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Homeomorphic Arrangements of Smooth Manifolds

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Abstract: Symmetry between mathematical constructions is a very desired phenomena in mathematics in general, and in algebraic geometry in particular. For line arrangements, symmetry between topological characterizations and the combinatorics of the arrangement has often been studied, and the first counterexample where symmetry breaks is in dimension 13. In the first part of this paper, we shall prove that two arrangements of smooth compact manifolds of any dimension that are connected through smooth functions are homeomorphic. In the second part, we prove this in the affine case in dimension 4.

Keywords: smooth manifold; lines; homeomorphism

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1. Introduction

An arrangement is a finite collection of affine subspaces (of possibly varying dimensions) in $\mathbb{R}^l$ or $\mathbb{C}^l$, or a collection of linear subspaces of a projective space $\mathbb{CP}^{l-1}$ or $\mathbb{RP}^{l-1}$. The topology of the complement of the union of the planes is of considerable interest. In 1989, Randell [1] proved a deep theorem, which shows that two arrangements have the same topology if they can be transferred from one to the other using a smooth one parameter family of arrangements. This resulted in the invention of a new invariant–the moduli space. Randell's study was very fruitful and resulted in many important theorems concerning symmetry rope. For instance, in [2] it was implemented for lines which in this case are a linear subspace of $\mathbb{C}^2$, for example, the solution in $\mathbb{C}^2$ to the equation $y = (5+i)x + 2 + 4i$. In [3,4], it was proven that the combinatorial structure determines the fundamental group of the complement for a six line arrangement. Using Van Kampen theorems [5] and the Moishezon-Teicher algorithm [6], it was extended to seven and eight lines [7,8]. Later, it was generalized to nine lines in [9], ten lines in [10] and eleven lines in [11].

In this paper, we are going to improve Randell's theorem and we move to diffeomorphisms of smooth manifolds in general, a contemporary topic (see, e.g., [12,13]). Whereas Randell talks about changing hyperplane arrangements through smooth families of hyperplanes, we are going to show that we can also transfer from one arrangement of smooth manifolds to another using smooth families of hyperplane arrangements. As a consequence, we show that we can transform arrangements of lines to other arrangements of lines through symmetry of arrangements to polynomials, families of polynomials of any degree. Since lines in the complex planes are homeomorphic to two dimensional sets in $\mathbb{R}^4$, our theorems will use that idea for another improvement of the theorem.

2. Definitions and Notations

The following theorems and definitions are well known.

Theorem 1 (Topological invariance of dimension [14])**.** *A non-empty n-dimensional topological manifold cannot be homeomorphic to an m-dimensional manifold unless $m = n$.*

Definition 1 ([14]). *For any smooth manifold M, we define an open submanifold of M to be any open subset with the subspace topology and with the smooth charts obtained by restricting those of M.*

Theorem 2 (Open sub-manifold [14]). *Suppose M is a smooth manifold. The embedded submanifolds of codimension 0 in M are exactly the open submanifolds.*

Proposition 1 ([14]). *Suppose M and N are smooth manifolds with or without boundary, and $F : M \to N$ is an injective smooth immersion. If any of the following holds, then F is a smooth embedding:*

(a) *F is an open or closed map;*
(b) *F is a proper map;*
(c) *M is compact;*
(d) *M has empty boundary and $\dim M = \dim N$.*

Theorem 3 (Proper continuous maps are closed [14]). *Suppose X is a topological space and Y is a locally compact Hausdorff space. Then every proper continuous map $F : X \to Y$ is closed.*

Definition 2. *A stratification of a manifold is a partitioning of the manifold into a finite collection of submanifolds $\{U\}$ (called the strata) so that the following frontier condition is satisfied: Whenever U and V are strata with $V \cap cl(U) \neq \emptyset$, then $V \subset cl(U)$.*

Definition 3 ([14]). *A stratification is called a Whitney stratification if it satisfies Whitney's condition (b): For all strata U, V, with $V \cap cl(U) \neq \emptyset$, and for all $x \in V$, whenever x_i and y_j are sequences in V and U, respectively, with $x_i \neq y_i$ so that x_i converges to x and y_i converges to x, so that the secants $\overline{x_i y_i}$ converge to $l \in \mathbb{RP}^{n-1}$ and so that $T_{y_i}U$ converges to τ in the Grassmannian of dimension U planes in $\mathbb{R}^n$, then $l \subset \tau$.*

Theorem 4 ([1]). *Any stratification in which the closure of every stratum is a smooth submanifold is a Whitney stratification.*

Theorem 5 (Thom's first isotopy theorem [15]). *Let $f : M \to \mathbb{R}$ be a proper, smooth map which is a submersion on each stratum of a Whitney stratification of M. Then there is a stratum-preserving homeomorphism $h : M \to \mathbb{R} \times (f^{-1}(0) \cap M)$ which is smooth on each stratum and commutes with the projection to R. In particular, the fibers of f are homeomorphic by a stratum-preserving homeomorphism.*

Stereographic Projection

The following definitions and claims are well known: the function

$$\varphi_n : S^n \setminus (0, 0, \ldots, 1) \to \mathbb{R}^n$$

defined by

$$\varphi_n(x_1, \ldots, x_{n+1}) = \left(\frac{x_1}{1 - x_{n+1}}, \ldots, \frac{x_n}{1 - x_{n+1}} \right)$$

is a homeomorphism and so is from

$$\psi_n : iS^n \setminus (0, 0, \ldots, -1)$$

to $\mathbb{R}^n$ defined by

$$\psi_n(x_1, \ldots, x_{n+1}) = \left(\frac{x_1}{1 + x_{n+1}}, \ldots, \frac{x_n}{1 + x_{n+1}} \right).$$

