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On Coefficient Problems for Functions Connected with the Sine Function

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Abstract: In this paper, some coefficient problems for starlike analytic functions with respect to symmetric points are considered. Bounds of several coefficient functionals for functions belonging to this class are provided. The main aim of this paper is to find estimates for the following: coefficients, logarithmic coefficients, some cases of the generalized Zalcman coefficient functional, and some cases of the Hankel determinant.

Keywords: functions starlike with respect to symmetric points; coefficients of analytic functions; generalized Zalcman coefficient functional; Hankel determinant

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1. Introduction

Let $\mathcal{A}$ be the family of all functions analytic in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ having the power series expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

and let $\mathcal{S}$ denote the class of univalent functions in $\mathcal{A}$. Recall that a function analytic in a domain D is said to be univalent (one-to-one) there, if it does not take the same value twice (for the definitions and properties of $\mathcal{S}$ and other classes see, e.g., [1]). One of the problems of the geometric theory of analytic functions is connected with the coefficients of these functions. For decades, the main motivation for studying function coefficients was the Bieberbach conjecture that $|a_n| \leq n$ for $f \in \mathcal{S}$ (first proposed in 1916). The problem was finally proved by de Branges in 1985 (see [2,3] for the proof). There are many papers in which the nth coefficient a_n is estimated for various subclasses of analytic functions. In 1960, as an approach to prove the Bieberbach conjecture, Zalcman hypothesized that $|a_{2n} - a_{2n-1}| \leq (n-1)^2$ for $f \in \mathcal{S}$. This led to several papers related to the Zalcman functional for various subclasses of $\mathcal{S}$ (see, e.g., [4,5]), but the Zalcman conjecture remained open for many years for the class $\mathcal{S}$. In 2010, Krushkal [6,7] proved the conjecture for the class $\mathcal{S}$, but only for some initial values of n. More general versions of the Zalcman functional, i.e., functionals $\lambda a_{2n} - a_{2n-1}$ and $\lambda a_m a_n - a_{m+n-1}$, have also been considered (see, e.g., [8–13]).

The research on function coefficients also focused on estimating the so-called Hankel determinants. In the 1960s, Pommerenke defined the qth Hankel determinant for a function f of the form (1) as

$$H_{q,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \dots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \dots & a_{n+q} \\ \dots & \dots & \dots & \dots \\ a_{n+q-1} & a_{n+q} & \dots & a_{n+2q-2} \end{vmatrix},$$

where $q, n \in \{1, 2, \dots\}$ (see [14,15]). The bound of $H_{q,n}(f)$ was investigated for various subfamilies of $\mathcal{A}$. The sharp bounds of $|H_{2,2}(f)| = |a_2 a_4 - {a_3}^2|$, which is known

as the second Hankel determinant, were found for almost all important subclasses of the class $\mathcal{S}$ (see, e.g., [16–23]). It is worth noting that we still do not know the exact bound of this expression for $\mathcal{S}$. The estimation of the third Hankel determinant $|H_{3,1}(f)| = |a_3a_5 + 2a_2a_3a_4 - {a_3}^3 - {a_4}^2 - {a_2}^2a_5|$ is much more difficult to obtain as compared to $|H_{2,2}(f)|$. Some of the results obtained even for the most important subclasses of the class $\mathcal{S}$ are still not sharp (see, e.g., [24–33]).

In this paper, we find bounds of several coefficient functionals for functions belonging to the class of analytic functions related with the sine function. Let us start with the notation and definitions. By $\mathcal{S}^*$, we denote the class of starlike functions, i.e., functions $f \in \mathcal{A}$ such that $\mathfrak{Re}\{zf'(z)/f(z)\} > 0$ for all $z \in \mathbb{D}$. Let $\mathcal{B}_0$ be the class of Schwarz functions, i.e., analytic functions $w : \mathbb{D} \to \mathbb{D}$, $w(0) = 0$. The function $w \in \mathcal{B}_0$ has the Taylor series expansion

$$w(z) = \sum_{n=1}^{\infty} c_n z^n \,. \tag{2}$$

Moreover, recall that, for given analytic functions f and g in $\mathbb{D}$, we say that the function f is subordinate to g in $\mathbb{D}$ and write $f \prec g$ if there exists $w \in \mathcal{B}_0$ such that $f(z) = g(w(z))$, $z \in \mathbb{D}$. Moreover, if the function g is univalent in $\mathbb{D}$, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$. Using subordination, different subclasses of starlike functions were introduced by Ma and Minda (see [34]), in which of the quantity $zf'(z)/f(z)$ is subordinate to a more general superordinate function.

Let $\mathcal{S}^*_S$ denote the class of functions which are starlike with respect to symmetric points, which was introduced by Sakaguchi [35]. Recall that a function f is said to be starlike with respect to symmetric points, if for every r less than and sufficiently close to 1 and every ζ on the circle $|z| = r$, the angular velocity of $f(z)$ about the point $f(-\zeta)$ is positive at $z = \zeta$ as z traverses the circle $|z| = r$ in the positive direction, i.e.,

$$\mathfrak{Re}\left\{\frac{zf'(z)}{f(z) - f(-\zeta)}\right\} > 0 \quad \text{for} \quad z = \zeta,\ |\zeta| = r \,.$$

Thus, a function f in the class $\mathcal{S}^*_S$ is characterized by

$$\frac{2zf'(z)}{f(z) - f(-z)} \prec \varphi_0(z), \quad z \in \mathbb{D} \,, \tag{3}$$

where $\varphi_0(z) = (1+z)/(1-z)$. If the function φ_0 is replaced by any analytic univalent function φ with positive real part in $\mathbb{D}$ and symmetric with respect to real axis, then we obtain the class $\mathcal{S}^*_S(\varphi(z))$.

