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On Symmetry Properties of Frobenius Manifolds and Related Lie-Algebraic Structures

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Abstract: The aim of this paper is to develop an algebraically feasible approach to solutions of the oriented associativity equations. Our approach was based on a modification of the Adler–Kostant–Symes integrability scheme and applied to the co-adjoint orbits of the diffeomorphism loop group of the circle. A new two-parametric hierarchy of commuting to each other Monge type Hamiltonian vector fields is constructed. This hierarchy, jointly with a specially constructed reciprocal transformation, produces a Frobenius manifold potential function in terms of solutions of these Monge type Hamiltonian systems.

Keywords: Witten–Dijkgraaf–Verlinde-Verlinde associativity equations; oriented associativity equations; loop lie algebras; Frobenius manifold potential function; Adler–Kostant–Symes scheme; Lie-algeberaic analysis; compatible Hamiltonian flows; reciprocal transformation

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1. The Introductory Setting

Let us start with an interesting mathematical structure, suggested in [1–5], on the space of smooth functions: consider a real-valued C^∞-smooth differentiable Frobenius manifold potential function $F \in C^\infty(\mathbb{R}^n;\mathbb{R})$ and denote their partial derivatives as

$$F_{ij}(t) := \frac{\partial^2 F(t)}{\partial t_i \partial t_j}, F_{ijk}(t) := \frac{\partial^3 F(t)}{\partial t_i \partial t_j \partial t_k} \tag{1}$$

for i, j, and $k = \overline{1,n}$, $n \in \mathbb{N}$. These partial derivatives are symmetrical, with respect to permutations of their indices. Let us assume additionally that the symmetric matrix $\eta := \{\eta_{ij}(t) := F_{ij1}(t) : i,j = \overline{1,n}\}$ is non-degenerate, and call it an induced *metric* on the $\mathbb{R}^n$. In addition,

$$F_{ijk}(t) = \sum_{s \in \overline{1,n}} \eta_{is}(t) C^s_{ij}(t), \tag{2}$$

where, by definition,

$$C^s_{ij}(t) := \sum_{k \in \overline{1,n}} F_{ijk}(t)\eta^{ks}(t), \qquad \sum_{k \in \overline{1,n}} \eta^{sk}(t)\eta_{kj}(t) = \delta^s_j \tag{3}$$

for all i, j, and $s \in \mathbb{N}$. Assume now that the set $\mathbb{R}^n$ represents a local coordinate frame [6,7] of an a finite-dimensional manifold M. Then its tangent space $T_t(M)$ at a point $t \in M$ is described by means of the local vector field system $\{\partial/\partial t_i \in T_t(M) : i = \overline{1,n}\}$, which *a priori* commute to each other: $[\partial/\partial t_i, \partial/\partial t_j] = 0$ for all $i, j = \overline{1,n}$. Let us now assume that the manifold M is a *Frobenius manifold* [8–10], i.e., its tangent space $T_t(M)$ at any point $t \in M$ forms *an associative Frobenius algebra* F_M with respect to some multiplication "$\circ$" on F_M :

$$\partial/\partial t_i \circ \partial/\partial t_j := \textstyle\sum_{s \in \overline{1,n}} C_{ij}^s(t)\partial/\partial t_s, \qquad (\partial/\partial t_i \circ \partial/\partial t_j) \circ \partial/\partial t_s = \partial/\partial t_i \circ (\partial/\partial t_j \circ \partial/\partial t_s) \tag{4}$$

for any i, j and $s = \overline{1,n}$ with the structure constants defined by the expression (3). Define now a set of matrices $C_i(t) := \{C_{ij}^k(t) = C_{ji}^k(t) : j,k \in \overline{1,n}\}, i = \overline{1,n}$. Then, as it easily follows from (4), the structure constants (3) should satisfy the following additional constraints:

$$[C_i(t), C_j(t)] = 0, \qquad \partial C_i(t)/\partial t_j = \partial C_j(t)/\partial t_i \tag{5}$$

for any $t \in M$ and all $i, j = \overline{1,n}$. (5) are called the Witten–Dijkgraaf–Verlinde–Verlinde, or oriented associativity WDVV equations. These equations were first investigated in [11–13] for problems related with topological and string quantum field theory of elementary particles. A nice introduction into the topic can be found in B. Dubrovin Lecture Notes [2]. Lie-algebraic aspects of these equations and related integrability properties can be found in recent works [14,15].

