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Hermite–Hadamard Inclusions for Co-Ordinated Interval-Valued Functions via Post-Quantum Calculus

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Abstract: In this paper, the notions of post-quantum integrals for two-variable interval-valued functions are presented. The newly described integrals are then used to prove some new Hermite–Hadamard inclusions for co-ordinated convex interval-valued functions. Many of the findings in this paper are important extensions of previous findings in the literature. Finally, we present a few examples of our new findings. Analytic inequalities of this nature and especially the techniques involved have applications in various areas in which symmetry plays a prominent role.

Keywords: Hermite–Hadamard inequality; Hermite–Hadamard inclusion; (p,q)-integral; quantum calculus; co-ordinated convexity; interval-valued functions

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1. Introduction

The modern name for the study of calculus without limits is quantum calculus, or q-calculus. It has been studied since the early eighteenth century. Euler, a prominent mathematician, invented q-calculus, and F. H. Jackson [1] discovered the definite q-integral known as the q-Jackson integral in 1910. Orthogonal polynomials, combinatorics, number theory, simple hypergeometric functions, quantum theory, dynamics, and theory of relativity are only a few of the applications of quantum calculus in mathematics and physics; see, for example, [2–19] and the references therein. V. Kac and P. Cheung's book [20] discusses the fundamentals of quantum calculus as well as the basic theoretical terms.

J. Tariboon and S. K. Ntouyas [21] described and proved some of the properties of the q-derivative and q-integral of a continuous functions on finite intervals in 2013. Moreover, they proved Hermite–Hadamard-type inequalities and many others for convex functions in the setting of quantum calculus; for more information, see [22].

M. Tunç and E. Göv [23] presented the (p,q)-derivative and (p,q)-integral on finite intervals in 2016, proved some of their properties, and proved a number of integral inequalities using the (p,q)-calculus. Many researchers have recently begun working in this direction, based on the works of M. Tunç and E. Göv, and some further findings on the analysis of (p,q)-calculus can be found in [24–27].

In [28], S. S. Dragomir proved the following inequalities, which are Hermite–Hadamard type inequalities for co-ordinated convex functions on the rectangle from the plane $\mathbb{R}^2$.

Theorem 1. *Suppose that $f : [a,b] \times [c,d] \to \mathbb{R}$ is co-ordinated convex; then we have the following inequalities:*

$$\begin{aligned} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\leq \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right)dx + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right)dy\right] \\ &\leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx \\ &\leq \frac{1}{4}\left[\frac{1}{b-a}\int_a^b f(x,c)dx + \frac{1}{b-a}\int_a^b f(x,d)dx \right. \\ &\quad \left. + \frac{1}{d-c}\int_c^d f(a,y)dy + \frac{1}{d-c}\int_c^d f(b,y)dy\right] \\ &\leq \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}. \end{aligned} \tag{1}$$

The above inequalities are sharp. The inequalities in (1) hold in the reverse direction if the mapping f is a co-ordinated concave mapping.

The quantum variant of above inequality (1) was given by M. Kunt et al. in [29], S. Bermudo et al. [30] recently used q-calculus to describe new q^b-derivative and q^b-integral, as well as to give the Hermite–Hadamard inequality. H. Budak et al. [31] defined some new q^b-integrals for co-ordinates and Hermite–Hadamard inequalities for co-ordinated convex functions as a result of this. F. Wannalookkhee et al. [32] in 2021 gave some new definitions of $(p,q)^b$-integrals and used them to prove the following Hermite–Hadamard inequalities:

$$\begin{aligned} & f\left(\frac{q_1 a + p_1 b}{[2]_{p_1,q_1}}, \frac{p_2 c + q_2 d}{[2]_{p_2,q_2}}\right) \\ \leq\ & \frac{1}{2}\left[\frac{1}{p_1(b-a)}\int_a^{p_1 b+(1-p_1)a} f\left(x, \frac{p_2 c + q_2 d}{[2]_{p_2,q_2}}\right)\, {}_a d_{p_1,q_1}x \right. \\ & \left. + \frac{1}{p_2(d-c)}\int_{p_2 c+(1-p_2)d}^{d} f\left(\frac{q_1 a + p_1 b}{[2]_{p_1,q_1}}, y\right)\, {}^d d_{p_2,q_2}y\right] \\ \leq\ & \frac{1}{p_1 p_2(b-a)(d-c)}\int_a^{p_1 b+(1-p_1)a}\int_{p_2 c+(1-p_2)d}^{d} f(x,y)\, {}^d d_{p_2,q_2}y\, {}_a d_{p_1,q_1}x \\ \leq\ & \frac{1}{2p_2[2]_{p_1,q_1}(d-c)}\left[q_1\int_{p_2 c+(1-p_2)d}^{d} f(a,y)\, {}^d d_{p_2,q_2}y + p_1\int_{p_2 c+(1-p_2)d}^{d} f(b,y)\, {}^d d_{p_2,q_2}y\right] \\ & + \frac{1}{2p_1[2]_{p_2,q_2}(b-a)}\left[p_2\int_a^{p_1 b+(1-p_1)a} f(x,c)\, {}_a d_{p_1,q_1}x + q_2\int_a^{p_1 b+(1-p_1)a} f(x,d)\, {}_a d_{p_1,q_1}x\right] \\ \leq\ & \frac{q_1 p_2 f(a,c) + q_1 q_2 f(a,d) + p_1 p_2 f(b,c) + q_2 p_1 f(b,d)}{[2]_{p_1,q_1}[2]_{p_2,q_2}}, \end{aligned} \tag{2}$$

$$\begin{aligned} & f\left(\frac{p_1 a + q_1 b}{[2]_{p_1,q_1}}, \frac{q_2 c + p_2 d}{[2]_{p_2,q_2}}\right) \\ \leq\ & \frac{1}{2}\left[\frac{1}{p_1(b-a)}\int_{p_1 a+(1-p_1)b}^{b} f\left(x, \frac{q_2 c + p_2 d}{[2]_{p_2,q_2}}\right)\, {}^b d_{p_1,q_1}x \right. \\ & \left. + \frac{1}{p_2(d-c)}\int_c^{p_2 d+(1-p_2)c} f\left(\frac{p_1 a + q_1 b}{[2]_{p_1,q_1}}, y\right)\, {}_c d_{p_2,q_2}y\right] \end{aligned} \tag{3}$$

$$\begin{aligned}
&\le \frac{1}{p_1p_2(b-a)(d-c)}\int_{p_1a+(1-p_1)b}^{b}\int_{c}^{p_2d+(1-p_2)c} f(x,y)\ {}_cd_{p_2,q_2}y\ {}^bd_{p_1,q_1}x \\
&\le \frac{1}{2p_2[2]_{p_1,q_1}(d-c)}\left[p_1\int_{c}^{p_2d+(1-p_2)c} f(a,y)\ {}_cd_{p_2,q_2}y + q_1\int_{c}^{p_2d+(1-p_2)c} f(b,y)\ {}_cd_{p_2,q_2}y\right] \\
&\quad +\frac{1}{2p_1[2]_{p_2,q_2}(b-a)}\left[q_2\int_{p_1a+(1-p_1)b}^{b} f(x,c)\ {}^bd_{p_1,q_1}x + p_2\int_{p_1a+(1-p_1)b}^{b} f(x,d)\ {}^bd_{p_1,q_1}x\right] \\
&\le \frac{p_1q_2f(a,c)+p_1p_2f(a,d)+q_1q_2f(b,c)+q_1p_2f(b,d)}{[2]_{p_1,q_1}[2]_{p_2,q_2}},
\end{aligned}$$

$$\begin{aligned}
& f\left(\frac{p_1a+q_1b}{[2]_{p_1,q_1}},\frac{p_2c+q_2d}{[2]_{p_2,q_2}}\right) \qquad (4)\\
&\le \frac{1}{2}\left[\frac{1}{p_1(b-a)}\int_{p_1a+(1-p_1)b}^{b} f\left(x,\frac{p_2c+q_2d}{[2]_{p_2,q_2}}\right)\ {}^bd_{p_1,q_1}x\right. \\
&\quad \left.+\frac{1}{p_2(d-c)}\int_{p_2c+(1-p_2)d}^{d} f\left(\frac{p_1a+q_1b}{[2]_{p_1,q_1}},y\right)\ {}^dd_{p_2,q_2}y\right] \\
&\le \frac{1}{p_1p_2(b-a)(d-c)}\int_{p_1a+(1-p_1)b}^{b}\int_{p_2c+(1-p_2)d}^{d} f(x,y)\ {}^dd_{p_2,q_2}y\ {}^bd_{p_1,q_1}x \\
&\le \frac{1}{2p_2[2]_{p_1,q_1}(d-c)}\left[p_1\int_{p_2c+(1-p_2)d}^{d} f(a,y)\ {}^dd_{p_2,q_2}y + q_1\int_{p_2c+(1-p_2)d}^{d} f(b,y)\ {}^dd_{p_2,q_2}y\right] \\
&\quad +\frac{1}{2p_1[2]_{p_2,q_2}(b-a)}\left[p_2\int_{p_1a+(1-p_1)b}^{b} f(x,c)\ {}^bd_{p_1,q_1}x + q_2\int_{p_1a+(1-p_1)b}^{b} f(x,d)\ {}^bd_{p_1,q_1}x\right] \\
&\le \frac{p_1p_2f(a,c)+p_1q_2f(a,d)+q_1p_2f(b,c)+q_1q_2f(b,d)}{[2]_{p_1,q_1}[2]_{p_2,q_2}}.
\end{aligned}$$

2. Interval Calculus

In this section, we provide notation and background information on interval analysis. The space of all closed intervals of $\mathbb{R}$ is denoted by I_c, and Δ is a bounded element of I_c. We have the representation

$$\Delta = \left[\underline{\Theta_1},\overline{\Theta_1}\right] = \left\{\ll\, \in \mathbb{R} : \underline{\Theta_1} \le \ll\, \le \overline{\Theta_1}\right\}$$

where $\underline{\Theta_1},\overline{\Theta_1} \in \mathbb{R}$ and $\underline{\Theta_1} \le \overline{\Theta_1}$. $L(\Delta) = \overline{\Theta_1} - \underline{\Theta_1}$ can be used to express the length of the interval $\Delta = \left[\underline{\Theta_1},\overline{\Theta_1}\right]$. The left and right endpoints of interval Δ are denoted by the numbers $\underline{\Theta_1}$ and $\overline{\Theta_1}$, respectively. The interval Δ is said to be degenerate when $\underline{\Theta_1} = \overline{\Theta_1}$, and the form $\Delta = \Theta_1 = [\Theta_1,\Theta_1]$ is used. In addition, if $\underline{\Theta_1} > 0$, we can say Δ is positive, and if $\overline{\Theta_1} < 0$, we can say Δ is negative. I_c^+ and I_c^- denote the sets of all closed positive intervals and closed negative intervals of $\mathbb{R}$, respectively. Between the intervals Δ and Λ, the Pompeiu–Hausdorff distance is defined by

$$d_H(\Delta,\Lambda) = d_H\left(\left[\underline{\Theta_1},\overline{\Theta_1}\right],\left[\underline{\Theta_2},\overline{\Theta_2}\right]\right) = \max\left\{\left|\underline{\Theta_1}-\underline{\Theta_2}\right|,\left|\overline{\Theta_1}-\overline{\Theta_2}\right|\right\}. \qquad (5)$$

(I_c,d) is a complete metric space, as far as we know (see, [33]).