The classes defined and studied in [36–38] motivate us to consider the functions in the class $\mathcal{S}^*_S(\varphi(z))$ with $\varphi(z) = 1 + \sin z$. Hence, we can write

$$\mathcal{S}^*_S(\sin z) = \left\{ f \in \mathcal{S} : \ \frac{2zf'(z)}{f(z) - f(-z)} = 1 + \sin w(z),\ w \in \mathcal{B}_0,\ z \in \mathbb{D} \right\} .$$

We obtain the bounds for coefficients, logarithmic coefficients, some cases of the generalized Zalcman coefficient functional, and some cases of the Hankel determinant for functions from the class $\mathcal{S}^*_S(\sin z)$.

The article is structured as follows. In Section 2, we cite some results concerning functions from the class $\mathcal{B}_0$ that are needed for the proofs. In Section 3, we give estimates of coefficients and logarithmic coefficients of functions from $\mathcal{S}^*_S(\sin z)$. In Section 4, we estimate the generalized Zalcman functional and Hankel determinants for functions from $\mathcal{S}^*_S(\sin z)$. In Section 5, we present the conclusions.

2. Preliminary Results

In the proofs of our results, we need the following sharp estimates for functions from the class $\mathcal{B}_0$. The first one is the well-known bound of the Schwarz function coefficients

(see, e.g., [1]); the second one is due to Prokhorov and Szynal [39]; and the third one is the result obtained by Carlson [40].

Lemma 1 ([1]). *If $w \in \mathcal{B}_0$ is given by (2), then the sharp estimate $|c_n| \leq 1$ holds for $n \geq 1$.*

Lemma 2 ([39]). *Let $w \in \mathcal{B}_0$ be an analytic function of the form (2). Then, for any real numbers μ and ν, the following sharp estimate holds*

$$|c_3 + \mu c_1 c_2 + \nu c_1^3| \leq 1, \quad \text{if} \quad (\mu, \nu) \in D_1 \cup D_2,$$

where

$$D_1 = \{(\mu, \nu) \in \mathbb{R}^2 : |\mu| \leq \tfrac{1}{2},\ -1 \leq \nu \leq 1\},$$
$$D_2 = \{(\mu, \nu) \in \mathbb{R}^2 : \tfrac{1}{2} \leq |\mu| \leq 2,\ \tfrac{4}{27}(|\mu| + 1)^3 - (|\mu| + 1) \leq \nu \leq 1\}.$$

The extremal function has the form $w(z) = z^3$.

Lemma 3 ([40]). *Let $w \in \mathcal{B}_0$ be given by (2). Then*

$$|c_3| \leq 1 - |c_1|^2 - \frac{|c_2|^2}{1 + |c_1|}, \quad |c_4| \leq 1 - |c_1|^2 - |c_2|^2, \quad |c_5| \leq 1 - |c_1|^2 - |c_2|^2 - \frac{|c_3|^2}{1 + |c_1|}.$$

From the Schwarz–Pick lemma, it follows that

$$|c_2| \leq 1 - |c_1|^2. \tag{4}$$

From (4) and Lemma 3, we can obtain the following result.

Lemma 4. *Let $w \in \mathcal{B}_0$ be given by (2). Then*

$$|c_1 c_3 - {c_2}^2| \leq 1 - |c_1|^2.$$

The lemma given below was proven by Keogh and Merkes.

Lemma 5 ([41]). *Let $w \in \mathcal{B}_0$ be given by (2). Then, for all $\mu \in \mathbb{C}$, we have*

$$|c_2 + \mu {c_1}^2| \leq \max\{1, |\mu|\}.$$

Based on Theorem 2 in [42] by Efraimidis, the following two lemmas can be obtained (see also [43]).

Lemma 6. *Let $w \in \mathcal{B}_0$ be given by (2). Then, for $\mu \in \mathbb{C}$, $|\mu| \leq 1$, we have*

$$|c_4 + 2\mu c_1 c_3 + \mu {c_2}^2 + 3\mu^2 {c_1}^2 c_2 + \mu^3 {c_1}^4| \leq 1.$$

Lemma 7. *Let $w \in \mathcal{B}_0$ be given by (2). Then, for $\mu \in \mathbb{C}$, $|\mu| \leq 1$, we have:*

$$|c_5 + (1+\mu)c_1 c_4 + (1+\mu)c_2 c_3 + 3\mu c_1 {c_2}^2 + (1+\mu+\mu^2){c_1}^2 c_3 + 2\mu(1+\mu){c_1}^3 c_2 + \mu^2 {c_1}^5| \leq 1,$$
$$|c_5 + 2\mu c_1 c_4 + 2\mu c_2 c_3 + 3\mu^2 c_1 {c_2}^2 + 3\mu^2 {c_1}^2 c_3 + 4\mu^3 {c_1}^3 c_2 + \mu^4 {c_1}^5| \leq 1.$$

3. Bounds of Function Coefficients and Logarithmic Coefficients

The coefficients of $f \in \mathcal{S}_S^*(\sin z)$ can be expressed as the coefficients of a relative function w from the class $\mathcal{B}_0$. Let f and w be given by (1) and (2). Then, from the formula

$$\frac{2zf'(z)}{f(z) - f(-z)} = 1 + \sin w(z), \tag{5}$$

we obtain:

$$\begin{aligned} a_2 &= \tfrac{1}{2}c_1\,, \\ a_3 &= \tfrac{1}{2}c_2\,, \\ a_4 &= \tfrac{1}{4}\left(c_3 + \tfrac{1}{2}c_1c_2 - \tfrac{1}{6}{c_1}^3\right)\,, \\ a_5 &= \tfrac{1}{4}\left(c_4 - \tfrac{1}{2}{c_1}^2c_2 + \tfrac{1}{2}{c_2}^2\right)\,, \\ a_6 &= \tfrac{1}{6}\left(c_5 + \tfrac{1}{4}c_1c_4 + \tfrac{1}{2}c_2c_3 - \tfrac{1}{2}{c_1}^2c_3 - \tfrac{3}{8}c_1{c_2}^2 - \tfrac{5}{24}{c_1}^3c_2 + \tfrac{1}{120}{c_1}^5\right)\,. \end{aligned} \tag{6}$$

Theorem 1. *If* $f \in \mathcal{S}_S^*(\sin z)$ *is given* (1)*, then*

$$|a_2| \le \tfrac{1}{2}\,, \quad |a_3| \le \tfrac{1}{2}\,, \quad |a_4| \le \tfrac{1}{4}\,, \quad |a_5| \le \tfrac{1}{4}, \quad |a_6| \le \tfrac{1}{6}\,.$$

The bounds are sharp.

