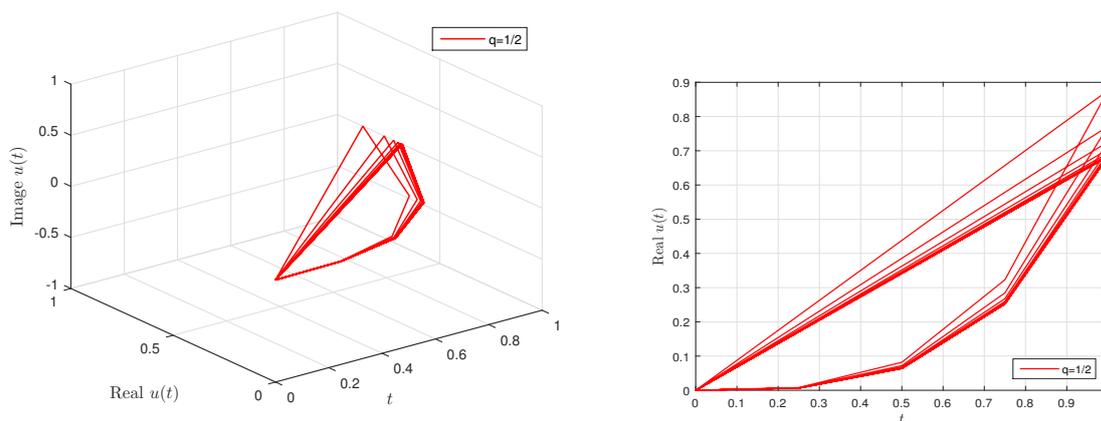


Figure 2. $u(t)$ with respect to t in Equation (5) in Example 1 for $q = \frac{1}{2}$ according to Table 2.

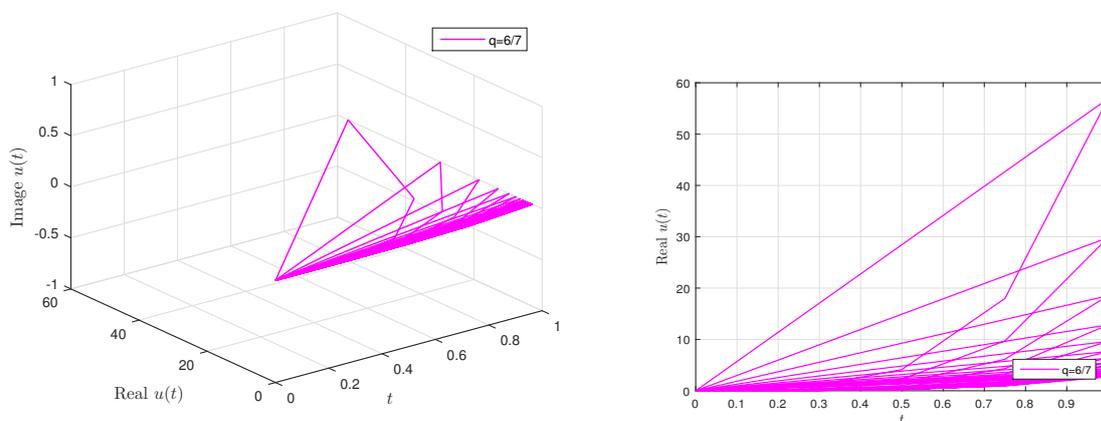


Figure 3. $u(t)$ with respect to t in Equation (5) in Example 1 for $\frac{6}{7}$ according to Table 2.

Example 2. We consider a constrained motion of a particle along a straight line restrained by two linear springs with equal spring constants (stiffness coefficient) under an external force and fractional damping along the t -axis (Figure 4).

The springs, unless subjected to force, are assumed to have free length (unstretched length) and resist a change in length. The motion of the system along the t -axis is independent of the initial spring tension. The springs are anchored on the t -axis at $t = -1$ and $t = 1$, and the vibration of the particle in this example is restricted to the t -axis only.

