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Applications of Generalized q-Difference Equations for General q-Polynomials

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Abstract: Andrews gave a remarkable interpretation of the Rogers–Ramanujan identities with the polynomials $\rho_e(N, y, x, q)$, and it was noted that $\rho_e(\infty, -1, 1, q)$ is the generation of the fifth-order mock theta functions. In the present investigation, several interesting types of generating functions for this q-polynomial using q-difference equations is deduced. Besides that, a generalization of Andrew's result in form of a multilinear generating function for q-polynomials is also given. Moreover, we build a transformation identity involving the q-polynomials and Bailey transformation. As an application, we give some new Hecke-type identities. We observe that most of the parameters involved in our results are symmetric to each other. Our results are shown to be connected with several earlier works related to the field of our present investigation.

Keywords: q-difference equations; q-polynomial; generating function; Hecke-type series; Bailey transformation

MSC: Primary 30C45; 30C50; 30C80; Secondary 11B65; 47B38

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1. Introduction and Motivation

Andrews [1,2] established and found a nice relationship of the fifth mock theta function with the q-Jacobi polynomials. To improve the development of Hecke-type series for the fifth order mock theta function, Andrews systematically considered the series:

$$\sum_{n=0}^{\infty} \frac{q^{n^2} x^n f_n(y;q)}{(q^2;q^2)_n}.$$

where

$$f_n(0;q) = 1 \quad \text{and} \quad f_n(1;q) = (-q;q)_n$$

and $(\alpha;q)_n$ stands for the q-shifted factorial.

If we take different choices for f, we obtain a variety of alternating parity questions connected with classical partition identities of Euler, Rogers, Ramanujan and Gordan.

Furthermore, Andrews [2] also gave a new natural interpretation of the fifth-order mock theta functions along with a new proof of the Hecke-type series representation. Recently, several well-known mathematicians and physicists studied these types of celebrated Hecke-type series from different mathematical view-points and perspectives. See, for example, [3–6] and the references cited therein.

Basic (or q-) polynomials and q-series, especially the q-hypergeometric polynomials, play an important rule in many diverse areas of mathematics and physics. Particularly, in the Theory of Partitions, Quantum Mechanics, Lie Theory, Mechanical Engineering, Combinatorial Analysis, Theory of Heat Conduction, Cosmology, Non-Linear Electric Circuit Theory, Particle Physics, Finite Vector Spaces and Statistics (see, for example, [7] (pp. 350, 351); see also [8–16]).

In this paper, motivated by Srivastava et al.'s published paper in *Symmetry* (see [17]), by using the method of q-difference equations, we consider and generalize the polynomials

$$\rho_e(N,y,x,q)=\sum_{k=0}^{N}\begin{bmatrix}N\\k\end{bmatrix}_{q^2}q^{k^2}(-yq;q)_k x^k, \tag{1}$$

and $L_{\bar{m},\bar{n}}(\alpha,x,z,a)$. We also give some new applications for these polynomials.

Throughout this paper, we use the following standard q-notations (see [1,18]). For $|q|<1$, we define the q-shifted factorials as:

$$(a;q)_0=1,\quad (a;q)_n=\prod_{k=0}^{n-1}(1-aq^k),\quad (a;q)_\infty=\prod_{k=0}^{\infty}(1-aq^k).$$

For convenience, we also adopt the following compact notation for the multiple q-shifted factorial:

$$(a_1,a_2,\ldots,a_m;q)_n=(a_1;q)_n(a_2;q)_n\ldots(a_m;q)_n,$$

where n is an integer or ∞. The basic hypergeometric series ${}_r\phi_s$ is defined as:

$${}_r\phi_s\left(\begin{matrix}a_1,a_2,\ldots,a_r\\ b_1,b_2,\ldots,b_s\end{matrix};q,z\right)=\sum_{n=0}^{\infty}\frac{(a_1,a_2,\ldots,a_r;q)_n}{(q,b_1,b_2,\ldots,b_s;q)_n}\left((-1)^n q^{n(n-1)/2}\right)^{1+s-r}z^n.$$

The q-binomial theorem is stated as:

$$\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}z^n=\frac{(az;q)_\infty}{(z;q)_\infty}.$$

One can see that, the following Euler identities:

$$\sum_{n=0}^{\infty}\frac{z^n}{(q;q)_n}=\frac{1}{(z;q)_\infty}\quad\text{and}\quad\sum_{n=0}^{\infty}\frac{(-z)^n q^{\binom{n}{2}}}{(q;q)_n}=(z;q)_\infty.$$

are the two important special and limiting cases of the q-binomial theorem.

In upcoming parts of this investigation, we need an important transformation formula for ${}_2\phi_1$, which is given by:

$${}_2\phi_1\left(\begin{matrix}a,b\\ c\end{matrix};q,c/ab\right)=\frac{(c/a,c/b;q)_\infty}{(c,c/ab;q)_\infty}. \tag{2}$$

Another useful formula is (see [19]):

$${}_2\phi_1\left(\begin{matrix}a,b\\ cq\end{matrix};q,c/ab\right)=\frac{(cq/a,cq/b;q)_\infty}{(cq,cq/ab;q)_\infty}\left[\frac{ab(1+c)-c(a+b)}{ab-c}\right]. \tag{3}$$

We also need the q-Pfaff-*Saalschütz* summation formula:

$${}_3\phi_2\left(\begin{matrix}q^{-n},a,b\\ c,abq^{1-n}/c\end{matrix};q,q\right)=\frac{(c/a,c/b;q)_n}{(c,c/ab;q)_n}. \tag{4}$$

Many researcher used the following beautiful transformation formula of Liu [20]:

$$
\begin{aligned}
{}_3\phi_2\left(\begin{matrix} q/a, q/b, v \\ c, d \end{matrix} ; q, uab/q\right) &= \frac{(ua, ub; q)_\infty}{(uq, uab/q; q)_\infty} \\
&\times \sum_{n=0}^{\infty} \frac{(u, q/a, q/b; q)_n (1-uq^{2n})}{(q, , ua, ub; q)_n (1-u)} (-uab)^n \\
&\times q^{n^2-2n} {}_3\phi_2\left(\begin{matrix} q^{-n}, uq^n, v \\ c, d \end{matrix} ; q, q\right)
\end{aligned} \tag{5}
$$

to obtain some kind of Hecke equation. In Section 6, we also get a transformation identity involving Hecke-type series for the generalized q-polynomials based on Bailey transformation.

Furthermore, the q-binomial coefficients are defined by:

$$
\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}, \quad \begin{bmatrix} \alpha \\ k \end{bmatrix}_{-q} = \frac{(-q;q)_\alpha}{(-q;q)_k (-q;q)_{\alpha-k}}, \quad \begin{bmatrix} n \\ k \end{bmatrix}_{q^2} = \frac{(q^2;q^2)_n}{(q^2;q^2)_k (q^2;q^2)_{n-k}}.
$$

Next, for any function $f(x)$, the q-derivative of $f(x)$ with respect to x, is defined as:

$$
\mathcal{D}_{q,x}\{f(x)\} = \frac{f(x) - f(qx)}{(1-q)x}.
$$

Furthermore, we define

$$
\mathcal{D}^0_{q,x}\{f(x)\} = f(x),
$$

and

$$
\mathcal{D}^n_{q,x}\{f(x)\} = \mathcal{D}_q\{\mathcal{D}^{n-1}_{q,x}\{f(x)\}\} \qquad (n \geq 1).
$$

The Rogers–Szegö polynomials are given by (see [21]):

$$
h_n(z, x|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} z^k x^{n-k} \tag{6}
$$

and

$$
g_n(z, x|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{n(k-n)} z^k x^{n-k}. \tag{7}
$$

Chen, Fu and Zhang [22] introduced the following homogeneous Rogers–Szegö polynomials:

$$
\bar{h}_n(x, y|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} p_k(x, y)
$$

and

$$
p_k(x, y) = (x-y)(x-yq)\ldots(x-yq^{k-1}).
$$

Furthermore, Chen et al. [22] consider some remarkable results for these polynomials.