Proof. The bounds of $|a_2|$ and $|a_3|$ follow from Lemma 1. The inequality for $|a_4|$ can be easily obtained from Lemma 2, with $\mu = \frac{1}{2}$ and $\nu = -\frac{1}{6}$.

From (6) for a_5, we have

$$4|a_5| \le |c_4| + \tfrac{1}{2}|c_1|^2|c_2| + \tfrac{1}{2}|c_2|^2\,. \tag{7}$$

Now, using (4) and Lemma 3 in (7), we get

$$\begin{aligned} 4|a_5| &\le 1 - |c_1|^2 - |c_2|^2 + \tfrac{1}{2}|c_1|^2(1 - |c_1|^2) + \tfrac{1}{2}|c_2|^2 \\ &= 1 - \tfrac{1}{2}|c_1|^2 - \tfrac{1}{2}|c_2|^2 - \tfrac{1}{2}|c_1|^4 \le 1\,. \end{aligned}$$

Thus, we have the fourth inequality in Theorem 1.

Formula (6) for a_6 can be written

$$6|a_6| = \tfrac{1}{4}|c_5 + c_1c_4 + c_2c_3 + {c_1}^2c_3 + 3c_5 + c_2c_3 - 3{c_1}^2c_3 - \tfrac{3}{2}c_1{c_2}^2 - \tfrac{5}{6}{c_1}^3c_2 + \tfrac{1}{30}{c_1}^5|\,.$$

Applying the triangle inequality, we get

$$24|a_6| \le |c_5 + c_1c_4 + c_2c_3 + {c_1}^2c_3| + 3|c_5 + \tfrac{1}{3}c_2c_3 - {c_1}^2c_3 - \tfrac{1}{2}c_1{c_2}^2 - \tfrac{5}{18}{c_1}^3c_2 + \tfrac{1}{90}{c_1}^5|\,.$$

From Lemma 7 for $\mu = 0$, we know that $|c_5 + c_1c_4 + c_2c_3 + {c_1}^2c_3| \le 1$, thus we have

$$24|a_6| \le 1 + 3W\,,$$

where

$$W = |c_5 + \tfrac{1}{3}c_2c_3 - {c_1}^2c_3 - \tfrac{1}{2}c_1{c_2}^2 - \tfrac{5}{18}{c_1}^3c_2 + \tfrac{1}{90}{c_1}^5|\,. \tag{8}$$

Now, we show that $W \le 1$. Applying the triangle inequality and Lemma 3 in (8), we obtain

$$W \le 1 - |c_1|^2 - |c_2|^2 - \frac{|c_3|^2}{1 + |c_1|} + \tfrac{1}{3}|c_2||c_3| + |c_1|^2|c_3| + \tfrac{1}{2}|c_1||c_2|^2 + \tfrac{5}{18}|c_1|^3|c_2| + \tfrac{1}{90}|c_1|^5\,.$$

The expression on the right side of the above inequality takes its greatest value with respect to $|c_3|$ when $|c_3| = \frac{1}{2}(1 + |c_1|)(\frac{1}{3}|c_2| + |c_1|^2)$, so

$$W \le h(c, d)\,,$$

where $|c_1| = c$, $|c_2| = d$ and

$$\begin{aligned} h(c,d) &= 1 - c^2 - d^2 - \tfrac{1}{4}(1+c)(\tfrac{1}{3}d + c^2)^2 + \tfrac{1}{6}d(1+c)(\tfrac{1}{3}d + c^2) \\ &\quad + \tfrac{1}{2}c^2(1+c)(\tfrac{1}{3}d + c^2) + \tfrac{1}{2}cd^2 + \tfrac{5}{18}c^3 d + \tfrac{1}{90}c^5 \\ &= 1 - c^2 + \tfrac{1}{4}c^4 + \tfrac{47}{180}c^5 + \tfrac{1}{6}c^2 d + \tfrac{4}{9}c^3 d - \tfrac{35}{36}d^2 + \tfrac{19}{36}cd^2 \,. \end{aligned}$$

The shape of the variability region of (c,d) is a simple consequence of the Schwarz–Pick lemma and coincides with $\Omega = \{(c,d) : \ 0 \le c \le 1, \ 0 \le d \le 1 - c^2\}$. A simple algebraic computation shows that the critical points of h satisfy

$$\begin{cases} \frac{1}{6}c^2 + \frac{4}{9}c^3 - \frac{35}{18}d + \frac{19}{18}cd = 0 \\ -2c + c^3 + \frac{47}{36}c^4 + \frac{1}{3}cd + \frac{4}{3}c^2 d + \frac{19}{36}d^2 = 0 \end{cases} ,$$

thus, in Ω, there are two critical points $(0,0)$ and (c_0, d_0), where $c_0 = 0.828\ldots$ and $d_0 = 0.343\ldots$. For these points, we have $h(0,0) = 1$ and $h(c_0, d_0) = 0.596\ldots$. On the boundary of Ω, we get:

$$\begin{aligned} h(c,0) &= 1 - c^2 + \tfrac{1}{4}c^4 + \tfrac{47}{180}c^5 \le 1 \,, \\ h(0,d) &= 1 - \tfrac{35}{36}d^2 \le 1 \,, \\ h(c, 1-c^2) &= \tfrac{1}{36} + \tfrac{10}{9}c^2 - \tfrac{8}{9}c^4 + \tfrac{19}{36}c - \tfrac{11}{18}c^3 + \tfrac{31}{90}c^5 \,. \end{aligned}$$