The inverse functions are:

$$\varphi_n^{-1}(y_1,\ldots,y_n) = \left(\frac{2y_1}{\sum\limits_{i=1}^{n} y_i^2+1},\ldots,\frac{2y_n}{\sum\limits_{i=1}^{n} y_i^2+1}, 1-\frac{2}{\sum\limits_{i=1}^{n} y_i^2+1}\right)$$

$$\psi_n^{-1}(y_1,\ldots,y_n) = \left(\frac{2y_1}{\sum\limits_{i=1}^{n} y_i^2+1},\ldots,\frac{2y_n}{\sum\limits_{i=1}^{n} y_i^2+1}, -1+\frac{2}{\sum\limits_{i=1}^{n} y_i^2+1}\right).$$

The composition defined from $\mathbb{R}^n$ minus the origin to itself is:

$$\psi_n \circ \varphi_n^{-1}(y_1,\ldots,y_n) = \varphi \circ \psi^{-1}(y_1,\ldots,y_n) = \left(\frac{y_1}{\sum\limits_{i=1}^{n} y_i^2},\ldots,\frac{y_n}{\sum\limits_{i=1}^{n} y_i^2}\right).$$

In this paper we denote $p := (0,0,1)$,$\varphi := \varphi_2$,$\psi := \psi_2$.

3. Topological and Geometric Aspects of Manifolds

With the previous theorems in mind we prove the following.

Lemma 1. *Let A, B and $C_1,\ldots,C_n$ be manifolds such that B is a closed submanifold of A and $C_1,\ldots,C_n$ are closed submanifolds of B. Assume that for all $1 \le i \le n$, $\dim(C_i) < \dim(B)$. Then $\bigcup\limits_{i=1}^{n} C_i$ is nowhere dense in the subspace topology of B and in addition $Cl(B \setminus \bigcup\limits_{i=1}^{n} C_i) = B$.*

Proof. First we prove that for any $1 \le i \le n$, C_i is nowhere dense. Indeed, since C_i is closed it is sufficient to show that it has an empty interior. Assume to the contrary that S is an open set of B contained in C_i, then from the definition of subspace topology it is an open set also in C_i by Theorem 2. We get that S is a manifold of dimension equal to the dimension of B and also a manifold with a dimension equal to C_i. The contradiction to Theorem 1 proves the statement. Since a finite union is nowhere dense so is $\bigcup\limits_{i=1}^{n} C_i$ and since $1 \le i \le n$ C_i is closed then $Cl(B \setminus \bigcup\limits_{i=1}^{n} C_i) = B$. □

Proposition 2. *Let A, B, C, D be Hausdorff spaces where A is compact. Let $f : A \times B \to C$ be a continuous function and $g : B \to D$ be continuous and proper. Let $h : A \times B \to C \times D$ be defined by sending (x, y) to $(f(x, y), g(y))$ then h is continuous and proper.*

Proof. Let p be the projection of $C \times D$ to D and let S be some set. Then, an element (a, b) is in $h^{-1}(S)$ if there exist $(c, d) \in C \times D$ such that $h((a, b)) = (c, d)$. By definition, $h((a, b) = (f(a, b), g(b))$ and $h((a, b)) = (c, d)$ if and only if $(f(a, b), g(b)) = (c, d))$. This implies that $f(a, b) = c$ and $g(b) = d$ so $h^{-1}(S) = f^{-1}(S) \cap g^{-1}(p(S)$. First we prove that h is indeed continuous. Let S_1 be an open set since p is an open function and f, g are continuous. Then $f^{-1}(S) \cap g^{-1}(p(S)$ is an intersection of the two open sets and, therefore, open. Next we will prove that h is proper: let S be a compact set. In particular it is closed so $h^{-1}(S)$ is closed. On the other hand, $h^{-1}(S) = f^{-1}(S) \cap g^{-1}(p(S) \subset A \times g^{-1}p(S)$ since p is continuous, $P(S)$ is compact and since h is proper $g^{-1}p(S)$ is also compact. Since A and $g^{-1}p(S)$ are compact so is $A \times g^{-1}p(S)$. To conclude, we get that $h^{-1}(S)$ is closed and a subset of compact subset and, therefore, compact. □

Proposition 3. *Let Q and M be smooth manifolds and $f : Q \times \mathbb{R} \to M$ a smooth function. For any $t \in \mathbb{R}$, let $f^t : Q \to M$ be the function sending q to $f(q,t)$. Let $F : Q \times \mathbb{R} \to M \times \mathbb{R}$ be the function that sends (g,t) to $(f(g,t),t)$. Then*

1. *if any $t \in \mathbb{R}$ f^t is injective then F is injective;*
2. *if f is smooth then F is smooth;*
3. *if f is smooth and for any $t \in \mathbb{R}$ f^t is an immersion then F is an immersion;*
4. *if f is smooth and $\dim(Q) = 0$ then F is an immersion;*
5. *if Q is compact,f is smooth, for any $t \in \mathbb{R}$ f^t is injective and if $\dim(Q) > 0$ it is also an immersion then F is an embedding with a closed image;*
6. *if F is an embedding and π is the projection of $M \times \mathbb{R}$ to the second factor then $\pi|_{Im(F)}$ is a submersion.*

Proof.

1. Let $(q_1,t_1),(q_2,t2) \in Q \times \mathbb{R}$ such that $F((q_1,t_1) = F(q_2,t_2)$. Then by the definition of F we get that $(f(q_1,t_1),t_1) = (f(q_2,t_2),t_2)$. Therefore, $t_1 = t_2$ and we denote it as t, so we get that $f(q_1,t) = f(q_2,t)$. By the definition of f^t this implies that $f^t(q_1) = f^t(q_2)$ and since f^t is injective, then $q_1 = q_2$.
2. Since f is a smooth function from $Q \times \mathbb{R}$ to M, there exist for every $p \in Q \times \mathbb{R}$ smooth charts (U,φ) containing p and (V,ψ) containing $f(p)$ such that $f(U) \subset V$ and $\psi \circ f \circ \varphi^{-1}$ is a smooth function from U to V. Since the domain is $Q \times \mathbb{R}$ then $(U,\varphi) = (U_1 \times \mathbb{R}, \varphi_1 \times Id_{\mathbb{R}})$. Since (V,ψ) is a chart of M then $(V \times \mathbb{R}, \psi \times Id_{\mathbb{R}})$ is a chart for $M \times \mathbb{R}$. If we take the point $q \in U$ it is equal to (q_1,t) where $q_1 \in U_1$ and $t \in \mathbb{R}$. We look at the charts $(U_1 \times \mathbb{R}, \varphi \times Id_{\mathbb{R}})$ and $(V \times Id_{\mathbb{R}}, \psi \times Id_{\mathbb{R}})$. Then