Motivated by Liu [20] and Cigler [23], Cao and Niu [24] studied the extension of Cigler's polynomials by the q-difference equations:

$$
C_n^{(\alpha)}(x, b) = \sum_{k=0}^{n} \begin{bmatrix} n+\alpha \\ k \end{bmatrix} \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k q^{\binom{k}{2}} (q;q)_k x^{n-k} b^k
$$

and

$$
D_n^{(\alpha)}(x, b) = \sum_{k=0}^{n} \begin{bmatrix} n+\alpha \\ k \end{bmatrix} \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k q^{k^2-kn} (q;q)_k x^{n-k} b^k.
$$

Based on Andrews [2] results, we now give a new q-polynomial as follows:

$$L_{\bar{m},\bar{n}}(\alpha,x,z,a)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\begin{bmatrix}\alpha\\k\end{bmatrix}_{-q}q^{\tau(\bar{m},\bar{n})+\binom{k}{2}}(a;q)_k z^k x^{n-k},\tag{8}$$

where $\tau(\bar{m},\bar{n})=\bar{m}\binom{k}{2}-\bar{n}\binom{k+1}{2}$ and $\bar{m},\bar{n}$ are real numbers.

Remark 1. *First of all, it is easy to see that, if in (8), we take*

$$\alpha=n\in\mathbb{Z},\quad \bar{m}=0,\quad \bar{n}=-1,\quad a=-yq,\quad x=1\quad \textit{and}\quad z\to x,$$

then we obtain (1). Secondly, for

$$\alpha=\infty,\qquad \bar{m}=-1,\qquad \bar{n}=0\qquad \textit{and}\qquad a=-q$$

in (8), we recover (6). Thirdly, if we put

$$\alpha=\infty,\quad \bar{m}=-1,\quad \bar{n}=0,\quad x=xq^{-n}\quad \textit{and}\quad a=-q$$

in (8), then we have (7). Finally, if

$$\alpha=n=\infty,\qquad x=1\quad \textit{and}\quad a=-q$$

in (8), we have the following results:

$$\sum_{j=0}^{\infty}\frac{q^{j^2}}{(q^2;q^2)_j}(-q;q)_j=\sum_{j=0}^{\infty}\frac{q^{j^2}}{(q;q)_j}=\frac{1}{(q,q^4;q^5)_\infty}\qquad(z=1)$$

and

$$\sum_{j=0}^{\infty}\frac{q^{j^2+j}}{(q^2;q^2)_j}(-q;q)_j=\sum_{j=0}^{\infty}\frac{q^{j^2+j}}{(q;q)_j}=\frac{1}{(q^2,q^3;q^5)_\infty}\qquad(z=q)$$

By using analytic function, Liu [25] established and considered the key relation between the q-exponential operator and q-difference equations. While using the homogeneous q-difference equations Cao [26] gave the generating functions for q-hypergeometric polynomials. Recently, Cao et al. [27] also deduced the generalized q-difference equations for general Al-Salam–Carlitz polynomials. It was observed that the method of q-difference operator is effective in solving generating functions for certain q-orthogonal polynomials (see for example [22,28–31]).

Liu [20] obtained several important results on Rogers–Szegö polynomials by q-difference equations with two variables as describe in Proposition 1. Furthermore, Liu and Zeng [32] further studied the relations between the q-difference equations and Rogers–Szegö polynomials. Indeed, they studied if an analytic function in several variables satisfies a system of q-difference equations, then it can be expanded in terms of the product of some polynomials.

Proposition 1. *Let $f(a,b)$ be a two-variable analytic function at $(0,0)\in\mathbb{C}^2$, then*

(A) f can be expanded in terms of $h_n(a,b|q)$ if and only if f satisfies the functional equation

$$bf(aq,b)-af(a,bq)=(b-a)f(a,b).$$

(B) f can be expanded in terms of $g_n(a,b|q)$ if and only if f satisfies the functional equation

$$af(aq,b)-bf(a,bq)=(a-b)f(aq,bq).$$

Our further investigation is organized as follows. In Section 2, we obtain the generalization of the q-polynomials (8). In Section 3, we give the generating functions for the generalized q-polynomials with six parameters by the method of q-difference equations. In Section 4, we gain a mixed generating functions for the generalized q-polynomials. In Section 5, we deduce a multilinear generating function for the generalized q-2D Hermite polynomials as a generalization of Andrews's result. In Section 6, we build a transformation identity involving the generalized q-polynomials by using our transformation via Bailey transform. As applications, a transformational identity is given in regard to some Hecke-type identities. We note that the parameters involved in the results of this section are symmetric to each other.

2. A Main Result

Theorem 1. *Let $f(a,b,c,d,e,f)$ be a 6-variable analytic function at $(0,0,0,0,0,0) \in \mathbb{C}^6$, then f can be expanded in terms of $L_{\bar{m},\bar{n}}(\alpha,x,z,a)$ if and only if f satisfies the functional equation*

$$\begin{aligned} &x\Big[f(\alpha,x,a,z,\bar{m},\bar{n}) - f(\alpha,x,a,zq^2,\bar{m},\bar{n})\Big] \\ &\quad = q^{\alpha-\bar{n}} z(1-fq^{1-\alpha})[f(\alpha,x,a,zq^{\bar{m}-\bar{n}},\bar{m},\bar{n}) \\ &\qquad - f(\alpha,xq,a,zq^{\bar{m}-\bar{n}},\bar{m},\bar{n})] + zq^{-\bar{n}}(1-fq^{\alpha+1-2\bar{n}}) \\ &\quad \times [f(\alpha,x,a,zq^{1+\bar{m}-\bar{n}},\bar{m},\bar{n}) - f(\alpha,xq,a,zq^{1+\bar{m}-\bar{n}},\bar{m},\bar{n})] \\ &\qquad - azq^{1-\bar{n}}[f(\alpha,x,a,zq^{\bar{m}-\bar{n}+2},\bar{m},\bar{n}) - f(\alpha,xq,a,zq^{\bar{m}-\bar{n}+2},\bar{m},\bar{n})]. \end{aligned} \tag{9}$$

Proof. From the theory of several complex variables, we assume that:

$$f(\alpha,x,a,z,\bar{m},\bar{n}) = \sum_{k=0}^{\infty} A_k(\alpha,x,a,\bar{m},\bar{n}) z^k. \tag{10}$$

Substituting (10) into (9), we have

$$\begin{aligned} &x\left\{\sum_{k=0}^{\infty} A_k(\alpha,x,a,\bar{m},\bar{n}) z^k - \sum_{k=0}^{\infty} A_k(\alpha,x,a,\bar{m},\bar{n})(zq^2)^k\right\} \\ &= q^{\alpha-\bar{n}} z \sum_{k=0}^{\infty}\left\{A_k(\alpha,x,a,\bar{m},\bar{n})(zq^{\bar{m}-\bar{n}})^k - A_k(\alpha,xq,a,\bar{m},\bar{n})(zq^{\bar{m}-\bar{n}})^k\right\} \\ &+ (-azq^{\alpha+1-\bar{n}}) \sum_{k=0}^{\infty}\left\{A_k(\alpha,x,a,\bar{m},\bar{n})(zq^{1+\bar{m}-\bar{n}})^k - A_k(\alpha,xq,a,\bar{m},\bar{n})(zq^{1+\bar{m}-\bar{n}})^k\right\} \\ &+ q^{-\bar{n}} z \sum_{k=0}^{\infty}\left\{A_k(\alpha,x,a,\bar{m},\bar{n})(zq^{1+\bar{m}-\bar{n}})^k - A_k(\alpha,xq,a,\bar{m},\bar{n})(zq^{1+\bar{m}-\bar{n}})^k\right\} \\ &+ (-aq^{1-\bar{n}}) z \sum_{k=0}^{\infty}\left\{A_k(\alpha,x,a,\bar{m},\bar{n})(zq^{2+\bar{m}-\bar{n}})^k - A_k(\alpha,xq,a,\bar{m},\bar{n})(zq^{2+\bar{m}-\bar{n}})^k\right\}. \end{aligned} \tag{11}$$

By direct calculation, equating coefficients of z^k on both sides of (11), we obtain

$$A_k(\alpha,x,a,\bar{m},\bar{n}) = \frac{(q^\alpha + q^{k-1})(1-aq^k)}{(1-q^k)(1+q^k)} q^{\bar{m}(k-1)-\bar{n}k} D_{q,x}\{A_{k-1}(\alpha,x,a,\bar{m},\bar{n})\}.$$

Repeating this process, we have

$$A_k(\alpha,x,a,\bar{m},\bar{n}) = \frac{q^{\alpha k}(-q^{-\alpha};q)_k (a;q)_k}{(q^2;q^2)_k} q^{\bar{m}\binom{k}{2}-\bar{n}\binom{k+1}{2}} D_{q,x}^k\{A_0(\alpha,x,a,\bar{m},\bar{n})\}.$$

