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Common Solution for a Finite Family of Equilibrium Problems, Quasi-Variational Inclusion Problems and Fixed Points on Hadamard Manifolds

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Abstract: In this paper, we introduce an iterative algorithm for finding a common solution of a finite family of the equilibrium problems, quasi-variational inclusion problems and fixed point problem on Hadamard manifolds. Under suitable conditions, some strong convergence theorems are proved. Our results extend some recent results in literature.

Keywords: equilibrium problem; fixed point problem; quasi-variational inclusion problem; firmly nonexpansive mapping; hadamard manifolds; maximal monotone vector fields; monotone vector fields; iterative algorithm

JEL Classification: 47H05; 47J25; 47J25; 58A05; 58C30; 90C33; 47J25

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Citation: Zhu, J.; Tang, J.; Chang, S.-s.; Liu, M.; Zhao, L. Common Solution for a Finite Family of Equilibrium Problems, Quasi-Variational Inclusion Problems and Fixed Points on Hadamard Manifolds. *Symmetry* **2021**, *13*, 1161. https://doi.org/10.3390/sym13071161

Academic Editor: Alexei Kanel-Belov

Received: 11 June 2021
Accepted: 24 June 2021
Published: 28 June 2021

1. Introduction

Recently, many convergence results by the proximal point algorithm have been extended from the classical linear spaces to the setting of manifolds (see, for example, refs. [1–18]. Li et al. [10] developed the proximal point method in the setting of Hadamard manifolds. Later, Li et al. [11] extended the Mann and Halpern iteration scheme for finding the fixed points of nonexpansive mappings from Hilbert spaces to Hadamard manifolds. Very recently, Ansari et al. [12] and Al-Homidan-Ansari-Babu [13] considered the problem of finding

$$x \in Fix(T) \bigcap (A+B)^{-1}(0) \tag{1}$$

in a Hadamard manifold, where T is a nonexpansive mapping, B is a set-valued maximal monotone mapping, and A is a single-valued continuous and monotone mapping. They proposed some Halpern-type and Mann-type iterative methods. Under suitable conditions they proved that the sequence generated by the algorithm converges strongly to a common element of the set of fixed points of the mapping T and the set of solutions of the inclusion problem.

Recently, Calao et al. [14] and Khammahawong et al. [19] studied the equilibrium problem on Hadamard manifolds. Let M be a Hadamard manifold, TM be the tangent bundle of M, K be a nonempty closed geodesic convex subset of M, and $F : K \times K \longrightarrow R$ be a bifunction satisfying $F(x,x) = 0$, for all $x \in K$. Then the equilibrium problem on the Hadamard manifold is to find $x^* \in K$ such that

$$F(x^*, y) \geq 0, \forall y \in K. \tag{2}$$

We denote the set of a equilibrium points of the equilibrium problem (2) by $EP(F)$.

Motivated and inspired by the above results, we consider the following common solution problem for a finite family of the equilibrium problem, quasi-variational inclusion problem and the fixed point on Hadamard manifolds: i.e., to find $x^* \in K$ such that

$$x^* \in \bigcap_{i=1}^{m} EP(F_i) \bigcap (A+B)^{-1}(0) \bigcap F(S). \tag{3}$$

In this paper, an iterative algorithm for finding a commmon solution of problem (3) is proposed. Under suitable conditions, some strong convergence is proven. Our results extend some recent results in literature.

2. Preliminaries

In this section, we recall some fundamental definitions, properties, useful results, and notations of Riemannian geometry. Readers can refer to the textbook [20].

Let M be a finite dimensional differentiable manifold, T_pM be the tangent space of M at $p \in M$, and we denote by $TM = \bigcup_{p\in M} T_pM$ the tangent bundle of M. An inner product $\mathcal{R}_p(\cdot,\cdot)$ on T_pM is called a Riemannian metric on T_pM. A tensor field $\mathcal{R}(\cdot,\cdot)$ is said to be a Riemannian metric on M if for every $p \in M$, the tensor $\mathcal{R}_p(\cdot,\cdot)$ is a Riemannian metric on T_pM. The corresponding norm to the inner product $\mathcal{R}(\cdot,\cdot)$ on T_pM is denoted by $||\cdot||_p$. A differentiable manifold M endowed with a Riemannian metric $\mathcal{R}(\cdot,\cdot)$ is called a Riemannian manifold. The length of a piecewise smooth curve $\gamma : [0,1] \to M$ joining p to q (i.e., $\gamma(0) = p$ and $\gamma(1) = q$) is defined as

$$L(\gamma) = \int_0^1 ||\gamma'(t)||dt. \tag{4}$$

the Riemannian distance $d(p,q)$ is the minimal length over the set of all such curves joining p to q, which induces the original topology on M.

A Riemannian manifold M is complete if for any $p \in M$, all geodesics emanating from p are defined for all $t \in \mathbb{R}$. A geodesic joining p to q in M is said to be a minimal geodesic if its length is equal to $d(p,q)$. A Riemannian manifold M equipped with Riemannian distance d is a metric space (M,d). By Hopf–Rinow Theorem [20], if M is complete, then any pair of points in M can be joined by a minimal geodesic. Moreover, (M,d) is a complete metric space and bounded closed subsets are compact.

Definition 1. *If M is a complete Riemannian manifold, then the exponential map $exp_p : T_pM \to M$ at $p \in M$ is defined by $exp_pv = \gamma_v(1,p)$ for all $v \in T_pM$, where $\gamma_v(\cdot,p)$ is the geodesic starting from p with velocity v, that is, $\gamma_v(0,p) = p$ and $\gamma'_v(0,p) = v$.*

It is known that $exp_ptv = \gamma_v(t,p)$ for each real number t. It is easy to see that $exp_p0 = \gamma_v(0,p) = p$, where 0 is the zero tangent vector. Moreover, for any $p,q \in M$ we have $d(p,q) = ||exp_p^{-1}q||$. Note that the exponential map exp_p is differentiable on T_pM for any $p \in M$.