The vibration of the system is represented by a system of equations with the first equation having similar form of a simple harmonic oscillator, which cannot produce instability. Hence, the existence solution of the system depends on the following equation represented as the SFqDEq with the B.C:

$$\begin{cases} {}^c \mathcal{D}_q^{\frac{10}{3}} [u](t) + \frac{1}{8} [2 - 2L - \theta^2 L - \theta^2 L \cos t] u(t) = v \sin(u(t)), \\ \frac{16}{9} u(1) = \mathcal{I}_q^{\frac{23}{6}} [u](1), \\ u(0) = u''(0) = u'''(0) = 0, \end{cases} \tag{6}$$

for all $t \in J = (0, 1)$, $q \in J$. Here, θ and v are constants, and L is the unstretched length of the spring. In Problem (1),

$$\alpha = \frac{10}{3} \geq 3, \quad n = \left[\frac{10}{3} \right] + 1 = 4, \quad c = \frac{16}{9} \geq 1, \quad \gamma = \frac{23}{6} \in [1, \infty).$$

Define the continuous map:

$$h(t, u(t)) = \frac{1}{8} [2 - 2L - \theta^2 L - \theta^2 L \cos t] u(t) - v \sin(u(t))$$

for $t \in (0, 1)$, such that

$$\lim_{t \rightarrow 0^+} h(t, \cdot) = +\infty,$$

that is, h is singular at $t = 0$. Consider particular values of the parameters $L = 1.5$ m, $\theta = 0.5$. We consider particular values of the parameter $v = 7.25$. Therefore, all conditions of Theorem 2 hold. Thus, the SFqDEq (6) has a solution.

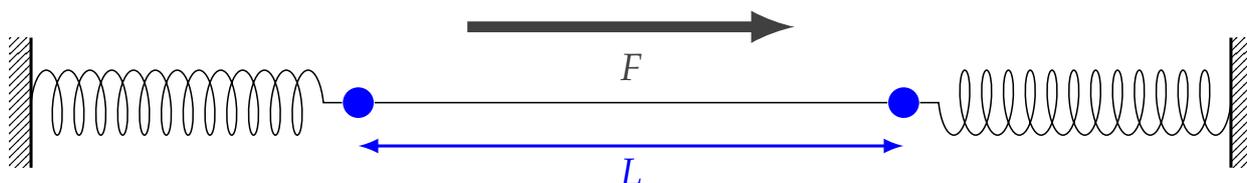


Figure 4. A particle along a straight line restrained by two linear springs with equal spring constants.

5. Conclusions

The existence of solutions was successfully investigated for a system of m -singular sum fractional q -differential equations under some integral B.Cs in the sense of CpFqDr. The positive solutions' existence was also studied with the help of a fixed point Arzelà–Ascoli theorem. Illustrative examples and numerical experiments were provided to validate our theoretical results.

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Appendix A. Supporting Information

Algorithm A1 The proposed method for calculating $\Gamma_q(x)$.

```

1: function g = qGamma(q, x, n)
2: %q-Gamma Function
3: p=1;
4: for k=0:n
5:   p=p*(1-q^(k+1))/(1-q^(x+k));
6: end;
7: g=p/(1-q)^(x-1);
8: end

```

Algorithm A2 The proposed method for calculating $(x - y)_q^{(\alpha)}$.

```

1: function p = qfunction1(x, y, q, sigma, n)
2: s=1;
3: if n==0
4:   p=1
5: else
6:   for k=1:n-1
7:     s = s*(x-y*q^k)/(x-y*q^(sigma+k));
8:   end;
9:   p=x^sigma * s;
10: end;
11: end

```

Algorithm A3 The proposed method for calculating $(D_q f)(x)$.

```

1: function g = Dq(q, x, n, fun)
2: if x==0
3:   g=limit ((fun(x)-fun(q*x))/((1-q)*x),x,0);
4: else
5:   g=(fun(x)-fun(q*x))/((1-q)*x);
6: end;
7: end

```

Algorithm A4 The proposed method for calculating $(D_q f)(x)$.

```

1: function g = Iq(q, x, n, fun)
2: p=1;
3: for k=0:n
4:   p=p+ q^k*fun(x*q^k);
5: end;
6: g=x*(1-q) * p;
7: end

```

Algorithm A5 The proposed method for calculating $I_q^\alpha[x]$.

```

1: function g = Iq_alpha(q, alpha, x, n, fun)
2: p=0;
3: for k=0:n
4:   s1=1;
5:   for i=0:k-1
6:     s1=s1*(1-q^(alpha+i));
7:   end
8:   s2=1;
9:   for i=0:k-1
10:    s2=s2*(1-q^(i+1));
11:  end
12:  p=p + q^k*s1*eval(subs(fun, t*q^k))/s2;
13: end;
14: g=round((t^alpha)* ((1-q)^alpha)* p, 6);
15: end
    
```