   $$(\psi \times Id_{\mathbb{R}}) \circ F \circ (\varphi \times Id_{\mathbb{R}})^{-1}(q_1,t) = (\psi \times Id_{\mathbb{R}}) \circ (f \times id_{\mathbb{R}}) \circ (\varphi^{-1}(q_1),t)$$

   which is equal to

   $$(\psi \times Id_{\mathbb{R}}) \circ (f \circ \varphi^{-1}(q_1,t),t) = (\psi \circ f \circ \varphi^{-1}(q_1,t),t).$$

   Let us assume that M is of order k. Then $\psi \circ f \circ \varphi^{-1}$ have k component functions $f_1,\ldots,f_k$ which are all smooth. So we can write F as $k+1$ components $f_1,\ldots,f_k,p$ where p is the projection from $Q \times \mathbb{R}$ to $\mathbb{R}$. We can see that all the components are smooth and, therefore, F is smooth.
3. Let p be a point as in the previous paragraph and that $\psi \times Id_{\mathbb{R}} \circ F \circ \varphi^{-1}$ has k + 1 components $f_1,\ldots,f_k,t$. Since $\varphi = \varphi_1 \times Id_{\mathbb{R}}$ then the coordinates of the domain of $\psi \times Id_{\mathbb{R}} \circ F \circ \varphi^{-1}$ are $x_1,\ldots,x_m,t$ when we assume that the order of Q is m. Let $1 \leq i \leq k$ and $1 \leq j \leq m$, then the partial derivative of f_i in the coordinate x_j is equal to

   $$\lim_{h\to 0} \frac{f_i(x_1,\ldots,x_{j-1},x_j+h,x_{j+1},\ldots,x_k,t) - f_i(x_1,\ldots x_k,t))}{h}$$

   which is equal to

   $$\lim_{h\to 0} \frac{f_i^t(x_1,\ldots,x_{j-1},x_j+h,x_{j+1},\ldots,x_k) - f_i^t((x_1,\ldots x_k,t))}{h},$$

   where f_i^t is the i-th component of $\psi \circ f^t \circ \varphi_1^{-1}$. We can see that the partial derivative of f_i in the coordinate x_j is equal to the partial derivative of f_i^t in the coordinate

x_j. The x_j partial derivative of t is 0 and the t-th derivative of t is 1. To conclude, the Jacobian of F in the point p is

$$\left[\begin{array}{ccc|c} & & & 0 \\ & A & & \vdots \\ & & & 0 \\ * & \cdots & * & 1 \end{array}\right]$$

where A is the Jacobian of f^t. Since f^t is an immersion so is F.

4. Since $\dim(Q) = 0$, it is a union of points with the discrete topology and every point has a homeomorphism to $\mathbb{R}^0$ which is precisely one point by a function that sends one point to the other. So if we take one point $a \times \mathbb{R}$ and the homeomorphism to $\mathbb{R}$ as the natural one, then we have that one function $f(t) = z$ where z is the variable of $\mathbb{R}$ in $M \times \mathbb{R}$ which is obviously an immersion.
5. By paragraphs (1)–(4) F is injective immersion. Since the identity is obviously proper, then by Lemma 3 F is proper. So by Proposition 1 it is a smooth embedding. By Theorem 3 it is closed; in particular, it has a closed image.
6. It is sufficient to prove that $F \circ \pi$ is a submersion but $F \circ \pi$ is a projection of the second factor of $Q \times \mathbb{R}$ which is known to be a submersion.

□

4. Main Theorem

In order to prove the main Theorem 4, we need the following lemma.

Lemma 2. *Let M, S be smooth manifolds. H is a finite set of closed smooth submanifolds of M. $s : M \to S$ is a submersion such that:*

1. *for every $h \in H$, $\dim(h) < \dim(M)$;*
2. *for every $h_1, h_2 \in H$ such that $h_1 \subsetneq h_2$, $\dim(h_1) < \dim(h_2)$;*
3. *for every $h_1, h_2 \in H$, there exist $H_1 \subset H$ such that $h_1 \cap h_2 = \cup H_1$. For all $h \in H$, $s|_h$ is a submersion.*

Let us denote

1. *for $h \in H$ $\overline{h} = \{h \setminus \bigcup_{h_1 \in H \wedge h_1 \subsetneq h} h_1 \mid h \in H\}$;*
2. *$\overline{H} := \bigcup_{h \in H} \{\overline{h}\}$;*
3. *$\overline{M} = M \setminus (\cup \overline{H})$;*
4. *$W := \overline{H} \cap \{\overline{M}\}$.*

Then $\overline{H} \cup M \setminus \cap \overline{H}$ is Whitney's stratification, and for all $w \in W$, $s|_w$ is a submersion.

Proof. Since every element in $\overline{H}$ is a submanifold minus a finite union of closed sets, it is an open submanifold of h and by Theorem 2 every $\overline{h} \in \overline{H}$ is a submanifold. It is also closed in M and, therefore, $\overline{M}$ is also an open submanifold. It is obvious that $\cup W = M$. Next we will prove that every two different sets in W are disjoint. It is sufficient to show for elements in $\overline{H}$. Let $\overline{h_1}, \overline{h_2} \in \overline{H}$ where $\overline{h_1} \neq \overline{h_2}$. Assume to the contrary that $x \in \overline{h_1} \cap \overline{h_2}$. From the definition of $\overline{H}$ there exist $h_1, h_2 \in H$ such that $\overline{h_1} = h_1 \setminus \bigcup_{h \in H \wedge h \subsetneq h_1} h$ and $\overline{h_2} = h_2 \setminus \bigcup_{h \in H \wedge h \subsetneq h_2} h$. We can see that $x \in h_1$ and $x \in h_2$ and, therefore, $x \in h_1 \cap h_2$. By our assumption there exist $H_1 \subset H$ such that $h_1 \cap h_2 = \cup H_1$. Therefore, $x \in \cup H_1$ which implies that there exist $h_3 \in H_1$ such that $x \in h_3$ and $h_3 \subset h_1 \cap h_2$. If h_1 were equal to h_2 then w_1 would be equal to w_2. Therefore, there exist $h_i (i = 1, 2)$ such that $h_1 \cap h_2 \subsetneq h_i$, hence $h_3 \subsetneq h_i$ and, therefore, $x \notin w_i$. This contradiction proves our statement.