$|\Delta|$ denotes the absolute value of Δ, which is the maximum of the absolute values of its endpoints:

$$|\Delta| = \max\left\{\left|\underline{\Theta_1}\right|,\left|\overline{\Theta_1}\right|\right\}.$$

The following are the concepts for fundamental interval arithmetic operations for the intervals Δ and Λ:

$$\Delta+\Lambda \;=\; \left[\underline{\Theta_1}+\underline{\Theta_2},\overline{\Theta_1}+\overline{\Theta_2}\right],$$

$$\begin{aligned}
\Delta - \Lambda &= [\underline{\Theta_1} - \overline{\Theta_2}, \overline{\Theta_1} - \underline{\Theta_2}], \\
\Delta \cdot \Lambda &= [\min U, \max U] \text{ where } U = \{\underline{\Theta_1}\,\underline{\Theta_2}, \underline{\Theta_1}\,\overline{\Theta_2}, \overline{\Theta_1}\underline{\Theta_2}, \overline{\Theta_1}\,\overline{\Theta_2}\}, \\
\Delta / \Lambda &= [\min V, \max V] \text{ where } V = \{\underline{\Theta_1}/\underline{\Theta_2}, \underline{\Theta_1}/\overline{\Theta_2}, \overline{\Theta_1}/\underline{\Theta_2}, \overline{\Theta_1}/\overline{\Theta_2}\} \text{ and } 0 \notin \Lambda.
\end{aligned}$$

The interval Δ's scalar multiplication is defined by

$$\mu\Delta = \mu[\underline{\Theta_1}, \overline{\Theta_1}] = \begin{cases} [\mu\underline{\Theta_1}, \mu\overline{\Theta_1}], & \mu > 0; \\ \{0\}, & \mu = 0; \\ [\mu\overline{\Theta_1}, \mu\underline{\Theta_1}], & \mu < 0, \end{cases}$$

where $\mu \in \mathbb{R}$.

The opposite of the interval Δ is

$$-\Delta := (-1)\Delta = [-\overline{\Theta_1}, -\underline{\Theta_1}],$$

where $\mu = -1$.

In general, $-\Delta$ is not additive inverse for Δ, i.e., $\Delta - \Delta \neq 0$.

Definition 1 ([2]). *For some kind of the intervals $\Delta, \Lambda \in I_c$, we denote the the H-difference of Δ and Λ as the $\Omega \in I_c$, and we have*

$$\Delta \ominus_g \Lambda = \Omega \Leftrightarrow \begin{cases} (i)\ \Delta = \Lambda + \Omega \\ \quad or \\ (ii)\ \Lambda = \Delta + (-\Omega). \end{cases}$$

It seems uncontroversial that

$$\Delta \ominus_g \Lambda = \begin{cases} [\underline{\Theta_1} - \underline{\Theta_2}, \overline{\Theta_1} - \overline{\Theta_2}], \text{ if } L(\Delta) \geq L(\Lambda) \\ [\overline{\Theta_1} - \overline{\Theta_2}, \underline{\Theta_1} - \underline{\Theta_2}], \text{ if } L(\Delta) \leq L(\Lambda), \end{cases}$$

where $L(\Delta) = \overline{\Theta_1} - \underline{\Theta_1}$ and $L(\Lambda) = \overline{\Theta_2} - \underline{\Theta_2}$.

The definitions of operations generate a large number of algebraic properties, enabling I_c to be a quasilinear space (see [34]). The following are some of these characteristics (see [33–36]):

(1) (Law of associative under $+$) $(\Delta + \Lambda) + \mathcal{C} = \Delta + (\Lambda + \mathcal{C})$ for all $\Delta, \Lambda, \mathcal{C} \in I_c$,
(2) (Additivity element) $\Delta + 0 = 0 + \Delta = \Delta$ for all $\Delta \in I_c$,
(3) (Law of commutative under $+$) $\Delta + \Lambda = \Lambda + \Delta$ for all $\Delta, \Lambda \in I_c$,
(4) (Law of cancellation under $+$) $\Delta + \mathcal{C} = \Lambda + \mathcal{C} \Longrightarrow \Delta = \Lambda$ for all $\Delta, \Lambda, \mathcal{C} \in I_c$,
(5) (Law of associative under $\times$) $(\Delta \cdot \Lambda) \cdot \mathcal{C} = \Delta \cdot (\Lambda \cdot \mathcal{C})$ for all $\Delta, \Lambda, \mathcal{C} \in I_c$,
(6) (Law of commutative under $\times$) $\Delta \cdot \Lambda = \Lambda \cdot \Delta$ for all $\Delta, \Lambda \in I_c$,
(7) (Multiplicativity element) $\Delta \cdot 1 = 1 \cdot \Delta$ for all $\Delta \in I_c$,
(8) (The first law of distributivity) $\lambda(\Delta + \Lambda) = \lambda\Delta + \lambda\Lambda$ for all $\Delta, \Lambda \in I_c$ and all $\lambda \in \mathbb{R}$,
(9) (The second law of distributivity) $(\lambda + \mu)\Delta = \lambda\Delta + \mu\Delta$ for all $\Delta \in I_c$ and all $\lambda, \mu \in \mathbb{R}$.

Aside from any of these characteristics, the distributive law does not always apply to intervals. As an example, $\Delta = [1, 2], \Lambda = [2, 3]$ and $\mathcal{C} = [-2, -1]$.

$$\Delta \cdot (\Lambda + \mathcal{C}) = [0, 4],$$

whereas

$$\Delta \cdot \Lambda + \Delta \cdot \mathcal{C} = [-2, 5].$$

Another distinct feature is the inclusion $\subseteq$, which is described by

$$\Delta \subseteq \Lambda \Longleftrightarrow \underline{\Theta_1} \geq \underline{\Theta_2} \text{ and } \overline{\Theta_1} \leq \overline{\Theta_2}.$$

In [37], Zhao et al. gave the notions about the co-ordinated convex interval-valued functions and inclusions of Hermite–Hadamard type.

Definition 2 ([37])**.** *A function $F = [\underline{F}, \overline{F}] : [a,b] \times [c,d] \to I_c^+$ is said to be co-ordinated convex interval-valued function if the following inclusion holds:*

$$\begin{aligned} & F(tx+(1-t)y, su+(1-s)w) \\ \supseteq \quad & tsF(x,u) + t(1-s)F(x,w) + s(1-t)F(y,u) + (1-s)(1-t)F(y,w), \end{aligned}$$

for all $(x,y),(u,w) \in [a,b] \times [c,d]$ and $s,t \in [0,1]$.

Remark 1. *A function $F = [\underline{F}, \overline{F}] : [a,b] \times [c,d] \to I_c^+$ is said to be co-ordinated convex interval-valued function if and only if $\underline{F}$ and $\overline{F}$ are co-ordinated convex and concave, respectively.*

Lemma 1 ([37])**.** *A function $F : [a,b] \times [c,d] \to I_c^+$ is an interval-valued convex on co-ordinates if and only if there exist two functions $F_x : [c,d] \to I_c^+$, $F_x(w) = F(x,w)$ and $F_y : [a,b] \to I_c^+$, $F_y(u) = F(y,u)$ are interval-valued convex.*

It is easy to prove that an interval-valued convex function is an interval-valued co-ordinated convex, but the converse may not be true. For this, we can see the following example.

Example 1. *An interval-valued function $F : [0,1]^2 \to I_c^+$ defined as $F(x,y) = [xy, (6-e^x)(6-e^y)]$ is an interval-valued convex on co-ordinates, but it is not an interval-valued convex on $[0,1]^2$.*

For more recent inclusions of Hermite–Hadamard type for co-ordinated convex interval-valued functions one can read [37,38].

3. Basics of Quantum and Post-Quantum Calculus

In this section, we review some necessary definitions about q and (p,q)-calculus for real-valued and interval-valued functions. Moreover, here and further, we use the following notations with $0 < q < p \leq 1$:

$$\begin{aligned} [n]_q &= \frac{q^n - 1}{q - 1} = 1 + q + q^2 + \ldots + q^{n-1}, \\ [n]_{p,q} &= \frac{p^n - q^n}{p - q} = p^{n-1} + p^{n-2}q + p^{n-3}q^2 + \ldots + q^{n-1}. \end{aligned}$$

Definition 3 ([23])**.** *For a function $f : [a,b] \to \mathbb{R}$, the definite $(p,q)_a$-integral of f is stated as:*

$$\int_a^x f(t) \,{}_a d_{p,q} t = (p-q)(x-a) \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} f\left(\frac{q^n}{p^{n+1}} x + \left(1 - \frac{q^n}{p^{n+1}}\right) a \right) \tag{6}$$

where $0 < q < p \leq 1$ and $x \in [a, pb + (1-p)a]$.

Definition 4 ([24])**.** *For a function $f : [a,b] \to \mathbb{R}$, the definite $(p,q)^b$-integral of f is stated as:*

$$\int_x^b f(t) \,{}^b d_{p,q} t = (p-q)(b-x) \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} f\left(\frac{q^n}{p^{n+1}} x + \left(1 - \frac{q^n}{p^{n+1}}\right) b \right) \tag{7}$$

with $0 < q < p \leq 1$ and $x \in [pa + (1-p)b, b]$.