Definition 2. *A complete simply connected Riemannian manifold of non-positive sectional curvature is called a Hadamard Manifold.*

Proposition 1 ([20])**.** *Let M be a Hadamard manifold. Then for any two points $x,y \in M$, there exists a unique normalized geodesic $\gamma : [0,1] \to M$ joining $x = \gamma(0)$ to $y = \gamma(1)$, which is in fact a minimal geodesic denoted by*

$$\gamma(t) = exp_xtexp_x^{-1}y, \forall t \in [0,1]. \tag{5}$$

Lemma 1 ([21]). *Let $\triangle(p,q,r)$ be a geodesic triangle in a Hadamard manifold M. Then there exist $\bar{p},\bar{q},\bar{r} \in \mathbb{R}^2$ such that*

$$d(p,q) = ||\bar{p}-\bar{q}||,\ d(q,r) = ||\bar{q}-\bar{r}||,\ d(r,p) = ||\bar{r}-\bar{p}||.$$

The triangle $\triangle(\bar{p},\bar{q},\bar{r})$ is called the comparison triangle of the geodesic-triangle $\triangle(p,q,r)$, which is unique up to isometry of M.

Lemma 2 ([11]). *Let $\triangle(p,q,r)$ be a geodesic triangle in a Hadamard manifold M, $\triangle(\bar{p},\bar{q},\bar{r})$ be its comparison triangle.*

(1) Let α,β,γ (respectively, $\bar{\alpha},\bar{\beta},\bar{\gamma}$) be the angles of $\triangle(p,q,r)$ (respectively, $\triangle(\bar{p},\bar{q},\bar{r})$ at the vertices p,q,r(respectively, $\bar{p},\bar{q},\bar{r}$). Then, the following inequalities hold:

$$\bar{\alpha} \geq \alpha,\ \bar{\beta} \geq \beta,\ \bar{\gamma} \geq \gamma.$$

(2) Let z be a point on the geodesic joining p to q and $\bar{z}$ be its comparison point in the interval $[\bar{p},\bar{q}]$. suppose that $d(z,p) = ||\bar{z}-\bar{p}||$ and $d(z,q) = ||\bar{z}-\bar{q}||$. Then

$$d(z,r) \leq ||\bar{z}-\bar{r}||. \tag{6}$$

The following inequalities can be proved easily.

Lemma 3 ([15]). *Let M be a finite dimensional Hadamard manifold.*

(i) *Let $\gamma : [0,1] \to M$ be a geodesic joining x to y. Then we have*

$$d(\gamma(t_1),\gamma(t_2)) = |t_1-t_2|d(x,y),\ \forall t_1,t_2 \in [0,1]; \tag{7}$$

(From now on $d(x,y)$ denotes the Riemannian distance).

(ii) *for any $x,y,z,u,w \in M$ and $t \in [0,1]$, the following inequalities hold:*

$$d(exp_x(1-t)exp_x^{-1}y,z) \leq td(x,z) + (1-t)d(y,z); \tag{8}$$

$$d^2(exp_x(1-t)exp_x^{-1}y,z) \leq td^2(x,z) + (1-t)d^2(y,z) - t(1-t)d^2(x,y); \tag{9}$$

$$d(exp_x(1-t)exp_x^{-1}y, exp_u(1-t)exp_u^{-1}w) \leq td(x,u) + (1-t)d(y,w). \tag{10}$$

Let M be a Hadamard manifold. A subset $K \subset M$ is said to be geodesic convex if for any two points x and y in K, the geodesic joining x to y is contained in K.

In the sequel, unless otherwise specified, we always assume that M is a finite dimensional Hadamard manifold, and K is a nonempty, bounded, closed and geodesic convex set in M and $Fix(S)$ is the fixed point set of a mapping S.

A function $f : K \to (-\infty,\infty]$ is said to be geodesic convex if, for any geodesic $\gamma(\lambda)(0 \leq \lambda \leq 1)$ joining $x,y \in C$, the function $f \circ \gamma$ is convex, i.e.,

$$f(\gamma(\lambda)) \leq \lambda f(\gamma(0)) + (1-\lambda)f(\gamma(1)) = \lambda f(x) + (1-\lambda)f(y).$$

Definition 3. *Let X be a complete metric space and $Q \subset X$ be a nonempty set. A sequence $\{x_n\} \subset X$ is called Fejér monotone with respect to Q if for any $y \in Q$ and $n \geq 0$,*

$$d(x_{n+1},y) \leq d(x_n,y).$$

Lemma 4 ([22,23]). *Let X be a complete metric space, $Q \subset X$ be a nonempty set. If $\{x_n\} \subset X$ is Fejér monotone with respect to Q, then $\{x_n\}$ is bounded. Moreover, if a cluster point x of $\{x_n\}$ belongs to Q, then $\{x_n\}$ converges to x.*

Definition 4. *A mapping $S : K \to K$ is said to be*

(1) *nonexpansive if*

$$d(Sx, Sy) \le d(x, y), \ \forall x, y \in K.$$

(2) *firmly nonexpansive [20] if for all $x, y \in K$, the function $\phi : [0,1] \to [0, \infty]$ defined by*

$$\phi(t) := d(exp_x texp_x^{-1} Sx, exp_y texp_y^{-1} Sy), \ \forall t \in [0,1]$$

is nonincreasing

Proposition 2 ([24]). *Let $S : K \to K$ be a mapping. Then the following statements are equivalent.*

