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Towards a Measurement Theory for Off-Shell Quantum Fields

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Abstract: In this study, we develop quantum measurement theory for quantum systems described by C*-algebras. This is the first step to establish measurement theory for interacting quantum fields with off-shell momenta. Unlike quantum mechanics (i.e., quantum systems with finite degrees of freedom), measurement theory for quantum fields is still in development because of the difficulty of quantum fields that are typical quantum systems with infinite degrees of freedom. Furthermore, the mathematical theory of quantum measurement is formulated in the von Neumann algebraic setting in previous studies. In the paper, we aim to extend the applicable area of quantum measurement theory to quantum systems described by C*-algebras from a mathematical viewpoint, referring to the sector theory that is related to symmetry and based on the theory of integral decomposition of states. In particular, we define central subspaces of the dual space of a C*-algebra and use them to define instruments. This attempt makes the connection between measurement theory and sector theory explicit and enables us to understand the macroscopic nature and the physical meaning of measurement.

Keywords: quantum measurement; C*-algebra; algebraic quantum field theory; local net; extension of local net; completely positive instrument; macroscopic distinguishability

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Citation: Okamura, K. Towards a Measurement Theory for Off-Shell Quantum Fields. *Symmetry* **2021**, *13*, 1183. https://doi.org/10.3390/sym13071183

Academic Editor: Motoichi Ohtsu

Received: 1 June 2021
Accepted: 28 June 2021
Published: 30 June 2021

1. Introduction

In this study, we develop a measurement theory for quantum systems described by C*-algebras. Interacting quantum fields assumed in this study are quantum systems with infinite degrees of freedom and with off-shell momenta, whose observables are given by self-adjoint elements of C*-algebras. The C*-algebraic approach to quantum fields is not unrelated to the usual approach by field operators. It is a powerful way to remove the difficulty of unbounded operators by making them bounded operators. For example, in a free real Bose field, the exponential $e^{i\phi(f)}$ (or resolvent) of the field operator $\phi(f)$, where f is a real function, is a bounded operator, and the collection of them generates a C*-algebra. This study is inspired by the measurement of the quantum field generated by the interaction between the electromagnetic field and electrons at the nanoscale, which is called the dressed photon (DP) phenomenon [1]. It is known to behave completely differently from electromagnetic waves propagating in free space or electromagnetic fields in a uniform medium, and has long been studied as near-field optics. The measurement theory for such systems is still unexplored, and we believe that a framework extending the current theory is necessary. For this reason, we adopt an approach based on both algebraic quantum field theory (AQFT) and quantum measurement theory and their mathematics. There are many examples of the contribution of mathematics to the progress of physical theories, and the introduction of new mathematics contributes greatly to the implementation of new physical concepts. In the study, we will actively use the mathematical framework for conceptual advancement.

In the algebraic formulation of quantum theory, the observable algebra of a quantum system is described by a $*$-algebra $\mathcal{X}$, and a state is described by an expectation functional ω on $\mathcal{X}$. From an algebraic point of view, Hilbert space is treated as a secondary one to be used in analysis as needed. For each state ω, a Hilbert space is given by the GNS representation $(\pi_\omega, \mathcal{H}_\omega, \Omega_\omega)$:

$$\omega(X) = \langle \Omega_\omega | \pi_\omega(X) \Omega_\omega \rangle \tag{1}$$

for all $X \in \mathcal{X}$. C*-algebras, a special case of $*$-algebra, are used in AQFT [2–4]. Various Hilbert spaces can be given by the GNS representation, and the fact that the representation has a physical meaning as well as the Hilbert space itself primarily promotes the conceptual understanding of the algebraic formulation. The contribution of Haag and Kastler [2] to this progress has been significant. Although there are studies on the algebraic formulation prior to their study, Ref. [2] is probably the first to successfully confront the fact that there are many different representations (depending on the choice of state). In [2], the "physical equivalence" of representations (also called weak equivalence) was used to give a clear meaning to the replacement between equivalent representations. In [5–8], a physical meaning was given to the situation in (A)QFT where different representations chosen by the DHR selection criterion coexist. It is a criterion that selects representations equivalent (through unitary transformations) to the vacuum representation (obtained from the GNS representation from the vacuum state) of the observable algebra on the domain which is spatial to some bounded domain. A representation satisfying this criterion describes a situation in which localized excitations of the quantum field exist. It was shown in [9] that a class (collection) of representations satisfying certain conditions corresponds to a situation where topological charges exist, and that, by using these representations, field algebra $\mathcal{F}$ and global gauge group G are reconstructed from observable algebra $\mathcal{A}$. This result is known as an iconic result in AQFT. Representations with different charges form their own sectors (with unitary equivalence), which are not only unitarily inequivalent but also mutually "disjoint", giving rise to the so-called "superselection rule". This result is closely related to the representation theory of field operators including the algebra of canonical commutation relations, where unitarily inequivalent representations arise (see [10–13] and references therein). Global gauge group G here is an unbroken symmetry, and the results of [9] are not valid for broken symmetries [14]. The extension of Ref. [9]'s results to broken symmetry situations was done in [14,15], and Ojima [16] defined the generalized sector as a "quasi-equivalence class of factor states", allowing for a unified treatment of macroscopic aspects in quantum systems in various contexts, including measurement.

To date, the instrument introduced by Davies and Lewis [17] has contributed greatly to the development of quantum measurement theory. They introduced instruments from a statistical viewpoint, and specified probability distributions and states after the measurement obtained by measuring a system using the measurement apparatus. However, because the relationship between the instrument and the usual quantum mechanical description was not clear at first, the analysis using the instrument did not progress until the investigation by Ozawa [18]. He introduced a completely positive instrument and a measuring process, the latter being used for quantum mechanical modeling of measurement. Every measuring process defines a completely positive instrument. The main result of [18] is the converse in a quantum system with finite degrees of freedom, i.e., every completely positive instrument in such a system is defined by a measuring process. This is a standard fact in quantum measurement theory now. Furthermore, the theory of completely positive instruments in quantum systems with infinite degrees of freedom described by the general von Neumann algebra has recently been developed in [19,20]. C*-algebras and von Neumann algebras can be viewed as non-commutative versions of topological and measurable spaces, respectively. The latter is a special case of the former, but their analysis methods are very different. In the current measurement theory, focusing on probability distributions and states after the measurement has led to the selection of components to be macroscopic by the measurement and the successful investigation of the relationship with quantum mechanical modeling.

In order to formulate the measurement theory for quantum systems described by C*-algebras, the more general case compared to von Neumann algebras, we believe that it is necessary to integrate a completely positive instrument and the sector theoretical treatment of the macroscopic aspect of the quantum system. The reason for this is that, because the concept of state is statistically characterized, we consider that the difference of values output by the measurement should be macroscopically distinguished by the disjointness of states of the composite system of the system and the measuring apparatus. In other words, a measurement is a physical process that leads to the situation wherein different output values of the measuring apparatus correspond to mutually disjoint states of the composite system. From this viewpoint, a measuring process, a quantum mechanical modeling of the measurement, is of course important historically and theoretically, but it should not necessarily be the first consideration in establishing the physical meaning and description of the measurement. On the other hand, this study is advantageous in that the identification of sectors by the measurement is justified by the measurement-theoretic description. We are convinced that the establishment of the measurement theory in quantum systems described by C*-algebras will open up new perspectives for the understanding of macroscopic aspects of quantum systems. Herein, we reexamine the result of [21]. While [21] focused on the use of measuring processes, we make thorough use of the instrument in this study.

In Section 2, the local net and open system are discussed and the description of dynamics as an open system in AQFT is stated. In Section 3, we review the sector theory and its mathematics. In Section 4, the central subspaces of the dual of a C*-algebra are defined. In the C*-algebraic setting, we define instruments in terms of central subspaces. Furthermore, we define and characterize central instruments in order to examine the differences between the C*-algebraic setting and the von Neumann algebraic setting. In Section 5, we summarize the results of the study and present the perspective.

2. Systems of Interest: Local Nets and Open System

2.1. C*-Algebraic Quantum Theory

All the statistical aspects of a physical system **S** are registered in a C*-probablity space $(\mathcal{X}, \omega)$, a pair of a C*-algebra $\mathcal{X}$, and a state ω on $\mathcal{X}$ [21]. Observables of **S** are described by self-adjoint elements of $\mathcal{X}$. On the other hand, the state ω is an expectation functional on $\mathcal{X}$ and statistically describes a physical situation (or an experimental setting) of **S**. We keep claiming that every quantum system is described in the language of noncommutative (quantum) probability theory (see [22] for an introduction to quantum probability theory). In Appendix A, the basic facts on operator algebras are summarized.

2.2. Local Net

Let M be a manifold or a (locally finite) graph. We suppose that M describes the space-time or the space under consideration. $\mathcal{R}$ denotes the set of bounded regions of M, which satisfies $\cup\mathcal{R} = M$. $M \in \mathcal{R}$ is assumed when M is bounded.

Definition 1 (local net)**.** *A family $\{\mathcal{A}(\mathcal{O})\}_{\mathcal{O}\in\mathcal{R}}$ of C*-algebras is called a local net on M if it satisfies the following conditions:*
(i) For every inclusion $\mathcal{O}_1 \subset \mathcal{O}_2$, we have $\mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2)$.
(ii) For any mutually causally separated (spatial) regions $\mathcal{O}_1$ and $\mathcal{O}_2$,

$$[\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2)] = \{AB - BA | A \in \mathcal{A}(\mathcal{O}_1), B \in \mathcal{A}(\mathcal{O}_2)\} = \{0\}. \tag{2}$$

For every local net $\{\mathcal{A}(\mathcal{O})\}_{\mathcal{O}\in\mathcal{R}}$ on M, there exists a C*-algebra

$$\mathcal{A} = \overline{\bigcup_{\mathcal{O}\in\mathcal{R}} \mathcal{A}(\mathcal{O})}^{\|\cdot\|}, \tag{3}$$

called the global algebra of $\{\mathcal{A}(\mathcal{O})\}_{\mathcal{O}\in\mathcal{R}}$. If M is bounded, then $\mathcal{A} = \mathcal{A}(M)$ since $M \in \mathcal{R}$ and $\mathcal{O} \subset M$ for all $\mathcal{O} \in \mathcal{R}$. When a group G acts on $\mathcal{R}$ as a symmetry, we assume the covariance condition for $\{\mathcal{A}(\mathcal{O})\}_{\mathcal{O}\in\mathcal{R}}$: there exists an automorphic action α of G on $\mathcal{A}$ such that

$$\alpha_g(\mathcal{A}(\mathcal{O})) = \mathcal{A}(g\mathcal{O}) \tag{4}$$

for all $g \in G$ and $\mathcal{O} \in \mathcal{R}$, where $g\mathcal{O} = \{gx | x \in \mathcal{O}\}$.

To describe the statistical aspect of quantum fields by a local net $\{\mathcal{A}(\mathcal{O})\}_{\mathcal{O}\in\mathcal{R}}$, states on the global algebra $\mathcal{A}$ or "local states" [23] are used.