Since the functions $g_1(c) = \frac{10}{9}c^2 - \frac{8}{9}c^4$ and $g_2(c) = \frac{19}{36}c - \frac{11}{18}c^3 + \frac{31}{90}c^5$ reach their greatest values for $c = \frac{\sqrt{10}}{4}$ and $c = 1$, respectively, $g_1(c) \le g_1(\frac{\sqrt{10}}{4}) = \frac{25}{72}$ and $g_2(c) \le g_2(1) = \frac{47}{180}$, and it follows that

$$h(c, 1-c^2) \le \tfrac{1}{36} + \tfrac{25}{72} + \tfrac{47}{180} < 1 \,.$$

Hence, $W \le 1$, and so $24|a_6| \le 4$, and we have the fifth inequality in Theorem 1.

Observe that, if $c_1 = 1$ and $c_k = 0$ for $k \ne 1$, then $a_2 = \frac{1}{2}$. Similarly, if $c_2 = 1$ and $c_k = 0$ for $k \ne 2$, then $a_3 = \frac{1}{2}$. If $c_3 = 1$ and $c_k = 0$ for $k \ne 3$, then $a_4 = \frac{1}{4}$. If $c_4 = 1$ and $c_k = 0$ for $k \ne 4$, then $a_5 = \frac{1}{4}$. If $c_5 = 1$ and $c_k = 0$ for $k \ne 5$, then $a_6 = \frac{1}{6}$. This means that the equalities in Theorem 1 hold for the functions f given by (5) with $w(z) = z$, $w(z) = z^2$, $w(z) = z^3$, $w(z) = z^4$ and $w(z) = z^5$, respectively. □

The logarithmic coefficients of $f \in \mathcal{S}$, denoted by $\gamma_n = \gamma_n(f)$, are defined with the following series expansion

$$\log \frac{f(z)}{z} = 2 \sum_{n=1}^{\infty} \gamma_n z^n \,.$$

For a function f given by (1), the logarithmic coefficients are as follows:

$$\begin{aligned} \gamma_1 &= \tfrac{1}{2}a_2 \,, \\ \gamma_2 &= \tfrac{1}{2}\left(a_3 - \tfrac{1}{2}{a_2}^2\right) \,, \\ \gamma_3 &= \tfrac{1}{2}\left(a_4 - a_2 a_3 + \tfrac{1}{3}{a_2}^3\right) \,, \\ \gamma_4 &= \tfrac{1}{2}\left(a_5 - a_2 a_4 + {a_2}^2 a_3 - \tfrac{1}{2}{a_3}^2 - \tfrac{1}{4}{a_2}^4\right) \,, \\ \gamma_5 &= \tfrac{1}{2}\left(a_6 - a_2 a_5 - a_3 a_4 + a_2 {a_3}^2 + {a_2}^2 a_4 - {a_2}^3 a_3 + \tfrac{1}{5}{a_2}^5\right) \,. \end{aligned} \tag{9}$$

Theorem 2. *If $f \in \mathcal{S}_S^*(\sin z)$ is given by (1), then*

$$|\gamma_1| \le \tfrac{1}{4} \,, \quad |\gamma_2| \le \tfrac{1}{4} \,, \quad |\gamma_3| \le \tfrac{1}{8} \,, \quad |\gamma_4| \le \tfrac{1}{8}, \quad |\gamma_5| \le \tfrac{1}{12} \,.$$

The bounds are sharp.

Proof. From (6) and (9), we get

$$\begin{aligned}
\gamma_1 &= \tfrac{1}{4}c_1\,,\\
\gamma_2 &= \tfrac{1}{4}\left(c_2 - \tfrac{1}{4}{c_1}^2\right),\\
\gamma_3 &= \tfrac{1}{8}\left(c_3 - \tfrac{1}{2}c_1c_2\right),\\
\gamma_4 &= \tfrac{1}{8}\left(c_4 - \tfrac{1}{2}c_1c_3 - \tfrac{1}{4}{c_1}^2c_2 + \tfrac{1}{48}{c_1}^4\right),\\
\gamma_5 &= \tfrac{1}{12}\left(c_5 - \tfrac{1}{2}c_1c_4 - \tfrac{1}{4}c_2c_3 - \tfrac{1}{8}{c_1}^2c_3 - \tfrac{3}{8}c_1{c_2}^2 + \tfrac{5}{48}{c_1}^3c_2 - \tfrac{1}{60}{c_1}^5\right).
\end{aligned} \tag{10}$$

The bounds of $|\gamma_1|$, $|\gamma_2|$, and $|\gamma_3|$ follow from Lemmas 1, 5 (with $\mu = -\frac{1}{4}$), and 2 (with $\mu = -\frac{1}{2}$ and $\nu = 0$), respectively.

Formula (10) for γ_4 can be written

$$\begin{aligned}
16|\gamma_4| &= |2c_4 - c_1c_3 - \tfrac{1}{2}{c_1}^2c_2 + \tfrac{1}{24}{c_1}^4|\\
&= |c_4 - c_1c_3 - \tfrac{1}{2}{c_2}^2 + \tfrac{3}{4}{c_1}^2c_2 - \tfrac{1}{8}{c_1}^4 + c_4 + \tfrac{1}{2}{c_2}^2 - \tfrac{5}{4}{c_1}^2c_2 + \tfrac{1}{12}{c_1}^4|\,.
\end{aligned}$$

Applying the triangle inequality, we get

$$16|\gamma_4| \le |c_4 - c_1c_3 - \tfrac{1}{2}{c_2}^2 + \tfrac{3}{4}{c_1}^2c_2 - \tfrac{1}{8}{c_1}^4| + |c_4 + \tfrac{1}{2}{c_2}^2 - \tfrac{5}{4}{c_1}^2c_2 + \tfrac{1}{12}{c_1}^4|\,.$$