(i) *S is firmly nonexpansive;*

(ii) *For any $x, y \in K$ and $t \in [0,1]$*

$$d(S(x), S(y)) \le d(exp_x texp_x^{-1} Sx, exp_y texp_y^{-1} Sy); \tag{11}$$

(iii) *For any $x, y \in K$*

$$\mathcal{R}(exp_{S(x)}^{-1} S(y), exp_{S(x)}^{-1} x) + \mathcal{R}(exp_{S(y)}^{-1} S(x), exp_{S(y)}^{-1} y) \le 0. \tag{12}$$

Lemma 5 ([16]). *If $S : K \to K$ is a firmly nonexpansive mapping and $Fix(S) \ne \emptyset$, then for any $x \in K$ and $p \in Fix(S)$ the following conclusion holds:*

$$d^2(Sx, p) \le d^2(x, p) - d^2(Sx, x). \tag{13}$$

In the sequel, we denote by $\Omega(M)$ the set of all single-valued vector fields $A : M \to TM$, such that $A(x) \in T_x M$ for each $x \in M$, and let the domain $D(A)$ of A be defined by

$$D(A) = \{x \in M : A(x) \in T_x M\}.$$

Definition 5 ([25]). *A single-valued vector field $A \in \Omega(M)$ is said to be monotone if*

$$\langle A(x), exp_x^{-1} y \rangle \le \langle A(y), -exp_y^{-1} x \rangle, \ \forall x, y \in M.$$

Let $\mathcal{X}(M)$ denote the set of all set-valued vector fields $B : M \to 2^{TM}$ such that $B(x) \subset T_x M$ for all $x \in M$, and let the domain $D(B)$ of B be defined by $D(B) = \{x \in M : B(x) \ne \emptyset\}$.

Definition 6 ([26]). *A set-valued vector field $B \in \mathcal{X}(M)$ is said to be*

(1) *monotone if for any $x, y \in D(B)$,*

$$\mathcal{R}(u, exp_x^{-1} y) \le \mathcal{R}(v, -exp_y^{-1} x), \ \forall u \in B(x) \text{ and } \forall v \in B(y);$$

(2) *maximal monotone if it is monotone and for all $x \in D(B)$ and $u \in T_x M$, the condition*

$$\mathcal{R}(u, exp_x^{-1} y) \le \mathcal{R}(v, -exp_y^{-1} x) \ \ \forall y \in D(B) \text{ and } \forall v \in B(y).$$

implies $u \in B(x)$.

(3) *For given $\lambda > 0$, the resolvent of B of order $\lambda > 0$ is a set-valued mapping $J_\lambda^B : M \to 2^{TM}$ defined by*

$$J_\lambda^B(x) := \{z \in M : x \in exp_z \lambda B(z)\}, \ \forall x \in M.$$

Theorem 1 ([24]). *Let $B \in \mathcal{X}(M)$. The following assertions hold for any $\lambda > 0$*

(1) *the vector field B is monotone if and only if J_λ^B is single-valued and firmly nonexpansive.*

(2) *if $D(B) = M$, the vector field B is maximal monotone if and only if J_λ^B is single-valued, firmly nonexpansive and the domain $D(J_\lambda^B) = M$.*

Proposition 3 ([24]). *Let K be a nonempty subset of M and $T : K \to M$ be a firmly nonexpansive mapping. Then*

$$\mathcal{R}(exp_{Ty}^{-1}x, exp_{Ty}^{-1}y) \leq 0$$

holds for any $x \in F(T)$ and $y \in K$.

Lemma 6 ([13]). *Let K be a nonempty closed subset of M and $B \in \mathcal{X}(M)$ be a maximal monotone vector field. Let $\{\lambda_n\} \subset (0, \infty)$ be a real sequence with $lim_{n\to\infty}\lambda_n = \lambda > 0$ and a sequence $\{x_n\} \subset K$ with $lim_{n\to\infty}x_n = x \in K$ such that $lim_{n\to\infty}J^B_{\lambda_n}(x_n) = y$. Then, $y = J^B_\lambda(x)$.*

Proposition 4 ([12]). *Let $A \in \Omega(M)$ be a single-valued monotone vector field, $B \in \mathcal{X}(M)$ be a set-valued maximal monotone vector field. For any $x \in K$, the following assertions are equivalent*

(1) $x \in (A + B)^{-1}(0)$;

(2) $x = J^B_\lambda(exp_x(-\lambda A(x)))$, $\forall\lambda > 0$.

Let K be a nonempty closed geodesic convex set in M and $F : K \times K \to \mathbb{R}$ be a bifunction satisfying the following assumptions:

(A_1) for all $x \in K, F(x, x) \geq 0$;

(A_2) F is monotone, that is, for all $x, y \in K, F(x, y) + F(y, x) \leq 0$;

(A_3) for every $y \in K, x \mapsto F(x, y)$ is upper semicontinuous;

(A_4) for every $x \in K, y \mapsto F(x, y)$ are geodesic convex and lower semicontinuous;

(A_5) $x \mapsto F(x, x)$ is lower semicontinuous;

(A_6) there exists a compact set $L \subseteq M$ such that $x \in K \setminus L \implies \exists y \in K \bigcap L$ such that $F(x, y) < 0$.