2.3. Open System

We shall discuss how to describe the dynamics of open systems. In the context of quantum statistical mechanics, open systems are a subject that has been discussed for a long time. Open systems are also fundamental in quantum field theory, and are closely related to scattering theory. In particular, it is a necessary description of the dynamics in the paper concerning the DP as a typical example of off-shell quantum fields. This is because the DP phenomena are known to involve the process of generation by incident light and annihilation that changes to scattered light. On the other hand, it is essential that the quantum field considered here is a quantum system with an infinite degree of freedom system, and we should pay attention to the description of its dynamics (see Section 4 for details). In the following, we introduce the mathematical concepts necessary to describe the dynamics of open systems.

The discussion below is based on the understanding that closed systems are a special case of open systems. We consider a quantum system **S** described by a C*-algebra $\mathcal{X}$. Every time evolution of **S** as a closed system is described by an automorphism of $\mathcal{X}$. Furthermore, when the time t is parametrized by $\mathbb{R}$, the time evolution of **S** as a closed system is described by a strongly continuous automorphism group $\alpha : \mathbb{R} \ni t \mapsto \alpha_t \in Aut(\mathcal{X})$ satisfying $\alpha_0 = \mathrm{id}_{\mathcal{X}}$, $\alpha_s \circ \alpha_t = \alpha_{s+t}$ and $\alpha_{-t} = \alpha_t^{-1}$ for all $s, t \in \mathbb{R}$. In contrast to a closed system, the time evolution of an open system is described by a completely positive map $T : \mathcal{X} \to \mathcal{X}$. The complete positivity of maps between C*-algebras is defined as follows:

Definition 2 (Complete positivity [24–27])**.** *Let $\mathcal{C}$ and $\mathcal{D}$ be C*-algebras. A linear map $T : \mathcal{C} \to \mathcal{D}$ is said to be completely positive (CP) if*

$$\sum_{i,j=1}^{n} D_i^* T(C_i^* C_j) D_j \geq 0 \tag{5}$$

for all $n \in \mathbb{N}$, $C_1, \cdots, C_n \in \mathcal{C}$ and $D_1, \cdots, D_n \in \mathcal{D}$.

It is known that a CP map is positive, but the converse is not true. Every homomorphism of a C*-algebra $\mathcal{C}$ into a C*-algebra $\mathcal{D}$ is CP. In particular, all automorphisms of a C*-algebra $\mathcal{C}$ are CP. For every C*-algebra $\mathcal{C}$ and $n \in \mathbb{N}$, $M_n(\mathcal{C})$ denotes the C*-algebra of square matrices of order n whose entries are elements of $\mathcal{C}$. For every linear map $T : \mathcal{C} \to \mathcal{D}$ and $n \in \mathbb{N}$, a linear map $T^{(n)} : M_n(\mathcal{C}) \to M_n(\mathcal{D})$ is defined by $T^{(n)}(C) = (T(C_{ij}))$ for all $C = (C_{ij}) \in M_n(\mathcal{C})$. A linear map $T : \mathcal{C} \to \mathcal{D}$ is said to be n-positive if $T^{(n)} : M_n(\mathcal{C}) \to M_n(\mathcal{D})$ is positive. A linear map $T : \mathcal{C} \to \mathcal{D}$ is CP if and only if it is n-positive for all $n \in \mathbb{N}$. The dual map $T^* : \mathcal{D}^* \to \mathcal{C}^*$ of $T : \mathcal{C} \to \mathcal{D}$ is defined by

$$(T^*\varphi)(C) = \varphi(T(C)) \tag{6}$$

for all $\varphi \in \mathcal{D}^*$ and $C \in \mathcal{C}$. T is CP if and only if the linear map $\mathcal{D}^* \ni \varphi \mapsto \sum_{i,j=1}^{n} C_i T^*(D_i \varphi D_j^*) C_j^* \in \mathcal{C}^*$ is positive for all $n \in \mathbb{N}$, $C_1, \cdots, C_n \in \mathcal{C}$ and $D_1, \cdots, D_n \in \mathcal{D}$. Here, for every $A, B \in \mathcal{D}$ and $\varphi \in \mathcal{D}^*$, $A\varphi, \varphi B, A\varphi B \in \mathcal{D}^*$ are defined by

$$(A\varphi)(D) = \varphi(DA), \tag{7}$$

$$(\varphi B)(E) = \varphi(BE), \tag{8}$$

$$(A\varphi B)(F) = \varphi(BFA), \tag{9}$$

respectively, for all $D, E, F \in \mathcal{D}$.

The following structure theorem for normal CP maps defined on $\boldsymbol{B}(\mathcal{H})$ is well-known.

Theorem 1. *Let $\mathcal{H}$ be a separable Hilbert space. Let T be a normal CP map on $\boldsymbol{B}(\mathcal{H})$.*
(1) *There exist a separable Hilbert space $\mathcal{K}$, an element ξ of $\mathcal{K}$, a positive operator R on $\mathcal{K}$, and a unitary operator U on $\mathcal{H} \otimes \mathcal{K}$ such that*

$$T(X) = \mathrm{Tr}_{\mathcal{K}}[U^*(X \otimes R)U(1 \otimes |\xi\rangle\xi|)] \tag{10}$$

for all $X \in \boldsymbol{B}(\mathcal{H})$.
(2) *There exists a family $\{K_i\}_{i=1}^{\infty}$ of bounded operators on $\mathcal{H}$ such that*

$$T(X) = \sum_{i=1}^{\infty} K_i^* X K_i \tag{11}$$

for all $X \in \boldsymbol{B}(\mathcal{H})$.

The proof of this theorem is given in Appendix B. The dynamics of open systems in the Heisenberg picture are described by a quantum stochastic process in the sense of Accardi–Frigerio–Lewis [28,29]. Following their study, measurement theory in the Heisenberg picture is formulated in [20].

3. Sector Theory

The concept of sector is defined by Ojima [16] as follows:

Definition 3. *A sector of $\mathcal{X}$ is a quasi-equivalence class of a factor state.*

A state on $\mathcal{X}$ is called a factor if the center $\mathcal{Z}_\omega(\mathcal{X}) = \pi_\omega(\mathcal{X})'' \cap \pi_\omega(\mathcal{X})'$ of $\pi_\omega(\mathcal{X})''$ is trivial, i.e., $\mathcal{Z}_\omega(\mathcal{X}) = \mathbb{C}1$. Let π be a representation of $\mathcal{X}$ on a Hilbert space $\mathcal{H}$. We say that a linear functional ω on $\mathcal{X}$ is π-normal if there exists a trace-class operator σ on $\mathcal{H}$ such that

$$\omega(X) = \mathrm{Tr}[\pi(X)\sigma] \tag{12}$$

for all $X \in \mathcal{X}$.

Definition 4. *Let π_1 and π_2 be a representation of $\mathcal{X}$ on Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively.*
(1) *π_1 and π_2 are quasi-equivalent, written as $\pi_1 \approx \pi_2$, if every π_1-normal state is π_2-normal and vice versa.*
(2) *π_1 and π_2 are mutually disjoint, written as $\pi_1 \between \pi_2$, if no π_1-normal state is π_2-normal and vice versa.*
Two states ω_1 and ω_2 on $\mathcal{X}$ are quasi-equivalent (mutually disjoint, resp.), written as $\omega_1 \approx \omega_2$ ($\omega_1 \between \omega_2$, resp.), if π_{ω_1} and π_{ω_2} are quasi-equivalent (mutually disjoint, resp.).

The sector theory based on sector defined above has already been discussed in [16,21]. However, we believe that mathematics related to sector theory should be reexamined in order to develop measurement theory for quantum systems described by C*-algebras. The following theorem mathematically justifies the definition of sector, which is obvious from [30] (Corollary 5.3.6).

Theorem 2. *Two factor states ω_1 and ω_2 are either quasi-equivalent or disjoint.*

By the above theorem, two factor states ω_1 and ω_2 belong to different sectors if and only if $\omega_1 \between \omega_2$. A sector corresponds to a macroscopic situation where order parameters of the system have definite values. Although the unitary equivalence of states is efficient for pure states, physically important states are not always pure. For example, KMS states in some quantum system with infinite degrees of freedom are of type III. We would like to stress that the unitary equivalence class of a pure state is not appropriate for a unit of the state space. The reason will be discussed later.

Next, we shall define the notion of orthogonality of states. The order relation $\omega_1 \leq \omega_2$ for two positive linear functionals ω_1 and ω_2 on $\mathcal{X}$ is defined by

$$\omega_1(X) \leq \omega_2(X) \tag{13}$$

for all $X \in \mathcal{X}_+$.

Definition 5. *Let ω_1, ω_2 be positive linear functionals on $\mathcal{X}$. We say that ω_1 and ω_2 are mutually orthogonal, written as $\omega_1 \perp \omega_2$, if there exists no non-zero positive linear functional ω' such that $\omega' \leq \omega_1$ and $\omega' \leq \omega_2$.*

The following theorem shows the gap between the disjointness and the orthogonality of states.

Theorem 3 ([31] (Lemma 4.1.19. and Lemma 4.2.8.))**.** *Let ω_1, ω_2 be positive linear functionals on $\mathcal{X}$. Put $\omega = \omega_1 + \omega_2$.*
(1) *If ω_1 and ω_2 are mutually orthogonal, then there exists an orthogonal projection $P \in \pi_\omega(\mathcal{X})'$ such that*

$$\omega_1(X) = \langle \Omega_\omega | P \pi_\omega(X) \Omega_\omega \rangle, \quad \omega_2(X) = \langle \Omega_\omega | (1-P) \pi_\omega(X) \Omega_\omega \rangle \tag{14}$$

for all $X \in \mathcal{X}$.
(2) *If ω_1 and ω_2 are mutually disjoint, then there exists an orthogonal projection $C \in \mathcal{Z}_\omega(\mathcal{X})$ such that*

$$\omega_1(X) = \langle \Omega_\omega | C \pi_\omega(X) \Omega_\omega \rangle, \quad \omega_2(X) = \langle \Omega_\omega | (1-C) \pi_\omega(X) \Omega_\omega \rangle \tag{15}$$

for all $X \in \mathcal{X}$.

The topology of $\mathcal{S}(\mathcal{X})$ used here is the restriction of the weak*-topology of $\mathcal{X}^*$ to $\mathcal{S}(\mathcal{X})$. That is to say, it is generated by the basis $\mathcal{B} = \{O_\omega(\{X_i, \varepsilon_i\}_{i=1}^n) \mid \omega \in \mathcal{S}(\mathcal{X}), n \in \mathbb{N}, X_1, \cdots, X_n \in \mathcal{X}, \varepsilon_1, \cdots, \varepsilon_n > 0\}$, where $O_\omega(\{X_i, \varepsilon_i\}_{i=1}^n) = \{\omega' \in S(X) \mid \forall i = 1, \cdots, n, \ |\omega(X_i) - \omega'(X_i)| < \varepsilon_i\}$. Then, $\mathcal{S}(\mathcal{X})$ is a compact convex set, and we use the Borel field $\mathcal{B}(\mathcal{S}(\mathcal{X}))$ of $\mathcal{S}(\mathcal{X})$ generated by this topology. A positive linear functional ω on $\mathcal{X}$ is called a barycenter of a regular Borel measure μ on $\mathcal{S}(\mathcal{X})$ if

$$\omega = \int_{\mathcal{S}(\mathcal{X})} \rho \, d\mu(\rho). \tag{16}$$

μ is then called a barycentric measure of ω.

