Facial Homogeneous Colouring of Graphs

Tomáš Madaras * and Mária Šurimová

Institute of Mathematics, Faculty of Science, P.J. Šafárik University in Košice, Jesenná 5, 04001 Košice, Slovakia; maria.surimova@student.upjs.sk

* Correspondence: tomas.madaras@upjs.sk

Abstract: A proper colouring of a plane graph G is called facially homogeneous if it uses the same number of colours for every face of G . We study various sufficient conditions of facial homogeneous colourability of plane graphs, its relation to other facial colourings, and the extension of this concept for embedded graphs in general.

Keywords: facial colouring; homogeneous colouring; polychromatic colouring

1. Introduction

Throughout this paper, we consider simple connected plane graphs (that is, graphs drawn in the plane such that no two edges cross). The faces of a plane graph G (that is, connected regions of $\mathbb{R}^2 \setminus G$) will be here represented in graph-theoretic sense by closed walks: the boundary walk of a face α of G is a closed walk $v_1, e_1, \dots, e_{k-1}, v_k, e_k, v_1$ whose vertices and edges are incident with α and, for each $i \in \{1, \dots, k\}$, e_i is followed by e_{i+1} (indices modulo k) in the clockwise order of edges around v_i in G . The boundary walk associated with unbounded region of $\mathbb{R}^2 \setminus G$ is the outerface of G ; G is called outerplanar if its outerface contains all vertices. A k -colouring of a graph $G = (V, E)$ is a mapping $c : V \rightarrow \{1, 2, \dots, k\}$; if, for every edge $uv \in E$, $c(u) \neq c(v)$ holds, c is called proper.

Within years of graph theory research, constant attention was paid to vertex or edge colourings of plane graphs defined by constraints on the faces—among the earliest examples, one can mention cyclic colourings introduced by Ore and Plummer [1] (where the vertices of each face are coloured differently. The corresponding chromatic invariant is the cyclic chromatic number χ_c ; for the obtained results, see the survey [2]). Many variants of facial colourings are discussed in recent surveys [3,4]. Here, we turn our attention to a kind of facial colouring inspired by recent notion of homogeneous colouring from [5]:

A facial ℓ -homogeneous colouring of a plane graph G is a proper vertex colouring of G such that each face of G has ℓ colours on its boundary (thus, from the point of view of colour spaces of faces, such a colouring shows a kind of global symmetry feature); if G admits such a colouring, it is called facially ℓ -homogeneous. The set of those positive integers ℓ for which there exists a facial ℓ -homogeneous colouring of G is the palette of facial homogeneity of G , $\text{palf}(G)$. For fixed $\ell \in \text{palf}(G)$, the scale of facial ℓ -homogeneity of G (denoted by $\text{scalf}_\ell(G)$) is defined as the set of all positive integers k for which there exists a facial ℓ -homogeneous proper vertex k -colouring of G . For graph G with a facial ℓ -homogeneous colouring, we assign

$$\begin{aligned}\chi_{fh[\ell]}^-(G) &= \min \text{scalf}_\ell(G) \\ \chi_{fh[\ell]}^+(G) &= \max \text{scalf}_\ell(G)\end{aligned}$$

Observe that plane triangulations and plane bipartite graphs are trivially facial homogeneously colourable. In general, for a plane graph G with faces of the same size Δ^* , $\chi_{fh[\Delta^*]}^+(G) = |V(G)|$ and $\chi_{fh[\Delta^*]}^-(G) = \chi_c(G)$ (as facial Δ^* -homogeneous colouring of G is simply its cyclic colouring); in addition, $\text{scalf}_{\Delta^*}(G)$ forms an integer interval (as one may modify an existing facial homogeneous k -colouring, $k < |V(G)|$, by assigning the