Definition 7 ([14]). *Let $F : K \times K \to \mathbb{R}$ be a bifunction. The resolvent of F is a multivalued operator $T^F_r : M \to 2^K$ defined by*

$$T^F_r(x) = \{z \in K : F(z, y) - \frac{1}{r}\langle exp_z^{-1}x, exp_z^{-1}y\rangle \geq 0, \ \forall y \in K\}.$$

Theorem 2 ([14]). *Let $F : K \times K \to \mathbb{R}$ be a bifunction satisfying the following conditions:*

(1) *F is monotone;*

(2) *for all $r > 0$, T^F_r is properly defined, that is, the domain $D(T^F_r) \neq \emptyset$.*

Then for any $r > 0$,

(i) *the resolvent T^F_r is single-valued;*

(ii) *the resolvent T^F_r is firmly nonexpansive;*

(iii) *the fixed point set of T^F_r is the equilibrium point set of F,*

$$F(T^F_r) = EP(F).$$

Theorem 3 ([24]). *Let $F : K \times K \to \mathbb{R}$ be a bifunction satisfying the assumptions $(A_1) - (A_3)$. Then $D(T^F_r) = M$.*

Lemma 7 ([24]). *Let $F : K \times K \to \mathbb{R}$ be a bifunction satisfying the assumptions $(A_1), (A_3), (A_4), (A_5)$ and (A_6). Then, there exists $z \in K$ such that*

$$F(z, y) - \frac{1}{r}\langle exp_z^{-1}x, exp_z^{-1}y\rangle \geq 0, \ \forall y \in K,$$

for all $r > 0$ and $x \in M$.

3. The Main Results

In the sequel, we always assume that

(1) K is a nonempty closed bounded geodesic convex subsets of a Hadamard manifold M;
(2) $B : K \longrightarrow 2^{TM}$ is a maximal monotone setvalued vector field;
(3) $A : K \longrightarrow TM$ is a continuous and monotone single-valued vector field satisfying the following condition [13].

(Assumption 3.1):

$$d(exp_x(-\lambda A(x)), exp_y(-\lambda A(y))) \leq (1-\rho)d(x,y), \forall x,y \in K, \lambda > 0 \ and \ \rho \in [0,1]. \quad (14)$$

(4) $S : K \to K$ is a nonexpansive mapping;
(5) $F_i : K \times K \longrightarrow \mathbb{R}, i = 1,2,\cdots,m$, is a bifunction satisfying the conditions $(A_1)-(A_6)$, and for given $r > 0$, the resolvent of F_i is a multivalued operator $T_r^{F_i} : M \longrightarrow 2^K$ such that for all $x \in M$

$$T_r^{F_i}(x) = \{z \in K : F_i(z,y) - \frac{1}{r}\langle exp_z^{-1}x, exp_z^{-1}y\rangle \geq 0, \forall y \in K\}, i = 1,2,\cdots,m.$$

(6) Denote by

$$S_r^l := T_r^{F_l} \circ T_r^{F_{l-1}} \circ \cdots \circ T_r^{F_2} \circ T_r^{F_1}, l = 1,2,\cdots,m.$$

Theorem 4. *Let K, M, A, B, $\{F_i\}_{i=1}^m$, $\{S_r^l\}_{l=1}^m$ and S be the same as above. Let $\{x_n\},\{u_n\},\{y_n\}$ and $\{z_n\}$ be the sequences generated by $x_0 \in K$*

$$\begin{cases} u_n = J_{\lambda_n}^B(exp_{x_n}(-\lambda_n A(x_n))), \\ y_n = exp_{x_n}\alpha_n exp_{x_n}^{-1} Su_n, \\ z_n = S_r^m(y_n), \\ x_{n+1} = exp_{x_n}\beta_n exp_{x_n}^{-1} z_n. \ \forall n \geq 0, \end{cases} \quad (15)$$

where $\{\alpha_n\},\{\beta_n\}$ and $\{\lambda_n\}$ are positive sequences satisfying the following conditions:
(i) $0 < a \leq \alpha_n, \beta_n \leq b < 1, \forall n \in \mathbb{N}$;
(ii) $0 < \hat{\lambda} \leq \lambda_n \leq \tilde{\lambda} < \infty, \forall n \in \mathbb{N}$;
(iii) $\sum_{n=1}^{\infty} \alpha_n\beta_n = \infty$
If $\Omega = \bigcap_{i=1}^m EP(F_i) \bigcap (A+B)^{-1}(0) \bigcap F(S) \neq \emptyset$, then $\{x_n\}$ converges strongly to a solution of problem (1.4).

Proof. We divide the proof of Theorem 4 into five steps.

For $n \geq 0$, let $\gamma_n : [0,1] \to M$ be the geodesic joining $\gamma_n(0) = x_n$ to $\gamma_n(1) = z_n$ and $\hat{\gamma_n} : [0,1] \to M$ be the geodesic joining $\hat{\gamma_n}(0) = x_n$ to $\hat{\gamma_n}(1) = Su_n$. Then $\{x_{n+1}\}$ and $\{y_n\}$ can be written as

$$x_{n+1} := \gamma_n(\beta_n), \ y_n = \hat{\gamma_n}(\alpha_n). \quad (16)$$

(I) We first prove that Ω is closed and geodesic convex.

Because every nonexpansive mapping is continuous, therefore $F(S)$ is closed. We can prove that $F(S)$ is geodesic convex.