Until now we proved that $\overline{W}$ is a partitioning of the manifold M into a finite collection of submanifolds. Let $w \in W$. Then it is equal to $h_1 \setminus \bigcup_{h \in H \wedge h \subsetneq h_1} h$ since by our assumption every h that is strictly contained in h_1 has a smaller dimension by Lemma 1 $Cl(w) = h_1$. Similarly, since for every $h \in H$ $\dim(h) < \dim(M)$ and $\overline{M} = M \setminus \cup H$. Then by Lemma 1 $Cl(\overline{M}) = M$. Assume now that we have two elements w_1, w_2 such that $Cl(w_1) \cap w_2 \neq \emptyset$. Then, by our last statement, $Cl(w_1) = h_1$ for some $h_1 \in H$ and $w_2 = h_2 \setminus \bigcup_{h \in H \wedge h \subsetneq h_2} h$ for $h_2 \in H$. If $h_1 \cap h_2 \neq h_2$, $h_1 \cap h_2 \subsetneq h_2$, and, therefore, $h_1 \cap h_2 \cap w_2 = \emptyset$. Since $w_2 \subseteq h_2$ we get that $h_2 \cap w_2 = w_2$ and, therefore, $h_1 \cap w_2 = \emptyset$. This contradiction forces us to say that $h_1 \cap h_2 = h_2$ which implies that $h_2 \subseteq h_1$ and, therefore, $w_2 \subseteq h_1$. Since $Cl(\overline{M}) = M$, it is obvious that every element in W is a subset of $cl(\overline{M})$. To conclude, we get that $\overline{W}$ is a stratification. Since for every $w \in \overline{W}$ there is $h \in H$ such that $Cl(w) = h$ and every $h \in H$ is a smooth submanifold and the closure of $\overline{M} = M$, which is also a smooth manifold, by Theorem 4 $\overline{W}$ is also a Whitney stratification. Since submersion is a local property and $s|h$ and are submersions, this is also true for their open subspaces, namely the elements in $\overline{W}$. □

Proposition 4. *Let M be a compact smooth manifold, J a finite set. For every $j \in J$, Q_j is a compact smooth manifold and $g_j : Q_j \times \mathbb{R} \to M$ is a smooth function. For every $j \in J$ and $t \in \mathbb{R}$, let $g_j^t : Q_j \to M$ be the function that sends q to $g_j(q,t)$. $f : J \times J \to P(J)$ is a function such that:*

1. *for any $(i,j) \in J \times J$ and for all $t \in \mathbb{R}$ $Im(g_i^t) \cap Im(g_j^t) = \bigcup_{k \in f((i,j))} Im(g_k^t)$;*
2. *for all $j \in J$ and $t \in \mathbb{R}$ g_j^t is injective;*
3. *for all $j \in J$ and $t \in \mathbb{R}$ if $\dim(Q_j) > 0$ g_t^j is an immersion;*
4. *for all $j \in J$ $\dim(Q_j) < \dim(M)$;*
5. *for all $j,k \in J$ if there exist $t \in \mathbb{R}$ such that $Im(g_t^k) \subsetneq Im(g_j^t)$ then $\dim(Q_k) < \dim(Q_j)$.*

   Then $M \setminus \bigcup_{j \in J} Im(g_j^0)$ is homeomorphic to $M \setminus \bigcup_{j \in J} Im(g_j^1)$.

Proof. We define for any $j \in J$, $G_j : Q_j \times \mathbb{R} \to M \times \mathbb{R}$ by $G_j(x,t) := (g_j^t(x), t)$. Since g_j is smooth, Q_j is compact. For all $t \in \mathbb{R}$, g_j^t is injective and if $\dim(Q_j) > 0$ then it is also an immersion. Then, by Proposition 3 G_j is an embedding with a closed image, and for π equal to the projection of $M \times \mathbb{R}$ to the second factor $\pi|_{Im(G_j)}$ is a submersion.

Now we are going to show that for any $(i,j) \in J \times J$,

$$Im(G_i) \cap Im(G_j) = \bigcup_{k \in f((i,j))} Im(G_k)$$

Indeed, $(x,t) \in Im(G_i)$ if and only if $x \in g_t^t$. Therefore, $(x,t) \in Im(G_i) \cap Im(G_j)$ if and only if $x \in Im(g_i^t) \cap Im(g_j^t)$ which by (1) this happens if and only if $x \in \bigcup_{i \in f(k)} Im(G_i^t)$ and happens if and only if $(x,t) \in \bigcup_{k \in f((i,j))} Im(G_k)$. Let $i,j \in J$. If $Im(G_i) \subsetneq Im(G_j)$. We would like to show that $\dim(Im(G_i)) < \dim(Im(G_i))$. We know that (x,t) is in $Im(G_i)$ if and only if $x \in g_j^t$ and the same is true for j. So if $(x,t) \in Im(G_j) \setminus Im(G_i)$, then $x \in Im(g_i^t) \setminus Im(g_j^t)$ and for this specific t if $x \in g_i^t$ then $(x,t) \in Im(G_i)$ which implies that $(x,t) \in Im(G_i)$ and, therefore, $x \in g_t^t$. Combining these two facts, we get that $Im(g_i^t) \subsetneq Im(g_j^t)$ which implies by our condition that $\dim(Q_j) < \dim(Q_l)$ which imply that $\dim(Q_j \times \mathbb{R}) < \dim(Q_l \times \mathbb{R})$ which means that $\dim(Im(G_j)) < \dim(Im(G_l))$, as needed. Let $H := \langle Im(G_j) \mid j \in J \rangle$. Then H is a finite subset of closed smooth manifolds. For all h_1, h_2, there exist H_1 such that $h_1 \cap h_2 = \cup H$, for all $h_1, h_2 \in H$ such that $h_1 \subsetneq h_2$ $\dim(h_1) < \dim(h_2)$ and for all $h \in H$ $\dim(h) < \dim(N)$.