The notion of a Frobenius manifold was first axiomatized and thoroughly studied by B. Dubrovin [2–5] in the early nineties, and plays a central role in mirror field theory symmetry [16–18], theory of unfolding spaces of singularities [19], quantization theory [20,21], quantum cohomology [8], and integrability theory [1,19,22–31] of dispersion-less many-dimensional systems.

A full Frobenius structure on M consists of the data $(\circ, e, \eta, E)$. Here $\circ : T(M) \otimes_S T(M) \to T(M)$ is an associative and commutative multiplication on the tangent sheaf, so that $T(M)$ becomes a sheaf of commutative algebras over the ring $\mathbb{R}\{t\}$ of convergent series with identity $e \in T(M), \eta$ is a metric on M (non-degenerate quadratic form $T(M) \otimes_S T(M)$), and E is a so called *Euler vector field*. These structures are connected by various constraints and compatibility conditions, and are presented in [2,3] and [32,33]. For example, the metric η must be flat and "$\circ$"–invariant, i.e., $\langle a|b \circ c\rangle_\eta = \langle a \circ b|c\rangle_\eta$ for the metric $\langle \cdot|\cdot\rangle_\eta$ on M and any a, b, and $c \in T(M)$. Various weaker versions of the Frobenius structure are interesting in themselves and also appear in [19–21] in different contexts.

Let us also mention an additional notion of a unital Frobenius manifold F_M, introduced in [10] and further studied in [9]. This structure consists of an associative and commutative multiplication "$\circ$" on the tangent sheaf as above, satisfying the following properties: 1^0) a flat structure $T(M)$ on M subject to a flat connection $d_\omega : \Gamma(\Lambda(M) \otimes T(M)) \to \Gamma(\Lambda(M) \otimes T(M))$, $d_\omega d_\omega = 0$, is compatible with a multiplication "$\circ$", if in a neighborhood of any point there exists a vector field $C \in \Gamma(T(M))$, such that for arbitrary local flat vector fields $X, Y \in \Gamma(T(M))$ one has

$$X \circ Y = [X, [Y, C]], \tag{6}$$

where $C \in \Gamma(T(M))$ is called *a local vector potential* for $\circ$; 2^0) $T(M)$ is called compatible with $(\circ, e)$, $e \in \Gamma(T(M))$ is an identity element, if 1^0) holds and moreover, the identity element $e := \partial/\partial t_1$ is flat, that is the corresponding covariant derivative $\nabla_X^\omega e = 0$ for any $X \in \Gamma(T(M))$. From (6) one easily ensues the relationships (5), where

$$C_{ij}^k(t) = \partial/\partial t_i \partial/\partial t_j C^k(t), \qquad \partial/\partial t_1 \circ \partial/\partial t_i = \partial/\partial t_i, \tag{7}$$

for any i, j, and $k = \overline{1,n}$ and $t \in M$.

As a very interesting example of the above construction can be obtained for the special case $n = 3$. We can take into account a reduction of the commuting matrices $C_j \in \mathrm{End}\,\mathbb{E}^3, j = \overline{1,3}$, presented in [1–3]. Namely, assume that a smooth Frobenius manifold potential function $F \in C^\infty(\mathbb{R}^n; \mathbb{R})$ is representable as

$$F(t) = \frac{1}{2}t_1^2 t_3 + \frac{1}{2}t_1 t_2^2 + f(t_1, t_2, t_3), \tag{8}$$

where a smooth mapping $f : \mathbb{R}^3 \to \mathbb{R}$ satisfies, following from (4) in the form $(\partial/\partial t_2 \circ \partial/\partial t_2) \circ \partial/\partial t_3 = \partial/\partial t_2 \circ (\partial/\partial t_2 \circ \partial/\partial t_3),\ \partial/\partial t_1 \circ \partial/\partial t_j = \partial/\partial t_j, j = \overline{1,3}$, such a partial differential equation:

$$f^2_{t_2t_2t_3} - f_{t_3t_3t_3} - f_{t_2t_2t_2}f_{t_2t_3t_3} = 0 \tag{9}$$

for any $(t_1, t_2, t_3) \in \mathbb{R}^3$. In particular, as it was shown by B. Dubrovin and Y. Manin [2,3,32,33], the Equation (9) allows the following system of compatible (for any parameter $p \in \mathbb{C}\backslash\{0\}$) linear differential equations:

$$\frac{\partial x}{\partial t_1} = \frac{1}{p}C_1 x,\ \frac{\partial x}{\partial t_2} = \frac{1}{p}\,C_2 x,\ \frac{\partial x}{\partial t_3} = \frac{1}{p}\,C_3 x \tag{10}$$

on vectors $x := (x_1, x_2, x_3) \in \mathbb{E}^3$, determined by matrices

$$C_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, C_2 = \begin{pmatrix} 0 & b & c \\ 1 & a & b \\ 0 & 1 & 0 \end{pmatrix}, C_3 = \begin{pmatrix} 0 & c & b^2 - ac \\ 0 & b & c \\ 1 & 0 & 0 \end{pmatrix}, \tag{11}$$

where $a := f_{t_2t_2t_2}, b := f_{t_2t_2t_3}, c := f_{t_2t_3t_3}$ and generating the corresponding loop $\widetilde{Diff}(\mathbb{R}^3)$-group diffeomorphisms. It is easy also to check that matrices (11) satisfy the matrix Equation (5), that is

$$\begin{aligned} [C_2, C_3] &= 0 = [C_1, C_j], \\ \partial C_3/\partial t_2 &= \partial C_2/\partial t_3, [C_2, C_3] = 0 = \partial C_j/\partial t_1, \end{aligned} \tag{12}$$

for $t \in M, j = \overline{1,3}$. An effective Lie-algebraic analysis of the Dubrovin–Manin linear system (10) was recently presented in [14,15].

In the present work, based on a modification of the Adler–Kostant–Symes integrability scheme, applied to the co-adjoint orbits of the loop diffeomorphism group of circle, a new two-parametric hierarchy of commuting to each other Monge type Hamiltonian vector fields

$$u_{t_1} = u_x, \quad v_{t_1} = v_x, \quad u_{t_2} = -(u^2 + 2v)_x, v_{t_2} = (v^2 - 2uv)_x, \tag{13}$$

and

$$u_{t_3} = (\frac{3}{2}v^2 - 6uv - u^3)_x, v_{t_3} = (-v^3 - 3u^2v + 3uv^2 - 3v^2)_x, ..., \tag{14}$$

on a pair of smooth functions $(u, v) \in C^\infty(M; \mathbb{R}^2)$ is constructed. Making use of a suitably constructed reciprocal transformation, applied to this hierarchy, one gives rise to constructing a Frobenius manifold potential function in terns of solutions to these Hamiltonian systems. In particular, we succeeded in describing a class of Frobenius manifold structures, generated by the non-linear Monge type evolution systems (13) and (14).

Proposition 1. *Let a function $F : M \to \mathbb{R}$ be defined by the following differential relationships*

$$\begin{aligned} \frac{\partial^2 F(t_1, t_2, t_3)}{\partial t_1 \partial t_2} &= v, \quad \frac{\partial^2 \mathbb{F}(t_1, t_2, t_3)}{\partial t_1 \partial t_3} = v(2u - v), \\ \frac{\partial^2 F(t_1, t_2, t_3)}{\partial t_1 \partial t_4} &= 2v[v^2 + 3v - 3u(u - v)], \end{aligned} \tag{15}$$

where the pair of functions $(u, v) \in C^\infty(M; \mathbb{R}^2)$ satisfies the evolution flows (13) and (14). Then this function $F : M \to \mathbb{R}$ is a potential function of the Frobenius manifold M, describing the related Frobenius manifold algebraic structures.

2. Frobenius Manifolds, the Related Compatible Co-Adjoint Loop Lie Algebra and Integrability

Consider now the functional Lie algebra $\mathcal{G} \simeq (C^\infty(T^*(\mathbb{S}^1);\mathbb{R});\{\cdot,\cdot\})$, generated by special Hamiltonian vector fields on the cotangent space $T^*(\mathbb{S}^1)$ to the circle $\mathbb{S}^1$ and endowed with the canonical Lie commutator

$$\{a,b\}(x;p) := \frac{\partial}{\partial p}a(x;p)\frac{\partial}{\partial x}b(x;p) - \frac{\partial}{\partial p}b(x;p)\frac{\partial}{\partial x}a(x;\lambda) \tag{16}$$

for any $a,b \in \mathcal{G}$ at point $(x,p) \in T^*(\mathbb{S}^1)$. This algebra possesses the following symmetric and non-degenerate bi-linear form:

$$(a|b) := \int_{\mathbb{R}} dp \int_{\mathbb{S}^1} dxa(x;p)b(x;p)dx, \tag{17}$$

with respect to which $\mathcal{G}^* \simeq \mathcal{G}$. Moreover, the Lie algebra is metrized with respect to the bilinear form (17) as it is ad-invariant: $(a|[b,c]) = ([a,b]|c)$ for any a, b, and $c \in \mathcal{G}$.