In fact, let $p,q \in F(S)$, we need to prove that a gecdesic $\gamma : [0,1] \to M$ joining p to q is contained in $F(S)$. It is well-known that in Hadamard manifold M, for any $p,q \in M$, and $t \in [0,1]$, there exists a unique point $\omega_t = \gamma(t) = exp_p tex_p^{-1}q$ such that

$$d(p,q) = d(p,\omega_t) + d(\omega_t,q).$$

By the nonexpansiveness of S and the geodesic convexity of Riemannian distance, we have

$$d(p,S(\omega_t)) = d(S(p),S(\omega_t)) \leq d(p,\omega_t) = d(p,\gamma(t)) \leq td(p,q).$$

Similarly, we get

$$d(S(\omega_t), q) \leq (1-t)d(p,q).$$

From the above, we have

$$d(p,q) \leq d(p, S(\omega_t)) + d(S(\omega_t), q) \leq d(p,q).$$

So

$$d(p,q) = (p, S(\omega_t)) + d(S(\omega_t), q).$$

By the uniqueness of ω_t, we have $\omega_t = S(\omega_t)$. Therefore, $\omega_t = \gamma(t) \in F(S)$. Thus, $F(S)$ is geodesic convex.

From Proposition 4, we have $(A+B)^{-1}(0) = F(J_\lambda^B(exp(-\lambda A)))$. Because J_λ^B is nonexpansive, this together with Assumption 3.1, $J_\lambda^B(exp(-\lambda A))$ is nonexpansive. Therefore, $(A+B)^{-1}(0)$ is closed and geodesic convex in M.

From Theorem 2, we have $T_r^{F_i}$ is firmly nonexpansive and $F(T_r^{F_i}) = EP(F_i)$. Therefore, $EP(F_i)$ is closed and geodesic convex in M. Hence, Ω is closed and geodesic convex.

(II) We prove that $\{x_n\}$ is Fejér monotone with respect to Ω.

Let $\omega \in \Omega$, Then $\omega \in (A+B)^{-1}(0)$ and $\omega \in \bigcap_{i=1}^m EP(F_i)$. By Proposition 4, we have $\omega = J_{\lambda_n}^B(exp_\omega(-\lambda_n A(\omega)))$. By nonexpansiveness of resolvent $J_{\lambda_n}^B$ of B and Assumption 3.1, we have

$$\begin{aligned} d(u_n, \omega) &= d(J_{\lambda_n}^B(exp_{x_n}(-\lambda_n A(x_n))), J_{\lambda_n}^B(exp_\omega(-\lambda_n A(\omega)))) \\ &\leq d(exp_{x_n}(-\lambda_n A(x_n)), exp_\omega(-\lambda_n A(\omega))) \\ &\leq (1-\rho)d(x_n, \omega) \\ &\leq d(x_n, \omega). \end{aligned} \tag{17}$$

Since $\omega \in \bigcap_{i=1}^M EP(F_i)$, by Theorem 2, for each $i = 1, 2, \cdots, m$, $T_r^{F_i}$ is non-expansive and $Fix(T_r^{F_i}) = EP(F_i)$, therefore S_r^m is also non-expansive and $\omega \in Fix(S_r^m)$, we have

$$d(z_n, \omega) = d(S_r^m(y_n), S_r^m(\omega)) \leq d(y_n, \omega). \tag{18}$$

Because $y_n = \hat{\gamma}_n(\alpha_n)$ for all $n \geq 0$, S is a nonexpansive mapping on K and $\omega \in F(S)$, by using the geodesic convexity of Riemannian distance and (17), we have

$$\begin{aligned} d(y_n, \omega) &= d(\hat{\gamma}_n(\alpha_n), \omega) \\ &\leq (1-\alpha_n)d(\hat{\gamma}_n(0), \omega) + \alpha_n d(\hat{\gamma}_n(1), \omega) \\ &\leq (1-\alpha_n)d(x_n, \omega) + \alpha_n d(Su_n, \omega) \\ &\leq (1-\alpha_n)d(x_n, \omega) + \alpha_n d(u_n, \omega) \\ &\leq d(x_n, \omega). \end{aligned} \tag{19}$$

Since $x_{n+1} := \gamma_n(\beta_n)$, from (18) and (19), we have

$$\begin{aligned} d(x_{n+1}, \omega) &= d(\gamma_n(\beta_n), \omega) \\ &\leq (1-\beta_n)d(\gamma_n(0), \omega) + \beta_n d(\gamma_n(1), \omega) \\ &\leq (1-\beta_n)d(x_n, \omega) + \beta_n d(z_n, \omega) \\ &\leq (1-\beta_n)d(x_n, \omega) + \beta_n d(y_n, \omega) \\ &\leq (1-\beta_n)d(x_n, \omega) + \beta_n d(x_n, \omega) \\ &\leq d(x_n, \omega). \end{aligned} \tag{20}$$

This shows that $\{d(x_n, \omega)\}$ is decreasing and bounded below. Hence, the limit $\lim_{n\to\infty} d(x_n, \omega)$ exists for each $\omega \in \Omega$. This implies that $\{x_n\}$ is Fejér monotone with respect to Ω. Hence, the sequence $\{x_n\}$ is bounded. So are the sequences $\{exp_{x_n}(-\lambda_n A(x_n))\}$, $\{S(u_n)\}$, $\{y_n\}$ and $\{z_n\}$.