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colour $k + 1$ a vertex previously coloured with a non-unique colour). Furthermore, a facial ℓ -homogeneous colouring of G that satisfies $\chi_{fh[\ell]}^-(G) = \ell$ is proper polychromatic colouring, investigated in [6] (relaxing the condition on colouring to be proper). In that paper, it was proved that the problem of polychromatic three-colourability (not necessarily proper) of plane graphs is NP-complete even for graphs with faces of sizes three and four only. On the other hand, Hoffmann and Kriegel proved in [7] that every two-connected plane bipartite graph admits a proper polychromatic three-colouring, and its existence was shown also for subcubic plane graphs different from K_4 and K_4 with one edge subdivided, see [8]. In [9], it was proved that every cubic bipartite plane graph has a polychromatic proper four-colouring. Note that a plane graph which has no proper ℓ -polychromatic colouring may still possess a facial ℓ -homogeneous colouring using more than ℓ colours; actually, all graphs mentioned in Section 2 are facially 3-homogeneous. In this section, we also address other questions connected with facially homogeneous colourings as well as computer assisted approaches to facial homogeneity testing. Section 3 contains several sufficient conditions for plane graphs to be facially 3-homogeneous. Finally, in Section 4, we discuss facial homogeneous colourability of graphs embedded into higher surfaces, providing some examples of embeddings which are not colourable in this manner.

2. Computational Results

In our study of facial homogeneous colourings, we tested several large collections of plane graphs to obtain partial answers to the following questions:

- Does there exist a plane graph G with $\text{palf}(G) = \emptyset$?
- For $\ell \in \text{palf}(G)$, what is the difference $\chi_{fh[\ell]}^-(G) - \chi(G)$?
- Does there exist a plane graph G for which $\text{palf}(G)$ is not an integer interval?
- Does there exist a plane graph G with some $\ell \in \text{palf}(G)$ such that $\text{scalf}_\ell(G)$ is not an integer interval?

For testing, several approaches may be considered. The one we used is based on the existing Wolfram Function Repository procedure `FindProperColorings[G,k]` (see [10]) which returns, for an input planar graph G and a positive integer k , the list of all proper k -colourings; taking a vertex colouring from the obtained list, it was then tested for the condition of facial homogeneity (a particular list of faces of a plane drawing of G can be obtained using the procedure `IGFaces[G]` from the `IGraphM` package, see [11]). Note that a planar graph of vertex connectivity one or two may have several different plane embeddings, of which `IGFaces` finds just one. Thus, for such graphs, nonequivalent plane embeddings with different nature of facial homogeneous colourability may exist. The smallest such example is the five-vertex graph obtained from a four-star by adding an extra edge: its plane embedding with two faces of size five is both facially 3- and 4-homogeneous, whereas the embedding with one triangular face and another one of size seven is not facially 4-homogeneous. On the other hand, polyhedral graphs—that is, planar and three-connected—have unique plane embedding.

Another approach which may come under consideration involves the integer linear programming formulation of the facial homogeneous colourability problem: for a plane graph $G = (V, E, F)$ with $V = \{v_1, \dots, v_n\}$, $F = \{f_1, \dots, f_s\}$ and two positive integers ℓ, k , does there exist a facial ℓ -homogeneous proper colouring of G which uses colours $1, \dots, k$? We introduce, for every $i \in \{1, \dots, k\}$, a binary variable y_i which equals one if the colour i was used in colouring of G (the objective function to be minimized is then $\sum_{i=1}^k y_i$), and, for $j \in \{1, \dots, n\}$, a binary variable $x_{j,i}$ indicating whether the vertex v_j was coloured with colour i . Furthermore, we consider several constraints:

- For each $j \in \{1, \dots, n\}$, $\sum_{t=1}^k x_{j,t} = 1$ (the vertex v_j is coloured with only one colour);
- For each $v_p v_q \in E$ and for each $i \in \{1, \dots, k\}$, $x_{p,i} + x_{q,i} \leq 1$ (the colour i is not used on both endvertices v_p, v_q —the properness condition);