From Lemma 6 for $\mu = -\frac{1}{2}$, we know that

$$|c_4 - c_1c_3 - \tfrac{1}{2}{c_2}^2 + \tfrac{3}{4}{c_1}^2c_2 - \tfrac{1}{8}{c_1}^3| \le 1\,,$$

thus we have

$$16|\gamma_4| \le 1 + W\,,$$

where

$$W = |c_4 + \tfrac{1}{2}{c_2}^2 - \tfrac{5}{4}{c_1}^2c_2 + \tfrac{1}{12}{c_1}^4|\,. \tag{11}$$

Now, we show that $W \le 1$. Applying the triangle inequality and Lemma 3 in (11), we obtain

$$W \le 1 - |c_1|^2 - |c_2|^2 + \tfrac{1}{2}|c_2|^2 + \tfrac{5}{4}|c_1|^2c_2 + \tfrac{1}{12}|c_1|^4,$$

thus

$$W \le h(c,d)\,,$$

where $|c_1| = c$, $|c_2| = d$ and

$$h(c,d) = 1 - c^2 - \tfrac{1}{2}d^2 + \tfrac{5}{4}c^2d + \tfrac{1}{12}c^4\,.$$

A simple algebraic computation shows that the critical points of h in $\Omega = \{(c,d) : 0 \le c \le 1,\ 0 \le d \le 1 - c^2\}$ satisfy

$$\begin{cases} \frac{5}{4}c^2 - d = 0\\ \frac{1}{3}c^3 + \frac{5}{2}cd - 2c = 0 \end{cases},$$

thus, in Ω, there are two critical points $(0,0)$ and (c_0,d_0), where $c_0 = \frac{4\sqrt{3}}{\sqrt{83}}$ and $d_0 = \frac{60}{83}$. For these points, we have $h(0,0) = 1$ and $h(c_0,d_0) = \frac{59}{83}$. On the boundary of Ω, we get:

$$\begin{aligned}
h(c,0) &= 1 - c^2 + \tfrac{1}{12}c^4 \le 1\,,\\
h(0,d) &= 1 - \tfrac{1}{2}d^2 \le 1\,,\\
h(c,1-c^2) &= \tfrac{1}{2} + \tfrac{5}{4}c^2 - \tfrac{5}{3}c^4 \le h(\tfrac{\sqrt{6}}{4},\tfrac{5}{8}) = \tfrac{47}{64} < 1\,.
\end{aligned}$$

Hence, $W \leq 1$, and so $16|\gamma_4| \leq 2$, and we have the fourth inequality in Theorem 2. Formula (10) for γ_5 can be written

$$\begin{aligned} 12|\gamma_5| = \tfrac{1}{2}|c_5 - c_1c_4 - c_2c_3 + \tfrac{3}{4}c_1{c_2}^2 + \tfrac{3}{4}{c_1}^2c_3 - \tfrac{1}{2}{c_1}^3c_2 + \tfrac{1}{16}{c_1}^5 \\ + c_5 + \tfrac{1}{2}c_2c_3 - \tfrac{3}{2}c_1{c_2}^2 - {c_1}^2c_3 + \tfrac{17}{24}{c_1}^3c_2 - \tfrac{23}{240}{c_1}^5| \,. \end{aligned}$$

Applying the triangle inequality, we get

$$\begin{aligned} 24|\gamma_5| \leq |c_5 - c_1c_4 - c_2c_3 + \tfrac{3}{4}c_1{c_2}^2 + \tfrac{3}{4}{c_1}^2c_3 - \tfrac{1}{2}{c_1}^3c_2 + \tfrac{1}{16}{c_1}^5| \\ + |c_5 + \tfrac{1}{2}c_2c_3 - \tfrac{3}{2}c_1{c_2}^2 - {c_1}^2c_3 + \tfrac{17}{24}{c_1}^3c_2 - \tfrac{23}{240}{c_1}^5| \,. \end{aligned}$$

From Lemma 7 for $\mu = -\frac{1}{2}$, we know that

$$|c_5 - c_1c_4 - c_2c_3 + \tfrac{3}{4}c_1{c_2}^2 + \tfrac{3}{4}{c_1}^2c_3 - \tfrac{1}{2}{c_1}^3c_2 + \tfrac{1}{16}{c_1}^5| \leq 1 \,,$$

thus we have

$$24|\gamma_5| \leq 1 + W \,,$$

where

$$W = |c_5 + \tfrac{1}{2}c_2c_3 - \tfrac{3}{2}c_1{c_2}^2 - {c_1}^2c_3 + \tfrac{17}{24}{c_1}^3c_2 - \tfrac{23}{240}{c_1}^5| \,. \tag{12}$$

Now, we show that $W \leq 1$. Applying the triangle inequality and Lemma 3 in (12), we obtain

$$W \leq 1 - |c_1|^2 - |c_2|^2 - \frac{|c_3|^2}{1+|c_1|} + \tfrac{1}{2}|c_2||c_3| + \tfrac{3}{2}|c_1||c_2|^2 + |c_1|^2|c_3| + \tfrac{17}{24}|c_1|^3|c_2| + \tfrac{23}{240}|c_1|^5 \,.$$

The expression on the right side of the above inequality takes its greatest value with respect to $|c_3|$ when $|c_3| = \frac{1}{2}(1+|c_1|)(\frac{1}{2}|c_2| + |c_1|^2)$, so

$$W \leq h(c,d) \,,$$

where $|c_1| = c$, $|c_2| = d$ and

$$\begin{aligned} h(c,d) &= 1 - c^2 - d^2 - \tfrac{1}{4}(1+c)(\tfrac{1}{2}d + c^2)^2 + \tfrac{1}{4}d(1+c)(\tfrac{1}{2}d + c^2) \\ &\quad + \tfrac{3}{2}cd^2 + \tfrac{1}{2}c^2(1+c)(\tfrac{1}{2}d + c^2) + \tfrac{17}{24}c^3d + \tfrac{23}{240}c^5 \\ &= 1 - c^2 - d^2 + \tfrac{1}{4}(1+c)(\tfrac{1}{2}d + c^2)^2 + \tfrac{3}{2}cd^2 + \tfrac{17}{24}c^3d + \tfrac{23}{240}c^5 \,. \end{aligned}$$