Setting
$$f(\alpha,0,x,a,\bar{m},\bar{n}) = A_0(\alpha,x,a,\bar{m},\bar{n}) = \sum_{n=0}^{\infty} \mu_n x^n,$$
we have
$$\begin{aligned} A_k(\alpha,x,a,\bar{m},\bar{n}) &= \frac{q^{\alpha k}(-q^{-\alpha};q)_k(a;q)_k}{(q^2;q^2)_k} q^{\bar{m}\binom{k}{2}-\bar{n}\binom{k+1}{2}}(D^k_{q,x})\{A_0(\alpha,x,a,\bar{m},\bar{n})\} \\ &= \frac{q^{\alpha k}(-q^{-\alpha};q)_k(a;q)_k}{(q^2;q^2)_k} q^{\bar{m}\binom{k}{2}-\bar{n}\binom{k+1}{2}} \sum_{n=0}^{\infty} \mu_n \left\{D^k_{q,x}(x^n)\right\} \\ &= \frac{q^{\alpha k}(-q^{-\alpha};q)_k(a;q)_k}{(q^2;q^2)_k} q^{\bar{m}\binom{k}{2}-\bar{n}\binom{k+1}{2}} \sum_{n=k}^{\infty} \mu_n \begin{bmatrix} n \\ k \end{bmatrix} (q;q)_k x^{n-k}. \end{aligned} \tag{12}$$

By using (10), we have
$$\begin{aligned} &f(\alpha,z,x,a,\bar{m},\bar{n}) \\ &\quad= \sum_{k=0}^{\infty} \frac{q^{\alpha k}(-q^{-\alpha};q)_k(a;q)_k}{(q^2;q^2)_k} q^{\bar{m}\binom{k}{2}-\bar{n}\binom{k+1}{2}} \sum_{n=k}^{\infty} \mu_n \{D^k_{q,x}(x^n)\} z^k \\ &\quad= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \mu_n \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} \alpha \\ k \end{bmatrix}_{-q} q^{\tau(\bar{m},\bar{n})+\binom{k}{2}} (a;q)_k z^k x^{n-k} \\ &\quad= \sum_{n=0}^{\infty} \mu_n L_{\bar{m},\bar{n}}(\alpha,x,z,a). \end{aligned} \tag{13}$$

Which completes the proof of Theorem 1. □

3. Generating Function for the Generalized q-Polynomials

Al-Salam and Carlitz [21] gave the generating function by using q-transformation as stated by Theorem 2. In this section, we give the generating function of the generalized q-polynomials by using q-difference equation.

Theorem 2 (see [21]). *If $h_n(z,x)$ is Rogers–Szegö polynomials, then we have*
$$\sum_{n=0}^{\infty} h_n(z,x)\frac{t^n}{(q;q)_n} = \frac{1}{(zt,xt;q)_\infty} \qquad (\max\{|zt|,|xt|\} < 1) \tag{14}$$

Theorem 3. *For* $\max\{|z|,|t|\} < 1$ *, we have*
$$\begin{aligned} &\sum_{n=0}^{\infty} L_{\bar{m},\bar{n}}(\alpha,x,z,a)\frac{t^n}{(q;q)_n} \\ &\qquad= \frac{1}{(tx;q)_\infty}\sum_{k=0}^{\infty}\frac{(-q^{-\alpha},a;q)_k}{(q^2;q^2)_k} q^{\bar{m}\binom{k}{2}-\bar{n}\binom{k+1}{2}+\alpha k}(zt)^k \end{aligned} \tag{15}$$

Proof. Denoting the right hand side of (15) by $f(\alpha,z,x,a,\bar{m},\bar{n})$, we verify that $f(\alpha,z,x,a,\bar{m},\bar{n})$ satisfies (9) as given below:
$$\begin{aligned} f(\alpha,z,x,a,\bar{m},\bar{n}) &= \frac{1}{(tx;q)_\infty}\sum_{k=0}^{\infty}\frac{(-q^{-\alpha},a;q)_k}{(q^2;q^2)_k} q^{\bar{m}\binom{k}{2}-\bar{n}\binom{k+1}{2}+\alpha k}(zt)^k \\ &= \sum_{k=0}^{\infty} \begin{bmatrix} \alpha \\ k \end{bmatrix}_{-q} q^{\tau(\bar{m},\bar{n})+\binom{k}{2}} \frac{(a;q)_k}{(q;q)_k}(tz)^k \sum_{s=0}^{\infty}\frac{(xt)^s}{(q;q)_s} \\ &= \sum_{n=0}^{\infty}\frac{t^n}{(q;q)_n}\sum_{k=0}^{n} \begin{bmatrix} \alpha \\ k \end{bmatrix}_{-q} q^{\tau(\bar{m},\bar{n})+\binom{k}{2}} \frac{(a;q)_k}{(q;q)_k}(z)^k(D_{q,x})^k(x^n), \end{aligned}$$

so we have

$$f(\alpha, z, x, a, \bar{m}, \bar{n}) = \sum_{n=0}^{\infty} \mu_n L_{\bar{m},\bar{n}}(\alpha, x, z, a)$$

and

$$f(\alpha, 0, x, a, \bar{m}, \bar{n}) = \sum_{n=0}^{\infty} \mu_n x^n = \frac{1}{(xt;q)_\infty} = \sum_{n=0}^{\infty} \frac{(xt)^n}{(q;q)_n}.$$

Thus, we have

$$f(\alpha, z, x, a, \bar{m}, \bar{n}) = \sum_{n=0}^{\infty} L_{\bar{m},\bar{n}}(\alpha, x, z, a) \frac{t^n}{(q;q)_n},$$

which completes the proof. □

Remark 2. *For $\alpha \to \infty$, $\bar{m} = -1$, $\bar{n} = 0$ and $a = -q$ in Theorem 3, Equation (15) reduces to (14).*

4. A Mixed Generating Function for the Generalized q-Polynomials

Using the q-binomial theorem, we have the following proposition.

Proposition 2. *If $|x| < 1$, we have*

$$\sum_{n=0}^{\infty} p_n(t, s|q) \frac{x^n}{(q;q)_n} = \frac{(sx;q)_\infty}{(xt;q)_\infty}. \tag{16}$$

In this section, we obtain the following mixed generating function for the generalized q-polynomials.

Theorem 4. *For $\max\{|z|, |x|, |t|\} < 1$, we have*

$$\sum_{n=0}^{\infty} \frac{p_n(t,s)}{(q;q)_n} L_{\bar{m},\bar{n}}(\alpha, x, z, a) = \frac{(sx;q)_\infty}{(xt;q)_\infty} \sum_{k=0}^{\infty} \begin{bmatrix} \alpha \\ k \end{bmatrix}_{-q} q^{\tau(\bar{m},\bar{n}) + \binom{k}{2}} \frac{(a, s/t;q)_k}{(q, xs;q)_k} (zt)^k \tag{17}$$

Proof. Denoting the right hand side of (17) as $f(\alpha, x, a, z, \bar{m}, \bar{n})$, we have

$$\begin{aligned} f(\alpha, x, a, z, \bar{m}, \bar{n}) &= \frac{(sx;q)_\infty}{(xt;q)_\infty} \sum_{k=0}^{\infty} \begin{bmatrix} \alpha \\ k \end{bmatrix}_{-q} q^{\tau(\bar{m},\bar{n}) + \binom{k}{2}} \frac{(a, s/t;q)_k}{(q, xs;q)_k} (tz)^k \\ &= \sum_{k=0}^{\infty} \begin{bmatrix} \alpha \\ k \end{bmatrix}_{-q} q^{\tau(\bar{m},\bar{n}) + \binom{k}{2}} \frac{(a;q)_k}{(q;q)_k} z^k \frac{(xsq^k;q)_\infty}{(xt;q)_\infty} t^k (s/t;q)_k \\ &= \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \begin{bmatrix} \alpha \\ k \end{bmatrix}_{-q} q^{\tau(\bar{m},\bar{n}) + \binom{k}{2}} \frac{(a;q)_k}{(q;q)_k} z^k x^{n-k} \frac{p_n(t,s)}{(q;q)_n (q;q)_{n-k}}. \end{aligned} \tag{18}$$

It is verified that (18) satisfies (9) and so, we have

$$f(\alpha, x, a, z, \bar{m}, \bar{n}) = \sum_{n=0}^{\infty} \mu_n L_{\bar{m},\bar{n}}(\alpha, x, z, a)$$

and

$$f(\alpha, x, a, 0, \bar{m}, \bar{n}) = \frac{(sx;q)_\infty}{(xt;q)_\infty} = \sum_{n=0}^{\infty} \frac{(x)^n (s/t;q)_n}{(q;q)_n} = \sum_{n=0}^{\infty} \frac{p_n(t,s) x^n}{(q;q)_n}.$$

Thus, we have

$$f(\alpha,x,a,z,\bar{m},\bar{n}) = \sum_{n=0}^{\infty} \frac{p_n(t,s)}{(q;q)_n} L_{\bar{m},\bar{n}}(\alpha,x,z,a).$$

This completes the proof. □

Remark 3. *For $z=0$ in Theorem 4, Equation (17) reduces to (16).*

If we take

$$\alpha \to \infty, \quad \bar{m}=-1, \quad \bar{n}=0 \quad \text{and} \quad a=-q,$$

Theorem 4 yields the following corollary.