Table A1. Some numerical results for the calculation of $\Gamma_q(x)$ with $q = \frac{1}{3}$ that is constant, $x = 4.5, 8.4, 12.7$ and $n = 1, 2, \dots, 15$ of Algorithm A1.

n	$x = 4.5$	$x = 8.4$	$x = 12.7$	n	$x = 4.5$	$x = 8.4$	$x = 12.7$
1	2.472950	11.909360	68.080769	9	<u>2.340263</u>	11.257158	64.351366
2	2.383247	11.468397	65.559266	10	2.340250	<u>11.257095</u>	64.351003
3	2.354446	11.326853	64.749894	11	2.340245	11.257074	<u>64.350881</u>
4	2.344963	11.280255	64.483434	12	2.340244	11.257066	64.350841
5	2.341815	11.264786	64.394980	13	2.340243	11.257064	64.350828
6	2.340767	11.259636	64.365536	14	2.340243	11.257063	64.350823
7	2.340418	11.257921	64.355725	15	2.340243	11.257063	64.350822
8	2.340301	11.257349	64.352456				

Table A2. Some numerical results for the calculation of $\Gamma_q(x)$ with $q = \frac{1}{3}, \frac{1}{2}, \frac{2}{3}$, $x = 5$ and $n = 1, 2, \dots, 35$ of Algorithm A1.

n	$q = \frac{1}{3}$	$q = \frac{1}{2}$	$q = \frac{2}{3}$	n	$q = \frac{1}{3}$	$q = \frac{1}{2}$	$q = \frac{2}{3}$
1	3.016535	6.291859	18.937427	18	2.853224	4.921884	8.476643
2	2.906140	5.548726	14.154784	19	2.853224	4.921879	8.474597
3	2.870699	5.222330	11.819974	20	2.853224	4.921877	8.473234
4	2.859031	5.069033	10.537540	21	2.853224	4.921876	8.472325
5	2.855157	4.994707	9.782069	22	2.853224	4.921876	8.471719
6	2.853868	4.958107	9.317265	23	2.853224	4.921875	8.471315
7	2.853438	4.939945	9.023265	24	2.853224	4.921875	8.471046
8	<u>2.853295</u>	4.930899	8.833940	25	2.853224	4.921875	8.470866
9	2.853247	4.926384	8.710584	26	2.853224	4.921875	8.470747
10	2.853232	4.924129	8.629588	27	2.853224	4.921875	8.470667
11	2.853226	4.923002	8.576133	28	2.853224	4.921875	8.470614
12	2.853224	4.922438	8.540736	29	2.853224	4.921875	<u>8.470578</u>
13	2.853224	4.922157	8.517243	30	2.853224	4.921875	8.470555
14	2.853224	4.922016	8.501627	31	2.853224	4.921875	8.470539
15	2.853224	4.921945	8.491237	32	2.853224	4.921875	8.470529
16	2.853224	4.921910	8.484320	33	2.853224	4.921875	8.470522
17	2.853224	<u>4.921893</u>	8.479713	34	2.853224	4.921875	8.470517

Table A3. Some numerical results for the calculation of $\Gamma_q(x)$ with $x = 8.4$, $q = \frac{1}{3}, \frac{1}{2}, \frac{2}{3}$ and $n = 1, 2, \dots, 40$ of Algorithm A1.