- For each $j \in \{1, \dots, n\}$ and each $i \in \{1, \dots, k\}$, $x_{j,i} - y_i \leq 0$ (if v_j is coloured with colour i , this colour is marked as used);
- For each $\{u_1, \dots, u_\ell\} \subset \{1, \dots, k\}$, $t \in \{1, \dots, \ell\}$ and for each $f \in F$, $\deg(f) = r$, $f = [v_{i_1}, \dots, v_{i_r}]$, consider the constraint set $U_{f, u_1, \dots, u_\ell}$ consisting of;
 - $\sum_{j \in \{i_1, \dots, i_r\}} x_{j, u_t} \geq 1$ (each of ℓ colours $u_1, \dots, u_\ell$ is used on the face f);
 - For each $i \in \{1, \dots, k\} \setminus \{u_1, \dots, u_\ell\}$, $\sum_{j \in \{i_1, \dots, i_r\}} x_{j,i} = 0$ (except for colours $u_1, \dots, u_\ell$, no other colour is used on f);

Now, a face $f \in F$ is ℓ -homogeneously colourable if and only if there exist $\{\tilde{u}_1, \dots, \tilde{u}_\ell\} \subset \{1, \dots, k\}$ for which all constraints in $U_{f, \tilde{u}_1, \dots, \tilde{u}_\ell}$ are satisfied.

(Note that, for polychromatic colouring, there is only one constraint set $U_{f, 1, \dots, \ell}$ for each face f of G).

The detailed results on numbers of small graphs having particular values $\chi_{fh[l]}^-$ of are contained in Tables 1 and 2. Table 1 contains the results on facial 3-homogeneity of polyhedral graphs up to 10 vertices. Table 2 contains the results on facial 4-homogeneity of non-bipartite cubic polyhedral graphs of girth of at least 4 up to 20 vertices (since these graphs are not considered in [9]).

Table 1. Facial 3-homogeneity of polyhedral graphs.

No. of Vertices	3	4	5	6	7	8	9	10
no. of polyhedral graphs	0	1	2	7	34	257	2606	32,300
no. of polyhedral graphs G with $\chi(G) = 3, \chi_{fh[3]}^-(G) = 4$	0	0	1	0	12	45	530	6445
no. of polyhedral graphs G with $\chi(G) = 3, \chi_{fh[3]}^-(G) = 5$	0	0	0	0	0	0	0	1
no. of polyhedral graphs G with $\chi(G) = 4, \chi_{fh[3]}^-(G) = 5$	0	0	0	0	1	2	8	66
no. of polyhedral graphs G with $\text{palf}(G) = \emptyset$	0	0	0	0	0	0	0	2

Table 2. Facial 4-homogeneity of cubic polyhedral graphs.

No. of Vertices	4	6	8	10	12	14	16	18	20
no. of non-bipartite cubic polyhedral graphs of girth ≥ 4	0	0	0	1	1	4	10	32	122
no. of non-bipartite cubic polyhedral graphs of girth ≥ 4 with $\chi_{fh[4]}^-(G) = 4$	0	0	0	0	1	3	10	31	122
no. of non-bipartite cubic polyhedral graphs of girth ≥ 4 with $\chi_{fh[4]}^-(G) = 5$	0	0	0	1	0	1	0	1	0

These tests showed that, among all polyhedral graphs up to 10 vertices, only two (see Figure 1) have empty palettes of facial homogeneity:

Figure 1. The only polyhedral graphs up to 10 vertices with an empty homogeneity palette.

Theorem 1. The graph G_1 does not admit a facial homogeneous colouring.