Remark 2. *If f is a symmetric function, that is, $f(t) = f(b + a - t)$, for $t \in [a, b]$, then we have*

$$\int_a^{pb+(1-p)a} f(t)_a d_{p.q} t = \int_{pa+(1-p)b}^{b} f(t)^b d_{p,q} t.$$

Definition 5 ([26,32]). *For a function $f : [a, b] \times [c, d] \to \mathbb{R}$,*

1. *The $(p, q)_a^d$ integral of f is given as:*

$$\int_a^x \int_y^d f(t,s)\ {}^d d_{p_2,q_2} s\ {}_a d_{p_1,q_1} t = (p_1 - q_1)(p_2 - q_2)(x - a)(d - y)$$
$$\times \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{q_1^n}{p_1^{n+1}} \frac{q_2^m}{p_2^{m+1}} f\left(\frac{q_1^n}{p_1^{n+1}} x + \left(1 - \frac{q_1^n}{p_1^{n+1}}\right) a, \frac{q_2^m}{p_2^{m+1}} y + \left(1 - \frac{q_2^m}{p_2^{m+1}}\right) d \right),$$

where $x, y \in [a, p_1 b + (1 - p_1)a] \times [p_2 c + (1 - p_2)d, d]$.

2. *The $(p, q)_c^b$ integral of f is given as:*

$$\int_x^b \int_c^y f(t,s)\ {}_c d_{p_2,q_2} s\ {}^b d_{p_1,q_1} t = (p_1 - q_1)(p_2 - q_2)(b - x)(y - c)$$
$$\times \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{q_1^n}{p_1^{n+1}} \frac{q_2^m}{p_2^{m+1}} f\left(\frac{q_1^n}{p_1^{n+1}} x + \left(1 - \frac{q_1^n}{p_1^{n+1}}\right) b, \frac{q_2^m}{p_2^{m+1}} y + \left(1 - \frac{q_2^m}{p_2^{m+1}}\right) c \right)$$

where $x, y \in [p_1 a + (1 - p_1)a, b] \times [c, p_2 d + (1 - p_2)c]$.

3. *The $(p, q)^{bd}$ integral of f is given as:*

$$\int_x^b \int_y^d f(t,s)\ {}^d d_{p_2,q_2} s\ {}^b d_{p_1,q_1} t = (p_1 - q_1)(p_2 - q_2)(b - x)(d - y)$$
$$\times \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{q_1^n}{p_1^{n+1}} \frac{q_2^m}{p_2^{m+1}} f\left(\frac{q_1^n}{p_1^{n+1}} x + \left(1 - \frac{q_1^n}{p_1^{n+1}}\right) b, \frac{q_2^m}{p_2^{m+1}} y + \left(1 - \frac{q_2^m}{p_2^{m+1}}\right) d \right),$$

where $x, y \in [p_1 a + (1 - p_1)b, b] \times [p_2 c + (1 - p_2)d, d]$.

4. *The $(p, q)_{ac}$ integral of f is given as:*

$$\int_a^x \int_c^y f(t,s)\ {}_c d_{p_2,q_2} s\ {}_a d_{p_1,q_1} t = (p_1 - q_1)(p_2 - q_2)(x - a)(y - c)$$
$$\times \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{q_1^n}{p_1^{n+1}} \frac{q_2^m}{p_2^{m+1}} f\left(\frac{q_1^n}{p_1^{n+1}} x + \left(1 - \frac{q_1^n}{p_1^{n+1}}\right) a, \frac{q_2^m}{p_2^{m+1}} y + \left(1 - \frac{q_2^m}{p_2^{m+1}}\right) c \right)$$

where $x, y \in [a, p_1 b + (1 - p_1)c] \times [c, p_2 d + (1 - p_2)c]$.

Recently, in [39], the authors gave the notions of quantum integral for the interval-valued functions and stated the following:

Definition 6 ([39]). *For an interval-valued function $F = \left[\underline{F}, \overline{F}\right] : [a, b] \to I_c$, the Iq_a-definite integral is defined by*

$$\int_a^x F(s)\ {}_a d_q^I s = (1 - q)(x - a) \sum_{n=0}^{\infty} q^n F(q^n x + (1 - q^n)a) \tag{8}$$

for all $x \in [a, b]$.

Definition 7 ([40]). *For an interval-valued function $F = [\underline{F}, \overline{F}] : [a,b] \to I_c$, the Iq^b-definite integral is defined by*

$$\int_x^b F(s)\ {}^b d_q^I s = (1-q)(b-x)\sum_{n=0}^{\infty} q^n F(q^n x + (1-q^n)b) \tag{9}$$

for all $x \in [a,b]$.

In [41], Ali et al. gave the post-quantum version of Definition 7 and defined it as:

Definition 8. *For an interval-valued function $F = [\underline{F}, \overline{F}] : [a,b] \to I_c$, the $I(p,q)^b$-definite integral is defined by*

$$\int_x^b F(s)\ {}^b d_{p,q}^I s = (p-q)(b-x)\sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} F\left(\frac{q^n}{p^{n+1}}x + \left(1 - \frac{q^n}{p^{n+1}}\right)b\right) \tag{10}$$

for all $x \in [pa + (1-p)b, b]$.

In [40], Ali et al. gave the co-ordinated version of the quantum integrals for interval-valued functions and defined it as:

Definition 9 ([40]). *Suppose that $F = [\underline{F}, \overline{F}] : [a,b] \times [c,d] \to I_c$ is an interval-valued function. Then, the definite q_{ac}, q_a^d, q_c^b and q^{bd} integrals on $[a,b] \times [c,d]$ are defined by*

$$\begin{aligned}
\int_a^x \int_c^y F(t,s)\ {}_c d_{q_2}^I s \quad {}_a d_{q_1}^I t \ &= (1-q_1)(1-q_2)(x-a)(y-c) \\
&\times \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} q_1^n q_2^m F(q_1^n x + (1-q_1^n)a, q_2^m y + (1-q_2^m)c), \\
\int_a^x \int_y^d F(t,s)\ {}^d d_{q_2}^I s \quad {}_a d_{q_1}^I t \ &= (1-q_1)(1-q_2)(x-a)(d-y) \\
&\times \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} q_1^n q_2^m F(q_1^n x + (1-q_1^n)a, q_2^m y + (1-q_2^m)d), \\
\int_x^b \int_c^y F(t,s)\ {}_c d_{q_2}^I s \quad {}^b d_{q_1}^I t \ &= (1-q_1)(1-q_2)(b-x)(y-c) \\
&\times \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} q_1^n q_2^m F(q_1^n x + (1-q_1^n)b, q_2^m y + (1-q_2^m)c)
\end{aligned}$$

and

$$\begin{aligned}
\int_x^b \int_y^d F(t,s)\ {}^d d_{q_2}^I s \quad {}^b d_{q_1}^I t \ &= (1-q_1)(1-q_2)(b-x)(d-y) \\
&\times \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} q_1^n q_2^m F(q_1^n x + (1-q_1^n)b, q_2^m y + (1-q_2^m)d)
\end{aligned}$$

respectively, for $(x,y) \in [a,b] \times [c,d]$.

Remark 3. *It is very easy to observe that*

$$\begin{aligned}\int_a^b\int_c^d F(t,s)\ {}_c d^I_{q_2}s\ \ {}_a d^I_{q_1}t &= \int_a^b\int_c^d F(t,s)\ {}^d d^I_{q_2}s\ \ {}_a d^I_{q_1}t = \int_a^b\int_c^d F(t,s)\ {}_c d^I_{q_2}s\ \ {}^b d^I_{q_1}t \\ &= \int_a^b\int_c^d F(t,s)\ {}^d d^I_{q_2}s\ \ {}^b d^I_{q_1}t = \int_a^b\int_c^d F(t,s)\ d^I s\ \ d^I t\end{aligned}$$

by taking the limits $q_1, q_2 \to 1^-$ (see, [42]).

Now, we define $I(p,q)$-integrals for the functions of two variables as:

Definition 10. *For an interval-valued function $F = [\underline{F}, \overline{F}] : [a,b] \times [c,d] \to I_c$,*

1. *The $I(p,q)_a^d$ integral of F is given as:*

$$\int_a^x \int_y^d F(t,s)\ {}^d d^I_{p_2,q_2}s\ {}_a d^I_{p_1,q_1}t = (p_1-q_1)(p_2-q_2)(x-a)(d-y) \\ \times \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{q_1^n}{p_1^{n+1}}\frac{q_2^m}{p_2^{m+1}} F\left(\frac{q_1^n}{p_1^{n+1}}x + \left(1-\frac{q_1^n}{p_1^{n+1}}\right)a, \frac{q_2^m}{p_2^{m+1}}y + \left(1-\frac{q_2^m}{p_2^{m+1}}\right)d\right),$$

   where $x, y \in [a, p_1 b + (1-p_1)a] \times [p_2 c + (1-p_2)d, d]$.

2. *The $I(p,q)_c^b$ integral of F is given as:*

$$\int_x^b \int_c^y F(t,s)\ {}_c d^I_{p_2,q_2}s\ {}^b d^I_{p_1,q_1}t = (p_1-q_1)(p_2-q_2)(b-x)(y-c) \\ \times \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{q_1^n}{p_1^{n+1}}\frac{q_2^m}{p_2^{m+1}} F\left(\frac{q_1^n}{p_1^{n+1}}x + \left(1-\frac{q_1^n}{p_1^{n+1}}\right)b, \frac{q_2^m}{p_2^{m+1}}y + \left(1-\frac{q_2^m}{p_2^{m+1}}\right)c\right)$$

   where $x, y \in [p_1 a + (1-p_1)a, b] \times [c, p_2 d + (1-p_2)c]$.

3. *The $I(p,q)_{ac}$ integral of F is given as:*

$$\int_a^x \int_c^y F(t,s)\ {}_c d^I_{p_2,q_2}s\ {}_a d^I_{p_1,q_1}t = (p_1-q_1)(p_2-q_2)(x-a)(y-c) \\ \times \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{q_1^n}{p_1^{n+1}}\frac{q_2^m}{p_2^{m+1}} F\left(\frac{q_1^n}{p_1^{n+1}}x + \left(1-\frac{q_1^n}{p_1^{n+1}}\right)a, \frac{q_2^m}{p_2^{m+1}}y + \left(1-\frac{q_2^m}{p_2^{m+1}}\right)c\right)$$

   where $x, y \in [a, p_1 b + (1-p_1)c] \times [c, p_2 d + (1-p_2)c]$.

4. *The $I(p,q)^{bd}$ integral of F is given as:*

$$\int_x^b \int_y^d F(t,s)\ {}^d d^I_{p_2,q_2}s\ {}^b d^I_{p_1,q_1}t = (p_1-q_1)(p_2-q_2)(b-x)(d-y) \\ \times \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{q_1^n}{p_1^{n+1}}\frac{q_2^m}{p_2^{m+1}} F\left(\frac{q_1^n}{p_1^{n+1}}x + \left(1-\frac{q_1^n}{p_1^{n+1}}\right)b, \frac{q_2^m}{p_2^{m+1}}y + \left(1-\frac{q_2^m}{p_2^{m+1}}\right)d\right),$$

   where $x, y \in [p_1 a + (1-p_1)b, b] \times [p_2 c + (1-p_2)d, d]$.