n	$q = \frac{1}{3}$	$q = \frac{1}{2}$	$q = \frac{2}{3}$	n	$q = \frac{1}{3}$	$q = \frac{1}{2}$	$q = \frac{2}{3}$
1	11.909360	63.618604	664.767669	21	11.257063	49.065390	260.033372
2	11.468397	55.707508	474.800503	22	11.257063	49.065384	260.011354
3	11.326853	52.245122	384.795341	23	11.257063	49.065381	259.996678
4	11.280255	50.621828	336.326796	24	11.257063	49.065380	259.986893
5	11.264786	49.835472	308.146441	25	11.257063	49.065379	259.980371
6	11.259636	49.448420	290.958806	26	11.257063	49.065379	259.976023
7	11.257921	49.256401	280.150029	27	11.257063	49.065379	259.973124
8	11.257349	49.160766	273.216364	28	11.257063	49.065378	259.971192
9	11.257158	49.113041	268.710272	29	11.257063	49.065378	259.969903
10	11.257095	49.089202	265.756606	30	11.257063	49.065378	259.969044
11	11.257074	49.077288	263.809514	31	11.257063	49.065378	259.968472
12	11.257066	49.071333	262.521127	32	11.257063	49.065378	259.968090
13	11.257064	49.068355	261.666471	33	11.257063	49.065378	259.967836
14	11.257063	49.066867	261.098587	34	11.257063	49.065378	259.967666
15	11.257063	49.066123	260.720833	35	11.257063	49.065378	259.967553
16	11.257063	49.065751	260.469369	36	11.257063	49.065378	259.967478
17	11.257063	49.065564	260.301890	37	11.257063	49.065378	259.967427
18	11.257063	49.065471	260.190310	38	11.257063	49.065378	259.967394
19	11.257063	49.065425	260.115957	39	11.257063	49.065378	259.967371
20	11.257063	49.065402	260.066402	40	11.257063	49.065378	259.967357

References

- Beyer, H.; Kempfle, S. Definition of physical consistent damping laws with fractional derivatives. *Z. Angew. Math. Mech.* **1995**, *75*, 623–635.
- He, J.H. Some applications of nonlinear fractional differential equations and their approximations. *Sci. Technol. Soc.* **1999**, *15*, 86–90.
- He, J.H. Approximate analytic solution for seepage flow with fractional derivatives in porous media. *Comput. Methods Appl. Mech. Eng.* **1998**, *167*, 57–68.
- Caputo, M. Linear models of dissipation whose Q is almost frequency independent-II. *Geophys. J. Int.* **1967**, *13*, 529–539.
- Yan, J.P.; Li, C.P. On chaos synchronization of fractional differential equations. *Chaos Solitons Fractals* **2007**, *32*, 725–735.
- Sommacal, L.; Melchior, P.; Dossat, A.; Petit, J.; Cabelguen, J.M.; Oustaloup, A.; Ijspeert, A.J. Improvement of the muscle fractional multimodel for low-rate stimulation. *Biomed. Signal Process. Control* **2007**, *2*, 226–233.
- Rezapour, S.; Mojammadi, H.; Samei, M.E. SEIR epidemic model for COVID-19 transmission by Caputo derivative of fractional order. *Adv. Differ. Equ.* **2020**, *2020*, 490, doi:10.1186/s13662-020-02952-y.
- Silva, M.F.; Machado, J.A.T.; Lopes, A.M. Fractional order control of a hexapod robot. *Nonlinear Dyn.* **2004**, *38*, 417–433.
- Mathieu, B.; Melchior, P.; Oustaloup, A.; Ceyral, C. Fractional differentiation for edge detection. *Signal Process* **2003**, *83*, 2421–2432.
- Diethelm, K.; Ford, N.J. Analysis of Fractional Differential Equations. *J. Math. Anal. Appl.* **2002**, *265*, 229–248.
- Mishra, S.K.; Samei, M.E.; Chakraborty, S.K.; Ram, B. On q -variant of Dai–Yuan conjugate gradient algorithm for unconstrained optimization problems. *Nonlinear Dyn.* **2021**, *2021*, 35, doi:10.1007/s11071-021-06378-3.
- Ntouyas, S.K.; Samei, M.E. Existence and uniqueness of solutions for multi-term fractional q -integro-differential equations via quantum calculus. *Adv. Differ. Equ.* **2019**, *2019*, 475, doi:10.1186/s13662-019-2414-8.
- Lakshmikantham, V.; Vatsala, A.S. Basic theory of fractional differential equations. *Nonlinear Anal. Theory Methods Appl.* **2008**, *69*, 2677–2682.
- Samei, M.E. Existence of solutions for a system of singular sum fractional q -differential equations via quantum calculus. *Adv. Differ. Equ.* **2020**, *2020*, 23, doi:10.1186/s13662-019-2480-y.
- Matar, M.M.; Abbas, M.I.; Alzabut, J.; Kaabar, M.K.A.; Etemad, S.; Rezapour, S. Investigation of the p -Laplacian nonperiodic nonlinear boundary value problem via generalized Caputo fractional derivatives. *Adv. Differ. Equ.* **2021**, *2021*, 68.
- Jackson, F. q -difference equations. *Am. J. Math.* **1910**, *32*, 305–314, doi:10.2307/2370183.
- Adams, C. The general theory of a class of linear partial q -difference equations. *Trans. Am. Math. Soc.* **1924**, *26*, 283–312.
- Adams, C. Note on the integro- q -difference equations. *Trans. Am. Math. Soc.* **1929**, *31*, 861–867.
- Ahmad, B.; Nieto, J.J.; Alsaedi, A.; Al-Hutami, H. Existence of solutions for nonlinear fractional q -difference integral equations with two fractional orders and nonlocal four-point boundary conditions. *J. Franklin Inst.* **2014**, *351*, 2890–2909.
- Ahmad, B.; Ntouyas, S.; Purnaras, I. Existence results for nonlocal boundary value problems of nonlinear fractional q -difference equations. *Adv. Differ. Equ.* **2012**, *2012*, 140, doi:10.1186/1687-1847-2012-140.