Corollary 1. *For $\max\{|z|,|x|,|t|\}<1$, we have*

$$\sum_{n=0}^{\infty} \frac{p_n(t,s)h_n(x,z)x^n}{(q;q)_n} = \frac{(xs;q)_\infty}{(xt;q)_\infty} \sum_{k=0}^{\infty} \frac{(tz)^k(s/t;q)_k}{(q,sx;q)_k}$$

As a special case of Theorem 4, if we take $\bar{m}=\bar{n}=0$, we have the following corollary.

Corollary 2. *For $\max\{|z|,|x|,|t|\}<1$, we have*

$$\sum_{n=0}^{\infty} \frac{p_n(t,s)}{(q;q)_n} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} \alpha \\ k \end{bmatrix}_{-q} q^{\binom{k}{2}} (a;q)_k z^k x^{n-k}$$
$$= \frac{(xs;q)_\infty}{(xt;q)_\infty} {}_3\phi_2\left(\begin{matrix} -q^{-\alpha}, a, s/t, \\ -q, xs \end{matrix} ; q, -ztq^{\alpha} \right)$$

5. A Multilinear Generating Function for the Generalized q-Polynomials

Andrews [2] proved that the coefficient of $q^n x^m y^j$ in

$$\sum_{n_1,n_2\ldots n_{k-1}=0}^{\infty} \frac{y_1^{N_1} y_2^{N_2} \ldots y_k^{N_k} q^{N_1^2+N_2^2+\ldots N_k^2} H_{n_1} H_{n_2} \ldots H_{n_{k-1}}}{(q^2;q^2)_{n_1}(q^2;q^2)_{n_2} \ldots (q^2;q^2)_{n_{k-1}}} \tag{19}$$

is the number of partitions enumerated with exactly m parts and exactly j different even parts that appear an odd number of times ($k=2$ or 3). The generalized q-Hermite polynomials, which is also the generalization of Andrews's result (19) due to Kursungoz, is given by

$$H_n = \sum_{j=0}^{n} y^j q^j \begin{bmatrix} n \\ j \end{bmatrix}_{q^2}, \tag{20}$$

also as $N_i = n_i + n_{i+1} + \ldots + n_{k-1}$. if $x=1, \alpha=n, \bar{m}=0, \bar{n}=1$. Therefore, we have

$$L_{\bar{0},\bar{1}}(n,1,y,0) = \sum_{j=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{q^2} q^j y^j = H_n.$$

In this section, we first give a multilinear generating function for certain q-polynomials. We then obtain some results for this multilinear generating function by using the homogeneous q-difference equation. Andrews [21] proved the following formula for the q-Lauricella function.

Proposition 3 (see [21])**.** *For* $\max\{|\alpha|, |r|, |y_1| \dots |y_k|\} < 1$ *, we have*

$$\sum_{n_1,n_2\dots n_k=0}^{\infty} \frac{(\alpha;q)_{n_1+n_2\dots+n_k}}{(r;q)_{n_1+n_2\dots+n_k}} \frac{(\beta_1;q)_{n_1}(\beta_2;q)_{n_2}\dots(\beta_k;q)_{n_k}}{(q;q)_{n_1}(q;q)_{n_2}\dots(q;q)_{n_k}} y_1^{n_1} y_2^{n_2}\dots y_k^{n_k}$$
$$= \frac{(\alpha,\beta_1 y_1,\beta_2 y_2,\dots\beta_k y_k;q)_\infty}{(r,y_1,y_2,\dots y_k;q)_\infty} {}_{k+1}\phi_k\left(\begin{matrix} r/\alpha, y_1, y_2, \dots y_k \\ \beta_1 y_1, \beta_2 y_2 \dots \beta_k y_k \end{matrix}; q, \alpha\right). \tag{21}$$

In the following, we obtain our main results about the multilinear generating function by using the q-difference equation.

Theorem 5. *For* $\max\{|\alpha|, |y_1| \dots |y_k|, |z_1| \dots |z_k|\} < 1$ *and* α_i *is any positive integer, we have*

$$\sum_{n_1,n_2\dots n_k=0}^{\infty} \frac{(\alpha;q)_{n_1+n_2\dots+n_k}}{(r;q)_{n_1+n_2\dots+n_k}} \frac{\sigma_1(q;L)}{(q;q)_{n_1}(q;q)_{n_2}\dots(q;q)_{n_k}}$$
$$\times (\beta_1;q)_{n_1}(\beta_2;q)_{n_2}\dots(\beta_k;q)_{n_k}$$
$$= \frac{(\alpha,\beta_1 y_1,\beta_2 y_2,\dots\beta_k y_k;q)_\infty}{(r,y_1,y_2,\dots y_k;q)_\infty} \sum_{s=0}^{\infty} \frac{(r/\alpha, y_1, y_2, \dots y_k;q)_s}{(\beta_1 y_1, \beta_2 y_2 \dots \beta_k y_k;q)_s} (\alpha)^s$$
$$\times \prod_{i=1}^{k} \sum_{n_i=0}^{\infty} \begin{bmatrix} \alpha \\ n_i \end{bmatrix}_{-q} q^{\tau(\bar{m}_i,\bar{n}_i)+\binom{n_i}{2}} \frac{(a_i;q)_{n_i}}{(q;q)_{n_i}} z_i^{n_i} q^{sn_i} \frac{(\beta_i;q)_{n_i}}{(\beta_i y_i q^s;q)_{n_i}}, \tag{22}$$

where

$$\sigma_1(q;L) = L_{\bar{m}_1,\bar{n}_1}(\alpha_1,y_1,z_1,a_1) L_{\bar{m}_2,\bar{n}_2}(\alpha_2,y_2,z_2,a_2)\dots L_{\bar{m}_k,\bar{n}_k}(\alpha_k,y_k,z_k,a_k) \tag{23}$$

Proof. Rewrite the Proposition 3 as:

$$\sum_{n_1,n_2\dots n_k=0}^{\infty} \frac{(\alpha;q)_{n_1+n_2\dots+n_k}}{(r;q)_{n_1+n_2\dots+n_k}} \frac{(\beta_1;q)_{n_1}(\beta_2;q)_{n_2}\dots(\beta_k;q)_{n_k}}{(q;q)_{n_1}(q;q)_{n_2}\dots(q;q)_{n_k}} y_1^{n_1} y_2^{n_2}\dots y_k^{n_k}$$
$$= \frac{(\alpha;q)_\infty}{(r;q)_\infty} \sum_{l=0}^{\infty} \frac{(r/\alpha;q)_l}{(q;q)_l} \frac{(\beta_1 y_1 q^l, \beta_2 y_2 q^l, \dots \beta_k y_k q^l;q)_\infty}{(y_1 q^l, y_2 q^l, \dots y_k q^l;q)_\infty} \alpha^l. \tag{24}$$