Definition 6. *A regular Borel measure μ on $\mathcal{S}(\mathcal{X})$ is orthogonal if*

$$\int_\Delta \rho \, d\mu(\rho) \ \perp \ \int_{\Delta^c} \rho \, d\mu(\rho) \tag{17}$$

for all $\Delta \in \mathcal{B}(\mathcal{S}(\mathcal{X}))$. $\mathcal{O}_\omega(\mathcal{S}(\mathcal{X}))$ denotes the set of orthogonal measures on $\mathcal{S}(\mathcal{X})$ with barycenter ω.

The following theorem characterizes orthogonal measures of a state.

Theorem 4 ([31] (Theorem 4.1.25.))**.** *Let $\mathcal{X}$ be a unital C^*-algebra and ω a state on $\mathcal{X}$. There is a one-to-one correspondence between the following three sets:*
(i) the orthogonal measures $\mu \in \mathcal{O}_\omega(\mathcal{S}(\mathcal{X}))$;
(ii) the abelian von Neumann subalgebras $\mathcal{B}$ of $\pi_\omega(\mathcal{X})'$;
(iii) the orthogonal projections P on $\mathcal{H}_\omega$ such that $P\Omega_\omega = \Omega_\omega$ and $P\pi_\omega(\mathcal{X})P \subseteq \{P\pi_\omega(\mathcal{X})P\}'$.
If μ, $\mathcal{B}$ and P are in correspondence, one has the following conditions:
(1) $\mathcal{B} = (\pi_\omega(\mathcal{X}) \cup \{P\})'$;

(2) *P is the orthogonal projection onto $\mathcal{B}\Omega_\omega$;*
(3) *$\mu(\widehat{X}_1 \cdots \widehat{X}_n) = \langle \Omega_\omega | \pi_\omega(X_1) P \pi_\omega(X_2) P \cdots P \pi_\omega(X_n) \Omega_\omega \rangle$;*
(4) *$\mathcal{B}$ is $*$-isomorphic to the range of the map $\kappa_\mu : L^\infty(\mathcal{S}(\mathcal{X}), \mu) \ni f \mapsto \kappa_\mu(f) \in \pi_\omega(\mathcal{X})'$ defined by*

$$\langle \Omega_\omega | \kappa_\mu(f) \pi_\omega(X) \Omega_\omega \rangle = \int_{\mathcal{S}(\mathcal{X})} f(\rho)\ \hat{X}(\rho)\ d\mu_\omega(\rho) \tag{18}$$

for all $X \in \mathcal{X}$ and $f \in L^\infty(\mathcal{S}(\mathcal{X}), \mu)$, where $\hat{X} \in C(\mathcal{S}(\mathcal{X}))$ is defined by $\hat{X}(\rho) = \rho(X)$ for all $\rho \in \mathcal{S}(\mathcal{X})$. κ_μ satisfies

$$\kappa_\mu(\widehat{X}) \pi_\omega(Y) \Omega_\omega = \pi_\omega(Y) P \pi_\omega(X) \Omega_\omega \tag{19}$$

for all $X, Y \in \mathcal{X}$.

By Theorems 3 and 4, we have the following theorem:

Theorem 5 ([31] (Proposition 4.2.9.))**.** *Let ω be a state on $\mathcal{X}$ and μ a barycentric measure of ω. The following conditions are equivalent.*
(1) *For every $\Delta \in \mathcal{B}(\mathcal{S}(\mathcal{X}))$,*

$$\int_\Delta \rho\ d\mu(\rho) \ \reflectbox{$\flat$}\ \int_{\Delta^c} \rho\ d\mu(\rho). \tag{20}$$

(2) *μ is orthogonal, and $\kappa_\mu(L^\infty(\mathcal{S}(\mathcal{X}), \mu))$ is a von Neumann subalgebra of the center $\mathcal{Z}_\omega(\mathcal{X})$ of $\pi_\omega(\mathcal{X})''$.*

For every $\omega \in \mathcal{S}(\mathcal{X})$, μ_ω denotes the orthogonal measure with barycenter ω corresponding to the center $\mathcal{Z}_\omega(\mathcal{X})$ of $\pi_\omega(\mathcal{X})''$. μ_ω is called the central measure of ω. The following theorem shows that the central measure gives the unique integral decomposition into mutually different sectors.

Theorem 6 ([31] (Theorem 4.2.11.))**.** *The central measure μ_ω of a state ω on $\mathcal{X}$ is pseudosupported by the set $\mathcal{S}_f(\mathcal{X})$ of factor states on $\mathcal{X}$, i.e., $\mu_\omega(\Delta) = 0$ for all $\Delta \in \mathcal{B}(\mathcal{S}(\mathcal{X}))$ such that $\Delta \cap \mathcal{S}_f(\mathcal{X}) = \emptyset$. If $\mathcal{X}$ is separable, then μ_ω is supported by $\mathcal{S}_f(\mathcal{X})$.*

That is to say, the concept of sector is applicable to any states via their central measures. $L^\infty(\mathcal{S}(\mathcal{X}), \mu_\omega)$ then describes the observable algebra that distinguishes sectors in ω and is $*$-isomorphic to $\mathcal{Z}_\omega(\mathcal{X})$. The $*$-isomorphism $\kappa_\omega := \kappa_{\mu_\omega} : L^\infty(\mathcal{S}(\mathcal{X}), \mu_\omega) \to \mathcal{Z}_\omega(\mathcal{X})$, defined by

$$\langle \Omega_\omega | \kappa_\omega(f) \pi_\omega(X) \Omega_\omega \rangle = \int_{\mathcal{S}(\mathcal{X})} f(\rho)\ \hat{X}(\rho)\ d\mu_\omega(\rho) \tag{21}$$

for all $X \in \mathcal{X}$ and $f \in L^\infty(\mathcal{S}(\mathcal{X}), \mu_\omega)$, justifies this statement. By the definition, all elements of the center $\mathcal{Z}_\omega(\mathcal{X})$ of $\pi_\omega(\mathcal{X})''$ are compatible with those of $\pi_\omega(\mathcal{X})''$. The following theorem is also shown.

Theorem 7 ([31] (Theorem 4.2.5.))**.** *Let ω be a state on $\mathcal{X}$ and μ an orthogonal measure with barycenter ω corresponding to a maximal abelian von Neumann subalgebra (MASA) of $\pi_\omega(\mathcal{X})'$. Then, μ is pseudosupported by the set $\mathcal{S}_e(\mathcal{X})$ of pure states on $\mathcal{X}$. If $\mathcal{X}$ is separable, then μ is supported by $\mathcal{S}_e(\mathcal{X})$.*

An orthogonal measure corresponding to a MASA of $\pi_\omega(\mathcal{X})'$ gives an irreducible decomposition of the state. In general, MASA of $\pi_\omega(\mathcal{X})'$ is not unique. The situation where MASA of $\pi_\omega(\mathcal{X})'$ is unique is special. This is the reason why the unitary equivalence class of a pure state is not appropriate for a unit of the state space. It is known that $\pi_\omega(\mathcal{X})''$ is a type I von Neumann algebra if $\pi_\omega(\mathcal{X})'$ is abelian. The following theorem characterizes such a situation in the context of orthogonal decompositions of states.

Theorem 8 ([31] (Theorem 4.2.3.)). *Let ω be a state on $\mathcal{X}$, and P the projection operator on $\mathcal{H}_\omega$ whose range is $\pi_\omega(\mathcal{X})'\Omega_\omega$. The following conditions are equivalent:*
(1) *$\pi_\omega(\mathcal{X})'$ is abelian;*
(2) *$P\pi_\omega(\mathcal{X})P$ generates an abelian algebra.*

4. Completely Positive Instrument

In this section, we analyze the concept of CP instrument in the C*-algebraic setting. In previous investigations [17–20], it has been examined in the von Neumann algebraic formulation of quantum theory. The generalization to C*-algebra is realized in terms of central subspaces of the dual of a C*-algebra. Our approach enables us to unify the measurement theory with sector theory.

4.1. Definition

Since the investigation [17] by Davies and Lewis, instruments have been defined on the predual of a von Neumann algebra. In order to define its C*-algebraic generalization, the dual space of a C*-algebra is too big in general. When a von Neumann algebra $\mathcal{M}$ on a Hilbert space $\mathcal{K}$ is not finite-dimensional, the predual $\mathcal{M}_*$ of $\mathcal{M}$ does not coincide with $\mathcal{M}^*$, i.e., $\mathcal{M}_* \subsetneq \mathcal{M}^*$. In addition, in the case where all physically relevant states are contained in $\mathcal{M}_*$, the whole space $\mathcal{M}^*$ is not needed. This does not depend on whether $\mathcal{M}$ is treated as a C*-algebra or a von Neumann algebra. In the C*-algebraic formulation introduced here, we can naturally use $\mathcal{M}_*$ as a domain of instruments.

Let $\mathcal{X}$ be a C*-algebra and π a representation of $\mathcal{X}$ on a Hilbert space $\mathcal{H}$. Let $\mathcal{M}$ be a von Neumann algebra on a Hilbert space $\mathcal{K}$. $\mathcal{Z}(\mathcal{M})$ denotes the center of $\mathcal{M}$. We define the subset $V(\pi)$ of $\mathcal{X}^*$ by

$$V(\pi) = \{\varphi \in \mathcal{X}^* \mid \exists \rho \in (\pi(\mathcal{X})'')_*, \forall X \in \mathcal{X}, \varphi(X) = \rho(\pi(X))\}. \tag{22}$$

A subspace $\mathcal{L}$ of $\mathcal{X}^*$ is said to be central if there exists a central projection C of $\mathcal{X}^{**}$, i.e., $C \in \mathcal{Z}(\mathcal{X}^{**})$, such that $\mathcal{L} = C\mathcal{X}^*$. Central subspaces of $\mathcal{X}^*$ are characterized as closed invariant subspaces (see [26] (Chapter III, Theorem 2.7.)). A central subspace $\mathcal{L}(= C\mathcal{X}^*)$ is said to be σ-finite if its dual $\mathcal{L}^*(\cong C\mathcal{X}^{**})$ is a σ-finite W^*-algebra. For every $M_1, M_2 \in \mathcal{V}^*$ and $\rho \in \mathcal{V}$, we define $M_1\rho, \rho M_2, M_1\rho M_2 \in \mathcal{V}$ by

$$\langle M, M_1\rho\rangle = \langle MM_1, \rho\rangle, \tag{23}$$
$$\langle M, \rho M_2\rangle = \langle M_2M, \rho\rangle, \tag{24}$$
$$\langle M, M_1\rho M_2\rangle = \langle M_2MM_1, \rho\rangle, \tag{25}$$

respectively, for all $M \in \mathcal{V}^*$. The usefulness of the central subspace can be seen in the following example:

Example 1 (See [26] (Chapter III) for example). (1) *Let $\mathcal{X}$ be a C*-algebra and π a representation of $\mathcal{X}$ on a Hilbert space $\mathcal{H}$. There exists a central projection $C(\pi)$ of $\mathcal{X}^{**}$ such that*

$$V(\pi) = C(\pi)\mathcal{X}^* = \{C(\pi)\varphi \mid \varphi \in \mathcal{X}^*\} = \{\varphi \in \mathcal{X}^* \mid C(\pi)\varphi = \varphi\}. \tag{26}$$

(2) *Let $\mathcal{M}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$. There exists a central projection C of $\mathcal{M}^{**}$ such that $\mathcal{M}_* = C\mathcal{M}^*$.*

The following theorem is known.