We denote $W = \{h \setminus \bigcup_{h_1 H \wedge h_1 \subsetneq h} \mid h \in H\}$, $N := (M \times \mathbb{R}) \setminus \cup H$ and $\overline{W} := W \cup \{N\}$. Then by Lemma 2 $\overline{W}$ is a Whitney stratification and for all $w \in \overline{W}$, $\pi|_w$ is a submersion, and since M is compact. π is proper on $M \times \mathbb{R}$. Therefore, by Theorem 5, there is a homeomorphism from $\pi^{-1}(0)$ to $\pi^{-1}(1)$ which is a stratum-preserving homeomorphism, so we can see that $\pi^{-1}(0) = M$. If $h \in \overline{H}$ then there exist $k \in P(I) \setminus \emptyset$ such that $h = H(k)$ $\pi^{-1}(1) \cap h = \bigcap_{i \in k} h_1^i \subset h_1^l$ where l is some element in k. In particular, for $i \in I$, if $h = H(\{i\})$, $\pi^{-1}(0) \cap h = h_1^i)$, so $\pi^{-1}(0) \cup \overline{H} = \bigcup_{i \in I} h_1^i$ which means that $\pi^{-1}(1) \cap \overline{N} = M \setminus \bigcup_{i \in I} h_1^i$. In the same way, $\pi^{-1}(0) \cap \overline{N} = M \setminus \bigcup_{i \in I} h_0^i$ and, therefore, they are homeomorphic. □

5. Adaptation of the Main Theorem to Curves

This section is a corollary of the main theorem in the case of curves. We are going to use the following definition for simplification.

Definition 4. *Let $F : X \times \mathbb{R} \to Y$ be a function. We define for every $t \in \mathbb{R}$, $F_t : X \to Y$ by $F_t(x) = F(x, t)$. Then we say that F satisfies condition 1 if it satisfies the following conditions:*

1. *F is smooth;*
2. *for every $t \in \mathbb{R}$, F_t is injective;*
3. *if $\dim(X) > 0$ then for every $t \in \mathbb{R}$, F_t is injective.*

Theorem 6. *Let $F : S^2 \times \mathbb{R} \to S^2 \times S^2$ be a function such that F on $S^2 \setminus p$ is $(\varphi \times Id) \circ g \circ (\varphi^{-1} \times \varphi)^{-1}$ where $g(x, y, t) = ((x, y), (H_1, H_2 + (x^2 + y^2)^k)$, such that H_1, H_2 are polynomials of the variables x, y over the smooth function with one variable, $k = max\{deg(H_1), deg(H_2))\}$ and (p, t) sends to $p \times p$. Then F satisfies condition 1.*

Proof. Let

$$H_1 := \sum_{i,j} h_{ij}^1 x^i y^j,$$

$$H_2 := \sum_{i,j} h_{ij}^2 x^i y^j$$

such that for all i, j and $k = 1, 2$, h_{ij}^k is a smooth function. First, we prove that F is a smooth function. Let $a \in S^2 \setminus p$, then we choose for the domain the chart $\varphi_1 \times Id$ and for the image $\varphi \times \varphi$, so we need to prove that

$$(\varphi \times Id)^{-1} \circ F \circ (\varphi \times \varphi)$$

is smooth. Indeed we get that

$$(\varphi \times Id)^{-1} \circ F \circ (\varphi \times \varphi) = (\varphi \times Id)^{-1} \circ (\varphi \times Id) \circ g \circ (\varphi \times \varphi)^{-1} \circ (\varphi \times \varphi) = g.$$

It is easy to see that "g" is smooth. For the points (p, t), we take the domain chart to be $\psi_2 \times Id$ and the image chart to be $\psi_2 \times \psi_2$. So we need to prove that $(\psi_2 \times Id)^{-1} \circ F \circ (\psi_2 \times \psi_2)$ is smooth. Indeed for a point (a, t) where $a \neq p$

$$(\psi_2 \times Id)^{-}1 \circ F \circ (\psi_2 \times \psi_2) = (\psi_2 \times Id)^{-}1 \circ (\varphi \times Id) \circ g \circ (\varphi^{-1} \times \varphi)^{-1} \circ (\psi_2 \times \psi_2)$$

$$= ((\psi_2)^{-}1 \circ (\varphi) \times Id) \circ g \circ (\varphi^{-1} \circ \psi_2 \times (\varphi^{-1} \circ \psi_2)).$$

Thus, ψ_2 is injective and $\psi_2(p_1) = (0, 0)$. So if $(x, t) \in \mathbb{R}^2 \setminus (0, 0) \times \mathbb{R}$ and we denote $z := x^2 + y^2$, we get that $((\psi_1)^{-}1 \circ (\varphi_1) \times Id)(x, y, t) = (\frac{x}{z}, \frac{y}{z}, t)$. If we apply g on the result we get:

$$\left(\frac{x}{z}, \frac{y}{z}\right), \left(\sum_{i,j} h_{ij}^1(t) \frac{x^i y^j}{z^{i+j}}, \sum_{i,j} h_{ij}^2 \frac{(x)^i (y)^j}{z^{i+j}} + \left(\left(\frac{(x)^2}{z^2} + \frac{(y)^2}{z^2}\right)^k\right)\right).$$