A simple algebraic computation shows that the critical points of h in $\Omega = \{(c,d) : 0 \leq c \leq 1,\ 0 \leq d \leq 1 - c^2\}$ satisfy

$$\begin{cases} \frac{1}{4}c^2 + \frac{23}{24}c^3 - \frac{15}{8}d + \frac{25}{8}cd = 0 \\ -2c + c^3 + \frac{83}{48}c^4 + \frac{1}{2}cd + \frac{23}{8}c^2 + \frac{25}{16}d^2 = 0 \end{cases} ,$$

thus, in Ω, there are two critical points $(0,0)$ and (c_0,d_0), where $c_0 = 0.484\ldots$ and $d_0 = 0.463\ldots$. For these points, we have $h(0,0) = 1$ and $h(c_0,d_0) = 0.827\ldots$. On the boundary of Ω we get:

$$\begin{aligned} h(c,0) &= 1 - c^2 + \tfrac{1}{4}c^4 + \tfrac{83}{240}c^5 \leq 1 \,, \\ h(0,d) &= 1 - d^2 \leq 1 \,, \\ h(c,1-c^2) &= \tfrac{1}{16} + \tfrac{9}{8}c^2 - \tfrac{15}{16}c^4 + \tfrac{25}{16}c - \tfrac{13}{6}c^3 + \tfrac{19}{20}c^5 \,. \end{aligned}$$

Since the functions $g_1(c) = \frac{9}{8}c^2 - \frac{15}{16}c^4$ and $g_2(c) = \frac{25}{16}c - \frac{13}{6}c^3 + \frac{19}{20}c^5$ reach their greatest values for $c = \frac{3}{\sqrt{15}}$ and $c = \frac{26-\sqrt{201}}{38}$, respectively, thus $g_1(c) \leq g_1(\frac{3}{\sqrt{15}}) = \frac{27}{80}$ and $g_2(c) \leq g_2(\frac{26-\sqrt{201}}{38}) = 0.5468\ldots$, and it follows that

$$h(c, 1-c^2) \leq \tfrac{1}{16} + \tfrac{27}{80} + 0.5468\ldots < 1\,.$$

Hence, $W \leq 1$, and so $24|\gamma_5| \leq 2$, and we have the fifth inequality in Theorem 2.

The equalities in Theorem 2 hold for the functions f given by (5) with $w(z) = z$, $w(z) = z^2$, $w(z) = z^3$, $w(z) = z^4$ and $w(z) = z^5$, respectively. □

4. Bounds of the Generalized Zalcman Functional and Hankel Determinants

Let us consider some cases of the generalized Zalcman functional $a_{n+m-1} - a_n a_m$ for functions from $\mathcal{S}_S^*(\sin z)$.

Theorem 3. *If $f \in \mathcal{S}_S^*(\sin z)$ is of the form (1), then*

$$|a_3 - {a_2}^2| \leq \tfrac{1}{2}\,, \quad |a_4 - a_2 a_3| \leq \tfrac{1}{4}\,, \quad |a_5 - {a_3}^2| \leq \tfrac{1}{4}\,, \quad |a_6 - a_3 a_4| \leq \tfrac{1}{6}\,.$$

The bounds are sharp.

Proof. From (6) and Lemma 5 with $\mu = -\frac{1}{2}$, we obtain

$$|a_3 - {a_2}^2| = \tfrac{1}{2}|c_2 - \tfrac{1}{2}{c_1}^2| \leq \tfrac{1}{2}\,.$$

From (6), we get

$$|a_4 - a_2 a_3| = \tfrac{1}{4}|c_3 - c_1 c_2 - \tfrac{1}{6}{c_1}^3|\,. \tag{13}$$

By applying Lemma 2 with $\mu = -1$ and $\nu = -\frac{1}{6}$ in (13), we obtain the second inequality in Theorem 3.

From (6), we have

$$|a_5 - {a_3}^2| = \tfrac{1}{4}|c_4 - \tfrac{1}{2}{c_1}^2 c_2 - \tfrac{1}{2}{c_2}^2|\,. \tag{14}$$

Now, using (4) and Lemma 3 in (14), we get

$$\begin{aligned} |a_5 - {a_3}^2| &\leq \tfrac{1}{4}\Big(1 - |c_1|^2 - |c_2|^2 + \tfrac{1}{2}|c_1|^2(1 - |c_1|^2) + \tfrac{1}{2}|c_2|^2\Big) \\ &= \tfrac{1}{4}\Big(1 - \tfrac{1}{2}|c_1|^2 - \tfrac{1}{2}|c_2|^2 - \tfrac{1}{2}|c_1|^4\Big) \leq \tfrac{1}{4}\,. \end{aligned}$$

Thus, we have the third result in Theorem 3.

From (6), we have

$$6|a_6 - a_3 a_4| = |c_5 + \tfrac{1}{4}c_1 c_4 - \tfrac{1}{4}c_2 c_3 - \tfrac{1}{2}{c_1}^2 c_3 - \tfrac{3}{4}c_1{c_2}^2 - \tfrac{1}{12}{c_1}^3 c_2 + \tfrac{1}{120}{c_1}^5|\,. \tag{15}$$

Applying the triangle inequality and Lemma 7 (for $\mu = 0$), in (15), we get

$$\begin{aligned} 24|a_6 - a_3 a_4| &\leq |c_5 + c_1 c_4 + c_2 c_3 + {c_1}^2 c_3| + 3|c_5 - \tfrac{2}{3}c_2 c_3 - {c_1}^2 c_3 - c_1{c_2}^2 - \tfrac{1}{9}{c_1}^3 c_2 + \tfrac{1}{90}{c_1}^5| \\ &\leq 1 + 3W\,, \end{aligned}$$

where

$$W = |c_5 - \tfrac{2}{3}c_2 c_3 - {c_1}^2 c_3 - c_1{c_2}^2 - \tfrac{1}{9}{c_1}^3 c_2 + \tfrac{1}{90}{c_1}^5|\,.$$

Similar to the proof of Theorem 1 for $|a_6|$, we can show that $W \leq 1$. Thus, we obtain the fourth result in Theorem 3.