Example 2. *Define an interval-valued mapping $F = [\underline{F}, \overline{F}] : [0,1] \times [0,1] \to I_c$ by $F(t,s) = [t^2 s^2, ts]$. Then, by Definition 10, for $p_1 = p_2 = \frac{3}{4}$ and $q_1 = q_2 = \frac{1}{2}$, we have*

1. *From $I(p,q)_a^d$-integral:*

$$\int_0^{\frac{3}{4}} \int_{\frac{1}{4}}^1 F(t,s)\ {}^1 d^I_{\frac{3}{4},\frac{1}{2}}s\ {}_0 d^I_{\frac{3}{4},\frac{1}{2}}t$$

$$
\begin{aligned}
&= \left(\frac{1}{4}\right)\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)\left(\frac{3}{4}\right) \\
&\quad \times \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(\frac{4}{3}\right)\left(\frac{4}{3}\right)\left(\frac{2}{3}\right)^{n}\left(\frac{2}{3}\right)^{m}\left[\left(\frac{2}{3}\right)^{2n}\cdot\left(1-\left(\frac{2}{3}\right)^{m}\right)^{2},\left(\frac{2}{3}\right)^{n}\left(1-\left(\frac{2}{3}\right)^{m}\right)\right] \\
&= [0.0729, 0.135].
\end{aligned}
$$

2. *From $I(p,q)_c^b$-integral:*

$$
\begin{aligned}
&\int_{\frac{1}{4}}^{1}\int_{0}^{\frac{3}{4}} F(t,s)\ {}_0d^I_{\frac{3}{4},\frac{1}{2}}s\ {}^1d^I_{\frac{3}{4},\frac{1}{2}}t \\
&= \left(\frac{1}{4}\right)\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)\left(\frac{3}{4}\right) \\
&\quad \times \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(\frac{4}{3}\right)\left(\frac{4}{3}\right)\left(\frac{2}{3}\right)^{n}\left(\frac{2}{3}\right)^{m}\left[\left(\frac{2}{3}\right)^{2m}\cdot\left(1-\left(\frac{2}{3}\right)^{n}\right)^{2},\left(\frac{2}{3}\right)^{m}\left(1-\left(\frac{2}{3}\right)^{n}\right)\right] \\
&= [0.0729, 0.135].
\end{aligned}
$$

3. *From $I(p,q)_{ac}$-integral:*

$$
\begin{aligned}
&\int_{0}^{\frac{3}{4}}\int_{0}^{\frac{3}{4}} F(t,s)\ {}_0d^I_{\frac{3}{4},\frac{1}{2}}s\ {}_0d^I_{\frac{3}{4},\frac{1}{2}}t \\
&= \left(\frac{1}{4}\right)\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)\left(\frac{3}{4}\right) \\
&\quad \times \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(\frac{4}{3}\right)\left(\frac{4}{3}\right)\left(\frac{2}{3}\right)^{n}\left(\frac{2}{3}\right)^{m}\left(\left(\frac{2}{3}\right)^{2n}\cdot\left(\frac{2}{3}\right)^{2m},\left(\frac{2}{3}\right)^{n}\cdot\left(\frac{2}{3}\right)^{m}\right) \\
&= [0.1262, 0.2025].
\end{aligned}
$$

4. *From $I(p,q)^{bd}$-integral*

$$
\begin{aligned}
&\int_{\frac{1}{4}}^{1}\int_{\frac{1}{4}}^{1} F(t,s)\ {}^1d^I_{\frac{3}{4},\frac{1}{2}}s\ {}^1d^I_{\frac{3}{4},\frac{1}{2}}t \\
&= \left(\frac{1}{4}\right)\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)\left(\frac{3}{4}\right) \\
&\quad \times \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(\frac{4}{3}\right)\left(\frac{4}{3}\right)\left(\frac{2}{3}\right)^{n}\left(\frac{2}{3}\right)^{m}\left(\left(1-\left(\frac{2}{3}\right)^{n}\right)^{2}\left(1-\left(\frac{2}{3}\right)^{m}\right)^{2},\left(1-\left(\frac{2}{3}\right)^{n}\right)\left(1-\left(\frac{2}{3}\right)^{n}\right)\right) \\
&= [0.0421, 0.09].
\end{aligned}
$$

4. Some New (p,q)-Hermite–Hadamard Inclusions

In this section, we deal with the Hermite–Hadamard-type inclusions for co-ordinated convex interval-valued functions using the newly defined interval-valued (p,q)-integrals in the last section.

Theorem 2. *Let $F = [\underline{F}, \overline{F}] : [a,b]\times[c,d] \to I_c^+$ be a co-ordinated convex interval-valued function on $[a,b]\times[c,d]$. Then, the following inclusions of Hermite–Hadamard type hold for $(p,q)_a^d$-integral:*

$$
\begin{aligned}
&F\left(\frac{q_1 a + p_1 b}{[2]_{p_1,q_1}}, \frac{p_2 c + q_2 d}{[2]_{p_2,q_2}}\right) \qquad (11) \\
&\supseteq \frac{1}{2}\left[\frac{1}{p_1(b-a)}\int_a^{p_1 b + (1-p_1)a} F\left(x, \frac{p_2 c + q_2 d}{[2]_{p_2,q_2}}\right)\ {}_a d^I_{p_1,q_1} x\right.
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{p_2(d-c)}\int_{p_2c+(1-p_2)d}^{d} F\left(\frac{q_1a+p_1b}{[2]_{p_1,q_1}},y\right)\,{}^{d}d^{I}_{p_2,q_2}y\Bigg] \\
\supseteq\;& \frac{1}{p_1p_2(b-a)(d-c)}\int_{a}^{p_1b+(1-p_1)a}\int_{p_2c+(1-p_2)d}^{d} F(x,y)\,{}^{d}d^{I}_{p_2,q_2}y\,{}_{a}d^{I}_{p_1,q_1}x \\
\supseteq\;& \frac{1}{2p_2[2]_{p_1,q_1}(d-c)}\left[q_1\int_{p_2c+(1-p_2)d}^{d} F(a,y)\,{}^{d}d^{I}_{p_2,q_2}y+p_1\int_{p_2c+(1-p_2)d}^{d} F(b,y)\,{}^{d}d^{I}_{p_2,q_2}y\right] \\
&+\frac{1}{2p_1[2]_{p_2,q_2}(b-a)}\left[p_2\int_{a}^{p_1b+(1-p_1)a} F(x,c)\,{}_{a}d^{I}_{p_1,q_1}x+q_2\int_{a}^{p_1b+(1-p_1)a} F(x,d)\,{}_{a}d^{I}_{p_1,q_1}x\right] \\
\supseteq\;& \frac{q_1p_2F(a,c)+q_1q_2F(a,d)+p_1p_2F(b,c)+q_2p_1F(b,d)}{[2]_{p_1,q_1}[2]_{p_2,q_2}}.
\end{aligned}
$$

Proof. Since $F=\left[\underline{F},\overline{F}\right]:[a,b]\times[c,d]\to I_c^+$ is a co-ordinated convex interval-valued function on co-ordinates $[a,b]\times[c,d]$, $\underline{F}$ and $\overline{F}$ are co-ordinated convex and concave on co-ordinates $[a,b]\times[c,d]$, respectively. Hence, from co-ordinated convexity of $\underline{F}$ and using (2), we have

$$
\begin{aligned}
&\underline{F}\left(\frac{q_1a+p_1b}{[2]_{p_1,q_1}},\frac{p_2c+q_2d}{[2]_{p_2,q_2}}\right) \\
\le\;& \frac{1}{2}\left[\frac{1}{p_1(b-a)}\int_{a}^{p_1b+(1-p_1)a}\underline{F}\left(x,\frac{p_2c+q_2d}{[2]_{p_2,q_2}}\right)\,{}_{a}d_{p_1,q_1}x\right. \\
&\left.+\frac{1}{p_2(d-c)}\int_{p_2c+(1-p_2)d}^{d}\underline{F}\left(\frac{q_1a+p_1b}{[2]_{p_1,q_1}},y\right)\,{}^{d}d_{p_2,q_2}y\right] \\
\le\;& \frac{1}{p_1p_2(b-a)(d-c)}\int_{a}^{p_1b+(1-p_1)a}\int_{p_2c+(1-p_2)d}^{d}\underline{F}(x,y)\,{}^{d}d_{p_2,q_2}y\,{}_{a}d_{p_1,q_1}x \\
\le\;& \frac{1}{2p_2[2]_{p_1,q_1}(d-c)}\left[q_1\int_{p_2c+(1-p_2)d}^{d}\underline{F}(a,y)\,{}^{d}d_{p_2,q_2}y+p_1\int_{p_2c+(1-p_2)d}^{d}\underline{F}(b,y)\,{}^{d}d_{p_2,q_2}y\right] \\
&+\frac{1}{2p_1[2]_{p_2,q_2}(b-a)}\left[p_2\int_{a}^{p_1b+(1-p_1)a}\underline{F}(x,c)\,{}_{a}d_{p_1,q_1}x+q_2\int_{a}^{p_1b+(1-p_1)a}\underline{F}(x,d)\,{}_{a}d_{p_1,q_1}x\right] \\
\le\;& \frac{q_1p_2\underline{F}(a,c)+q_1q_2\underline{F}(a,d)+p_1p_2\underline{F}(b,c)+q_2p_1\underline{F}(b,d)}{[2]_{p_1,q_1}[2]_{p_2,q_2}}.
\end{aligned}
\tag{12}
$$