If we use $f(\sigma_2(Y))$ where $\sigma_2(Y)$ is given by

$$\sigma_2(Y) = y_1, y_2, \dots, y_k, z_1, z_2, \dots, z_k, a_1, a_2, \dots,$$
$$a_k, \bar{m}_1, \bar{m}_2, \dots, \bar{m}_k, \bar{n}_1, \bar{n}_2, \dots, \bar{n}_k, \alpha_1, \alpha_2, \dots, \alpha_k,$$

which denote the right hand side of (22), then, by direct computation, we can verify that f satisfies (9) and so:

$$f(\sigma_2(Y)) = \frac{(\alpha;q)_\infty}{(r;q)_\infty} \sum_{l=0}^{\infty} \frac{(r/\alpha;q)_l}{(q;q)_l} \sum_{n_1=0}^{\infty} \begin{bmatrix} \alpha \\ n_1 \end{bmatrix}_{-q} q^{\tau(\bar{m}_1,\bar{n}_1)+\binom{n_1}{2}}$$
$$\times \frac{(a_1;q)_{n_1}}{(q;q)_{n_1}} z_1^{n_1} \{D_{q,y_1}\}^{n_1} \left\{ \frac{(\beta_1 y_1 q^l, ;q)_\infty}{(y_1 q^l, y_2 q^l;q)_\infty} \right\}$$
$$\times \sum_{n_2=0}^{\infty} \begin{bmatrix} \alpha \\ n_2 \end{bmatrix}_{-q} q^{\tau(\bar{m}_2,\bar{n}_2)+\binom{n_2}{2}} \frac{(a_2;q)_{n_2}}{(q;q)_{n_2}} z_2^{n_2} \{D_{q,y_2}\}^{n_2}$$
$$\times \left\{ \frac{(\beta_2 y_2 q^l;q)_\infty}{(y_2 q^l;q)_\infty} \right\} \dots \sum_{n_k=0}^{\infty} \begin{bmatrix} \alpha \\ n_k \end{bmatrix}_{-q} q^{\tau(\bar{m}_k,\bar{n}_k)+\binom{n_k}{2}}$$
$$\times \frac{(a_k;q)_{n_k}}{(q;q)_{n_k}} z_k^{n_k} \{D_{q,y_k}\}^{n_k} \left\{ \frac{(\beta_k y_k q^l;q)_\infty}{(y_k q^l;q)_\infty} \right\}.$$

By using Theorem 1, there exists a sequence $\lambda_{n_1,\dots n_k}$ independent of $\sigma_2(Y)$ and that:

$$f(\sigma_2(Y)) = \sum_{n_1,n_2\dots n_k=0}^{\infty} \lambda_{n_1,\dots n_k}\sigma_1(q;L). \tag{25}$$

Setting $z_1 = z_2 \dots z_k = 0$ in (25) and using (24), we have

$$\begin{aligned} f(\sigma_3(Y)) &= \sum_{n_1,n_2\dots n_k=0}^{\infty} \lambda_{n_1,\dots n_k} y_1^{n_1} y_2^{n_2} \dots y_k^{n_k} \\ &= \frac{(\alpha, \beta_1 y_1, \beta_2 y_2, \dots \beta_k y_k; q)_\infty}{(r, y_1, y_2, \dots y_k; q)_\infty} \\ &\qquad \times \sum_{l=0}^{\infty} \frac{(r/a, y_1, y_2, \dots y_k; q)_l}{(q, \beta_1 y_1, \beta_2 y_2, \dots \beta_k y_k; q)_l} \alpha^l \\ &= \sum_{n_1,n_2\dots n_k=0}^{\infty} \frac{(\alpha;q)_{n_1+n_2\dots+n_k}}{(r;q)_{n_1+n_2\dots+n_k}} \\ &\qquad \frac{(\beta_1;q)_{n_1}(\beta_2;q)_{n_2}\dots(\beta_k;q)_{n_k}}{(q;q)_{n_1}(q;q)_{n_2}\dots(q;q)_{n_k}} y_1^{n_1} y_2^{n_2} \dots y_k^{n_k}, \end{aligned}$$

where

$$\begin{aligned} (\sigma_3(Y)) = y_1, y_2, \dots, y_k, 0, 0, \dots, 0, a_1, a_2, \dots, \\ a_k, \bar{m}_1, \bar{m}_2, \dots, \bar{m}_k, \bar{n}_1, \bar{n}_2, \dots, \bar{n}_k, \alpha_1, \alpha_2, \dots, \alpha_k. \end{aligned}$$

We deduce that $f(\sigma_2(Y))$ is equal to the left hand side of (22), so we have

$$\begin{aligned} f(\sigma_2(Y)) = &\sum_{n_1,n_2\dots n_{2k}=0}^{\infty} \frac{(\alpha;q)_{n_1+n_2\dots+n_{2k}}}{(r;q)_{n_1+n_2\dots+n_{2k}}} \\ &\times \frac{(\beta_1;q)_{n_1}(\beta_2;q)_{n_2}\dots(\beta_k;q)_{n_{2k}}}{(q;q)_{n_1}(q;q)_{n_2}\dots(q;q)_{n_k}} \sigma_1(q;L), \end{aligned}$$

where $\sigma_1(q;L)$ is given by (23).

Thus, we have

$$\begin{aligned} &\sum_{n_1,n_2\dots n_k=0}^{\infty} \frac{(\alpha;q)_{n_1+n_2\dots+n_k}}{(r;q)_{n_1+n_2\dots+n_k}} \frac{\sigma_1(q;L)}{(q;q)_{n_1}(q;q)_{n_2}\dots(q;q)_{n_k}} \\ &\qquad \times (\beta_1;q)_{n_1}(\beta_2;q)_{n_2}\dots(\beta_k;q)_{n_k} \\ &= \frac{(\alpha, \beta_1 y_1, \beta_2 y_2, \dots \beta_k y_k; q)_\infty}{(r, y_1, y_2, \dots y_k; q)_\infty} {}_{k+1}\phi_k \left(\begin{matrix} r/\alpha, y_1, y_2, \dots y_k \\ \beta_1 y_1, \beta_2 y_2 \dots \beta_k y_k \end{matrix} ; q, \alpha \right) \\ &\qquad \times \prod_{i=1}^{k} \sum_{n_i=0}^{\infty} \begin{bmatrix} \alpha_i \\ n_i \end{bmatrix}_{-q} q^{\tau(\bar{m}_i,\bar{n}_i)+\binom{n_i}{2}} \frac{(a_i;q)_{n_i}}{(q;q)_{n_i}} z_i^{n_i} q^{kn_i} \frac{(\beta_i;q)_{n_i}}{(\beta_i y_i q^k)_{n_i}}. \end{aligned}$$

Which completes the proof. □

Remark 4. *When we take $z_i = 0$, (22) can reduce to (21).*

If we take

$$\alpha_i = \infty, \quad \bar{m}_i = -1, \quad \bar{n}_i = 0 \quad \text{and} \quad a_i = -q$$

in Theorem 5, we have the following result.

Corollary 3. *For* $\max\{|\alpha|, |r|, |y_1| \dots |y_k|, |z_1| \dots |z_k|\} < 1$ *, we have*

$$\sum_{n_1,n_2\dots n_k=0}^{\infty} \frac{(\alpha;q)_{n_1+n_2\dots+n_k}}{(r;q)_{n_1+n_2\dots+n_k}} \frac{h_{n_1}(y_1,z_1|q)h_{n_2}(y_2,z_2|q)\dots h_{n_k}(y_k,z_k|q)}{(q;q)_{n_1}(q;q)_{n_2}\dots(q;q)_{n_k}}$$
$$\times (\beta_1;q)_{n_1}(\beta_2;q)_{n_2}\dots(\beta_k;q)_{n_k}$$
$$= \frac{(\alpha,\beta_1y_1,\beta_2y_2,\dots\beta_ky_k;q)_\infty}{(r,y_1,y_2,\dots y_k;q)_\infty} \sum_{s=0}^{\infty} \frac{(r/\alpha,y_1,y_2,\dots y_k;q)_s}{(\beta_1y_1,\beta_2y_2\dots\beta_ky_k;q)_s}(\alpha)^s$$
$$\times \prod_{i=1}^{k}\sum_{n_i=0}^{\infty} z_i^{n_i} q^{sn_i} \frac{(\beta_i;q)_{n_i}}{(q,\beta_iy_iq^s;q)_{n_i}}.$$

If we take

$$\beta_i = 0, \quad y_i = x_i \quad \text{and} \quad z_i = x_i,$$

in Theorem 5, we get the generalized Andrew's result as follows, which is also derived by Liu [20].