Observe that the equalities in Theorem 3 hold for the functions f given by (5) with $w(z) = z^2$, $w(z) = z^3$, $w(z) = z^4$ and $w(z) = z^5$, respectively. □

Let us consider some cases of the Hankel determinant for functions from $\mathcal{S}_S^*(\sin z)$.

Theorem 4. *If $f \in \mathcal{S}_S^*(\sin z)$ is of the form (1), then*

$$|H_{2,2}(f)| \leq \tfrac{1}{4}\,, \quad |H_{2,3}(f)| \leq \tfrac{49}{432}\,, \quad |H_{3,1}(f)| \leq \tfrac{3}{16}\,.$$

The first bound is sharp.

Proof. From (6), we have

$$|H_{2,2}(f)| = |a_2a_4 - {a_3}^2| = \tfrac{1}{8}|c_1c_3 - {c_2}^2 + \tfrac{1}{2}{c_1}^2c_2 - {c_2}^2 - \tfrac{1}{6}{c_1}^4|$$

and hence, applying the triangle inequality, we get

$$|H_{2,2}(f)| \leq \tfrac{1}{8}\Big(|c_1c_3 - {c_2}^2| + |\tfrac{1}{2}{c_1}^2c_2 - {c_2}^2 - \tfrac{1}{6}{c_1}^4|\Big)\,. \tag{16}$$

From Lemma 4, we have

$$|c_1c_3 - {c_2}^2| \leq 1\,. \tag{17}$$

Moreover, from (4), we obtain

$$\begin{aligned}|\tfrac{1}{2}{c_1}^2c_2 - {c_2}^2 - \tfrac{1}{6}{c_1}^4| &\leq \tfrac{1}{2}{c_1}^2(1-|c_1|^2) + (1-|c_1|^2)^2 + \tfrac{1}{6}|c_1|^4\\ &= 1 - \tfrac{3}{2}|c_1|^2 + \tfrac{2}{3}|c_1|^4\,.\end{aligned}$$

Since the function $g(x) = 1 - \frac{3}{2}x + \frac{2}{3}x^2$, $x \in [0,1]$ is decreasing, for all $x \in [0,1]$, we have $g(x) \leq g(0) = 1$. Thus,

$$|\tfrac{1}{2}{c_1}^2c_2 - {c_2}^2 - \tfrac{1}{6}{c_1}^4| \leq 1\,. \tag{18}$$

Using (17) and (18) in (16), we get the bound of $|H_{2,2}(f)|$.

Now, we prove the second inequality from Theorem 4. From (6), we have

$$|H_{2,3}(f)| = |a_3a_5 - {a_4}^2| = \tfrac{1}{16}|2c_2c_4 - c_3(c_3 + c_1c_2 - \tfrac{1}{3}{c_1}^3) + {c_2}^3 - \tfrac{5}{4}{c_1}^2{c_2}^2 + \tfrac{1}{6}{c_1}^4c_2 - \tfrac{1}{36}{c_1}^6|$$

and hence, applying the triangle inequality, we get

$$16|H_{2,3}(f)| \leq 2|c_2||c_4| + |c_3||c_3 + c_1c_2 - \tfrac{1}{3}{c_1}^3| + |c_2|^3 + \tfrac{5}{4}|c_1|^2|c_2|^2 + \tfrac{1}{6}|c_1|^4|c_2| + \tfrac{1}{36}|c_1|^6\,. \tag{19}$$

From Lemma 3, we can obtain

$$|c_3| \leq 1 - |c_1|^2 - \frac{|c_2|^2}{1+|c_1|} \leq 1 - |c_1|^2 - \frac{|c_2|^2}{2}\,. \tag{20}$$

Applying (20) and Lemmas 2 (with $\mu = 1$ and $\nu = -\frac{1}{3}$) and 3 in (19), we have

$$16|H_{2,3}(f)| \leq 2|c_2|(1-|c_1|^2-|c_2|^2) + 1 - |c_1|^2 - \tfrac{1}{2}|c_2|^2 + |c_2|^3 + \tfrac{5}{4}|c_1|^2|c_2|^2 + \tfrac{1}{6}|c_1|^4|c_2| + \tfrac{1}{36}|c_1|^6\,,$$

thus

$$16|H_{2,3}(f)| \leq h(c,d)\,,$$

where $|c_1| = c$, $|c_2| = d$ and

$$h(c,d) = 2d(1-c^2-d^2) + 1 - c^2 - \tfrac{1}{2}d^2 + d^3 + \tfrac{5}{4}c^2d^2 + \tfrac{1}{6}c^4d + \tfrac{1}{36}c^6\,.$$