From co-ordinated concavity of $\overline{F}$ and again using (2), we have

$$
\begin{aligned}
&\overline{F}\left(\frac{q_1a+p_1b}{[2]_{p_1,q_1}},\frac{p_2c+q_2d}{[2]_{p_2,q_2}}\right) \\
\ge\;& \frac{1}{2}\left[\frac{1}{p_1(b-a)}\int_{a}^{p_1b+(1-p_1)a}\overline{F}\left(x,\frac{p_2c+q_2d}{[2]_{p_2,q_2}}\right)\,{}_{a}d_{p_1,q_1}x\right. \\
&\left.+\frac{1}{p_2(d-c)}\int_{p_2c+(1-p_2)d}^{d}\overline{F}\left(\frac{q_1a+p_1b}{[2]_{p_1,q_1}},y\right)\,{}^{d}d_{p_2,q_2}y\right] \\
\ge\;& \frac{1}{p_1p_2(b-a)(d-c)}\int_{a}^{p_1b+(1-p_1)a}\int_{p_2c+(1-p_2)d}^{d}\overline{F}(x,y)\,{}^{d}d_{p_2,q_2}y\,{}_{a}d_{p_1,q_1}x \\
\ge\;& \frac{1}{2p_2[2]_{p_1,q_1}(d-c)}\left[q_1\int_{p_2c+(1-p_2)d}^{d}\overline{F}(a,y)\,{}^{d}d_{p_2,q_2}y+p_1\int_{p_2c+(1-p_2)d}^{d}\overline{F}(b,y)\,{}^{d}d_{p_2,q_2}y\right] \\
&+\frac{1}{2p_1[2]_{p_2,q_2}(b-a)}\left[p_2\int_{a}^{p_1b+(1-p_1)a}\overline{F}(x,c)\,{}_{a}d_{p_1,q_1}x+q_2\int_{a}^{p_1b+(1-p_1)a}\overline{F}(x,d)\,{}_{a}d_{p_1,q_1}x\right]
\end{aligned}
\tag{13}
$$

$$\geq \quad \frac{q_1 p_2 \overline{F}(a,c) + q_1 q_2 \overline{F}(a,d) + p_1 p_2 \overline{F}(b,c) + q_2 p_1 \overline{F}(b,d)}{[2]_{p_1,q_1}[2]_{p_2,q_2}}.$$

Now, from the inequalities (12) and (13), we have following inclusions:

$$\begin{aligned}
& F\left(\frac{q_1 a + p_1 b}{[2]_{p_1,q_1}}, \frac{p_2 c + q_2 d}{[2]_{p_2,q_2}}\right) \\
= & \left[\underline{F}\left(\frac{q_1 a + p_1 b}{[2]_{p_1,q_1}}, \frac{p_2 c + q_2 d}{[2]_{p_2,q_2}}\right), \overline{F}\left(\frac{q_1 a + p_1 b}{[2]_{p_1,q_1}}, \frac{p_2 c + q_2 d}{[2]_{p_2,q_2}}\right)\right] \\
\supseteq & \frac{1}{2}\left[\frac{1}{p_1(b-a)}\left[\int_a^{p_1 b + (1-p_1)a} \underline{F}\left(x, \frac{p_2 c + q_2 d}{[2]_{p_2,q_2}}\right) {}_a d_{p_1,q_1} x, \int_a^{p_1 b + (1-p_1)a} \overline{F}\left(x, \frac{p_2 c + q_2 d}{[2]_{p_2,q_2}}\right) {}_a d_{p_1,q_1} x\right]\right. \\
& \left. + \frac{1}{p_2(d-c)}\left[\int_{p_2 c + (1-p_2)d}^{d} \underline{F}\left(\frac{q_1 a + p_1 b}{[2]_{p_1,q_1}}, y\right) {}^d d_{p_2,q_2} y, \int_{p_2 c + (1-p_2)d}^{d} \overline{F}\left(\frac{q_1 a + p_1 b}{[2]_{p_1,q_1}}, y\right) {}^d d_{p_2,q_2} y\right]\right] \\
= & \frac{1}{2}\left[\frac{1}{p_1(b-a)} \int_a^{p_1 b + (1-p_1)a} F\left(x, \frac{p_2 c + q_2 d}{[2]_{p_2,q_2}}\right) {}_a d^I_{p_1,q_1} x \right. \\
& \left. + \frac{1}{p_2(d-c)} \int_{p_2 c + (1-p_2)d}^{d} F\left(\frac{q_1 a + p_1 b}{[2]_{p_1,q_1}}, y\right) {}^d d^I_{p_2,q_2} y\right],
\end{aligned} \tag{14}$$

$$\begin{aligned}
& \frac{1}{2}\left[\frac{1}{p_1(b-a)} \int_a^{p_1 b + (1-p_1)a} F\left(x, \frac{p_2 c + q_2 d}{[2]_{p_2,q_2}}\right) {}_a d^I_{p_1,q_1} x \right. \\
& \left. + \frac{1}{p_2(d-c)} \int_{p_2 c + (1-p_2)d}^{d} F\left(\frac{q_1 a + p_1 b}{[2]_{p_1,q_1}}, y\right) {}^d d^I_{p_2,q_2} y\right] \\
= & \frac{1}{2}\left[\frac{1}{p_1(b-a)}\left[\int_a^{p_1 b + (1-p_1)a} \underline{F}\left(x, \frac{p_2 c + q_2 d}{[2]_{p_2,q_2}}\right) {}_a d_{p_1,q_1} x, \int_a^{p_1 b + (1-p_1)a} \overline{F}\left(x, \frac{p_2 c + q_2 d}{[2]_{p_2,q_2}}\right) {}_a d_{p_1,q_1} x\right]\right. \\
& \left. + \frac{1}{p_2(d-c)}\left[\int_{p_2 c + (1-p_2)d}^{d} \underline{F}\left(\frac{q_1 a + p_1 b}{[2]_{p_1,q_1}}, y\right) {}^d d_{p_2,q_2} y, \int_{p_2 c + (1-p_2)d}^{d} \overline{F}\left(\frac{q_1 a + p_1 b}{[2]_{p_1,q_1}}, y\right) {}^d d_{p_2,q_2} y\right]\right] \\
\supseteq & \frac{1}{p_1 p_2 (b-a)(d-c)} \\
& \times \left[\int_a^{p_1 b + (1-p_1)a} \int_{p_2 c + (1-p_2)d}^{d} \underline{F}(x,y) {}^d d_{p_2,q_2} y \, {}_a d_{p_1,q_1} x, \int_a^{p_1 b + (1-p_1)a} \int_{p_2 c + (1-p_2)d}^{d} \overline{F}(x,y) {}^d d_{p_2,q_2} y \, {}_a d_{p_1,q_1} x\right] \\
= & \frac{1}{p_1 p_2 (b-a)(d-c)} \int_a^{p_1 b + (1-p_1)a} \int_{p_2 c + (1-p_2)d}^{d} F(x,y) {}^d d^I_{p_2,q_2} y \, {}_a d^I_{p_1,q_1} x,
\end{aligned} \tag{15}$$

$$\begin{aligned}
& \frac{1}{p_1 p_2 (b-a)(d-c)} \int_a^{p_1 b + (1-p_1)a} \int_{p_2 c + (1-p_2)d}^{d} F(x,y) {}^d d^I_{p_2,q_2} y \, {}_a d^I_{p_1,q_1} x \\
= & \frac{1}{p_1 p_2 (b-a)(d-c)} \\
& \times \left[\int_a^{p_1 b + (1-p_1)a} \int_{p_2 c + (1-p_2)d}^{d} \underline{F}(x,y) {}^d d_{p_2,q_2} y \, {}_a d_{p_1,q_1} x, \int_a^{p_1 b + (1-p_1)a} \int_{p_2 c + (1-p_2)d}^{d} \overline{F}(x,y) {}^d d_{p_2,q_2} y \, {}_a d_{p_1,q_1} x\right] \\
\supseteq & \frac{1}{2 p_2 [2]_{p_1,q_1} (d-c)} \left[q_1 \left[\int_{p_2 c + (1-p_2)d}^{d} \underline{F}(a,y) {}^d d_{p_2,q_2} y, \int_{p_2 c + (1-p_2)d}^{d} \overline{F}(a,y) {}^d d_{p_2,q_2} y\right]\right. \\
& \left. + p_1 \left[\int_{p_2 c + (1-p_2)d}^{d} \underline{F}(b,y) {}^d d_{p_2,q_2} y, \int_{p_2 c + (1-p_2)d}^{d} \overline{F}(b,y) {}^d d_{p_2,q_2} y\right]\right]
\end{aligned} \tag{16}$$

$$
\begin{aligned}
&+\frac{1}{2p_1[2]_{p_2,q_2}(b-a)}\left[p_2\left[\int_a^{p_1b+(1-p_1)a}\underline{F}(x,c)\ {}_ad_{p_1,q_1}x,\int_a^{p_1b+(1-p_1)a}\overline{F}(x,c)\ {}_ad_{p_1,q_1}x\right]\right.\\
&\left.+q_2\left[\int_a^{p_1b+(1-p_1)a}\underline{F}(x,d)\ {}_ad_{p_1,q_1}x,\int_a^{p_1b+(1-p_1)a}\overline{F}(x,d)\ {}_ad_{p_1,q_1}x\right]\right]\\
=\ &\frac{1}{2p_2[2]_{p_1,q_1}(d-c)}\left[q_1\int_{p_2c+(1-p_2)d}^{d}F(a,y)\ {}^dd^I_{p_2,q_2}y+p_1\int_{p_2c+(1-p_2)d}^{d}F(b,y)\ {}^dd^I_{p_2,q_2}y\right]\\
&+\frac{1}{2p_1[2]_{p_2,q_2}(b-a)}\left[p_2\int_a^{p_1b+(1-p_1)a}F(x,c)\ {}_ad^I_{p_1,q_1}x+q_2\int_a^{p_1b+(1-p_1)a}F(x,d)\ {}_ad^I_{p_1,q_1}x\right]
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{1}{2p_2[2]_{p_1,q_1}(d-c)}\left[q_1\int_{p_2c+(1-p_2)d}^{d}F(a,y)\ {}^dd^I_{p_2,q_2}y+p_1\int_{p_2c+(1-p_2)d}^{d}F(b,y)\ {}^dd^I_{p_2,q_2}y\right] \qquad (17)\\
&+\frac{1}{2p_1[2]_{p_2,q_2}(b-a)}\left[p_2\int_a^{p_1b+(1-p_1)a}F(x,c)\ {}_ad^I_{p_1,q_1}x+q_2\int_a^{p_1b+(1-p_1)a}F(x,d)\ {}_ad^I_{p_1,q_1}x\right]\\
=\ &\frac{1}{2p_2[2]_{p_1,q_1}(d-c)}\left[q_1\left[\int_{p_2c+(1-p_2)d}^{d}\underline{F}(a,y)\ {}^dd_{p_2,q_2}y,\int_{p_2c+(1-p_2)d}^{d}\overline{F}(a,y)\ {}^dd_{p_2,q_2}y\right]\right.\\
&\left.+p_1\left[\int_{p_2c+(1-p_2)d}^{d}\underline{F}(b,y)\ {}^dd_{p_2,q_2}y,\int_{p_2c+(1-p_2)d}^{d}\overline{F}(b,y)\ {}^dd_{p_2,q_2}y\right]\right]\\
&+\frac{1}{2p_1[2]_{p_2,q_2}(b-a)}\left[p_2\left[\int_a^{p_1b+(1-p_1)a}\underline{F}(x,c)\ {}_ad_{p_1,q_1}x,\int_a^{p_1b+(1-p_1)a}\overline{F}(x,c)\ {}_ad_{p_1,q_1}x\right]\right.\\
&\left.+q_2\left[\int_a^{p_1b+(1-p_1)a}\underline{F}(x,d)\ {}_ad_{p_1,q_1}x,\int_a^{p_1b+(1-p_1)a}\overline{F}(x,d)\ {}_ad_{p_1,q_1}x\right]\right]\\
\supseteq\ &\frac{1}{[2]_{p_1,q_1}[2]_{p_2,q_2}}\left[q_1p_2\left[\underline{F}(a,c),\overline{F}(a,c)\right]+q_1q_2\left[\underline{F}(a,d),\overline{F}(a,d)\right]\right.\\
&\left.+p_1p_2\left[\underline{F}(b,c),\overline{F}(b,c)\right]+q_2p_1\left[\underline{F}(b,d),\overline{F}(b,d)\right]\right]\\
=\ &\frac{q_1p_2F(a,c)+q_1q_2F(a,d)+p_1p_2F(b,c)+q_2p_1F(b,d)}{[2]_{p_1,q_1}[2]_{p_2,q_2}}.
\end{aligned}
$$