(III) Next, we prove that
$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$$

We fix $n \geq 0$ and let $\Delta(x_n, z_n, \omega)$ be a geodesic triangle with vertices x_n, z_n and ω, and $\Delta(\overline{x_n}, \overline{z_n}, \overline{\omega}) \subseteq \mathbb{R}^2$ be a corresponding comparison triangle. Then, we have

$$d(x_n, \omega) = ||\overline{x_n} - \overline{\omega}||, \; d(z_n, \omega) = ||\overline{z_n} - \overline{\omega}|| \;\; and \;\; d(z_n, x_n) = ||\overline{z_n} - \overline{x_n}||.$$

Recall $x_{n+1} := exp_{x_n}\beta_n exp_{x_n}^{-1} z_n$, so it comparison point is $\overline{x_{n+1}} = (1-\beta_n)\overline{x_n} + \beta_n \overline{z_n}$. Using (6), (18) and (19), we get

$$\begin{aligned}
d^2(x_{n+1}, \omega) &\leq ||\overline{x_{n+1}} - \overline{\omega}||^2 \\
&= ||(1-\beta_n)\overline{x_n} + \beta_n \overline{z_n} - \overline{\omega}||^2 \\
&= ||(1-\beta_n)(\overline{x_n} - \overline{\omega}) + \beta_n(\overline{z_n} - \overline{\omega})||^2 \\
&= (1-\beta_n)||\overline{x_n} - \overline{\omega}||^2 + \beta_n||\overline{z_n} - \overline{\omega}||^2 - \beta_n(1-\beta_n)||\overline{x_n} - \overline{z_n}||^2 \\
&= (1-\beta_n)d^2(x_n, \omega) + \beta_n d^2(z_n, \omega) - \beta_n(1-\beta_n)d^2(x_n, z_n) \\
&\leq (1-\beta_n)d^2(x_n, \omega) + \beta_n d^2(y_n, \omega) - \beta_n(1-\beta_n)d^2(x_n, z_n) \\
&\leq (1-\beta_n)d^2(x_n, \omega) + \beta_n d^2(x_n, \omega) - \beta_n(1-\beta_n)d^2(x_n, z_n) \\
&= d^2(x_n, \omega) - \beta_n(1-\beta_n)d^2(x_n, z_n).
\end{aligned} \tag{21}$$

From (21), we also obtain

$$\beta_n(1-\beta_n)d^2(x_n, z_n) \leq d^2(x_n, \omega) - d^2(x_{n+1}, \omega),$$

and we further have

$$\begin{aligned}
d^2(x_n, z_n) &\leq \frac{1}{\beta_n(1-\beta_n)}(d^2(x_n, \omega) - d^2(x_{n+1}, \omega)) \\
&\leq \frac{1}{a(1-b)}(d^2(x_n, \omega) - d^2(x_{n+1}, \omega)).
\end{aligned}$$

Since $\{x_n\}$ is a Fejér monotone with respect to Ω which implies that $\lim_{n\to\infty} d(x_n, \omega)$ exists. By letting $n \to \infty$, we have

$$\lim_{n\to\infty} d(x_n, z_n) = 0. \tag{22}$$

Recall that $x_{n+1} := \gamma_n(\beta_n)$ for all $n \in \mathbb{N}$, using the geodesic convexity of Riemannian distance, we obtain

$$\begin{aligned}
d(x_{n+1}, x_n) &= d(\gamma_n(\beta_n), x_n) \\
&\leq (1-\beta_n)d(\gamma_n(0), x_n) + \beta_n d(\gamma_n(1), x_n) \\
&= (1-\beta_n)d(x_n, x_n) + \beta_n d(z_n, x_n) \\
&= \beta_n d(x_n, z_n) \\
&\leq b d(x_n, z_n).
\end{aligned}$$

Letting $n \to \infty$ and using (22), we get

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{23}$$

(IV) Next we prove that $\lim_{n\to\infty} d(x_n, y_n) = 0$ and $\lim_{n\to\infty} d(x_n, Su_n) = 0$.

Because $\{x_n\}$ is bounded, there exists a constant L such that $d(x_n, \omega) \leq L$ for all $n \geq 0$. By (17) and (20), Assumption 3.1 and geodesic convexity of Riemannian distance, we have

$$\begin{aligned} d(x_n, \omega) &\leq (1-\beta_{n-1})d(x_{n-1}, \omega) + \beta_{n-1}d(y_{n-1}, \omega) \\ &\leq (1-\beta_{n-1})d(x_{n-1}, \omega) + \beta_{n-1}((1-\alpha_{n-1})d(x_{n-1}, \omega) + \alpha_{n-1}d(u_{n-1}, \omega)) \\ &\leq (1-\beta_{n-1})d(x_{n-1}, \omega) + \beta_{n-1}((1-\alpha_{n-1})d(x_{n-1}, \omega) + \alpha_{n-1}(1-\rho)d(x_{n-1}, \omega)) \\ &= (1-\beta_{n-1})d(x_{n-1}, \omega) + \beta_{n-1}(1-\rho\alpha_{n-1})d(x_{n-1}, \omega) \\ &= (1-\rho\alpha_{n-1}\beta_{n-1})d(x_{n-1}, \omega), \end{aligned}$$

where ρ is the same as in Assumption 3.1. Let $0 \leq m \leq n$, then

$$d(x_n, \omega) \leq L \prod_{j=m}^{n-1} (1-\rho\alpha_j\beta_j). \tag{24}$$

On the other hand, by (19) and (20), we have

$$\begin{aligned} d(x_n, y_n) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, \omega) + d(y_n, \omega) \\ &\leq d(x_n, x_{n+1}) + 2d(x_n, \omega). \end{aligned}$$

Therefore, by using (24), the above inequality becomes

$$d(x_n, y_n) \leq d(x_n, x_{n+1}) + 2L \prod_{j=m}^{n-1} (1-\rho\alpha_j\beta_j).$$

By taking limit as $n \to \infty$, we get

$$\lim_{n\to\infty} d(x_n, y_n) \leq \lim_{n\to\infty} d(x_n, x_{n+1}) + 2L \lim_{n\to\infty} \prod_{j=m}^{n-1} (1-\rho\alpha_j\beta_j). \tag{25}$$