The function h is a decreasing function of the variable c, thus

$$h(c,d) \leq h(0,d) = 1 + 2d - \tfrac{1}{2}d^2 - d^3\,.$$

The function $h(0,d)$ reaches its greatest value in $[0,1]$ for $d=\frac{2}{3}$, thus

$$h(0,d) \leq h(0,\tfrac{2}{3}) = \tfrac{49}{27}\,.$$

Hence,

$$|H_{2,3}(f)| \leq \tfrac{1}{16}\cdot\tfrac{49}{27} = \tfrac{49}{432}\,.$$

Now, we prove the third inequality from Theorem 4. From (6), we have

$$\begin{aligned}|H_{3,1}(f)| &= \tfrac{1}{8}|c_2c_4 - \tfrac{1}{2}{c_1}^2c_4 - \tfrac{1}{2}{c_3}^2 + \tfrac{1}{2}c_1c_2c_3 + \tfrac{1}{6}{c_1}^3c_3 - \tfrac{1}{2}{c_2}^3 - \tfrac{3}{8}{c_1}^2{c_2}^2 + \tfrac{1}{6}{c_1}^4c_2 - \tfrac{1}{72}{c_1}^6|\\ &= \tfrac{1}{8}|c_4(c_2-\tfrac{1}{2}{c_1}^2) - \tfrac{1}{2}c_3(c_3 - c_1c_2 - \tfrac{1}{3}{c_1}^3) - \tfrac{1}{2}{c_2}^3 - \tfrac{3}{8}{c_1}^2{c_2}^2 + \tfrac{1}{6}{c_1}^4c_2 - \tfrac{1}{72}{c_1}^6|\end{aligned}$$

and hence, applying the triangle inequality, we get

$$8|H_{3,1}(f)| \leq |c_4||c_2-\tfrac{1}{2}{c_1}^2| + \tfrac{1}{2}|c_3||c_3 - c_1c_2 - \tfrac{1}{3}{c_1}^3| + \tfrac{1}{2}|c_2|^3 + \tfrac{3}{8}|c_1|^2|c_2|^2 + \tfrac{1}{6}|c_1|^4|c_2| + \tfrac{1}{72}|c_1|^6\,. \tag{21}$$

Using (20) and Lemmas 2 (with $\mu=-1$ and $\nu=-\frac{1}{3}$), 3, and 5 (with $\mu=-\frac{1}{2}$) in (21), we obtain

$$8|H_{3,1}(f)| \leq 1 - {c_1}^2 - {c_2}^2 + \tfrac{1}{2}(1-{c_1}^2-\tfrac{1}{2}{c_2}^2) + \tfrac{1}{2}|c_2|^3 + \tfrac{3}{8}|c_1|^2|c_2|^2 + \tfrac{1}{6}|c_1|^4|c_2| + \tfrac{1}{72}|c_1|^6\,, \tag{22}$$

thus

$$8|H_{3,1}(f)| \leq h(c,d)\,,$$

where $|c_1|=c$, $|c_2|=d$ and

$$h(c,d) = 1 - c^2 - d^2 + \tfrac{1}{2}(1-c^2-\tfrac{1}{2}d^2) + \tfrac{1}{2}d^3 + \tfrac{3}{8}c^2d^2 + \tfrac{1}{6}c^4d + \tfrac{1}{72}c^6\,.$$

The function h is a decreasing function of the variable c, thus

$$h(c,d) \leq h(0,d) = \tfrac{3}{2} - \tfrac{5}{4}d^2 + \tfrac{1}{2}d^3\,.$$

The function $h(0,d)$ reaches its greatest value in $[0,1]$ for $d=0$, thus

$$h(0,d) \leq h(0,0) = \tfrac{3}{2}\,.$$

Hence,

$$|H_{3,1}(f)| \leq \tfrac{1}{8}\cdot\tfrac{3}{2} = \tfrac{3}{16}\,.$$

Note that the first equality in Theorem 4 holds for the function f given by (5) with $w(z)=z^2$. The second and the third results are not sharp. It is expected that the sharp bounds of $|H_{2,3}(f)|$ and $|H_{3,1}(f)|$ are equal to $\frac{1}{16}$. Note that, for the functions f given by (5) with $w(z)=z^2$ and $w(z)=z^3$, in both cases, we obtain $|H_{2,3}(f)|=\frac{1}{16}$ and $|H_{3,1}(f)|=\frac{1}{16}$. □

Example 1. *For the function $w(z)=z\in\mathcal{B}_0$, we have $f(z)=z+\frac{1}{2}z^2-\frac{1}{24}z^4+\frac{1}{720}z^6+\ldots\in\mathcal{S}_S^*(\sin z)$ and thus*

$$|H_{2,3}(f)| = \tfrac{1}{576} < \tfrac{49}{432}, \quad |H_{3,1}(f)| = \tfrac{1}{576} < \tfrac{3}{16}\,.$$

For the function $w(z)=\frac{1}{2}(z+z^2)$, we get $f(z)=z+\frac{1}{4}z^2+\frac{1}{4}z^3+\frac{5}{192}z^4+\frac{1}{64}z^5\ldots$ and then

$$|H_{2,3}(f)| = \tfrac{119}{36864} < \tfrac{49}{432}, \quad |H_{3,1}(f)| = \tfrac{373}{36864} < \tfrac{3}{16}\,.$$

5. Conclusions

The problem of finding coefficient bounds plays an important role in studying the geometry of complex-valued functions. The logarithmic coefficients of functions can be used to find sharp estimations for the coefficients of an inverse function. This is of great significance because reaching a complete solution to the problem of finding bounds for the inverse function is usually more difficult than finding bounds for the function itself.

The generalized Zalcman functionals are important because they frequently appear in coefficient formulas for inversion transformation in the theory of univalent functions. Furthermore, the second coefficient provides information about the growth and distortion theorems for univalent function. Similarly, the Hankel determinants are very useful in the investigation of singularities and power series with integral coefficients. The Hankel determinant can also be used to study meromorphic functions. Descriptions of its many properties and applications can be found in the literature.

The bounds of various coefficient functionals in the class $\mathcal{S}_S^*(\sin z)$ presented in this paper were obtained due to connecting this class with the class $\mathcal{B}_0$ of Schwarz functions. It is worth noting that knowing everything about $\mathcal{B}_0$, including estimates of coefficient functionals, is a good tool in studies of other classes of analytic functions. Moreover, the class $\mathcal{S}_S^*(\varphi(z))$ can be investigated for some other cases of the function φ.

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