We obtain the required result (11) by combining the inclusions (14)–(17). □

Remark 4. *In Theorem 2, if $\underline{F}=\overline{F}$, then inclusions (11) reduce to inequalities (2).*

Remark 5. *In Theorem 2, if we set $p_1=p_2=1$, then Theorem 2 becomes ([40], Theorem 12).*

Theorem 3. *Let $F=[\underline{F},\overline{F}]:[a,b]\times[c,d]\to I_c^+$ be a co-ordinated convex interval-valued function on $[a,b]\times[c,d]$. The following inclusions of Hermite–Hadamard type hold for $I(p,q)_c^b$-integral:*

$$
\begin{aligned}
&F\left(\frac{p_1a+q_1b}{[2]_{p_1,q_1}},\frac{q_2c+p_2d}{[2]_{p_2,q_2}}\right) \qquad (18)\\
\supseteq\ &\frac{1}{2}\left[\frac{1}{p_1(b-a)}\int_{p_1a+(1-p_1)b}^{b}F\left(x,\frac{q_2c+p_2d}{[2]_{p_2,q_2}}\right)\ {}^bd^I_{p_1,q_1}x\right.\\
&\left.+\frac{1}{p_2(d-c)}\int_c^{p_2d+(1-p_2)c}F\left(\frac{p_1a+q_1b}{[2]_{p_1,q_1}},y\right)\ {}_cd^I_{p_2,q_2}y\right]\\
\supseteq\ &\frac{1}{p_1p_2(b-a)(d-c)}\int_{p_1a+(1-p_1)b}^{b}\int_c^{p_2d+(1-p_2)c}F(x,y)\ {}_cd^I_{p_2,q_2}y\ {}^bd^I_{p_1,q_1}x
\end{aligned}
$$

$$
\begin{aligned}
&\supseteq \frac{1}{2p_2[2]_{p_1,q_1}(d-c)}\left[p_1\int_c^{p_2d+(1-p_2)c}F(a,y)\ {}_cd^I_{p_2,q_2}y+q_1\int_c^{p_2d+(1-p_2)c}F(b,y)\ {}_cd^I_{p_2,q_2}y\right]\\
&\quad+\frac{1}{2p_1[2]_{p_2,q_2}(b-a)}\left[q_2\int_{p_1a+(1-p_1)b}^{b}F(x,c)\ {}^bd^I_{p_1,q_1}x+p_2\int_{p_1a+(1-p_1)b}^{b}F(x,d)\ {}^bd^I_{p_1,q_1}x\right]\\
&\supseteq \frac{p_1q_2F(a,c)+p_1p_2F(a,d)+q_1q_2F(b,c)+q_1p_2F(b,d)}{[2]_{p_1,q_1}[2]_{p_2,q_2}}.
\end{aligned}
$$

for all $q_1, q_2 \in (0,1)$.

Proof. If we follow the concepts used in the proof of Theorem 2 by taking into account the inclusions (3), the desired inclusions (18) can be attained. □

Remark 6. *In Theorem 3, if $\underline{F} = \overline{F}$, then inclusions (18) reduce to inequalities (3).*

Remark 7. *In Theorem 3, if we set $p_1 = p_2 = 1$, then Theorem 3 becomes ([40], Theorem 13).*

Theorem 4. *Let $F = [\underline{F}, \overline{F}] : [a,b] \times [c,d] \to I_c^+$ be a co-ordinated convex interval-valued function on $[a,b] \times [c,d]$. The following inclusions of Hermite–Hadamard type hold for $I(p,q)^{bd}$-integral:*

$$
\begin{aligned}
&F\left(\frac{p_1a+q_1b}{[2]_{p_1,q_1}},\frac{p_2c+q_2d}{[2]_{p_2,q_2}}\right) \qquad (19)\\
&\supseteq \frac{1}{2}\left[\frac{1}{p_1(b-a)}\int_{p_1a+(1-p_1)b}^{b}F\left(x,\frac{p_2c+q_2d}{[2]_{p_2,q_2}}\right)\ {}^bd^I_{p_1,q_1}x\right.\\
&\quad\left.+\frac{1}{p_2(d-c)}\int_{p_2c+(1-p_2)d}^{d}F\left(\frac{p_1a+q_1b}{[2]_{p_1,q_1}},y\right)\ {}^dd^I_{p_2,q_2}y\right]\\
&\supseteq \frac{1}{p_1p_2(b-a)(d-c)}\int_{p_1a+(1-p_1)b}^{b}\int_{p_2c+(1-p_2)d}^{d}F(x,y)\ {}^dd^I_{p_2,q_2}y\ {}^bd^I_{p_1,q_1}x\\
&\supseteq \frac{1}{2p_2[2]_{p_1,q_1}(d-c)}\left[p_1\int_{p_2c+(1-p_2)d}^{d}F(a,y)\ {}^dd^I_{p_2,q_2}y+q_1\int_{p_2c+(1-p_2)d}^{d}F(b,y)\ {}^dd^I_{p_2,q_2}y\right]\\
&\quad+\frac{1}{2p_1[2]_{p_2,q_2}(b-a)}\left[p_2\int_{p_1a+(1-p_1)b}^{b}F(x,c)\ {}^bd^I_{p_1,q_1}x+q_2\int_{p_1a+(1-p_1)b}^{b}F(x,d)\ {}^bd^I_{p_1,q_1}x\right]\\
&\supseteq \frac{p_1p_2F(a,c)+p_1q_2F(a,d)+q_1p_2F(b,c)+q_1q_2F(b,d)}{[2]_{p_1,q_1}[2]_{p_2,q_2}}.
\end{aligned}
$$

Proof. Following arguments similar to those in the proof of Theorem 2 by taking into account the inclusions (4), the desired inclusion (19) can be attained. □

Remark 8. *In Theorem 4, if $\underline{F} = \overline{F}$, then inclusions (19) reduce to inequalities (4).*

Remark 9. *In Theorem 4, if we set $p_1 = p_2 = 1$, then Theorem 4 becomes ([40], Theorem 14).*

Theorem 5. *Let $F = [\underline{F}, \overline{F}] : [a,b] \times [c,d] \to I_c^+$ be a co-ordinated convex interval-valued function on $[a,b] \times [c,d]$. The following inclusions of Hermite–Hadamard type hold for $I(p,q)_{ac}$-integral:*

$$
\begin{aligned}
&F\left(\frac{q_1a+p_1b}{[2]_{p_1,q_1}},\frac{q_2c+p_2d}{[2]_{p_2,q_2}}\right)\\
&\supseteq \frac{1}{2}\left[\frac{1}{p_1(b-a)}\int_a^{p_1b+(1-p_1)a}F\left(x,\frac{q_2c+p_2d}{[2]_{p_2,q_2}}\right)\ {}_ad^I_{p_1,q_1}x\right.
\end{aligned}
$$

$$
\begin{aligned}
& +\frac{1}{p_2(d-c)} \int_c^{p_2 d+(1-p_2)c} F\left(\frac{q_1 a+p_1 b}{[2]_{p_1,q_1}}, y\right) {}_c d^I_{p_2,q_2} y \Bigg] \\
\supseteq \quad & \frac{1}{p_1 p_2(b-a)(d-c)} \int_a^{p_1 b+(1-p_1)a} \int_c^{p_2 d+(1-p_2)c} F(x,y)\, {}_c d^I_{p_2,q_2} y\, {}_a d^I_{p_1,q_1} x \\
\supseteq \quad & \frac{1}{2p_2[2]_{p_1,q_1}(d-c)} \left[q_1 \int_c^{p_2 d+(1-p_2)c} F(a,y)\, {}_c d^I_{p_2,q_2} y + p_1 \int_c^{p_2 d+(1-p_2)c} F(b,y)\, {}_c d^I_{p_2,q_2} y \right] \\
& +\frac{1}{2p_1[2]_{p_2,q_2}(b-a)} \left[q_2 \int_a^{p_1 b+(1-p_1)a} F(x,c)\, {}_a d^I_{p_1,q_1} x + p_2 \int_a^{p_1 b+(1-p_1)a} F(x,d)\, {}_a d^I_{p_1,q_1} x \right] \\
\supseteq \quad & \frac{q_1 q_2 F(a,c)+q_1 p_2 F(a,d)+p_1 q_2 F(b,c)+p_1 p_2 F(b,d)}{[2]_{p_1,q_1}[2]_{p_2,q_2}}.
\end{aligned}
$$

Proof. Following arguments similar to those in the proof of Theorem 2 and the concepts of inequalities (2)–(4), by taking into account the $I(p,q)_{ac}$-integral, the desired inclusion can be attained. □

Remark 10. *In Theorem 5, if $\underline{F} = \overline{F}$, then we have the following inequality:*

$$
\begin{aligned}
& F\left(\frac{q_1 a+p_1 b}{[2]_{p_1,q_1}}, \frac{q_2 c+p_2 d}{[2]_{p_2,q_2}}\right) \\
\leq \quad & \frac{1}{2}\left[\frac{1}{p_1(b-a)} \int_a^{p_1 b+(1-p_1)a} F\left(x, \frac{q_2 c+p_2 d}{[2]_{p_2,q_2}}\right) {}_a d_{p_1,q_1} x \right. \\
& \left. +\frac{1}{p_2(d-c)} \int_c^{p_2 d+(1-p_2)c} F\left(\frac{q_1 a+p_1 b}{[2]_{p_1,q_1}}, y\right) {}_c d_{p_2,q_2} y \right] \\
\leq \quad & \frac{1}{p_1 p_2(b-a)(d-c)} \int_a^{p_1 b+(1-p_1)a} \int_c^{p_2 d+(1-p_2)c} F(x,y)\, {}_c d_{p_2,q_2} y\, {}_a d_{p_1,q_1} x \\
\leq \quad & \frac{1}{2p_2[2]_{p_1,q_1}(d-c)} \left[q_1 \int_c^{p_2 d+(1-p_2)c} F(a,y)\, {}_c d_{p_2,q_2} y + p_1 \int_c^{p_2 d+(1-p_2)c} F(b,y)\, {}_c d_{p_2,q_2} y \right] \\
& +\frac{1}{2p_1[2]_{p_2,q_2}(b-a)} \left[q_2 \int_a^{p_1 b+(1-p_1)a} F(x,c)\, {}_a d_{p_1,q_1} x + p_2 \int_a^{p_1 b+(1-p_1)a} F(x,d)\, {}_a d_{p_1,q_1} x \right] \\
\leq \quad & \frac{q_1 q_2 F(a,c)+q_1 p_2 F(a,d)+p_1 q_2 F(b,c)+p_1 p_2 F(b,d)}{[2]_{p_1,q_1}[2]_{p_2,q_2}}.
\end{aligned}
$$

This can be found as a special case in [26].