Theorem 9. *Let $\mathcal{X}$ be a C*-algebra and π_1 and π_2 representations of $\mathcal{X}$ on Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. The following conditions are equivalent:*
(1) $\pi_1 \approx \pi_2$. (2) $V(\pi_1) = V(\pi_2)$. (3) $C(\pi_1) = C(\pi_2)$.
Similarly, the following conditions are equivalent:
(4) $\pi_1 \between \pi_2$ (5) $V(\pi_1) \cap V(\pi_2) = \{0\}$. (6) $C(\pi_1)C(\pi_2) = 0$.

The former part of this theorem is shown in [26] (Chapter III, Proposition 2.12). We can show the latter part in a similar way.

We shall define instruments in terms of central subspaces in the fully C*-algebraic setting. Let $\mathcal{M}$ and $\mathcal{N}$ be W^*-algebras. $P(\mathcal{M}_*, \mathcal{N}_*)$ denotes the set of positive linear maps of $\mathcal{M}_*$ into $\mathcal{N}_*$. In addition, for any Banach space $\mathcal{L}$, $\langle \cdot, \cdot \rangle$ denotes the pairing of $\mathcal{L}^*$ and $\mathcal{L}$.

Definition 7 (instrument)**.** *Let $\mathcal{V}_{\rm in}$ and $\mathcal{V}_{\rm out}$ be σ-finite central subspaces of C*-algebras $\mathcal{X}$ and $\mathcal{Y}$, respectively, and $(S, \mathcal{F})$ a measurable space. $\mathcal{I}$ is called an instrument for $(\mathcal{X}, \mathcal{V}_{\rm in}, \mathcal{Y}, \mathcal{V}_{\rm out}, S)$ if it satisfies the following three conditions:*
(1) $\mathcal{I}$ is a map of $\mathcal{F}$ into $P(\mathcal{V}_{\rm in}, \mathcal{V}_{\rm out})$.
(2) $\langle 1, \mathcal{I}(S)\rho \rangle = \langle 1, \rho \rangle$ for all $\rho \in \mathcal{V}_{\rm in}$.
(3) For every $\rho \in \mathcal{V}_{\rm in}$, $M \in \mathcal{V}_{\rm out}^*$ and mutually disjoint sequence $\{\Delta_j\}_{j \in \mathbb{N}}$ of $\mathcal{F}$,

$$\langle M, \mathcal{I}(\cup_j \Delta_j)\rho \rangle = \sum_{j=1}^{\infty} \langle M, \mathcal{I}(\Delta_j)\rho \rangle. \tag{27}$$

When $\mathcal{X} = \mathcal{Y}$, an instrument $\mathcal{I}$ for $(\mathcal{X}, \mathcal{V}_{\rm in}, \mathcal{Y}, \mathcal{V}_{\rm out}, S)$ is called that, for $(\mathcal{X}, \mathcal{V}_{\rm in}, \mathcal{V}_{\rm out}, S)$. Furthermore, when $\mathcal{V}_{\rm in} = \mathcal{V}_{\rm out} = \mathcal{V}$, an instrument $\mathcal{I}$ for $(\mathcal{X}, \mathcal{V}_{\rm in}, \mathcal{V}_{\rm out}, S)$ is called for $(\mathcal{X}, \mathcal{V}, S)$. In particular, an instrument for $(\mathcal{M}, \mathcal{M}_*, S)$ is called for $(\mathcal{M}, S)$. For every instrument $\mathcal{I}$ for $(\mathcal{V}_{\rm in}, \mathcal{V}_{\rm out}, S)$ and normal state φ on $\mathcal{V}_{\rm in}^*$, we define the probability measure $\|\mathcal{I}\varphi\|$ on $(S, \mathcal{F})$ by $\|\mathcal{I}\varphi\|(\Delta) = \|\mathcal{I}(\Delta)\varphi\|$ for all $\Delta \in \mathcal{F}$. For every instrument $\mathcal{I}$ for $(\mathcal{X}, \mathcal{V}_{\rm in}, \mathcal{Y}, \mathcal{V}_{\rm out}, S)$, the dual map $\mathcal{I}^* : \mathcal{V}_{\rm out}^* \times \mathcal{F} \to \mathcal{V}_{\rm in}^*$ of $\mathcal{I}$ is defined by

$$\langle M, \mathcal{I}(\Delta)\rho \rangle = \langle \mathcal{I}^*(M, \Delta), \rho \rangle \tag{28}$$

for all $\rho \in \mathcal{V}_{\rm in}$, $M \in \mathcal{V}_{\rm out}^*$ and $\Delta \in \mathcal{F}$.

Definition 8. *An instrument $\mathcal{I}$ for $(\mathcal{X}, \mathcal{V}_{\rm in}, \mathcal{Y}, \mathcal{V}_{\rm out}, S)$ is said to be completely positive (CP) if the map $\mathcal{V}_{\rm out}^* \ni M \mapsto \mathcal{I}^*(M, \Delta) \in \mathcal{V}_{\rm in}^*$ is CP for all $\Delta \in \mathcal{F}$.*

For every map $\mathcal{J} : \mathcal{V}_{\rm out}^* \times \mathcal{F} \to \mathcal{V}_{\rm in}^*$ satisfying the following three conditions, there uniquely exists an instrument $\mathcal{I}$ for $(\mathcal{X}, \mathcal{V}_{\rm in}, \mathcal{Y}, \mathcal{V}_{\rm out}, S)$ such that $\mathcal{J} = \mathcal{I}^*$:
(1) For every $\Delta \in \mathcal{F}$, the map $\mathcal{V}_{\rm out}^* \ni M \mapsto \mathcal{J}(M, \Delta) \in \mathcal{V}_{\rm in}^*$ is normal, positive, and linear.
(2) $\mathcal{J}(1, S) = 1$.
(3) For every $\rho \in \mathcal{V}_{\rm in}$, $M \in \mathcal{V}_{\rm out}^*$ and mutually disjoint sequence $\{\Delta_j\}_{j \in \mathbb{N}}$ of $\mathcal{F}$,

$$\langle \mathcal{J}(M, \cup_j \Delta_j), \rho \rangle = \sum_{j=1}^{\infty} \langle \mathcal{J}(M, \Delta_j), \rho \rangle. \tag{29}$$

From now on, $\mathcal{I}$ denotes the dual map $\mathcal{I}^*$ of an instrument $\mathcal{I}$ for $(\mathcal{X}, \mathcal{V}_{\rm in}, \mathcal{Y}, \mathcal{V}_{\rm out}, S)$. The dual map of an instrument for $(\mathcal{X}, \mathcal{V}_{\rm in}, \mathcal{Y}, \mathcal{V}_{\rm out}, S)$ is also called an instrument for $(\mathcal{X}, \mathcal{V}_{\rm in}, \mathcal{Y}, \mathcal{V}_{\rm out}, S)$.

4.2. Central Decomposition of State via CP Instrument

Let $\mathcal{V}$ be a σ-finite central subspace of the dual space of a C*-algebra $\mathcal{X}$ and $(S, \mathcal{F})$ a measurable space. Let $C : \mathcal{F} \to \mathcal{Z}(\mathcal{V}^*)$ be a projection valued measure (PVM). A CP instrument $\mathcal{I}_C$ for $(\mathcal{X}, \mathcal{V}, S)$ is defined by

$$\mathcal{I}_C(\Delta)\rho = C(\Delta)\rho \tag{30}$$

for all $\rho \in \mathcal{V}$ and $\Delta \in \mathcal{F}$.

Theorem 10. *$\mathcal{I}_C$ satisfies the following conditions:*
(1) *$\mathcal{I}_C(S)\rho = \rho$ for all $\rho \in \mathcal{V}$.*
(2) *It is repeatable, i.e., it satisfies*

$$\mathcal{I}_C(\Delta)\mathcal{I}_C(\Gamma) = \mathcal{I}_C(\Delta \cap \Gamma) \tag{31}$$

for all $\Delta, \Gamma \in \mathcal{F}$.
(3) *For every $\rho \in \mathcal{V}_+ := \mathcal{V} \cap \mathcal{X}_+^*$ and $\Delta \in \mathcal{F}$, $\mathcal{I}_C(\Delta)\rho$ and $\mathcal{I}_C(\Delta^c)\rho$ are mutually disjoint.*
(4) *For every $\Delta \in \mathcal{F}$, $\mathcal{I}_C(\Delta)$ is $\mathcal{V}^*$-bimodule map, i.e., for every $\Delta \in \mathcal{F}$, $\rho \in \mathcal{V}$ and $M_1, M_2 \in \mathcal{V}^*$,*

$$\mathcal{I}_C(\Delta)(M_1\rho M_2) = M_1(\mathcal{I}_C(\Delta)\rho)M_2. \tag{32}$$

Conversely, if an instrument $\mathcal{I}$ for $(\mathcal{V}, S)$ satisfies the conditions (2) and (4), then there exists a spectral measure $C : \mathcal{F} \to \mathcal{Z}(\mathcal{V}^*)$ such that $\mathcal{I} = \mathcal{I}_C$.

Proof. We can easily check (1), (2), and (4). (3) is shown by using Theorem 9.

The converse is also obvious as follows. We define a map $C : \mathcal{F} \to \mathcal{V}^*$ by $C(\Delta) = \mathcal{I}(1, \Delta)$ for all $\Delta \in \mathcal{F}$. For every $\Delta \in \mathcal{F}$, $\rho \in \mathcal{V}$ and $M \in \mathcal{V}^*$, we have

$$\langle M, \mathcal{I}(\Delta)\rho\rangle = \langle 1, \mathcal{I}(\Delta)(\rho M)\rangle = \langle C(\Delta), \rho M\rangle = \langle MC(\Delta), \rho\rangle. \tag{33}$$

$\langle M, \mathcal{I}(\Delta)\rho\rangle = \langle C(\Delta)M, \rho\rangle$ is also shown in the same way. Therefore, we have $\langle [C(\Delta), M], \rho\rangle = 0$ for all $\Delta \in \mathcal{F}$, $\rho \in \mathcal{V}$ and $M \in \mathcal{V}^*$. When φ is normal faithful state on $\mathcal{V}^*$ and $\rho = \varphi([C(\Delta), M])^*$, $\langle ([C(\Delta), M])^*[C(\Delta), M], \varphi\rangle = 0$, so that $[C(\Delta), M] = 0$ for all $\Delta \in \mathcal{F}$ and $M \in \mathcal{V}^*$. We obtain $C(\Delta) \in \mathcal{Z}(\mathcal{V}^*)$ for all $\Delta \in \mathcal{F}$.