Now $\left(\frac{x}{z}\right)^2 + \left(\frac{y}{z}\right)^2 = \frac{x^2+y^2}{z^2} = \frac{z}{z^2} = \frac{1}{z}$, so we get

$$\left(\frac{x}{z}, \frac{y}{z}\right), \left(\sum_{i,j} h^1_{ij}(t)\frac{(x)^i(y)^j}{z^{i+j}}, \sum_{i,j} h^2_{ij}\frac{(x)^i(y)^j}{z^{i+j}} + \left(\left(\frac{1}{z}\right)^k\right)\right).$$

This is equal to

$$\left(\frac{x}{z}, \frac{y}{z}\right), \left(\sum_{i,j} h^1_{ij}(t)\frac{(x)^i(y)^j z^{k-i-j}}{z^k}, \sum_{i,j} h^2_{ij}\frac{x^i y^j z^{k-i-j}}{z^k} + \left(\frac{1}{z}\right)^k\right).$$

If we denote $A := \sum_{i,j} h^1_{ij} x^i y^j z^{k-i-j}$ and $B := \sum_{i,j} h^2_{ij} x^i y^j z^{k-i-j}$, we note that since k is always larger than $i + j$ in every summand, then when (x, y) approaches $(0, 0)$ then A and B approach 0. So we get $(\frac{x}{z}, \frac{y}{z}), (\frac{A}{z^k}, \frac{B+1}{z^k})$. If we apply $(\varphi^{-1} \circ \psi_2) \times (\varphi^{-1} \circ \psi_2)$, we get

$$\left(\frac{\frac{x}{z}}{(\frac{x}{z})^2 + (\frac{y}{z})^2}, \frac{\frac{y}{z}}{(\frac{x}{z})^2 + (\frac{y}{z})^2}\right), \left(\frac{\frac{A}{z^k}}{(\frac{A}{z^k})^2 + (\frac{B+1}{z^k})^2}, \frac{\frac{B+1}{z^k}}{(\frac{A}{z^k})^2 + (\frac{B+1}{z^k})^2}\right).$$

This is equal to $(x, y), \left(\frac{Az^k}{A^2+(B+1)^2}, \frac{(B+1)z^k}{A^2+(B+1)^2}\right)$.

We can see that when x, y converges to $(0, 0)$ the expression converges to $(0, 0), (0, 0)$. If we apply $(\psi_2 \times Id)^{-1} \circ F \circ (\psi_2 \times \psi_2)$ on $(0, 0, t)$, we get $((0, 0), (0, 0)$ because applying $(\psi_2 \times Id)^{-1}$ will give us (p, t) Then, applying F will give us $p \times p$ and, finally, applying $\psi_2 \times \psi_2$ will give us $((0, 0), (0, 0))$. So the function is continuous and a smooth function divided by a smooth function other than 0 in a small neighborhood is smooth. Let $t \in \mathbb{R}$. We will show that F_t is injective and is an immersion for all $t \in \mathbb{R}$. First we show it is injective for an element in $S^2 \setminus p$. Let $R := S^2 \setminus p \times t$ then $F = (\varphi \times Id)|_R \circ g \circ (\varphi \times \varphi)$. Since $(\varphi_1 \times Id)|_R, g, \varphi \times \varphi$ are injective, F_t is injective on $S^2 \setminus p$. Since the $p \times p$ is not in the image of $\varphi \times \varphi$ then $S^1 \setminus p_1$ is not going to $p \times p$ since p_1 is going to $p \times p$, then F_t is injective. Now we will show that the function is an immersion. For a point $a \in S^2 \setminus p$, we choose for the domain the chart α sending X to $(\varphi(X), t)$ and for the image $\varphi \times \varphi$ we get $\alpha \circ F \circ \varphi \times \varphi = \alpha \circ (\varphi \times Id) \circ g \circ (\varphi \times \varphi)^{-1} \circ (\varphi \times \varphi)$. This function sends (x, y) to $((x, y), (*, *))$. If we look at the minor of the Jacobian, we get

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{11} & a_{12} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

so these points have rank 2. For the point "p" we choose for the domain the chart β sending X to $(\psi(X), t)$. For the image $(\psi \times \psi)$ for similar reasons as before, the image is $((x, y), (*, *))$ which is also of rank 2. Therefore, on these points the function has rank 2, hence it is an immersion. □

The next theorem is an adaptation to lines in $\mathbb{C}^2$ which are topologically equivalent to sets in $\mathbb{R}^4$.

Theorem 7. *Let $g_i : \mathbb{R} \to P(\mathbb{R}^4)$ $(i = 1..n)$ be functions defined by $t \to \{(x, y, H_1, H_2) \mid x, y \in \mathbb{R}\}$ such that H_1, H_2 are polynomials of the variables x, y over the smooth function with one variable. Let $\kappa_i : \mathbb{R} \to \mathbb{R}^4$(for i=1..m) be a smooth function and let $f : [1..n] \times [1..n] \to [1..m]$ be a function such that the following conditions hold:*

1. *for all $t \in \mathbb{R}$ and $1 \le i, j \le n$, $g_i(t) \cap g_j(t) = \kappa_{f(i,j)}(t)$;*
2. *for all $i \ne j$ and $t \in \mathbb{R}$ $\kappa_i(t) \cap \kappa_j(t) = \emptyset$;*
3. *if there exist $t \in \mathbb{R}$ and i, r such that $\kappa_i(t) \in g_r(t)$ then there exists j such that $f(i, j) = r$.*

Then

$$R^4 \setminus \left(\bigcup_{i=1}^{n} Im(g_i(0)) \cup \bigcup_{i=1}^{m} Im(\kappa_i(0)) \right)$$

is homeomorphic to

$$R^4 \setminus \left(\bigcup_{i=1}^{n} Im(g_i(1)) \cup \bigcup_{i=1}^{m} Im(\kappa_i(0)) \right).$$