Proof. Note first that G_1 is three-colourable, but not in polychromatic way (considering the cycle $abeig$, it has to contain unique colour—say, three—but not on e , a or g , as d would be uncolourable or $gdei$ or $abed$ would bear only two colours, respectively. If three is used on b , then it must appear also on j and f , but then $bcfe$ or $fhje$ would bear two colours; the similar argument holds also if three is used on i). Assume that there exists a facial 3-homogeneous colouring of G_1 using at least four colours, with three colours one, two and three on the outerface. Up to the symmetry, there are just two possibilities for the colour four: either it is used on d (then g, e must have the same colour as well as a, e), or on e (then d, i must have the same colour and f, j must also have the same colour, forcing b to have the same colour as i and j); both lead to a contradiction. $\square$

Theorem 2. The graph G_2 does not admit a facial homogeneous colouring.

Proof. It is easy to check that $\chi(G_2) = 3$ and, in addition, none of its proper three-colourings are facially 3-homogeneous (colouring the vertices g, h, e by one, two, three, respectively, two vertices on the face $ghji$ have the same colour. Without loss of generality, let i has colour two; then j has to be coloured by three and both d, f by one, yielding that also c has colour one—but then one of the faces $bcef, aced$ bears only two colours, a contradiction). Assume that there exists a facial 3-homogeneous colouring of G_2 using at least four colours, with three colours one, two and three on the outerface. Then, by symmetry, we have three possibilities for the appearance of the colourfour:

- c has colour four. Then d, f cannot be coloured by four, hence e has the same colour as a and b , a contradiction.
- e has colour four. Let a, b have colours one and two, respectively. Then c must have colour three and, subsequently, d and f have colour three. As i, j cannot have colour four, both g, h are coloured by three, a contradiction.
- h has colour four. Now, if c has colour at least four, then e must have the same colour as a and b , which is impossible. Thus c has colour three. Then both e, j must have the same colour as well as j, g , a contradiction.

$\square$

It is interesting that, for majority of non-bipartite graphs of the mentioned collections, $\chi_{fh[3]}^-(G) = \chi(G)$; notable exceptions include wheels W_{2k+1} , the graph on Figure 2 or odd prisms. The smallest graph G for which $\chi_{fh[3]}^-(G) \neq \chi(G)$ is the graph W_5 (of four-sided pyramid) without one spoke edge. Moreover, we found just few graphs with $\chi_{fh[3]}^-(G) = 5$, and only one of them has $\chi(G) = 3$, see Figure 3.

Figure 2. A plane graph G with $\chi(G) = 4, \chi_{fh[3]}^-(G) = 5$.

Figure 3. A plane graph G with $\chi(G) = 3, \chi_{fh[3]}^-(G) = 5$.

That this graph has no polychromatic three-colouring is easy to see: a and j would have the same colour, say one, yielding that d, g would be also coloured by one, but then at least two quadrangular faces bear two colours, a contradiction. Assume further that there exists its facial 3-homogeneous colouring using four colours, with three colours one, two and three on the outerface. Up to the symmetry, there are just two possibilities for the colour four: either it is used on d (then b, h must have the same colour as e , which is impossible), or on e . If b, h have colours one and two, respectively, then a, d, j must have the same colour, namely, three. The vertex f then has one of colours one or two—according to the symmetry, we may consider it has colour one—and c, i do not have colour three. Hence g must have colour three, implying that one of faces $acgf, fgij$ bears just two colours, a contradiction.

3. Sufficient Conditions for Facial Homogeneity

In this section, we present here several sufficient conditions which guarantee facial homogeneous colourability of plane graphs (concentrating mainly on facial three-homogeneity). Note first that the relation between facial homogeneous colourings of graph blocks and the graph itself is only one-way.