By the conditions (2) and (4),

$$\begin{aligned}\langle C(\Delta \cap \Gamma), \rho\rangle &= \langle 1, \mathcal{I}(\Delta \cap \Gamma)\rho\rangle = \langle 1, \mathcal{I}(\Delta)\mathcal{I}(\Gamma)\rho\rangle = \langle C(\Delta), \mathcal{I}(\Gamma)\rho\rangle \\ &= \langle 1, \mathcal{I}(\Gamma)(\rho C(\Delta))\rangle = \langle C(\Gamma), \rho C(\Delta)\rangle = \langle C(\Delta)C(\Gamma), \rho\rangle.\end{aligned} \tag{34}$$

Thus, $C : \mathcal{F} \to \mathcal{Z}(\mathcal{V}^*)$ is a PVM, and we have $\mathcal{I} = \mathcal{I}_C$. □

An instrument $\mathcal{I}$ for $(\mathcal{X}, \mathcal{V}_{\mathrm{in}}, \mathcal{Y}, \mathcal{V}_{\mathrm{out}}, S)$ is said to be subcentral if, for every $\rho \in \mathcal{V}_{\mathrm{in},+}$ and $\Delta \in \mathcal{F}$, $\mathcal{I}_C(\Delta)\rho$ and $\mathcal{I}_C(\Delta^c)\rho$ are mutually disjoint. The condition (3) in Theorem 10 is a special case of the subcentrality of instruments. $\mathcal{P}(\mathcal{X}, \mathcal{V})$ denotes the subset $\{\mathcal{I}_C | C : \mathcal{F} \to \mathcal{Z}(\mathcal{V}^*)$ is a PVM.$\}$ of the set of instruments defined on $\mathcal{V}$. An instrument $\mathcal{I}$ for $(\mathcal{X}, \mathcal{V}, S)$ is said to be central if it is an element of $\mathcal{P}(\mathcal{X}, \mathcal{V})$ and is the maximum in $\mathcal{P}(\mathcal{X}, \mathcal{V})$, where the maximum is due to the (pre)order $\prec$ on instruments defined as follows: For instruments $\mathcal{I}_1, \mathcal{I}_2$ for $(\mathcal{X}, \mathcal{V}_{\mathrm{in}}, \mathcal{Y}, \mathcal{V}_{\mathrm{out}}, S_1)$ and $(\mathcal{X}, \mathcal{V}_{\mathrm{in}}, \mathcal{Y}, \mathcal{V}_{\mathrm{out}}, S_2)$, respectively, $\mathcal{I}_1 \prec \mathcal{I}_2$ if $\mathcal{I}_1(\mathcal{F})\rho \subset \mathcal{I}_2(\mathcal{F})\rho$ for all $\rho \in \mathcal{S}(\mathcal{X}) \cap \mathcal{V}_{\mathrm{in}}$, where $\mathcal{I}_i(\mathcal{F}_i)\rho$, $i = 1, 2$, is the subset of $(\mathcal{V}_{\mathrm{in}})_+$ defined by $\mathcal{I}_i(\mathcal{F}_i)\rho = \{\mathcal{I}_i(\Delta_i)\rho \mid \Delta_i \in \mathcal{F}_i\}$. By Theorem 10, we have the following theorem.

Theorem 11. *Let $(S, \mathcal{F})$ be a measurable space, $\mathcal{V}$ a σ-finite central subspace of the dual of a C^*-algebra $\mathcal{X}$, and $C : \mathcal{F} \to \mathcal{Z}(\mathcal{V}^*)$ a PVM. $\mathcal{I}_C$ is central if and only if the abelian W^*-algebra generated by $\{C(\Delta) | \Delta \in \mathcal{F}\}$ is isomorphic to $\mathcal{Z}(\mathcal{V}^*)$.*

5. Operational Requirement and Macroscopic Distinguishability

In this section, we discuss the characterization of CP instruments. We deepen our conceptual understanding of measurement theory by referring to the mathematics of sector theory. In sector theory, we explained that a sector is a macroscopic unit. As an application of sector theory to measurement theory, we follow the macroscopic distinction made by the disjointness of states. That is, in contrast to the usual understanding of measurement, our understanding is that a measurement is a physical process that realizes macroscopically distinguishable situations when different values are output. In past investigations, the concept of CP instrument has been justified by clarifying the statistical properties that a measuring apparatus should satisfy from an operational point of view in the (extended)

Schrödinger picture. We first review this here. Next, we proceed to characterize CP instruments from the perspective of the macroscopic distinguishability of states, which is related to sector theory.

Here, we assume that the system **S** is described by a C^*-algebra $\mathcal{X}$ and that $\mathcal{V}_{\text{in}}$ a σ-finite central subspace of $\mathcal{X}^*$. We consider a measuring apparatus $\mathbf{A}(\boldsymbol{x})$ with output variable $\boldsymbol{x}$ to measure the system **S**, where $\boldsymbol{x}$ takes values in a measurable space $(S, \mathcal{F})$. In the following, we consider three assumptions from an operational point of view. They are modified from [19,32] in the C^*-algebraic setting.

Assumption 1. $\mathbf{A}(\boldsymbol{x})$ *statistically specifies the following two components:*
(1) *the probability measure* $\Pr\{\boldsymbol{x} \in \Delta \| \omega\}$, $\Delta \in \mathcal{F}$, *on* $(S, \mathcal{F})$ *for every initial state* $\omega \in \mathcal{S}(\mathcal{X}) \cap \mathcal{V}_{\text{in}}$.
(2) *the state* $\omega_{\{\boldsymbol{x} \in \Delta\}}$ *(on a* C^**-algebra* $\mathcal{Y}$*) after the measurement under the condition that* ω *is an initial state and output values not contained in* Δ *are ignored. For every* $\omega \in \mathcal{S}(\mathcal{X}) \cap \mathcal{V}_{\text{in}}$ *and* $\Delta \in \mathcal{F}$, $\omega_{\{\boldsymbol{x} \in \Delta\}}$ *is unique whenever* $\Pr\{\boldsymbol{x} \in \Delta \| \omega\} \neq 0$, *or is indefinite otherwise.*

From now on, we consider only the case of $\mathcal{X} = \mathcal{Y}$ for simplicity. The joint probability distribution of the successive measurement of $\mathbf{A}(\boldsymbol{x})$ and $\mathbf{A}(\boldsymbol{y})$ in this order in a state $\omega \in \mathcal{V}_{\text{in}} \cap \mathcal{S}(\mathcal{X})$ is given by

$$\Pr\{\boldsymbol{x} \in \Delta, \boldsymbol{y} \in \Gamma \| \omega\} = \Pr\{\boldsymbol{x} \in \Delta \| \omega\} \Pr\{\boldsymbol{y} \in \Gamma \| \omega_{\{\boldsymbol{x} \in \Delta\}}\} \tag{35}$$

for all $\Delta \in \mathcal{F}$ and $\Gamma \in \mathcal{F}'$.

Assumption 2. *For every* $\Delta \in \mathcal{F}$, *measuring apparatus* $\mathbf{A}(\boldsymbol{y})$ *whose output variable* $\boldsymbol{y}$ *takes values in a measurable space* $(S', \mathcal{F}')$, *and* $\Gamma \in \mathcal{F}'$, *the map* $\mathcal{S}(\mathcal{X}) \cap \mathcal{V}_{\text{in}} \ni \omega \mapsto \Pr\{\boldsymbol{x} \in \Delta, \boldsymbol{y} \in \Gamma \| \omega\}$ *is affine, that is,*

$$\Pr\{\boldsymbol{x} \in \Delta, \boldsymbol{y} \in \Gamma \| \alpha\omega_1 + (1-\alpha)\omega_2\} = \alpha \Pr\{\boldsymbol{x} \in \Delta, \boldsymbol{y} \in \Gamma \| \omega_1\} + (1-\alpha) \Pr\{\boldsymbol{x} \in \Delta, \boldsymbol{y} \in \Gamma \| \omega_2\} \tag{36}$$

for all $\alpha \in [0,1]$ *and* $\omega_1, \omega_2 \in \mathcal{S}(\mathcal{X}) \cap \mathcal{V}_{\text{in}}$.

The affine property of joint distributions of successive measurements characterizes the instrument as shown in the following theorem.

Theorem 12. *Let* $\mathbf{A}(\boldsymbol{x})$ *be a measuring apparatus satisfying Assumption 1. Suppose that there exists a* σ*-finite central subspace* $\mathcal{V}_{\text{out}}$ *of* $\mathcal{X}$ *such that* $\{\omega_{\{\boldsymbol{x} \in \Delta\}} | \omega \in \mathcal{S}(\mathcal{X}) \cap \mathcal{V}_{\text{in}}, \Delta \in \mathcal{F}\} \subset \mathcal{V}_{\text{out}}$. *The following conditions are equivalent:*
(1) $\mathbf{A}(\boldsymbol{x})$ *satisfies Assumption 2.*
(2) *There exists an instrument* $\mathcal{I}$ *for* $(\mathcal{V}_{\text{in}}, \mathcal{V}_{\text{out}}, S)$ *such that*

$$\Pr\{\boldsymbol{x} \in \Delta \| \omega\} = \|\mathcal{I}(\Delta)\omega\| \tag{37}$$

for all $\omega \in \mathcal{S}(\mathcal{X}) \cap \mathcal{V}_{\text{in}}$ *and* $\Delta \in \mathcal{F}$, *and that*

$$\omega_{\{\boldsymbol{x} \in \Delta\}} = \frac{\mathcal{I}(\Delta)\omega}{\|\mathcal{I}(\Delta)\omega\|} \tag{38}$$

whenever $\Pr\{\boldsymbol{x} \in \Delta \| \omega\} \neq 0$.

The complete positivity of instrument is based on the general description of the dynamics of open systems. In Section 2, we discussed the dynamics of open systems state/representation-independently. We consider the following assumption that is called the trivial extendability.