Proof. Let $\phi : \mathbb{R}^4 \to \mathbb{R}^4$ be the homeomorphism sending (x, y, z, w) to $(x, y, z, w + (x^2 + y^2)^k)$ where $k = max(deg(H_1), deg(H_2))$. Let $J_1 := \{a_i \mid 1 \leq i \leq n\}$, $J_2 := \{b_1, b_2\}$, $J_3 := \{c, d_i \mid 1 \leq i \leq m\}$ and $J := \bigcup_{i=1}^{3} J_i$. To a_i we attach the manifold S^2 and the function $G_i : S^2 \times \mathbb{R} \to S^2 \times S^2$ which we defined piecewise on $S^2 \setminus p$. It will be defined as $(\varphi \times Id) \circ (\varphi \times Id) \circ g_i \circ \phi \circ (\varphi \times \varphi)^{-1}$ and $p \times \mathbb{R}$ will be sent to $p \times p$. By Theorem 6 they satisfy condition (1). For b_i we attach the manifold S^2 and the functions $\beta_i : S^2 \times \mathbb{R} \to S^2 \times S^2$ are defined as follows: $\beta_1(a, t) = (a, p)$ and $\beta_2(a, t) = (p, a)$. It is easy to see that these functions satisfy condition (1). For c and d_i we attach a manifold with a single point which we denote by Pt. We define the following function: for c we define the constant function P which sends every element to $p \times p$ and for d_i we define the function $D_i : \mathbb{R} \to S^2 \times S^2$ which is defined as $\kappa_i \circ \varphi^{-1} \times \varphi^{-1}$. Next we define $\tilde{f} : J \times J \to P(J)$ as follows:

1. $\tilde{f}(a_i, a_j) = \{c, f(a_i, a_j)\}$;
2. $\tilde{f}(a_i, b_j) = c$;
3. $\tilde{f}(a_i, c) = c$;
4. $\tilde{f}(a_i, d_r) = \begin{cases} d_r & \exists j \text{ s.t.} f(i, j) = r \\ \emptyset & otherwise \end{cases}$
5. $\tilde{f}(b_1, b_2) = c$;
6. $\tilde{f}(b_i, c) = c$;
7. $\tilde{f}(b_i, d_j) = \emptyset$;
8. $\tilde{f}(c, d_i) = \emptyset$;
9. $\tilde{f}(d_i, d_j) = \emptyset$.

To complete the definition, we add that $\tilde{f}(X, X) := X$ and if $\tilde{f}(Y, X)$ is defined then $\tilde{f}(X, Y) = \tilde{f}(Y, X)$. Now we would like to prove that for any $(i, j) \in J \times J$ and for any $t \in \mathbb{R}$ if we denote the function attached to k as r_k we get that $r_i^t \cap r_j^t = \bigcup_{k \in f((i,j))} r_k^t$.

1. For every k and $t \in \mathbb{R}$ the image of G_k^t is equal to $G_k(S^2 \times t)$, the image of $p \times t$ is $p \times p$ and the image of $S^2 \setminus p \times t$ is equal to $(\varphi \times Id) \circ (\phi \times Id) g_k \circ (\varphi \times \varphi)^{-1}(S^2 \setminus p \times t)$. Since φ is surjective on $\mathbb{R}^2$, this is equal to $\phi \times ID(g_k \circ (\varphi \times \varphi)^{-1}(\mathbb{R}^2 \times t))$. So the intersection of the images of G_i^t and G_j^t is equal to a $p \times p$ union with

$$(\phi \times Id) \circ g_i \circ (\varphi \times \varphi)^{-1}(\mathbb{R}^2 \times t) \cap (\phi \times Id) \circ g_j \circ (\varphi \times \varphi)^{-1}(\mathbb{R}^2 \times t)$$

which is equal

$$(\varphi \times \varphi)^{-1}((\phi \times Id) \circ g_i(\mathbb{R}^2 \times t)) \cap (\varphi \times \varphi)^{-1}((\phi \times Id) \circ (g_j)(\mathbb{R}^2 \times t)).$$

Since $(\varphi \times \varphi)^{-1}$ and $(\phi \times Id)$ are injective, this is equal to $(\varphi \times \varphi)^{-1} \circ (\phi \times Id)((g_i(\mathbb{R}^2 \times t)) \cap (g_j(\mathbb{R}^2 \times t)))$. From our assumption, we find that $g_i(\mathbb{R}^2 \times t)) \cap (g_j(\mathbb{R}^2 \times t)) = \kappa_f(i, j)(t)$. So applying $(\varphi \times \varphi)^{-1} \circ (\phi \times Id)$ will give us $(\varphi \times \varphi)^{-1} \circ (\phi \times Id)(\kappa_{f(i,j)}(t))$ as needed.

2. As in paragraph (1) $Im(G_i^t) = p \times p \cup (\varphi \times \varphi)^{-1} \circ (\phi \times Id) \circ g_k \circ (\varphi \times \varphi)^{-1}(\mathbb{R}^2 \times t)$. Since $p \times p$ is also in b_j and the image of $\varphi \times \varphi$ is $S^2 \setminus p \times S^2 \setminus p$, the intersection is precisely $p \times p$.
3. As in paragraph (1) $Im(G_i^t) = p \times p \cup g_k \circ (\varphi \times \varphi)^{-1}(\mathbb{R}^2 \times t)$. So the intersection with $p \times p$ is $p \times p$.
4. First, if there exists j we already proved that $(G_i(t)) \cap (G_j(t)) = \kappa_r(t)$ and $\kappa_r(t)$ contain only one point, then $G_i(t) \cap \kappa_r(t) = \kappa_r(t)$. Next if j do not exist then as in paragraph (1) $Im(G_i^t) = p \times p \cup (\varphi \times \varphi)^{-1} \circ (\phi \times Id) \circ g_k \circ (\varphi \times \varphi)^{-1}(\mathbb{R}^2 \times t)$ and $Im(D_i^t) = \varphi^{-1} \times \varphi^{-1} \circ (\phi \times Id) \circ \kappa_i$. Since $(\varphi \times \varphi)^{-1}$ and $\phi \times Id$ are injective and do not contain $p \times p$ in their image the intersection is equal to $(\varphi \times \varphi)_{-1} \circ (\phi \times Id)(\mathbb{R}^2 \times t) \cap (\varphi \times \varphi)^{-1} \circ (\phi \times Id)(\kappa_r(t))$, so we calculate $(g_i(\mathbb{R}^2 \times t) \cap \kappa_r(t))$. By our assumption, this is equal to the empty set. Applying $(\varphi \times \varphi)^{-1} \circ (\phi \times Id)$ will give us the desired conclusion.
5. Trivial.
6. Trivial.
7. Since $Im(D_i^t)$ is a subset of the image of $\varphi^{-1} \times \varphi^{-1}$ which is $S^2 \setminus p \times S^2 \setminus p$, we get what is needed.
8. Same proof as paragraph (7).
9. Since $Im(D_i^t) = \varphi^{-1} \times \varphi^{-1} \circ (\phi \times Id) \circ \kappa_i$, then