Corollary 4. *For* $\max\{|\alpha|, |r|, |x_1| \dots |x_k|, |y_1| \dots |y_k|\} < 1$*, we have*

$$\sum_{n_1,n_2\dots n_k=0}^{\infty} \frac{(\alpha;q)_{n_1+n_2\dots+n_k}}{(r;q)_{n_1+n_2\dots+n_k}} \frac{h_{n_1}(x_1,y_1|q)h_{n_2}(x_2,y_2|q)\dots h_{n_k}(x_k,y_k|q)}{(q;q)_{n_1}(q;q)_{n_2}\dots(q;q)_{n_k}}$$
$$= \frac{(\alpha;q)_\infty}{(r,x_1,y_1,x_2,y_2,\dots x_k,y_k;q)_\infty} {}_{2k+1}\phi_{2k}\left(\begin{matrix} r/\alpha,x_1,y_1,x_2,y_2,\dots x_k,y_k \\ 0,0\dots 0\end{matrix};q,\alpha\right).$$

In an application of Theorem 5, we proved the multilinear summation by using some special variable substitution. The following notation and substitutions are systematically adopted.

For $i = 1 \dots k-1$, we have

$$T_i = y_1y_2\dots y_i,$$
$$S_i = N_1 + N_2 + \dots + N_i,$$

and

$$N_i = n_i + n_{i+1} + \dots + n_{k-1}.$$

Theorem 6. *For* $\max\{|y_1| \dots |y_{k-1}|, |z_1|, |z_2|, \dots, |z_{k-1}|\} < 1$ *, we have*

$$\sum_{n_1,n_2\dots n_{k-1}=0}^{\infty} \frac{y_1^{N_1}y_2^{N_2}\dots y_k^{N_{k-1}}q^{N_1^2+N_2^2+\dots N_{k-1}^2}H_{n_1}H_{n_2}\dots H_{n_{k-1}}}{(q^2;q^2)_{n_1}(q^2;q^2)_{n_2}\dots(q^2;q^2)_{n_{k-1}}}$$
$$= \prod_{i=1}^{k} \frac{1}{(T_iq^{S_i},T_iz_iq^{S_i+1};q^2)_\infty}.$$

Proof. In Theorem 5, taking $\beta_i = 0$, $\alpha = \gamma$ and $n_k = 0$, we have

$$\sum_{n_1,n_2\dots n_{k-1}=0}^{\infty} \frac{\varsigma_q(L;n)}{(q;q)_{n_1}(q;q)_{n_2}\dots(q;q)_{n_{k-1}}}$$
$$= \prod_{i=1}^{k-1}\sum_{n_i=0}^{\infty} \begin{bmatrix}\alpha \\ n_i\end{bmatrix}_{-q} q^{\tau(\bar{m}_i,\bar{n}_i)+\binom{n_i}{2}} \frac{(a_i;q)_{n_i}}{(q;q)_{n_i}} z_i^{n_i} \frac{1}{(y_i;q)_\infty}. \quad (26)$$

where

$$\varsigma_q(L;n) = L_{\bar{m}_1,\bar{n}_1}(\alpha_1,y_1,z_1,a_1)L_{\bar{m}_2,\bar{n}_2}(\alpha_2,y_2,z_2,a_2)\dots L_{\bar{m}_{k-1},\bar{n}_{k-1}}(\alpha_{k-1},y_{k-1},z_{k-1},a_{k-1})$$

Making the following substations

$$\alpha_i = \infty, \quad a_i = -q, \quad \bar{m}_i = -3/2, \quad \bar{n}_i = -1/2, \quad \text{and} \quad z_i \to z_i y_i$$

in (26), we obtain

$$\begin{aligned}
&\sum_{n_1,n_2\ldots n_{k-1}=0}^{\infty} \frac{y_1^{n_1} y_2^{n_2} \ldots y_{k-1}^{n_{k-1}}}{(q;q)_{n_1}(q;q)_{n_2}\ldots(q;q)_{n_{k-1}}} \sum_{s_1=0}^{n_1} z_1^{s_1} q^{\frac{s_1}{2}} \begin{bmatrix} n_1 \\ s_1 \end{bmatrix} \\
&\sum_{s_2=0}^{n_2} z_2^{s_2} q^{\frac{s_2}{2}} \begin{bmatrix} n_2 \\ s_2 \end{bmatrix} \ldots \sum_{s_{k-1}=0}^{n_{k-1}} z_{k-1}^{s_{k-1}} q^{\frac{s_{k-1}}{2}} \begin{bmatrix} n_{k-1} \\ s_{k-1} \end{bmatrix} \\
&= \prod_{i=1}^{k-1} \sum_{n_i=0}^{\infty} \frac{1}{(y_i;q)_\infty} \frac{(z_i y_i)^{n_i} q^{\frac{n_i}{2}}}{(q;q)_{n_i}} \\
&= \prod_{i=1}^{k} \frac{1}{(y_i, z_i y_i q^{1/2}; q)_\infty}.
\end{aligned} \tag{27}$$

We need to specialize some variables in (27) as:

$$y_1 \to y_1 q^{N_1/2}$$

$$y_2 \to y_1 q^{N_1/2} y_2 q^{N_2/2}$$

$$y_3 \longrightarrow y_1 q^{N_1/2} y_2 q^{N_2/2} y_3 q^{N_3/2}$$

and

$$y_{k-1} \longrightarrow y_1 q^{N_1/2} y_2 q^{N_2/2} \ldots y_{k-1} q^{N_{k-1}/2}.$$

Hence, we have

$$q^{\frac{N_1 n_1 + \ldots N_1 n_{k-1}}{2}} = q^{N_1^2/2}$$

$$q^{\frac{N_{k-1} n_{k-1}}{2}} = q^{N_{k-1}^2/2}$$

and

$$y_1^{n_1} y_2^{n_2} \ldots y_{k-1}^{n_{k-1}} \to y_1^{N_1} y_2^{N_2} \ldots y_{k-1}^{N_{k-1}} q^{[N_1^2 + N_2^2 \ldots N_{k-1}^2]/2}.$$

Then, setting $q \to q^2$, we have

$$\begin{aligned}
&\sum_{n_1,n_2\ldots n_{k-1}=0}^{\infty} \frac{y_1^{N_1} y_2^{N_2} \ldots y_k^{N_{k-1}} q^{N_1^2 + N_2^2 + \ldots N_{k-1}^2} H_{n_1} H_{n_2} \ldots H_{n_{k-1}}}{(q^2;q^2)_{n_1}(q^2;q^2)_{n_2}\ldots(q^2;q^2)_{n_{k-1}}} \\
&= \prod_{i=1}^{k} \frac{1}{(T_i q^{S_i}, T_i z_i q^{S_i+1}; q^2)_\infty}.
\end{aligned}$$

The proof of our theorem is now completed. □

Further, replacing $\beta_i = 0$, $\alpha = \gamma$, $\alpha_i = \infty$,$a_i = -q$, $\bar{m}_i = -3/2$, $\bar{n}_i = -1/2$ $z_i = y_i^2$ and $q \to q^2$ in Theorem 5, we have the following corollary.

Corollary 5. *For* $\max\{|x_1|\ldots|x_k|, |y_1|\ldots|y_k|\} < 1$, *we have*

$$\sum_{n_1,n_2\ldots n_k=0}^{\infty} \frac{y_1^{n_1} y_2^{n_2} \ldots y_k^{n_k} H_{n_1} H_{n_2} \ldots H_{n_k}}{(q^2;q^2)_{n_1}(q^2;q^2)_{n_2}\ldots(q^2;q^2)_{n_k}} = \frac{1}{(y_1,\ldots,y_k,y_1^2 q,\ldots,y_k^2 q;q^2)_\infty}.$$

6. A Transformation Identity Involving Hecke-Type Series for the Generalized q-Polynomials

The following general expansion formula in Askey–Wilson polynomials, by using Bailey transform and Bressoud inversion, is proved in [33].

Proposition 4 (see [33]). *We have the following*

$$\sum_{n=0}^{\infty}\beta_n\delta_n=\sum_{n=0}^{\infty}\frac{(1-aq^{2n})(a,a/b;q)_n(b/a)^n}{(1-a)(bq,q;q)_n}\times\sum_{k=0}^{n}\frac{(1-bq^{2k})(aq^n,q^{-n};q)_k}{(1-b)(bq^{n+1},bq^{1-n}/a;q)_k}q^k\beta_k\sum_{r=0}^{\infty}\frac{(b/a;q)_r(b;q)_{r+2n}}{(q;q)_r(aq;q)_{r+2n}}\delta_{r+n} \tag{28}$$

As the application, we can get the following results directly.