Assumption 3. *For any quantum system $\mathbf{S}'$ that is described by a C^*-algebra $\mathcal{Y}$ and does not interact with an apparatus $\mathbf{A}(\boldsymbol{x})$ nor $\mathbf{S}$, $\mathbf{A}(\boldsymbol{x})$ can be extended into an apparatus $\mathbf{A}(\boldsymbol{x}')$ measuring the composite system $\mathbf{S}+\mathbf{S}'$ with the following statistical properties:*

$$\Pr\{\boldsymbol{x}' \in \Delta \| \omega \otimes \varphi\} = \Pr\{\boldsymbol{x} \in \Delta \| \omega\}, \tag{39}$$

$$(\omega \otimes \varphi)_{\{\boldsymbol{x}' \in \Delta\}} = \omega_{\{\boldsymbol{x} \in \Delta\}} \otimes \varphi \tag{40}$$

for all $\omega \in \mathcal{V}_{\rm in} \cap \mathcal{S}(\mathcal{X})$, $\varphi \in \mathcal{W} \cap \mathcal{S}(\mathcal{Y})$ and $\Delta \in \mathcal{F}$, where $\mathcal{W}$ is a central subspace of $\mathcal{Y}^*$.

Let $\mathcal{M}$ and $\mathcal{N}$ be von Neumann algebras. For every $\sigma \in \mathcal{N}_*$, we define a map $\mathrm{id} \otimes \sigma : \mathcal{M} \,\overline{\otimes}\, \mathcal{N} \to \mathcal{M}$ by $\langle \rho \otimes \sigma, X\rangle = \langle \rho, (\mathrm{id} \otimes \sigma)(X)\rangle$ for all $\rho \in \mathcal{M}_*$ and $X \in \mathcal{M} \,\overline{\otimes}\, \mathcal{N}$.

A measuring apparatus that satisfies Assumption 3 is described by a CP instrument. In the von Neumann algebraic setting, a measuring process is defined as follows.

Definition 9 (Measuring process [19] (Definition 3.2))**.** *Let $\mathcal{M}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$, and $(S, \mathcal{F})$ a measurable space. A 4-tuple $\mathbb{M} = (\mathcal{K}, \sigma, E, U)$ is called a measuring process for $(\mathcal{M}, S)$ if it satisfies the following conditions:*
(1) *$\mathcal{K}$ is a Hilbert space,*
(2) *σ is a normal state on $\boldsymbol{B}(\mathcal{K})$,*
(3) *$E : \mathcal{F} \to \boldsymbol{B}(\mathcal{K})$ is a spectral measure,*
(4) *U is a unitary operator on $\mathcal{H} \otimes \mathcal{K}$,*
(5) *$\{\mathcal{I}_{\mathbb{M}}(M, \Delta) \mid M \in \mathcal{M}, \Delta \in \mathcal{F}\} \subset \mathcal{M}$, where $\mathcal{I}_{\mathbb{M}} : \boldsymbol{B}(\mathcal{H}) \times \mathcal{F} \to \boldsymbol{B}(\mathcal{H})$ is defined by*

$$\mathcal{I}_{\mathbb{M}}(X, \Delta) = (\mathrm{id} \otimes \sigma)[U^*(X \otimes E(\Delta))U] \tag{41}$$

for all $X \in \boldsymbol{B}(\mathcal{H})$ and $\Delta \in \mathcal{F}$.

As shown in [18], every CP instrument for $(\boldsymbol{B}(\mathcal{H}), S)$ is defined by a measuring process. By contrast, in the case where $\mathcal{M}$ is a non-atomic injective von Neumann algebra, it is shown in [19] that there exist CP instruments for $(\mathcal{M}, S)$ which cannot be defined by any measuring processes. Furthermore, a necessary and sufficient condition for a CP instrument to be defined by a measuring process is given in [19].

In the context of measurement, we do not always care about sectors as a macroscopic unit, but we actively utilize the macroscopic distinction based on the disjointness. We introduce two kinds of subcentral lifting property for instruments as follows.

Definition 10. *An instrument $\mathcal{I}$ for $(\mathcal{X}, \mathcal{V}, S)$ is said to have the first subcentral lifting property if there exists a central subspace $\mathcal{W}$ of the dual space of a C^*-algebra $\mathcal{Y}(\supset \mathcal{X})$ and an instrument $\widetilde{\mathcal{I}}$ for $(\mathcal{X}, \mathcal{V}, \mathcal{Y}, \mathcal{W}, S)$ satisfying the following two conditions:*
(1) *For every $\omega \in \mathcal{S}(\mathcal{X}) \cap \mathcal{V}$ and $\Delta \in \mathcal{F}$, $\widetilde{\mathcal{I}}(\Delta)\omega \between \widetilde{\mathcal{I}}(\Delta^c)\omega$.*
(2) *For every $\omega \in \mathcal{S}(\mathcal{X}) \cap \mathcal{V}$, $X \in \mathcal{X}$ and $\Delta \in \mathcal{F}$, $[\widetilde{\mathcal{I}}(\Delta)\omega](X) = [\mathcal{I}(\Delta)\omega](X)$.*

Definition 11. *An instrument $\mathcal{I}$ for $(\mathcal{X}, \mathcal{V}, S)$ is said to have the second subcentral lifting property if there exists a central subspace $\mathcal{W}$ of the dual space of a C^*-algebra $\mathcal{Y}(\supset \mathcal{X})$ and an instrument $\widetilde{\mathcal{I}}$ for $(\mathcal{Y}, \mathcal{W}, S)$ satisfying the following two conditions:*
(1) *For every $\varphi \in \mathcal{S}(\mathcal{Y}) \cap \mathcal{W}$ and $\Delta \in \mathcal{F}$, $\widetilde{\mathcal{I}}(\Delta)\varphi \between \widetilde{\mathcal{I}}(\Delta^c)\varphi$.*
(2) *For every $\omega \in \mathcal{S}(\mathcal{X}) \cap \mathcal{V}$, there exists $\widetilde{\omega} \in \mathcal{S}(\mathcal{Y}) \cap \mathcal{W}$ such that $\widetilde{\omega}(X) = \omega(X)$ and $[\widetilde{\mathcal{I}}(\Delta)\widetilde{\omega}](Y) = [\mathcal{I}(\Delta)\omega](Y)$ for all $X, Y \in \mathcal{X}$ and $\Delta \in \mathcal{F}$.*

Both subcentral lifting properties characterize the measurement obtained by restricting a measurement, which realizes the disjointness of states (after the measurement) of a larger system corresponding to different output values, to the target system. On the other hand, the difference between these two properties may be obvious from the definitions.

An instrument $\mathcal{I}$ for $(\mathcal{X}, \mathcal{V}_{\mathrm{in}}, \mathcal{Y}, \mathcal{V}_{\mathrm{out}}, S)$ is said to be finite if there exists a finite subset S_0 of S and a map $T : S_0 \to P(\mathcal{V}_{\mathrm{in}}, \mathcal{V}_{\mathrm{out}})$ such that

$$\mathcal{I}(\Delta) = \sum_{s \in S_0 \cap \Delta} T(s) \tag{42}$$

for all $\Delta \in \mathcal{F}$.

Theorem 13. *Every finite instrument for* $(\mathcal{X}, \mathcal{V}, S)$ *has the first subcentral lifting property and the second subcentral lifting property.*

Proof. Let $\mathcal{I}$ be a finite instrument for $(\mathcal{X}, \mathcal{V}, S)$, a finite subset S_0 of S, and a map $T : S_0 \to P(\mathcal{V})$ satisfying Equation (42) for all $\Delta \in \mathcal{F}$. For every $\Delta \in \mathcal{F}$, a linear map $\widetilde{\mathcal{I}}(\Delta) : \mathcal{V} \to \mathcal{V} \otimes l^1(S_0)$ is defined by

$$\widetilde{\mathcal{I}}(\Delta)\omega = \sum_{s \in S_0 \cap \Delta} T(s)\omega \otimes \delta_s \tag{43}$$

for all $\omega \in \mathcal{V}$. Then, $\widetilde{\mathcal{I}}$ is a finite instrument for $(\mathcal{X}, \mathcal{V}, \mathcal{X} \otimes_{\min} l^\infty(S_0), \mathcal{V} \otimes l^1(S_0), S)$. Then, $\widetilde{\mathcal{I}}$ satisfies $\widetilde{\mathcal{I}}(\Delta)\omega \mathrel{\mathring{\mid}} \widetilde{\mathcal{I}}(\Delta^c)\omega$ for all $\omega \in \mathcal{S}(\mathcal{X}) \cap \mathcal{V}$ and $\Delta \in \mathcal{F}$. Furthermore, every $\omega \in \mathcal{S}(\mathcal{X}) \cap \mathcal{V}$, $X \in \mathcal{X}$ and $\Delta \in \mathcal{F}$, $[\widetilde{\mathcal{I}}(\Delta)\omega](X \otimes 1) = [\mathcal{I}(\Delta)\omega](X)$. Therefore, $\mathcal{I}$ has the first subcentral lifting property.

Next, we define a finite instrument $\widehat{\mathcal{I}}$ for $(\mathcal{X} \otimes_{\min} l^\infty(S_0), \mathcal{V} \otimes l^1(S_0), S)$ by

$$\widehat{\mathcal{I}}(\Delta)\varphi = \widetilde{\mathcal{I}}(\Delta)(j(\varphi)) \tag{44}$$

for all $\Delta \in \mathcal{F}$ and $\varphi \in \mathcal{V} \otimes l^1(S_0)$, where $j : \mathcal{V} \otimes l^1(S_0) \to \mathcal{V}$ is a linear map defined by

$$[j(\varphi)](X) = \varphi(X \otimes 1) \tag{45}$$

for all $X \in \mathcal{X}$. For every $\varphi \in \mathcal{S}(\mathcal{X} \otimes_{\min} l^\infty(S_0)) \cap (\mathcal{V} \otimes l^1(S_0))$ and $\Delta \in \mathcal{F}$, $\widehat{\mathcal{I}}(\Delta)\varphi \mathrel{\mathring{\mid}} \widehat{\mathcal{I}}(\Delta^c)\varphi$. For every $\omega \in \mathcal{S}(\mathcal{X}) \cap \mathcal{V}$, $\widetilde{\omega} = \omega \otimes \delta_{s_0}$, where $s_0 \in S_0$ satisfies $\widetilde{\omega}(X \otimes 1) = \omega(X)$ and $[\widehat{\mathcal{I}}(\Delta)\widetilde{\omega}](Y \otimes 1) = [\mathcal{I}(\Delta)\omega](Y)$ for all $X, Y \in \mathcal{X}$ and $\Delta \in \mathcal{F}$. Therefore, $\mathcal{I}$ has the second subcentral lifting property. □

We conjecture that every *CP instrument* has both subcentral lifting properties.

6. Discussion and Perspectives

In the study, we have defined instruments by using central subspaces of the dual of a C*-algebra. We have checked its consistency with the definition in the von Neumann algebraic setting. This result means that the extension of the measurement theory to C*-algebra in the paper is valid. Furthermore, we have proposed a unification of the measurement theory and the sector theory: we have defined and characterized the centrality of instruments. In addition, we have discussed the operational characterization and macroscopic nature of quantum measurement. In the context, we have actively used the disjointness of states to distinguish different output values of the meter. Our results are, of course, applicable to systems described by C*-algebras generated from field operators, and the macroscopic aspects of quantum fields can now be discussed in terms of measurement theory.