Lemma 1. *Let G be a plane graph consisting of blocks $B_1, \dots, B_r$. If each $B_i, i \in \{1, \dots, r\}$ is facially ℓ -homogeneous, then G is also facially ℓ -homogeneous.*

Proof. By induction on r , the number of blocks. Since the claim trivially holds for $r = 1$, we assume $r \geq 2$. Then, among blocks of G , there is a block—say, B_1 —which contains exactly one cut-vertex x . Let α be the outerface of B_1 and β be the face of G different from faces of B_1 which contains, as a boundary subwalk, the boundary of α . By assumption and the induction, both B_1 and the graph $G - B_1 + \{x\}$ are facially ℓ -homogeneous. Then, it is possible merge facial ℓ -homogeneous colourings of these two graphs to a joint facial ℓ -homogeneous colouring of G by choosing a permutation of colours in a facial ℓ -homogeneous colouring of B_1 such that β and α contain the same ℓ colours and the colour of x in B_1 is the same as in $G - B_1 + \{x\}$. $\square$

The converse of this lemma, however, is not true: see the facially 4-homogeneous graph on Figure 4 whose both blocks do not admit facially 4-homogeneous colouring.

Figure 4. Facially 4-homogeneous graph whose blocks are not facially 4-homogeneous.

Theorem 3. Let G be a connected plane graph with a girth of at least 6. Then G has the facial 3-homogeneous colouring and $\chi_{fh[3]}^-(G) \leq 4$.

Proof. By induction on the number of vertices of G . Note first that, from Euler's formula $n - f + m = 2$ (where n, m, f are the numbers of vertices, edges and faces of G , respectively), we obtain $\sum_{\alpha \in F(G)} (\deg(\alpha) - 6) + \sum_{v \in V(G)} (2 \deg(v) - 6) = -12$. According to the girth assumption, $\deg(\alpha) \geq 6$ for every face $\alpha \in F(G)$ and so the first sum is nonnegative but then the negativity of the right-hand side of the sum-equation implies the existence of vertices of degrees less than three, thus $\delta(G) \leq 2$. Furthermore, without loss of generality, we may assume that $\delta(G) = 2$ (a vertex x of degree one can be easily coloured by one of three colours on that face of $G - x$ which is incident with the pendant edge xy) and that G is not a cycle.

Let $P = x, u_1, u_2, \dots, u_k, y, k \geq 1$, be a trail of G such that $u_1, \dots, u_k$ are of degree 2 and x, y are of degree ≥ 3 , and let $G^- = G - \{u_1, \dots, u_k\}$. Consider first the case when $x = y$. Then, by girth assumption, $k \geq 5$; in addition, P is a cycle and x is a cut-vertex of G . The graph G^- is a plane graph of girth at least six which is smaller than G , thus, it has facial three-homogeneous colouring using at most four colours. Let α be a face of G^- which is incident to x and let $P_1 = x, v_1, \dots, v_l, x$ and $P_2 = x, w_1, \dots, w_n, x$ be facial subwalks of α such that their union is the boundary of α , P_1 lies in the interior of P and P_2 in the exterior of P . If P bounds a face in G , then P_1 or P_2 is empty; otherwise, the walks P_1P and P_2P are boundary walks of two distinct faces α_1, α_2 of G . Anyway, since $k \geq 5$, the vertices $u_1, \dots, u_k$ in G can be regularly coloured using all three colours used on α in G^- . Thus the cycle P uses three colours and, subsequently, each face of G which has P as a part of its boundary, uses three colours as well.

Hence, we can assume that P is a path. Now, if G^- is not connected, then it has two components G_x and G_y where $x \in V(G_x)$, $y \in V(G_y)$; let α_x, α_y be faces of G_x and G_y incident with x and y , respectively, which resulted from the face of G which contained P . Then G_x, G_y are plane graphs of girth of at least six which are smaller than G , hence both have facial 3-homogeneous colouring using at most four colours. In addition, it is possible to choose colours in G_y in such way that three colours on α_y are the same as three colours used on α_x . Then it is easy to colour vertices $u_1, \dots, u_k$ in G using three colours of α_x in a way that the colouring of G obtained from colourings of G_x, G_y is facially 3-homogeneous.

If G^- is connected, then the path P belongs to two faces of G and its deletion results in a face α with boundary walk $x, v_1, \dots, v_l, y, w_1, \dots, w_n, x = P_1P_2$ of G^- . G^- is a plane graph of girth of at least six; hence, it has facial 3-homogeneous four-colouring with, say, colours one, two and three on α .