Remark 11. *In Theorem 5, if we set $p_1 = p_2 = 1$, then Theorem 4 becomes ([40], Theorem 11).*

Corollary 1. *Let $F = [\underline{F}, \overline{F}] : [a,b] \times [c,d] \to I_c^+$ be a co-ordinated convex interval-valued function on $[a,b] \times [c,d]$. The following inclusions of Hermite–Hadamard type hold for $I(p,q)_{ac}$, $I(p,q)_a^d$, $I(p,q)_c^b$ and $I(p,q)^{bd}$-integrals:*

$$
\begin{aligned}
& \frac{1}{4}\left[F\left(\frac{q_1 a+p_1 b}{[2]_{p_1,q_1}}, \frac{p_2 c+q_2 d}{[2]_{p_2,q_2}}\right) + F\left(\frac{p_1 a+q_1 b}{[2]_{p_1,q_1}}, \frac{q_2 c+p_2 d}{[2]_{p_2,q_2}}\right) \right. \\
& \left. +F\left(\frac{p_1 a+q_1 b}{[2]_{p_1,q_1}}, \frac{p_2 c+q_2 d}{[2]_{p_2,q_2}}\right) + F\left(\frac{q_1 a+p_1 b}{[2]_{p_1,q_1}}, \frac{q_2 c+p_2 d}{[2]_{p_2,q_2}}\right) \right] \\
\supseteq \quad & \frac{1}{8p_1(b-a)} \left[\int_a^{p_1 b+(1-p_1)a} \left[F\left(x, \frac{q_2 c+p_2 d}{[2]_{p_2,q_2}}\right) + F\left(x, \frac{p_2 c+q_2 d}{[2]_{p_2,q_2}}\right) \right] {}_a d^I_{p_1,q_1} x \right.
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{p_1a+(1-p_1)b}^{b}\left[F\left(x,\frac{q_2c+p_2d}{[2]_{p_2,q_2}}\right)+F\left(x,\frac{p_2c+q_2d}{[2]_{p_2,q_2}}\right)\right]\,{}^{b}d^{I}_{p_1,q_1}x\Bigg]\\
&+\frac{1}{8p_2(d-c)}\left[\int_{c}^{p_2d+(1-p_2)c}\left[F\left(\frac{q_1a+p_1b}{[2]_{p_1,q_1}},y\right)+F\left(\frac{p_1a+q_1b}{[2]_{p_1,q_1}},y\right)\right]\,{}_{c}d^{I}_{p_2,q_2}y\right.\\
&\left.+\int_{p_2c+(1-p_2)b}^{d}\left[F\left(\frac{q_1a+p_1b}{[2]_{p_1,q_1}},y\right)+F\left(\frac{p_1a+q_1b}{[2]_{p_1,q_1}},y\right)\right]\,{}^{d}d^{I}_{p_2,q_2}y\right]\\
\supseteq\;&\frac{1}{4p_1p_2(b-a)(d-c)}\left[\int_{a}^{p_1b+(1-p_1)a}\int_{c}^{p_2d+(1-p_2)c}F(x,y)\,{}_{c}d^{I}_{p_2,q_2}y\,{}_{a}d^{I}_{p_1,q_1}x\right.\\
&+\int_{a}^{p_1b+(1-p_1)a}\int_{p_2c+(1-p_2)b}^{d}F(x,y)\,{}^{d}d^{I}_{p_1,q_1}y\,{}_{a}d^{I}_{p_1,q_1}x\\
&+\int_{p_1a+(1-p_1)b}^{b}\int_{c}^{p_2d+(1-p_2)c}F(x,y)\,{}_{c}d^{I}_{p_2,q_2}y\,{}^{b}d^{I}_{p_1,q_1}x\\
&\left.+\int_{p_1a+(1-p_1)b}^{b}\int_{p_2c+(1-p_2)b}^{d}F(x,y)\,{}^{d}d^{I}_{p_2,q_2}y\,{}^{b}d^{I}_{p_1,q_1}x\right]\\
\supseteq\;&\frac{q_2}{8p_1[2]_{p_1,q_1}(b-a)}\left[\int_{a}^{p_1b+(1-p_1)a}F(x,c)\,{}_{a}d^{I}_{p_1,q_1}x+\int_{a}^{p_1b+(1-p_1)a}F(x,d)\,{}_{a}d^{I}_{p_1,q_1}x\right.\\
&\left.+\int_{p_1a+(1-p_1)b}^{b}F(x,c)\,{}^{b}d^{I}_{p_1,q_1}x+\int_{p_1a+(1-p_1)b}^{b}F(x,d)\,{}^{b}d^{I}_{p_1,q_1}x\right]\\
&+\frac{p_2}{8p_1[2]_{p_1,q_1}(b-a)}\left[\int_{a}^{p_1b+(1-p_1)a}F(x,c)\,{}_{a}d^{I}_{p_1,q_1}x+\int_{a}^{p_1b+(1-p_1)a}F(x,d)\,{}_{a}d^{I}_{p_1,q_1}x\right.\\
&\left.+\int_{p_1a+(1-p_1)b}^{b}F(x,c)\,{}^{b}d^{I}_{p_1,q_1}x+\int_{p_1a+(1-p_1)b}^{b}F(x,d)\,{}^{b}d^{I}_{p_1,q_1}x\right]\\
&+\frac{q_1}{8p_2[2]_{p_2,q_2}(d-c)}\left[\int_{c}^{p_2d+(1-p_2)c}F(a,y)\,{}_{c}d^{I}_{p_2,q_2}y+\int_{c}^{p_2d+(1-p_2)c}F(b,y)\,{}_{c}d^{I}_{p_2,q_2}y\right.\\
&\left.+\int_{p_2c+(1-p_2)d}^{d}F(a,y)\,{}^{d}d^{I}_{p_2,q_2}x+\int_{p_2c+(1-p_2)d}^{d}F(b,y)\,{}^{d}d^{I}_{p_2,q_2}x\right]\\
&+\frac{p_1}{8p_2[2]_{p_2,q_2}(d-c)}\left[\int_{c}^{p_2d+(1-p_2)c}F(a,y)\,{}_{c}d^{I}_{p_2,q_2}y+\int_{c}^{p_2d+(1-p_2)c}F(b,y)\,{}_{c}d^{I}_{p_2,q_2}y\right.\\
&\left.+\int_{p_2c+(1-p_2)d}^{d}F(a,y)\,{}^{d}d^{I}_{p_2,q_2}x+\int_{p_2c+(1-p_2)d}^{d}F(b,y)\,{}^{d}d^{I}_{p_2,q_2}x\right]\\
&\supseteq\frac{p_1p_2+q_1q_2+p_1q_2+p_2q_1}{4[2]_{p_1,q_1}[2]_{p_2,q_2}}[F(a,c)+F(a,d)+F(b,c)+F(b,d)].
\end{aligned}
$$

Remark 12. *In Corollary 1, if we set $\underline{F}=\overline{F}$, then Corollary 1 becomes ([32], Corollary 1).*

Remark 13. *In Corollary 1, if we set $p_1=p_2=1$, then Corollary 1 reduces to ([40], Corollary 3).*

5. Examples

Example 3. *We define a convex interval-valued function $F=[\underline{F},\overline{F}]:\to I_c^+$ by $F(t,s)=[t^2s^2,ts]$. From Theorem 2, for $p_1=p_2=\frac{3}{4}$ and $q_1=q_2=\frac{3}{4}$, we have*

$$F\left(\frac{q_1a+p_1b}{p_1+q_1},\frac{p_2c+q_2d}{p_2+q_2}\right)=\left[\frac{36}{625},\frac{6}{25}\right],$$

$$\frac{1}{2}\left[\frac{1}{p_1(b-a)}\int_{a}^{p_1b+(1-p_1)a}F\left(x,\frac{p_2c+q_2d}{p_2+q_2}\right)\,{}_{a}d^{I}_{p_1,q_1}x\right.$$

$$+\frac{1}{p_2(d-c)}\int_{p_2c+(1-p_2)d}^{d} F\left(\frac{q_1a+p_1b}{p_1+q_1},y\right)\,{}^{d}d^{I}_{p_1,q_1}x\Bigg]$$
$$=\left[\frac{207}{2375},\frac{6}{25}\right],$$

$$\frac{1}{p_1p_2(b-a)(d-c)}\int_{a}^{p_1b+(1-p_1)a}\int_{p_2c+(1-p_2)d}^{d} F(x,y)\,{}^{d}d_{p_2,q_2}y\,{}_{a}d_{p_1,q_1}x$$
$$=\left[\frac{234}{1805},\frac{6}{25}\right],$$

$$\frac{1}{2p_2[2]_{p_1,q_1}(d-c)}\left[q_1\int_{p_2c+(1-p_2)d}^{d} F(a,y)\,{}^{d}d^{I}_{p_2,q_2}y+p_1\int_{p_2c+(1-p_2)d}^{d} F(b,y)\,{}^{d}d^{I}_{p_2,q_2}y\right]$$
$$+\frac{1}{2p_1[2]_{p_2,q_2}(b-a)}\left[p_2\int_{a}^{p_1b+(1-p_1)a} F(x,c)\,{}_{a}d^{I}_{p_1,q_1}x+q_2\int_{a}^{p_1b+(1-p_1)a} F(x,d)\,{}_{a}d^{I}_{p_1,q_1}x\right]$$
$$=\left[\frac{252}{1425},\frac{6}{25}\right]$$

and

$$\frac{q_1p_2F(a,c)+q_1q_2F(a,d)+p_1p_2F(b,c)+q_2p_1F(b,d)}{[2]_{p_1,q_1}[2]_{p_2,q_2}}=\left[\frac{6}{25},\frac{6}{25}\right].$$

It is obvious that

$$\left[\frac{36}{625},\frac{6}{25}\right]\supset\left[\frac{207}{2375},\frac{6}{25}\right]\supset\left[\frac{234}{1805},\frac{6}{25}\right]\supset\left[\frac{252}{1425},\frac{6}{25}\right]\supset\left[\frac{6}{25},\frac{6}{25}\right]$$

which shows that the results of Theorem 2 are correct.