In the setting of AQFT, we use a local net $\{\mathcal{A}(\mathcal{O})\}_{\mathcal{O} \in \mathcal{R}_1}$ on a space M_1 in order to describe the DP phenomena. In describing the measurement of DPs, only the use of the local net first adopted is not enough. In fact, to detect (the effect of) DPs, we need an operation wherein some probe is brought closer to the spatial scale at which DPs are generated. We introduced an extension of a local net to mathematically describe the operation at the level of observable algebras.

Definition 12. *Let $\{\mathcal{A}(\mathcal{O})\}_{\mathcal{O}\in\mathcal{R}_1}$ and $\{\mathcal{B}(\mathcal{O})\}_{\mathcal{O}\in\mathcal{R}_2}$ be local nets on M_1 and M_2, respectively. $\{\mathcal{B}(\mathcal{O})\}_{\mathcal{O}\in\mathcal{R}_2}$ is an extension of $\{\mathcal{A}(\mathcal{O})\}_{\mathcal{O}\in\mathcal{R}_1}$ if it satisfies the following three conditions:*
(i) $M_1 \subset M_2$.
(ii) $\mathcal{R}_1 \subset \mathcal{R}_2$.
(iii) For every $\mathcal{O} \in \mathcal{R}_1$, $\mathcal{A}(\mathcal{O}) \subset \mathcal{B}(\mathcal{O})$.

We use the extensions of a local net because the construction of the composite system of the system of interest and a measuring apparatus is not so simple. In particular, the construction of the composite system by the tensor product is not always applicable to quantum fields.

Let $\{\mathcal{B}(\mathcal{O})\}_{\mathcal{O}\in\mathcal{R}_2}$ be a local net on M_2 and an extension of a local net $\{\mathcal{A}(\mathcal{O})\}_{\mathcal{O}\in\mathcal{R}_1}$ on M_1. We suppose that M_1 is bounded. The composite system of the original system and a probe, which is close to the original system on the spatial scale where DPs are generated, is described by $\{\mathcal{B}(\mathcal{O})\}_{\mathcal{O}\in\mathcal{R}_2}$ as a quantum field. Furthermore, the material system, which is a part of the composite system, is assumed to be localized in the neighborhood of M_1. In the composite system, the generation and annihilation of DPs constantly occur near non-uniform materials in the unstable situation where light continues to incident constantly. By measuring the emitted light at regions far from M_1, we check (or estimate) the effect of DPs generated in M_1.

Constructing a concrete model of DPs as a quantum field in order to correlate experiments of DPs with the theory is a future task. We hope to describe the DP phenomena as open systems at the next stage. In the future, clarification of the relationship between this study and the recent trends in DP research [33] is required. Moreover, the mathematical theory of quantum measurement for quantum systems described by C*-algebras should be further developed.

Author Contributions: The author designed research, performed research, and wrote the paper. The author has read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Acknowledgments: The author thanks anonymous reviewers for their comments to improve the quality of this paper.

Conflicts of Interest: The author declares no conflict of interest.

Sample Availability: Not applicable.

Abbreviations

The following abbreviations are used in this manuscript:

AQFT	algebraic quantum field theory
CP	completely positive
DP	dressed photon
PVM	projection valued measure

Appendix A. Operator Algebra

We introduce the basic facts on operator algebras. See [26,30,31,34–37] for more details on operator algebras. A set $\mathcal{X}$ is called a C*-algebra if it satisfies the following conditions:
(1) $\mathcal{X}$ is a Banach space over $\mathbb{C}$.
(2) $\mathcal{X}$ is a $*$-algebra, i.e., it is an algebra with involution. The involution $^* : \mathcal{X} \to \mathcal{X}$ satisfies $(aX + bY)^* = \bar{a}X^* + \bar{b}Y^*$, $(XY)^* = Y^*X^*$, and $X^{**} := (X^*)^* = X$ for all $a, b \in \mathbb{C}$ and $X, Y \in \mathcal{X}$.
(3) The norm of $\mathcal{X}$ satisfies $\|X^*X\| = \|X\|^2$ for all $X \in \mathcal{X}$.
We assume that C*-algebras are unital.

Let $\mathcal{X}$ and $\mathcal{Y}$ be C*-algebras. A map $j : \mathcal{X} \to \mathcal{Y}$ is called a $*$-homomorphism if it satisfies the following conditions:

(i) $j(aX_1 + bX_2) = aj(X_1) + bj(X_2)$ for all $a, b \in \mathbb{C}$ and $X_1, X_2 \in \mathcal{X}$.
(ii) $j(X_1X_2) = j(X_1)j(X_2)$ for all $X_1, X_2 \in \mathcal{X}$.
(iii) $j(X^*) = j(X)^*$ for all $X \in \mathcal{X}$.
(iv) $j(1) = 1$.
A $*$-homomorphims β of $\mathcal{X}$ is called a $*$-automorphism of $\mathcal{X}$ if there exists a $*$-homomorphims γ of $\mathcal{X}$ such that $\beta \circ \gamma = \mathrm{id}_{\mathcal{X}}$ and $\gamma \circ \beta = \mathrm{id}_{\mathcal{X}}$. $Aut(\mathcal{X})$ denotes the set of automorphisms of $\mathcal{X}$. A $*$-homomorphism and a $*$-automorphism are simply called a homomorphism and an automorphism, respectively.

Let ω be a linear functional on $\mathcal{X}$.
(i) ω is positive if $\omega(X^*X) \geq 0$ for all $X \in \mathcal{X}$.
(ii) ω is normalized if $\omega(1) = 1$.
$\mathcal{X}^*$ denotes the set of (complex) linear functionals on $\mathcal{X}$. $\mathcal{X}^*_+$ denotes the set of positive linear functionals on $\mathcal{X}$. A linear functional on $\mathcal{X}$ is called a state on $\mathcal{X}$ if it is positive and normalized. $\mathcal{S}(\mathcal{X})$ denotes the set of states on $\mathcal{X}$. A state ω on $\mathcal{X}$ is faithful if $\omega(X^*X) = 0$ implies $X = 0$. A C*-algebra $\mathcal{W}$ is called a W^*-algebra if it is the dual of a Banach space $\mathcal{W}_*$, called the predual of $\mathcal{W}$. The second dual $\mathcal{X}^{**} = (\mathcal{X}^*)^*$ of a C*-algebra $\mathcal{X}$ is a W^*-algebra and is called the universal enveloping algebra of $\mathcal{X}$. A W^*-algebra $\mathcal{W}$ is said to be σ-finite if it admits at most countably many orthogonal projections. A positive linear functional φ on $\mathcal{W}$ is said to be normal if $\{\varphi(A_\gamma)\}_{\gamma\in\Gamma}$ converges to $\varphi(A)$ for all non-decreasing nets $\{A_\gamma\}_{\gamma\in\Gamma}$ of positive operators in $\mathcal{W}$ convergent to a positive operator $A \in \mathcal{W}$. A positive linear functional φ on $\mathcal{W}$ is normal if and only if $\varphi \in \mathcal{W}_*$. $\boldsymbol{B}(\mathcal{H})$ denotes the set of bounded linear operators on a Hilbert space $\mathcal{H}$. A W^*-algebra $\mathcal{M}$ is called a von Neumann algebra on a Hilbert space $\mathcal{H}$ if it is a subset of $\boldsymbol{B}(\mathcal{H})$, and the involution of $\mathcal{M}$ coincides with the adjoint operation on $\boldsymbol{B}(\mathcal{H})$. The predual $\mathcal{M}_*$ of a von Neumann algebra $\mathcal{M}$ on a Hilbert space $\mathcal{H}$ satisfies

$$\mathcal{M}_* = \{\varphi \in \mathcal{M}^* | \exists\rho \in \boldsymbol{T}(\mathcal{H}) \text{ s.t. } \varphi(M) = \mathrm{Tr}[M\rho] \text{ for all } M \in \mathcal{M}\}, \tag{A1}$$

where $\boldsymbol{T}(\mathcal{H})$ denotes the set of trace-class operators on $\mathcal{H}$.

For every state ω on $\mathcal{X}$, there exist a Hilbert space $\mathcal{H}_\omega$, a representation π_ω of $\mathcal{X}$ on $\mathcal{H}_\omega$ and a unit vector Ω_ω of $\mathcal{H}_\omega$ such that

$$\omega(X) = \langle \Omega_\omega | \pi_\omega(X)\Omega_\omega \rangle, \qquad X \in \mathcal{X}, \tag{A2}$$

and $\mathcal{H}_\omega = \overline{\pi_\omega(\mathcal{X})\Omega_\omega}$. Here, a map $\pi : \mathcal{X} \to \boldsymbol{B}(\mathcal{H})$ is called a representation of $\mathcal{X}$ on a Hilbert space $\mathcal{H}$ if it satisfies $\pi(aX + bY) = a\pi(X) + b\pi(Y)$, $\pi(XY) = \pi(X)\pi(Y)$, and $\pi(X^*) = \pi(X)^*$ for all $a, b \in \mathbb{C}$ and $X, Y \in \mathcal{X}$. The triple $(\pi_\omega, \mathcal{H}_\omega, \Omega_\omega)$ is called the GNS representation of ω and is unique up to unitary equivalence.

For any subset S of $\boldsymbol{B}(\mathcal{H})$, we define the commutant S' of S by $S' = \{A \in \boldsymbol{B}(\mathcal{H}) \mid \forall B \in S, AB = BA\}$ and the double commutant S'' of S by $S'' = (S')'$. $\pi_\omega(\mathcal{X})''$ and $\pi_\omega(\mathcal{X})'$ are then von Neumann algebras on $\mathcal{H}_\omega$.

Appendix B. The Proof of Theorem 1

First, we present theorems used to show Theorem 1.

Theorem A1 ([24–27,31])**.** *Let $\mathcal{X}$ be a C^*-algebra and $\mathcal{H}$ a Hilbert space. For every CP map $T : \mathcal{X} \to \boldsymbol{B}(\mathcal{H})$, there exist a Hilbert space $\mathcal{K}$, a representation π of $\mathcal{X}$ on $\mathcal{K}$, and $V \in \boldsymbol{B}(\mathcal{H}, \mathcal{K})$ such that*

$$T(X) = V^*\pi(X)V \tag{A3}$$

for all $X \in \mathcal{X}$, and that $\mathcal{K} = \overline{\mathrm{span}}(\pi(\mathcal{X})V\mathcal{H})$. If $\mathcal{X}$ and $\mathcal{H}$ are separable, then so is $\mathcal{K}$.

The triplet $(\pi, \mathcal{K}, V)$ is called a Stinespring representation of T, and is unique up to unitary equivalence.