If $k \geq 2$, then, regardless of colours of x, y in G^- , it is always possible to colour vertices $u_1, \dots, u_k$ with colours from $\{1, 2, 3\}$ such that P uses all three colours 1, 2, 3; this yields the desired facial three-homogeneous colouring of G .

Finally, let $k = 1$. With respect to symmetry, we discuss three possibilities how to extend facial three-homogeneous colouring of G^- to G :

- 1: In a facial three-homogeneous colouring of G^- , both subwalks $P_1 = x, v_1, \dots, v_l, y$, $P_2 = y, w_1, \dots, w_n, x$ use all three colours one, two, three. Then u_1 can be coloured with colour from $\{1, 2, 3\}$ which is different from colours of x, y .
- 2: The subwalk P_1 uses only colours one and two while P_2 uses all three colours one, two and three. Then u_1 can be coloured with colour three.
- 3: The subwalk P_1 uses only colours one, two, and P_2 uses only colours one, three; hence x, y are coloured with colour one. Then u_1 can be coloured with colour four.

□

Theorem 4. Let G be a plane graph with girth ≥ 11 . Then G has the facial three-homogeneous colouring and $\chi_{fh[3]}^-(G) = 3$.

Proof. The proof runs in the same way as the proof of the previous theorem, with the following modification: by [12], it is known that plane graphs with girth ≥ 11 contain two neighbouring vertices of degree two; thus, after the reduction in the path $x, v_1, \dots, v_k, y$ with inner vertices of degree two it always holds that $k \geq 2$, so three last cases of previous discussion can be omitted. □

Note that the results of [6] (Theorem 2) imply that each plane graph of girth of at least six has a polychromatic three-colouring; however, it need not be necessarily proper.

We leave, as an open problem, the question whether there exists a plane graph G of girth at least six for which $\chi_{fh[3]}^-(G) = 4$. We also believe that, for facial three-homogeneity, the bound on the girth in Theorem 1 can be decreased to five.

The following theorem extends the results of [6] (Proposition 7) on polychromatic three-colourability of outerplane graphs:

Theorem 5. Let G be a subdivision of an outerplane graph. Then G has the facial three-homogeneous colouring and $\chi_{fh[3]}^-(G) = 3$.

Proof. By induction on number of vertices of G . Since G is a subdivision of an outerplane graph (which contains a vertex of degree two), G also contains a vertex of degree two incident with the outerface. Without loss of generality, we may assume that G is two-connected (as it is always possible to permute colours of facial three-homogeneous three-colourings of blocks of G such that the colours of all cut-vertices of G match) and not a cycle.

Let $P = x, u_1, \dots, u_k, y$ be a subpath of the boundary walk of the outerface of G such that $u_1, \dots, u_k$ have degree 2, x and y have degree ≥ 3 . Let H be a graph obtained from G by removing vertices $u_1, \dots, u_k$ and, eventually, by adding the edge xy if x, y were not incident in G . Then H is a subdivision of an outerplane graph and is smaller than G ; thus, it has facial 3-homogeneous colouring using three colours, and, in addition, the vertices x, y receive different colours, say 1 and 2. Then u_1 may be coloured with colour 3 and $u_2, \dots, u_k$ may be coloured properly using three colours such that facial three-homogeneous colouring of G is obtained. □

Lemma 2. Let G be a plane graph and $F' \subset F(G)$ is a set of faces of size ≥ 4 in G . If each component of the induced subgraph $G[F']$ is facially three-homogeneous, then G is facially three-homogeneous.