21. Balkani, N.; Rezapour, S.; Haghi, R.H. Approximate solutions for a fractional q -integro-difference equation. *J. Math. Ext.* **2019**, *13*, 201–214.
22. Ferreira, R. Nontrivial solutions for fractional q -difference boundary value problems. *Electron. J. Qual. Theory Differ. Equ.* **2010**, *70*, 1–101.
23. Samei, M.E.; Ranjbar, G.K.; Hedayati, V. Existence of solutions for equations and inclusions of multi-term fractional q -integro-differential with non-separated and initial boundary conditions. *J. Inequalities Appl.* **2019**, *2019*, 273, doi:10.1186/s13660-019-2224-2.
24. Zhao, Y.; Chen, H.; Zhang, Q. Existence results for fractional q -difference equations with nonlocal q -integral boundary conditions. *Adv. Differ. Equ.* **2013**, *2013*, 48, doi:10.1186/1687-1847-2013-48.
25. Agarwal, R.P.; Baleanu, D.; Hedayati, V.; Rezapour, S. Two fractional derivative inclusion problems via integral boundary condition. *Appl. Math. Comput.* **2015**, *257*, 205–212.
26. Akbari Kojabad, E.; Rezapour, S. Approximate solutions of a sum-type fractional integro-differential equation by using Chebyshev and Legendre polynomials. *Adv. Differ. Equ.* **2017**, *2017*, 351.
27. Liang, S.; Samei, M.E. New approach to solutions of a class of singular fractional q -differential problem via quantum calculus. *Adv. Differ. Equ.* **2020**, *2020*, 14, doi:10.1186/s13662-019-2489-2.
28. Aydogan, S.M.; Baleanu, D.; Mousalou, A.; Rezapour, S. On high order fractional integro-differential equations including the Caputo-Fabrizio derivative. *Bound. Value Probl.* **2018**, *2018*, 90.
29. Baleanu, D.; Rezapour, S.; Etemad, S.; Alsaedi, A. On a time-fractional integro-differential equation via three-point boundary value conditions. *Math. Probl. Eng.* **2015**, *2015*, 12.
30. He, C.Y. *Almost Periodic Differential Equations*; Higher Education Press: Beijing, China, 1992. (In Chinese)
31. Barbălat, I. Systems d'equations differential d'oscillations nonlineaires. *Rev. Roumaine Math. Pure Appl.* **1959**, *4*, 267–270.
32. Samei, M.E.; Hedayati, V.; Rezapour, S. Existence results for a fractional hybrid differential inclusion with Caputo-Hadamard type fractional derivative. *Adv. Differ. Equ.* **2019**, *2019*, 163.
33. Annaby, M.; Mansour, Z. *q -Fractional Calculus and Equations*; Springer: Heidelberg, Germany; Cambridge, UK, 2012; doi:10.1007/978-3-642-30898-7.
34. Samko, S.; Kilbas, A.; Marichev, O. *Fractional Integrals and Derivatives: Theory and Applications*; Gordon and Breach Science Publishers: Lausanne, Switzerland; Philadelphia, PA, USA, 1993.
35. Krasnoselskii, M.A. *Positive Solution of Operator Equation*; Noordhoff: Groningen, The Netherlands, 1964.
36. Garrappa, R. Numerical Solution of Fractional Differential Equations: A Survey and a Software Tutorial. *Mathematics* **2018**, *6*, 16, doi:10.3390/math6020016.