By condition (iii), we have

$$\lim_{n\to\infty} \prod_{j=m}^{n-1} (1-\rho\alpha_j\beta_j) = 0.$$

This together with (23) and (25) implies that

$$\lim_{n\to\infty} d(x_n, y_n) = 0. \tag{26}$$

From (17) and (20), we have

$$\begin{aligned} d(x_n, Su_n) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, \omega) + d(Su_n, \omega) \\ &\leq d(x_n, x_{n+1}) + d(x_{n+1}, \omega) + d(u_n, \omega) \\ &\leq d(x_n, x_{n+1}) + 2d(x_n, \omega). \end{aligned}$$

Similarly, we can prove that

$$\lim_{n\to\infty} d(x_n, Su_n) = 0, \ \textit{and} \ \lim_{n\to\infty} d(x_n, u_n) = 0. \tag{27}$$

Since

$$\begin{aligned} d(S_r^m(x_n), x_n) &\leq d(S_r^m(x_n), S_r^m(y_n)) + d(S_r^m(y_n), x_n) \\ &\leq d(x_n, y_n) + d(z_n, x_n). \end{aligned}$$

Letting $n \to \infty$ and using (22) and (26), we have

$$\lim_{n\to\infty} d(S_r^m(x_n), x_n) = 0. \tag{28}$$

(V) Next we prove that the cluster point ν of the sequence $\{x_n\}$ belongs to Ω.

We have already proved in step 1 that the sequence $\{x_n\}$ is bounded. Therefore, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges to a cluster point ν of $\{x_n\}$. Thus, (27) implies that $u_{n_j} \to \nu$ as $j \to \infty$. By nonexpansiveness of S, we have

$$\begin{aligned} d(\nu, S(\nu)) &\le d(x_{n_j}, \nu) + d(x_{n_j}, S(u_{n_j})) + d(S(u_{n_j}), S(\nu)) \\ &\le d(x_{n_j}, \nu) + d(x_{n_j}, S(u_{n_j})) + d(u_{n_j}, \nu). \end{aligned}$$

By (27) and taking limit as $j \to \infty$, we have

$$d(\nu, S(\nu)) = 0,$$

which means that $\nu \in$Fix(S).

Next, we prove that $\nu \in \bigcap_{i=1}^m EP(F_i)$.

By (28), we have $lim_{j\to\infty} d(S_r^m(x_{n_j}), x_{n_j}) = 0$. Since S_r^m is a non-expansive mapping, it is demi-closed at zero, so $\nu \in Fix(S_r^m)$. In order to prove that $\nu \in \bigcap_{i=1}^m EF(F_i)$ it should be proved that $Fix(S_r^m) = \bigcap_{i=1}^m Fix(T_r^{F_i})$.

It is obvious that $\bigcap_{i=1}^m Fix(T_r^{F_i}) \subseteq Fix(S_r^m)$. Next we prove that

$$Fix(S_r^m) \subseteq \bigcap_{i=1}^m Fix(T_r^{F_i}).$$

Let $q \in Fix(S_r^m)$ and $p \in \bigcap_{i=1}^m Fix(T_r^{F_i})$, we have

$$\begin{aligned} d(q,p) &= d(S_r^m q, p) = d(T_r^{F_m} S_r^{m-1} q, p) \le d(S_r^{m-1} q, p) \\ &\le d(S_r^{m-2} q, p) \le \cdots \le d(S_r^1 q, p) = d(T_r^{F_1} q, p) \le d(q,p). \end{aligned}$$

This implies that

$$\begin{aligned} d(q,p) &= d(S_r^m q, p) = d(S_r^{m-1} q, p) = d(S_r^{m-2} q, p) = \cdots = d(S_r^1 q, p) \\ &= d(T_r^{F_1} q, p). \end{aligned} \tag{29}$$

It follows from (29) and Lemma 5 that for each $i = 1, 2, \cdots, m$, we have

$$d^2(S_r^i q, p) + d^2(S_r^i q, S_r^{i-1} q) \le d^2(S_r^{i-1} q, p) = d^2(q,p).$$

Since $d(S_r^i q, p) = d(q,p)$, this implies that for each $i = 1, 2, \cdots, m$

$$d(S_r^i q, S_r^{i-1} q) = 0, \text{ i.e., } S_r^{i-1} q \in Fix(T_r^{F_i}). \tag{30}$$

Taking $i = 1$ in (2), we have $q = T_r^{F_1}(q)$. Taking $i = 2$ in (3.17), we have that

$$q = T_r^{F_1}(q) = T_r^{F_2}(q).$$

Taking $i = 1, 2, \cdots, m$ in (2) we can prove that

$$q = T_r^{F_1}(q) = T_r^{F_2}(q) = \cdots = T_r^{F_{m-1}}(q) = T_r^{F_m}(q), \text{ i.e., } q \in \bigcap_{i=1}^m Fix(T_r^{F_i}).$$

Finally, we prove that $\nu \in (A+B)^{-1}(0)$.

Because $\hat{\lambda} \le \lambda_n \le \tilde{\lambda}$ then we can choose $\lambda > 0$ such that the subsequence $\{\lambda_{n_j}\}$ of $\{\lambda_n\}$ converges to λ. Because $u_n = J^B_{\lambda_n}(exp_{x_n}(-\lambda_n A(x_n)))$. Then, by (27) and Lemma 6, we have

$$\begin{aligned} 0 &= \lim_{j\to\infty} d(x_{n_j}, u_{n_j}) \\ &= \lim_{j\to\infty} d(x_{n_j}, J^B_{\lambda_{n_j}}(exp_{x_{n_j}}(-\lambda_{n_j} A(x_{n_j})))) \\ &= d(\nu, J^B_{\lambda}(exp_{\nu}(-\lambda A(\nu)))). \end{aligned}$$

From Proposition 4, we have $\nu \in (A+B)^{-1}(0)$. Hence, $\nu \in \Omega$. This completes the proof. □

In Theorem 4, take $F_i \equiv 0$ $(i = 1, 2, \cdots, m)$, then the following Corollary can be obtained from Theorem 4 immediately.