Theorem A2 ([26] (Chapter IV, Theorem 5.5.)). *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be von Neumann algebras on Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. If π is a normal homomorphism of $\mathcal{M}_1$ onto $\mathcal{M}_2$, then there exist a Hilbert space $\mathcal{L}$, a projection E of $\mathcal{M}_1' \overline{\otimes} \boldsymbol{B}(\mathcal{L})$, and an isometry U of $E(\mathcal{H}_1 \otimes \mathcal{L})$ onto $\mathcal{H}_2$ such that*

$$\pi(M) = U j_E(M \otimes 1) U^* \tag{A4}$$

for all $M \in \mathcal{M}_1$, where $j_E : \boldsymbol{B}(\mathcal{H}_1 \otimes \mathcal{L}) \to E\boldsymbol{B}(\mathcal{H}_1 \otimes \mathcal{L})E$ is defined by $j_E(X) = EXE$ for all $X \in \boldsymbol{B}(\mathcal{H}_1 \otimes \mathcal{L})$. $\mathcal{M}_1 \overline{\otimes} \mathbb{C}1$ is then a multiplicative domain of j_E.

As a corollary of Theorem A2, the following holds:

Corollary A1. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces. If π is a normal homomorphism of $\boldsymbol{B}(\mathcal{H}_1)$ onto $\boldsymbol{B}(\mathcal{H}_2)$, then there exist a Hilbert space $\mathcal{K}$ and a unitary W of $\mathcal{H}_1 \otimes \mathcal{K}$ onto $\mathcal{H}_2$ such that*

$$\pi(X) = W(X \otimes 1)W^* \tag{A5}$$

for all $X \in \boldsymbol{B}(\mathcal{H}_1)$.

Let $\mathcal{X}$ and $\mathcal{Y}$ be C*-algebras. We define a partial order $T_1 \leq T_2$ on $\mathrm{CP}(\mathcal{X}, \mathcal{Y})$ by $T_2 - T_1 \in \mathrm{CP}(\mathcal{X}, \mathcal{Y})$.

Theorem A3 ([25] (Theorem 1.4.2.)). *Let T_1, T_2 be elements of $\mathrm{CP}(\mathcal{X}, \boldsymbol{B}(\mathcal{H}))$ such that $T_1 \leq T_2$, and $(\pi, \mathcal{K}, V)$ is the Stinespring representation of T_2. There exists a positive operator R of $\pi(\mathcal{X})'$ such that*

$$T_1(X) = V^* R \pi(X) V \tag{A6}$$

for all $X \in \mathcal{X}$.

By using the above theorems, we show Theorem 1.

The proof of Theorem 1. Put $P = T(1)$. Suppose $P \neq 0$ without loss of generality. We define a unital normal CP map T' on $\boldsymbol{B}(\mathcal{H})$ by

$$T'(X) = \frac{1}{\|P\|} T(X) + \left(1 - \frac{P}{\|P\|}\right)^{\frac{1}{2}} X \left(1 - \frac{P}{\|P\|}\right)^{\frac{1}{2}} \tag{A7}$$

for all $X \in \boldsymbol{B}(\mathcal{H})$. By Theorem A1, there exist a separable Hilbert space $\mathcal{K}'$, a normal representation π' of $\mathcal{X}$ on $\mathcal{K}'$, and an isometry $V' \in \boldsymbol{B}(\mathcal{H}, \mathcal{K})$ such that $\mathcal{K}' = \overline{\mathrm{span}}(\pi'(\mathcal{X})V'\mathcal{H})$ and that

$$T'(X) = (V')^* \pi'(X) V' \tag{A8}$$

for all $X \in \boldsymbol{B}(\mathcal{H})$. Since

$$\frac{1}{\|P\|} T(X^*X) \leq T'(X^*X) \tag{A9}$$

for all $X \in \boldsymbol{B}(\mathcal{H})$, by Theorem A3, there exists a positive operator R' of $\pi'(\mathcal{X})'$ such that

$$\frac{1}{\|P\|} T(X) = (V')^* \pi'(X) R' V' \tag{A10}$$

for all $X \in \boldsymbol{B}(\mathcal{H})$. By Corollary A1, there exist a separable Hilbert space $\mathcal{L}_1$ and a unitary operator $W \in \boldsymbol{B}(\mathcal{H} \otimes \mathcal{L}_1, \mathcal{K}')$ such that

$$\pi'(X) = W'(X \otimes 1) W'^* \tag{A11}$$

for all $X \in \boldsymbol{B}(\mathcal{H})$. There then exists a positive operator R'' on $\mathcal{L}_1$ such that $R'W' = W'(1 \otimes R'')$.

Let $\mathcal{L}_2$ be an infinite-dimensional separable Hilbert space, v a unit vector in $\mathcal{L}_2$, and y a unit vector in $\mathcal{L}_1$. We define an isometry $U : \mathcal{H} \otimes \mathbb{C}y \otimes \mathbb{C}v \to \mathcal{H} \otimes \mathcal{L}_1 \otimes \mathcal{L}_2$ by

$$U_0(x \otimes y \otimes v) = (W')^* V' x \otimes v \tag{A12}$$

for all $x \in \mathcal{H}$. Since $\mathcal{H} \otimes \mathbb{C}y \otimes \mathbb{C}v$ and $U_0(\mathcal{H} \otimes \mathbb{C}y \otimes \mathbb{C}v)$ satisfy $\dim((\mathcal{H} \otimes \mathbb{C}y \otimes \mathbb{C}v)^\perp) = \dim((U_0(\mathcal{H} \otimes \mathbb{C}y \otimes \mathbb{C}v))^\perp)$ as subspaces of $\mathcal{H} \otimes \mathcal{L}_1 \otimes \mathcal{L}_2$, there exists a unitary operator U on $\mathcal{H} \otimes \mathcal{L}_1 \otimes \mathcal{L}_2$ such that $U|_{\mathcal{H}\otimes\mathbb{C}y\otimes\mathbb{C}v} = U_0$. We put $\mathcal{K} = \mathcal{L}_1 \otimes \mathcal{L}_2$ and $\xi = y \otimes v$, and define a positive operator R on $\mathcal{K}$ by $R = \|P\| R'' \otimes 1$. For every $X \in \boldsymbol{B}(\mathcal{H})$ and $x_1, x_2 \in \mathcal{H}$, we obtain

$$\begin{aligned}
\langle x_1 | T(X) x_2 \rangle &= \|P\| \langle x_1 | (V')^* W' (X \otimes R'')(W')^* V' x_2 \rangle \\
&= \|P\| \langle (W')^* V' x_1 \otimes v | (X \otimes R'' \otimes 1)[(W')^* V' x_2 \otimes v] \rangle \\
&= \langle U(x_1 \otimes y \otimes v) | (X \otimes R)[U(x_2 \otimes y \otimes v)] \rangle \\
&= \mathrm{Tr}[U^*(X \otimes R) U(|x_2\rangle\langle x_1| \otimes |\xi\rangle\langle\xi|)] \\
&= \mathrm{Tr}[\mathrm{Tr}_{\mathcal{K}}[U^*(X \otimes R) U(1 \otimes |\xi\rangle\langle\xi|)] |x_2\rangle\langle x_1|] \\
&= \langle x_1 | \mathrm{Tr}_{\mathcal{K}}[U^*(X \otimes R) U(1 \otimes |\xi\rangle\langle\xi|)] x_2 \rangle,
\end{aligned} \tag{A13}$$

which completes the proof of (1).

Next, we show (2). By Theorem A1, there exist a separable Hilbert space $\mathcal{K}_1$, a normal representation π of $\mathcal{X}$ on $\mathcal{K}$ and $V \in \boldsymbol{B}(\mathcal{H}, \mathcal{K})$ such that $\mathcal{K}_1 = \overline{\mathrm{span}}(\pi(\mathcal{X}) V \mathcal{H})$ and that

$$T(X) = V^* \pi(X) V \tag{A14}$$

for all $X \in \boldsymbol{B}(\mathcal{H})$. By Corollary A1, there exist a separable Hilbert space $\mathcal{K}_2$ and a unitary operator $W \in \boldsymbol{B}(\mathcal{K}_1, \mathcal{H} \otimes \mathcal{K}_2)$ such that

$$\pi(X) = W(X \otimes 1) W^* \tag{A15}$$

for all $X \in \boldsymbol{B}(\mathcal{H})$. Let $\{y_i\}_{i=1}^{\dim(\mathcal{K}_2)}$ be a complete orthonormal system of $\mathcal{K}_2$. For every $1 \le i \le \dim(\mathcal{K}_2)$, we define $K_i \in \boldsymbol{B}(\mathcal{H})$ by

$$\langle x_1 | K_i x_2 \rangle = \langle x_1 \otimes y_i | W^* V x_2 \rangle \tag{A16}$$

for all $x_1, x_2 \in \mathcal{H}$. For every $1 \le i \le \dim(\mathcal{K}_2)$, $X \in \boldsymbol{B}(\mathcal{H})$ and $x_1, x_2 \in \mathcal{H}$, we have

$$\begin{aligned}
\langle x_1 | K_i^* X K_i x_2 \rangle = \langle K_i x_1 | X K_i x_2 \rangle &= \sum_{j=1}^{\dim(\mathcal{H})} \langle K_i x_1 | z_j \rangle \langle z_j | X K_i x_2 \rangle \\
&= \sum_{j=1}^{\dim(\mathcal{H})} \langle K_i x_1 | z_j \rangle \langle X^* z_j | K_i x_2 \rangle \\
&= \sum_{j=1}^{\dim(\mathcal{H})} \langle W^* V x_1 | z_j \otimes y_i \rangle \langle X^* z_j \otimes y_i | W^* V x_2 \rangle \\
&= \sum_{j=1}^{\dim(\mathcal{H})} \langle W^* V x_1 | (|z_j\rangle\langle z_j| \otimes |y_i\rangle\langle y_i|)(X \otimes 1) W^* V x_2 \rangle \\
&= \langle W^* V x_1 | (X \otimes |y_i\rangle\langle y_i|) W^* V x_2 \rangle = \langle x_1 | V^* W (X \otimes |y_i\rangle\langle y_i|) W^* V x_2 \rangle.
\end{aligned} \tag{A17}$$

Therefore, for every $X \in \boldsymbol{B}(\mathcal{H})$ and $x_1, x_2 \in \mathcal{H}$, we obtain

$$\begin{aligned}\langle x_1 | T(X) x_2 \rangle &= \langle x_1 | V^* W (X \otimes 1) W^* V x_2 \rangle \\ &= \sum_{i=1}^{\dim(\mathcal{K}_2)} \langle x_1 | V^* W (X \otimes |y_i\rangle\langle y_i|) W^* V x_2 \rangle \\ &= \sum_{i=1}^{\dim(\mathcal{K}_2)} \langle x_1 | K_i^* X K_i x_2 \rangle = \langle x_1 | \left(\sum_{i=1}^{\dim(\mathcal{K}_2)} K_i^* X K_i \right) x_2 \rangle, \end{aligned} \tag{A18}$$

which completes the proof of (2).

□

The proof of (1) in the above theorem refers to that of [18] (Theorem 5.1). The results of this appendix are related to the theory of Hilbert modules [38–43].

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