$$Im(D_i^t) \cap Im(D_j^t) = \varphi^{-1} \times \varphi^{-1} \circ (\phi \times Id) \circ \kappa^i \cap \varphi^{-1} \times \varphi^{-1} \circ (\phi \times Id) \circ \kappa_j$$

$(\varphi \times \varphi)^{-1}$ and $(\phi \times Id)$ are injective. This is equal to $\varphi^{-1} \times \varphi^{-1} \circ (\phi \times Id)(\kappa_i(t) \cap \kappa_j(t))$. Since $(\kappa_i(t) \cap \kappa_j(t)) = \emptyset$ then $Im(D_i^t) \cap Im(D_j^t)$ is also empty.

It is easy to see that for every $j \in J$ the dimension of the manifold we attach to it is smaller than the dimension of $S^2 \times S^2$. Now we will prove that for all $j, k \in J$ if there exist $t \in \mathbb{R}$ such that $Im(r_t^k) \subsetneq Im(r_j^t)$, then $\dim(Q_k) < \dim(Q_j)$ such that r_k is the function attached to k and G_k is the manifold attached to k. We note that if $Im(r_t^k) \subsetneq Im(r_j^t)$ then $Im(r_t^k) \cap Im(r_j^t) = Im(r_t^k)$. Since we know that for any $(i,j) \in J \times J$ and for any $t \in \mathbb{R}$, if we denote the function attached to k as r_k we get that $r_i^t \cap r_j^t = \bigcup_{k \in f((i,j))} r_k^t$. If we have i, j that contradicts our statement there must be k where $k \in f((i,j))$ where $\dim(Q_i) = \dim(Q_j) = \dim(Q_k)$. It is easy to verify that this is not the case.

So by Theorem 4, $S^2 \times S^2 \setminus (\bigcup_{i=1}^n Im(G_i^0) \cup \bigcup_{i=1}^n Im(D_i^0) \cup (p \times S^2) \cup (S^2 \times p) \cup (p \times p))$ is homeomorphic to $S^2 \times S^2 \setminus (\bigcup_{i=1}^n Im(G_i^1) \cup \bigcup_{i=1}^n Im(D_i^1) \cup (p \times S^2) \cup (S^2 \times p) \cup (p \times p))$. We know that $A \setminus (B \cup C) = (A \setminus C) \setminus (B \setminus C)$. Then it turns out that

$$S^2 \times S^2 \setminus (p \times S^2 \cup S^2 \times p) \setminus \left(\bigcup_{i=1}^n Im(G_i^0) \cup \bigcup_{i=1}^n Im(D_i^0) \cup p \times S^2 \cup S^2 \times p \cup p \times p \right)$$

is homeomorphic to

$$S^2 \times S^2 \setminus (p \times S^2 \cup S^2 \times p) \setminus \left(\bigcup_{i=1}^n Im(G_i^1) \cup \bigcup_{i=1}^n Im(D_i^1) \right) \cup p \times S^2 \cup S^2 \times p \cup p \times p.$$

We can see that $S^2 \times S^2 \setminus (p \times S^2 \cup S^2 \times p) = S^2 \setminus p \times S^2 \times p$. We know from (1) that $Im(G_j^t) = p \times p \cup \phi(g_k \circ (\varphi \times \varphi)^{-1}(\mathbb{R}^2 \times t))$, so if we subtract $p \times p$ we get $Im(G_j^t) \setminus (p \times p) = (g_k \circ \phi \circ (\varphi \times \varphi)^{-1}(\mathbb{R}^2 \times t))$. We can see that the image is inside $((S^2 \setminus p) \times (S^2 \setminus p) \cup p \times p)$ and, therefore,

$$\varphi \times \varphi(Im(G_j^t) \setminus ((S^2 \setminus p) \times (S^2 \setminus p) \cup p \times p)) = Im(g_k \circ \phi).$$

Additionally, since the image of D_i^t is inside

$$((S^2 \setminus p) \times (S^2 \setminus p) \cup p \times p),$$

$\varphi \times \varphi(Im(D_i^t)) = Im(\kappa_i^t \circ \phi)$ and since $\varphi \times \varphi$ is a homeomorphism on $S^2 \setminus p \times S^2 \times p$, if we apply it we get that $\mathbb{R}^4 \setminus (\bigcup_{i=1}^{n} Im(g_i^0 \circ \phi) \cup \bigcup_{i=1}^{n} Im(\kappa_i^0 \circ \phi))$ is homeomorphic to $\mathbb{R}^4 \setminus (\bigcup_{i=1}^{n} Im(g_i^1 \circ \phi) \cup \bigcup_{i=1}^{n} Im(\kappa_i^1 \circ \phi))$. Applying the homeomorphism ϕ will give us that $\mathbb{R}^4 \setminus (\bigcup_{i=1}^{n} Im(g_i^0) \cup \bigcup_{i=1}^{n} Im(\kappa_i^0))$ is homeomorphic to $\mathbb{R}^4 \setminus (\bigcup_{i=1}^{n} Im(g_i^1) \cup \bigcup_{i=1}^{n} Im(\kappa_i^1))$, as needed. $\square$

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