Proof. Let $H_1, \dots, H_k$ be components of $G[F']$. Each of these components is facially three-homogeneous using, say, h_i colours. Then the facial three-homogeneous colouring of G may be obtained by following way: for each $i = 1, \dots, k$ we take facial three-homogeneous

colouring of H_i using colours $c_1^i, \dots, c_{h_i}^i$, and the remaining vertices of G (which lie on triangle faces) we colour with new different colours. $\square$

Theorem 6. Let G be a plane graph with all faces of odd size. Then G has the facial three-homogeneous colouring and $\chi_{fh[3]}^-(G) \leq 4$; the bound 4 is best possible.

Proof. Let H be the kleeptope of G (that is, H is obtained from G by inserting a new vertex x_α into each face α of G and joining x_α with new edges to all vertices incident with α). Then H is a plane multigraph with triangular faces, and is four-colourable. Furthermore, the condition of odd faces sizes for G gives that, for each vertex x_α of H , the subgraph of H induced by x_α and its neighbours requires four colours for its proper colouring, with one colour used solely on x_α . Thus any proper four-colouring of H induces a facial three-homogeneous colouring of G . The sharpness of bound four follows from proper colouring of the wheel graph W_6 . $\square$

4. Concluding Remarks

The concept of facial homogeneous vertex colourings may be also considered for graphs embedded in surfaces of higher genera. Here, to find a facial non-homogeneous colouring of an embedding is easier than in the plane case. For example, the embedding of the wheel graph W_6 on a torus on Figure 5 admits no facial homogeneous colouring: due to the presence of triangular face, it should be facial three-homogeneous colouring, but all six vertices are incident with a common face (the grey one) which hosts four colours as $\chi(W_6) = 4$.

Figure 5. Facial non-homogeneous embedding of a wheel.

There exist also graphs for which no orientable embedding into a particular orientable surface is facial homogeneously colourable. For example, it is known (see, for example, [13]) that there exist exactly four embeddings of K_6 on torus and, by Euler formula for torus, each of them has nine faces. The Euler formula for torus also yields that, for an embedding G of K_6 , $\sum_{v \in V(G)} (\deg(v) - 6) + \sum_{f \in F(G)} (2 \deg(f) - 6) = 0$, so $\sum_{f \in F(K_6)} (2 \deg(f) - 6) = 6$, giving $\sum_{f \in F(G)} (\deg(f) - 3) = 3$. Thus G necessarily contains a nontriangular face (where each its vertex has single incidence with that face) as well as a triangular one. Since all vertices of K_6 have to be coloured differently, G cannot admit a facial (three-)homogeneous colouring. In addition, the maximum orientable genus of K_6 is equal to five (the value five yields a single-face embedding). If $\tilde{G} = (V, E, F)$ is an orientable embedding of K_6 into a surface of genus $g \in [1, 4]$, then, by the Euler formula, $|V| - |E| + |F| = 2 - 2g$, from which we obtain that $|F| = 11 - 2g$ and $\sum_{\alpha \in F} \deg(\alpha) = 30 - 2g$. These equations imply that $\tilde{G}$ contains two faces of unequal size. However, it is not clear whether, in $\tilde{G}$, these two faces see different numbers of colours (note that some vertexes of a face might be incident several times with it, but its colour is counted just once). This question could be resolved by enumerating all orientable embeddings of K_6 ; anyway, so far, we could not find any information on it. However, we believe that, apart of single-face embedding of K_6 into orientable surface of genus five, none of its embedding is facially homogeneously colourable. We also believe that there exists a “super-example” of a graph such that its maximum orientable genus embedding yields two faces, and none of its orientable embeddings (regardless of genus) is facial homogeneously colourable.

In addition, one may consider—as a “true” generalization of polychromatic colourings—facially homogeneous colourings which are not necessarily proper. This relaxation may lead to higher number of colours used to facially homogeneously colour a graph than for the proper case: as an example, consider the graph G in Figure 6: there exists an improper facially three-homogeneous colouring of G using five colours, whereas every proper five-colouring of G which keeps three colours on five faces of G necessarily induces four colours on the outerface of G .

Figure 6. Proper vs. improper facial 3-homogeneity of G .

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