Theorem 7. *The following assertions holds true:*

$$\begin{aligned}&\sum_{n=0}^{\infty}\begin{bmatrix}N\\n\end{bmatrix}_q\begin{bmatrix}\bar{\alpha}\\n\end{bmatrix}_{-q}q^{\tau(\bar{m},\bar{n})+\binom{n}{2}}x^{N-n}(\tilde{a};q)_n(\alpha,\beta;q)_n(aq/\alpha\beta)^n\\&=\frac{(aq/\alpha,aq/\alpha;q)_\infty}{(aq,aq/\alpha\beta;q)_\infty}\sum_{n=0}^{\infty}\frac{(1-aq^{2n})(a;q)_n(-1)^nq^{\binom{n}{2}}(aq/\alpha\beta)^n(\alpha,\beta;q)_n}{(1-a)(q,aq/\alpha,aq/\beta;q)_n}\\&\qquad\sum_{k=0}^{n}\begin{bmatrix}N\\k\end{bmatrix}_q\begin{bmatrix}\alpha\\k\end{bmatrix}_{-q}(aq^n,q^{-n},\tilde{a};q)_kq^kq^{\tau(\bar{m},\bar{n})+\binom{k}{2}}x^{N-k}\end{aligned}$$

Proof. we take

$$b=0,$$
$$\beta_n=\begin{bmatrix}N\\n\end{bmatrix}_q\begin{bmatrix}\bar{\alpha}\\n\end{bmatrix}_{-q}q^{\tau(\bar{m},\bar{n})+\binom{n}{2}}x^{N-n}(\tilde{a};q)_n,$$
$$\delta_n=(\alpha,\beta;q)_n(aq/\alpha\beta)^n$$

in (28), and

$$\begin{aligned}&\sum_{n=0}^{\infty}\begin{bmatrix}N\\n\end{bmatrix}_q\begin{bmatrix}\bar{\alpha}\\n\end{bmatrix}_{-q}q^{\tau(\bar{m},\bar{n})+\binom{n}{2}}x^{N-n}(\tilde{a};q)_n(\alpha,\beta;q)_n(aq/\alpha\beta)^n\\&=\sum_{n=0}^{\infty}\frac{(1-aq^{2n})(a;q)_n(-1)^nq^{\binom{n}{2}}(aq/\alpha\beta)^n(\alpha,\beta;q)_n}{(1-a)(q,;q)_n(aq;q)_{2n}}\\&\qquad\sum_{k=0}^{n}\begin{bmatrix}N\\k\end{bmatrix}_q\begin{bmatrix}\alpha\\k\end{bmatrix}_{-q}(aq^n,q^{-n},\tilde{a};q)_kq^kq^{\tau(\bar{m},\bar{n})+\binom{k}{2}}x^{N-k}\sum_{r=0}^{\infty}\frac{(\alpha q^n,\beta q^n;q)_r(aq/\alpha\beta)^r}{(q;q)_r(aq^{2n+1};q)_r}\end{aligned}$$

By using (2) in the third summation of the above equation, we can get this main result. □

In its special case, if we let

$$N\to\infty,\quad \bar{\alpha}\to\infty,\quad \tilde{a}=-yq,\quad \bar{m}=0,\quad \bar{n}=-1\quad\text{and}\quad \alpha,\beta\to\infty$$

in Theorem 7 and make use of (2), we obtain the following result, which is related to $\rho_e(\infty,y,a,q)$ identity involving the little q-Jacobi polynomials (see [2]).

Corollary 6. *We have the following identity:*

$$\rho_e(\infty, y, a, q) = \sum_{n=0}^{\infty} \frac{(-yq;q)_n a^n q^{n^2}}{(q^2;q^2)_n}$$
$$= \frac{1}{(aq;q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n a^n q^{2n^2}(a^2;q^2)_n(1-aq^{2n})}{(q^2;q^2)_n(1-a)} p_n(y;-a/q,-1;q), \tag{29}$$

where the little q-Jacobi polynomial is given by

$$p_n(y;A,B;q) = {}_2\phi_1\left(\begin{matrix} q^{-n}, ABq^{n+1} \\ Aq \end{matrix}; q, qy\right)$$

If we take

$$b = 0, \quad \beta_n \to \frac{A_n}{(\alpha,\beta;q)_n} \quad \text{and} \quad \delta_n = (\alpha,\beta;q)_n (t/\alpha\beta)^n$$

and using (3) in the Proposition 4, we have the Zhang and Song's result (see [6]).

Corollary 7. *If A_n is a complex sequence, then, under suitable convergence conditions, we have*

$$\sum_{n=0}^{\infty} A_n (t/\alpha\beta)^n = \frac{(tq/\alpha, tq/\beta;q)_\infty}{(tq, tq/\alpha\beta;q)_\infty}$$
$$\times \sum_{n=0}^{\infty} \frac{(t,\alpha,\beta;q)_n(1-tq^{2n})(-1)^n q^{\binom{n}{2}}}{(q,tq/\alpha,tq/\beta;q)_n(1-t)} (t/\alpha\beta)^n$$
$$\times \left\{ \frac{\alpha\beta(1+tq^{2n}) - tq^n(\alpha+\beta)}{\alpha\beta - t} \right\} \sum_{j=0}^{n} \frac{(q^{-n}, tq^n;q)_j q^j}{(\alpha,\beta;q)_j} A_j$$

Recently, Chan and Liu [28,29] derived some Hecke-type identities by using Liu's transformation formulas for q-series (5) [31]. Wang and Yee [4], essentially motivated by the works of Liu [20], gave a double series of Hecke–Rogers type formulas. After that, Wang [3] provided new proofs to five of Ramanujan's intriguing identities on false functions by using (5). Zhang and Song [6] also gave some Hecke-types identities by using two q-series expansion formulas. In this section, we shall give some new identities of q-polynomials which are some new Hecke-type series.

Theorem 8. *The following assertions hold true:*

$$(q;q^2)_\infty (q;q)_\infty = \sum_{n=0}^{\infty} \sum_{j=-n}^{n} (1-q^{4n+2})(-1)^j q^{2n^2-j^2}$$
$$\sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}}{(-q;q)_n} = \sum_{n=0}^{\infty} \sum_{j=-n}^{n} (-1)^j q^{n^2-j^2}(1-q^n-q^{4n+2}+q^{3n+1})$$

Proof. We take

$$b = 0, \quad \beta_n = \frac{(\bar{a};q)_n m^n}{(q,s,x;q)_n} \quad \text{and} \quad \delta_n = (\alpha,\beta;q)_n (a/\alpha\beta)^n$$

in the Proposition 4, we have the following result:

$$
\begin{aligned}
&{}_3\phi_2\left(\begin{matrix}\alpha,\beta,\bar{a}\\ s,x\end{matrix};q,am/\alpha\beta\right)\\
&\quad=\frac{(aq/\alpha,aq/\beta;q)_\infty}{(aq,aq/\alpha\beta;q)_\infty}\sum_{n=0}^{\infty}\frac{(a,\alpha,\beta;q)_n(1-aq^{2n})(-1)^n q^{\binom{n}{2}}}{(q,aq/\alpha,aq/\beta;q)_n(1-a)}(a/\alpha\beta)^n\\
&\qquad\times\left\{\frac{\alpha\beta(1+aq^{2n})-aq^n(\alpha+\beta)}{\alpha\beta-a}\right\}{}_3\phi_2\left(\begin{matrix}q^{-n},aq^n,\bar{a}\\ s,x\end{matrix};q,mq\right),
\end{aligned}
\tag{30}
$$

where

$$
\max\{|m|,|a/\alpha\beta|\}<1.
$$

Furthermore, by setting

$$
a=q,\quad s=-q,\quad x=0,\quad m=-1\quad\text{and}\quad\bar{a}=0
$$

in (30), we have

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\frac{(\alpha,\beta;q)_n}{(q,-q;q)_n}\left(\frac{-q}{\alpha\beta}\right)^n\\
&\quad=\frac{(q^2/\alpha,q^2/\beta;q)_\infty}{(q^2,q^2/\alpha\beta;q)_\infty}\sum_{n=0}^{\infty}\frac{(\alpha,\beta;q)_n(1-q^{2n+1})(-1)^n q^{\binom{n}{2}}}{(q^2/\alpha,q^2/\beta;q)_n(1-q)}(q/\alpha\beta)^n\\
&\qquad\left\{\frac{\alpha\beta(1+q^{2n+1})-q^{n+1}(\alpha+\beta)}{\alpha\beta-q}\right\}{}_2\phi_1\left(\begin{matrix}q^{-n},q^{n+1}\\ -q\end{matrix};q,-q\right)
\end{aligned}
\tag{31}
$$