Corollary 1. *Let K, M, A, B and S be the same as in Theorem 4. Let $\{x_n\}, \{u_n\}$ and $\{y_n\}$ be sequences generated by $x_0 \in K$ and*

$$\begin{cases} u_n = J^B_{\lambda_n}(exp_{x_n}(-\lambda_n A(x_n))), \\ y_n = exp_{x_n}\alpha_n exp^{-1}_{x_n} S(u_n), \\ x_{n+1} = exp_{x_n}\beta_n exp^{-1}_{x_n} y_n. \ \forall n \ge 0, \end{cases} \tag{31}$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\lambda_n\}$ are positive sequences satisfying the conditions (i), (ii) and (iii) in Theorem 4. If $\Omega_1 = (A+B)^{-1}(0) \bigcap F(S) \ne \emptyset$, then $\{x_n\}$ converges strongly to some point $\nu \in \Omega_1$.

In Theorem 4 take $F_i \equiv 0 (i = 1, 2, \cdots, m)$ and $S = I$ (identity mapping on K), then the following corollary can be obtained from Theorem 4.

Corollary 2. *Let K, M, A and B be the same as in Theorem 4. Let $\{x_n\}, \{u_n\}$ and $\{y_n\}$ be the sequences generated by $x_0 \in K$ and*

$$\begin{cases} u_n = J^B_{\lambda_n}(exp_{x_n}(-\lambda_n A(x_n))), \\ y_n = exp_{x_n}\alpha_n exp^{-1}_{x_n} u_n, \\ x_{n+1} = exp_{x_n}\beta_n exp^{-1}_{x_n} y_n. \ \forall n \ge 0, \end{cases} \tag{32}$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\lambda_n\}$ are positive sequences satisfying the conditions (i), (ii) and (iii) in Theorem 4. If $\Omega_2 = (A+B)^{-1}(0) \ne \emptyset$, then $\{x_n\}$ converges strongly to some point $\nu \in \Omega_2$.

In Theorem 4, take $A \equiv 0$ and $S = I$ (the identity mapping on K), then the following result can be obtained from Theorem 4 immediately.

Corollary 3. *Let K, M, B, $\{F_i\}^m_{i=1}$ and $\{S^l_{r_n}\}^m_{l=1}$ be the same as in Theorem 4. Let $\{x_n\}, \{u_n\}, \{y_n\}$ and $\{z_n\}$ be the sequences generated by $x_0 \in K$:*

$$\begin{cases} u_n = J^B_{\lambda_n}(x_n), \\ y_n = exp_{x_n}\alpha_n exp^{-1}_{x_n} u_n, \\ z_n = S^m_r(y_n), \\ x_{n+1} = exp_{x_n}\beta_n exp^{-1}_{x_n} z_n. \ \forall n \ge 0, \end{cases} \tag{33}$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\lambda_n\}$ are positive sequences satisfying the the conditions (i), (ii) and (iii) in Theorem 4. If $\Omega_3 = \bigcap^m_{i=1} EP(F_i) \bigcap B^{-1}(0) \ne \emptyset$, then $\{x_n\}$ converges strongly to some point $\nu \in \Omega_3$.

In Theorem 4 take $A \equiv 0, B \equiv 0$ and $S = I$ (identity mapping on K), then the following result can be obtained from Theorem 4 immediately.

Corollary 4. *Let K, M, $\{F_i\}_{i=1}^m$ and $\{S_{r_n}^l\}_{l=1}^m$ be the same as in Theorem 4. Let $\{x_n\}$ and $\{z_n\}$ be sequences generated by $x_0 \in K$ and*

$$\begin{cases} z_n = S_r^m(x_n), \\ x_{n+1} = exp_{x_n}\beta_n exp_{x_n}^{-1} z_n. \ \forall n \geq 0, \end{cases} \tag{34}$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\lambda_n\}$ are positive sequences satisfying the conditions (i), (ii) and (iii) in Theorem 4. If $\Omega_4 = \bigcap_{i=1}^m EP(F_i) \neq \emptyset$, then $\{x_n\}$ converges to some point $v \in \Omega_4$.

Author Contributions: Conceptualization, J.Z., J.T., M.L., and L.Z.; Methodology, J.Z., J.T., M.L., and L.Z.; Software, J.Z. and J.T.; Validation, J.T., M.L., and L.Z.; Formal analysis, J.Z.; Investigation, M.L.; Resources, J.Z., S.-s.C., J.T.; Data curation, J.T., M.L., and L.Z.; Writing, original draft preparation, J.Z., S.-s.C., J.T., M.L., and L.Z.; Writing, review and editing, J.Z., S.-s.C. and J.T.; Visualization, L.Z.; Supervision, M.L.; Project administration, J.Z., J.T., M.L., and L.Z.; Funding acquisition, J.Z., J.T., M.L., and L.Z. All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported by the Scientic Research Fund of Science and Technology Department of Sichuan Provincial (2018JY0340,2018JY0334) and the Scientic Research Fund of SiChuan Provincial Education De-partment (16ZA0331). This work was also supported by the Scientic Research Fund of Yibin University (2021YY03) and the Center for General Education, China Medical University, Taichung,Taiwan.

Data Availability Statement: Not applicable.

Acknowledgments: The authors would like to express their thanks to the Editor and the reviewers for their helpful comments.

Conflicts of Interest: The authors declare no conflict of interest.

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