Example 4. *We define a convex interval-valued function* $F=[\underline{F},\overline{F}]:\to I_c^+$ *by* $F(t,s)=[t^2s^2,ts]$. *From Theorem 3, for* $p_1=p_2=\frac{3}{4}$ *and* $q_1=q_2=\frac{3}{4}$, *we have*

$$F\left(\frac{p_1a+q_1b}{[2]_{p_1,q_1}},\frac{q_2c+p_2d}{[2]_{p_2,q_2}}\right)=\left[\frac{36}{625},\frac{6}{25}\right],$$

$$\frac{1}{2}\left[\frac{1}{p_1(b-a)}\int_{p_1a+(1-p_1)b}^{b} F\left(x,\frac{q_2c+p_2d}{[2]_{p_2,q_2}}\right)\,{}^{b}d^{I}_{p_1,q_1}x\right.$$
$$\left.+\frac{1}{p_2(d-c)}\int_{c}^{p_2d+(1-p_2)c} F\left(\frac{p_1a+q_1b}{[2]_{p_1,q_1}},y\right)\,{}_{c}d^{I}_{p_2,q_2}y\right]$$
$$=\left[\frac{207}{2375},\frac{6}{25}\right],$$

$$\frac{1}{p_1p_2(b-a)(d-c)}\int_{p_1a+(1-p_1)b}^{b}\int_{c}^{p_2d+(1-p_2)c} F(x,y)\,{}_{c}d^{I}_{p_2,q_2}y\,{}^{b}d^{I}_{p_1,q_1}x=\left[\frac{234}{1805},\frac{6}{25}\right],$$

$$\frac{1}{2p_2[2]_{p_1,q_1}(d-c)}\left[p_1\int_{c}^{p_2d+(1-p_2)c} F(a,y)\,{}_{c}d^{I}_{p_2,q_2}y+q_1\int_{c}^{p_2d+(1-p_2)c} F(b,y)\,{}_{c}d^{I}_{p_2,q_2}y\right]$$
$$+\frac{1}{2p_1[2]_{p_2,q_2}(b-a)}\left[q_2\int_{p_1a+(1-p_1)b}^{b} F(x,c)\,{}^{b}d^{I}_{p_1,q_1}x+p_2\int_{p_1a+(1-p_1)b}^{b} F(x,d)\,{}^{b}d^{I}_{p_1,q_1}x\right]$$
$$=\left[\frac{252}{1425},\frac{6}{25}\right]$$

and

$$\frac{p_1q_2F(a,c) + p_1p_2F(a,d) + q_1q_2F(b,c) + q_1p_2F(b,d)}{[2]_{p_1,q_1}[2]_{p_2,q_2}} = \left[\frac{6}{25}, \frac{6}{25}\right].$$

It is obvious that

$$\left[\frac{36}{625}, \frac{6}{25}\right] \supset \left[\frac{207}{2375}, \frac{6}{25}\right] \supset \left[\frac{234}{1805}, \frac{6}{25}\right] \supset \left[\frac{252}{1425}, \frac{6}{25}\right] \supset \left[\frac{6}{25}, \frac{6}{25}\right]$$

which shows that the results of Theorem 3 are correct.

Example 5. *We define a convex interval-valued function $F = [\underline{F}, \overline{F}] :\to I_c^+$ by $F(t,s) = [t^2s^2, ts]$. From Theorem 4, for $p_1 = p_2 = \frac{3}{4}$ and $q_1 = q_2 = \frac{3}{4}$, we have*

$$F\left(\frac{p_1a + q_1b}{[2]_{p_1,q_1}}, \frac{p_2c + q_2d}{[2]_{p_2,q_2}}\right) = \left[\frac{16}{625}, \frac{4}{25}\right],$$

$$\begin{aligned} &\frac{1}{2}\left[\frac{1}{p_1(b-a)} \int_{p_1a+(1-p_1)b}^{b} F\left(x, \frac{p_2c + q_2d}{[2]_{p_2,q_2}}\right) {}^{b}d^{I}_{p_1,q_1}x \right. \\ &\left. + \frac{1}{p_2(d-c)} \int_{p_2c+(1-p_2)d}^{d} F\left(\frac{p_1a + q_1b}{[2]_{p_1,q_1}}, y\right) {}^{d}d^{I}_{p_2,q_2}y \right] \\ = &\left[\frac{104}{2375}, \frac{4}{25}\right], \end{aligned}$$

$$\frac{1}{p_1p_2(b-a)(d-c)} \int_{p_1a+(1-p_1)b}^{b} \int_{p_2c+(1-p_2)d}^{d} F(x,y) \, {}^{d}d^{I}_{p_2,q_2}y \, {}^{b}d^{I}_{p_1,q_1}x = \left[\frac{676}{9025}, \frac{4}{25}\right],$$

$$\begin{aligned} &\frac{1}{2p_2[2]_{p_1,q_1}(d-c)}\left[p_1 \int_{p_2c+(1-p_2)d}^{d} F(a,y) \, {}^{d}d^{I}_{p_2,q_2}y + q_1 \int_{p_2c+(1-p_2)d}^{d} F(b,y) \, {}^{d}d^{I}_{p_2,q_2}y\right] \\ &+ \frac{1}{2p_1[2]_{p_2,q_2}(b-a)}\left[p_2 \int_{p_1a+(1-p_1)b}^{b} F(x,c) \, {}^{b}d^{I}_{p_1,q_1}x + q_2 \int_{p_1a+(1-p_1)b}^{b} F(x,d) \, {}^{b}d^{I}_{p_1,q_1}x\right] \\ = &\left[\frac{52}{475}, \frac{4}{25}\right] \end{aligned}$$

and

$$\frac{p_1p_2F(a,c) + p_1q_2F(a,d) + q_1p_2F(b,c) + q_1q_2F(b,d)}{[2]_{p_1,q_1}[2]_{p_2,q_2}} = \left[\frac{4}{25}, \frac{4}{25}\right].$$

It is obvious that

$$\left[\frac{16}{625}, \frac{4}{25}\right] \supset \left[\frac{104}{2375}, \frac{4}{25}\right] \supset \left[\frac{676}{9025}, \frac{4}{25}\right] \supset \left[\frac{52}{475}, \frac{4}{25}\right] \supset \left[\frac{4}{25}, \frac{4}{25}\right]$$

which shows that the results of Theorem 4 are correct.

Example 6. *We define a convex interval-valued function $F = [\underline{F}, \overline{F}] :\to I_c^+$ by $F(t,s) = [t^2s^2, ts]$. From Theorem 5, for $p_1 = p_2 = \frac{3}{4}$ and $q_1 = q_2 = \frac{3}{4}$, we have*

$$F\left(\frac{q_1a + p_1b}{[2]_{p_1,q_1}}, \frac{q_2c + p_2d}{[2]_{p_2,q_2}}\right) = \left[\frac{81}{625}, \frac{9}{25}\right],$$

$$\frac{1}{2}\left[\frac{1}{p_1(b-a)} \int_{a}^{p_1b+(1-p_1)a} F\left(x, \frac{q_2c + p_2d}{[2]_{p_2,q_2}}\right) {}_{a}d^{I}_{p_1,q_1}x\right.$$

$$+\frac{1}{p_2(d-c)}\int_c^{p_2d+(1-p_2)c} F\left(\frac{q_1a+p_1b}{[2]_{p_1,q_1}},y\right)\,{}_cd^I_{p_2,q_2}y\Bigg]$$
$$=\left[\frac{81}{475},\frac{9}{25}\right],$$

$$\frac{1}{p_1p_2(b-a)(d-c)}\int_a^{p_1b+(1-p_1)a}\int_c^{p_2d+(1-p_2)c} F(x,y)\,{}_cd^I_{p_2,q_2}y\,{}_ad^I_{p_1,q_1}x=\left[\frac{81}{361},\frac{9}{25}\right],$$

$$\frac{1}{2p_2[2]_{p_1,q_1}(d-c)}\left[q_1\int_c^{p_2d+(1-p_2)c} F(a,y)\,{}_cd^I_{p_2,q_2}y+p_1\int_c^{p_2d+(1-p_2)c} F(b,y)\,{}_cd^I_{p_2,q_2}y\right]$$
$$+\frac{1}{2p_1[2]_{p_2,q_2}(b-a)}\left[q_2\int_a^{p_1b+(1-p_1)a} F(x,c)\,{}_ad^I_{p_1,q_1}x+p_2\int_a^{p_1b+(1-p_1)a} F(x,d)\,{}_ad^I_{p_1,q_1}x\right]$$
$$=\left[\frac{27}{95},\frac{9}{25}\right]$$

and
$$\frac{q_1q_2F(a,c)+q_1p_2F(a,d)+p_1q_2F(b,c)+p_1p_2F(b,d)}{[2]_{p_1,q_1}[2]_{p_2,q_2}}=\left[\frac{9}{25},\frac{9}{25}\right].$$

It is obvious that
$$\left[\frac{81}{625},\frac{9}{25}\right]\supset\left[\frac{81}{475},\frac{9}{25}\right]\supset\left[\frac{81}{361},\frac{9}{25}\right]\supset\left[\frac{27}{95},\frac{9}{25}\right]\supset\left[\frac{9}{25},\frac{9}{25}\right]$$

which shows that the results of Theorem 5 are right.

6. Conclusions

In this work, for interval-valued functions of two variables, we defined (p,q)-integrals. We have used newly described integrals to prove the Hermite–Hadamard-type inclusions for co-ordinated convex interval-valued functions. Other researchers' previously reported findings are deduced as special cases of our results for $p=1$, $q\to 1^-$ and $\underline{F}=\overline{F}$. Finally, some examples are given to demonstrate the findings of this article. Results for the case of symmetric interval-valued functions can be obtained by applying the concept in Remark 2, which will be studied in future work. We will look at some further refinements of the Hermite–Hadamard inclusions as well as other well-known mathematical inclusions using (p,q)-integrals in the future.

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