Andrews [2] gave the following result concerning the little q-Jacobi polynomials.

$$
\begin{aligned}
p_n(-1;-1,-1;q)&={}_2\phi_1\left(\begin{matrix}q^{-n},q^{n+1}\\ -q\end{matrix};q,-q\right)\\
&=(-1)^n q^{\binom{n+1}{2}}\sum_{j=-n}^{n}(-1)^j q^{-j^2}
\end{aligned}
$$

Submitting the above transformation in (31), we obtain

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\frac{(\alpha,\beta;q)_n}{(q,-q;q)_n}\left(\frac{-q}{\alpha\beta}\right)^n\\
&\quad=\frac{(q^2/\alpha,q^2/\beta;q)_\infty}{(q^2,q^2/\alpha\beta;q)_\infty}\sum_{n=0}^{\infty}\frac{(\alpha,\beta;q)_n(1-q^{2n+1})(-1)^n q^{\binom{n}{2}}}{(q^2/\alpha,q^2/\beta;q)_n(1-q)}(q/\alpha\beta)^n\\
&\qquad\left\{\frac{\alpha\beta(1+q^{2n+1})-q^{n+1}(\alpha+\beta)}{\alpha\beta-q}\right\}(-1)^n q^{\binom{n+1}{2}}\sum_{j=-n}^{n}(-1)^j q^{-j^2}.
\end{aligned}
\tag{32}
$$

We can now get the required result easily by setting

$$
(\alpha,\beta)\to(\infty,\infty)\quad\text{and}\quad(\alpha,\beta)\to(q,\infty)
$$

in (32), respectively. □

Theorem 9. *The following assertion holds true:*

$$
\sum_{n=0}^{\infty}\frac{(q;q^2)_n q^{2n^2}}{(q^2;q^2)_n(-q;q)_{2n+1}}=\frac{1}{(q^2;q^2)_\infty}\sum_{n=0}^{\infty}(-1)^n q^{3n^2}(1-q^{2n+1}+q^{4n+2}-q^{6n+3})
$$

Proof. Setting $q \to q^2, s \to -q^2, a \to q^2, x \to -q^3, m \to 1$ and $\bar{a} = q$ in (30), we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(\alpha, \beta, q; q^2)_n}{(q^2, -q^2, -q^3; q^2)_n} (q^2/\alpha\beta)^n \\
&\quad = \frac{(q^4/\alpha, q^4/\beta; q^2)_\infty}{(q^4, q^4/\alpha\beta; q^2)_\infty} \sum_{n=0}^{\infty} \frac{(\alpha, \beta; q^2)_n (1 - q^{4n+2})(-1)^n q^{n^2-n}}{(q^4/\alpha, q^4/\beta; q^2)_n (1 - q^2)} (q^2/\alpha\beta)^n \\
&\quad \left\{ \frac{\alpha\beta(1 + q^{4n+2}) - q^{2n+2}(\alpha + \beta)}{\alpha\beta - q^2} \right\} {}_3\phi_2 \left(\begin{matrix} q^{-2n}, q^{2n+2}, q \\ -q^2, -q^3 \end{matrix} ; q^2, q^2 \right).
\end{aligned} \tag{33}$$

By using q-Pfaff-*Saalschütz* (4), we have

$${}_3\phi_2 \left(\begin{matrix} q^{-2n}, q^{2n+2}, q \\ -q^2, -q^3 \end{matrix} ; q^2, q^2 \right) = \frac{(1+q)q^n}{1 + q^{2n+1}}$$

Submitting the ${}_3\phi_2$ transformation and taking $(\alpha, \beta) \to (\infty, \infty)$ in (33), we have

$$\sum_{n=0}^{\infty} \frac{(q; q^2)_n q^{2n^2}}{(q^2; q^2)_n (-q; q)_{2n+1}} = \frac{1}{(q^2; q^2)_\infty} \sum_{n=0}^{\infty} (-1)^n q^{3n^2} (1 - q^{2n+1} + q^{4n+2} - q^{6n+3})$$

□

Theorem 10. *Each of the following identities holds true:*

$$\sum_{n=0}^{\infty} \frac{(q; q^2)_n q^{2n^2-2n}}{(q^2; q^2)_n (-q; q)_{2n}} = \frac{1}{(q^2; q^2)_\infty} \sum_{n=0}^{\infty} \sum_{j=-n}^{n} (1 - q^{8n+4})(-1)^j q^{2n^2-2n+j^2} \tag{34}$$

$$\sum_{n=0}^{\infty} \frac{(q; q^2)_n^2 (-1)^n q^{n^2-2n}}{(q^2; q^2)_n (-q; q)_{2n}} = \frac{(q; q^2)_\infty}{(q^2; q^2)_\infty} \sum_{n=0}^{\infty} \sum_{j=-n}^{n} (1 + q^{6n+3})(-1)^{n+j} q^{n^2-2n+j^2} \tag{35}$$

$$\sum_{n=0}^{\infty} \frac{(q; q^2)_n}{(q^4; q^4)_n} q^{n^2-2n} = \frac{(-q; q^2)_\infty}{(q^2; q^2)_\infty} \sum_{n=0}^{\infty} \sum_{j=-n}^{n} (1 - q^{6n+3})(-1)^j q^{n^2-2n+j^2} \tag{36}$$

and

$$\sum_{n=0}^{\infty} \frac{(q; q^2)_n (-1)^n q^{n^2-3n}}{(-q; q)_{2n}} = \sum_{n=0}^{\infty} \sum_{j=-n}^{n} (1 - q^{2n} + q^{6n+2} - q^{8n+4})(-1)^{n+j} q^{n^2-3n-j^2} \tag{37}$$

Proof. Letting $a = q^2, s = -q, x = -q^2, m = 1/q^2, \bar{a} = q$ and $q \to q^2$ in (30), we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(\alpha, \beta, q; q^2)_n}{(q^2, -q, -q^2; q^2)_n} \left(\frac{1}{\alpha\beta}\right)^n \\
&= \frac{(q^4/\alpha, q^4/\beta; q^2)_\infty}{(q^4, q^4/\alpha\beta; q^2)_\infty} \sum_{n=0}^{\infty} \frac{(\alpha, \beta; q^2)_n (1 - q^{4n+2})(-1)^n q^{n^2-n}}{(q^4/\alpha, q^4/\beta; q^2)_n (1 - q^2)} (q^2/\alpha\beta)^n \\
&\left\{ \frac{\alpha\beta(1 + q^{4n+2}) - q^{2n+2}(\alpha + \beta)}{\alpha\beta - q^2} \right\} {}_3\phi_2 \left(\begin{matrix} q^{-2n}, q^{2n+2}, q \\ -q, -q^2 \end{matrix} ; q^2, 1 \right)
\end{aligned}$$

Wang and Yee [4] provided the following identity which is also equivalent to the identity in Andrews' paper [1].

$$(-1)^n q^{n(n+1)}{}_3\phi_2\left(\begin{matrix} q^{-2n}, q^{2+2n}, q \\ -q, -q^2 \end{matrix}; q^2, 1\right) = 1 + 2\sum_{j=1}^{n}(-1)^j q^{j^2}$$
$$= \sum_{j=-n}^{n}(-1)^j q^{j^2}$$

Now letting $(\alpha, \beta) \to (\infty, \infty)$, $(\alpha, \beta) \to (q, \infty)$, $(\alpha, \beta) \to (-q, \infty)$, $(\alpha, \beta) \to (q^2, \infty)$ and using the above equation, respectively, we can get (34)–(37). □

7. Concluding Remarks and Observations

In our present investigation, by using the method of q-difference equations, we have systematically deduced several types of generating functions for certain q-polynomial. Furthermore, we have given a multilinear generating function for the q-polynomials as a generalization of Andrew's result. Moreover, we have built a transformation identity involving the q-polynomials and Bailey transformation. As an application, we have to study some new Hecke-type identities. We have also highlighted some known and new consequences of our main results.

We have observed that the q-operator identity and q-difference equation are equivalent with two variables, so we have focused on the expansion of a function of many variables and on some orthogonal polynomials. Therefore we have given the expansion of six variables (2D-hermite polynomials and Andrew's polynomials). Beside it, we have also got the single, double and even multiple generating functions of these polynomials. Furthermore, we have got their new applications in Combination Theory. Next, it is believed that the works presented here in this paper, along with the recent works cited here, will be a motivation for further researches to study this kind of orthogonal polynomial and its applications in many other areas of mathematics and physics.

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