Below, we will consider the case when the Lie algebra $\mathcal{G}$ allows splitting into the direct sum of two sub-algebras: $\mathcal{G} = \mathcal{G}_+ \oplus \mathcal{G}_-$, where

$$\mathcal{G}_+ := \{a(x;p) = \sum_{j\in\mathbb{N}} a_j(x)p^j \in \mathcal{G}\} \tag{18}$$

and

$$\mathcal{G}_- := \{b(x;p) = \sum_{0\leq j\ll\infty} b_j(x)p^{-j} \in \mathcal{G}\}, \tag{19}$$

as $p \to \infty$, for which the following dual isomorphisms $\mathcal{G}_+^* \simeq \mathcal{G}_-$, $\mathcal{G}_-^* \simeq \mathcal{G}_+$ hold.

Proceed now to describing via the classical Adler–Kostant–Symes scheme [34–39] commuting co-adjoint orbits of the Lie algebra $\mathcal{G}$ on the adjoint space $\mathcal{G}^* \simeq \mathcal{G}$, generated by smooth Casimir functionals $h \in I(\mathcal{G}^*)$ with respect to the classical Lie-Poisson bracket on $\mathcal{G}^* \simeq \mathcal{G}$:

$$\{h(l),(l|a)\} := (l|[\nabla h(l),a]) = 0 \tag{20}$$

for $l \in \mathcal{G}^*$ and arbitrary $a \in \mathcal{G}$, where, by definition, $\frac{d}{d\varepsilon}h(l+\varepsilon b)|_{\varepsilon=0} := (\nabla h(l)|b)$ for any $b \in \mathcal{G}$. Namely, the following Hamiltonian flows on $\mathcal{G}^*$

$$\partial l/\partial t_k = -ad^*_{\nabla h_+^{(k)}(l)} l = [\nabla h_+^{(k)}(l), l] = [l, \nabla h_-^{(k)}(l)], \tag{21}$$

where, by definition, $\nabla h_\pm^{(k)}(l) := \nabla h^{(k)}(l)|_{\mathcal{G}_\pm}$, are commuting to each other subject to the corresponding evolution parameters $t_k \in \mathbb{R}, k \in \mathbb{Z}_+$, for arbitrary infinite hierarchy of smooth functionally independent Casimir functionals $h^{(k)} \in I(\mathcal{G}^*), k \in \mathbb{Z}_+$. The latter is, evidently, equivalent to the following Lax-Sato type vector field representations:

$$[\partial/\partial t_k + \widetilde{\nabla h_+^{(k)}}(l), \partial/\partial t_m + \widetilde{\nabla h_+^{(m)}}(l)] = 0 \tag{22}$$

for all $k, m \in \mathbb{Z}_+$, where, by definition, any element $a \in \mathcal{G}$ via the expression $\tilde{a}(x;p) := \frac{\partial a}{\partial p}\frac{\partial}{\partial x} - \frac{\partial a}{\partial x}\frac{\partial}{\partial p} \in \Gamma(T_{(x,p)}(T^*(\mathbb{S}^1)))$ generates a canonical Hamiltonian vector field on $T^*(\mathbb{S}^1)$ at point $(x;p) \in T^*(\mathbb{S}^1)$.

Take now an analytic at the momentum $p \in \mathbb{R}$ element $l \in \mathcal{G}^* \simeq \mathcal{G}$ in the following asymptoptic as $p \to \infty$ form:

$$l(x;p) = p + u(x) + \sum_{j\in\mathbb{N}} l_j(x)p^{-j}, \tag{23}$$

where the element $p \in \mathcal{G}^*$ is considered here as an infinitesimal Lie algebra $\mathcal{G}$ character, satisfying the conditions $[\mathcal{G}_{\pm}, p] \in \mathcal{G}_{\pm}$, that can be easily checked by direct computations. The flows (21) are equivalent to the following co-adjoint action

$$\partial l_-/\partial t_k = -ad^*_{\nabla h_+^{(k)}(l)} l_- = [\nabla h_+^{(k)}(l), l_-]_- \tag{24}$$

on $\mathcal{G}^* \simeq \mathcal{G}$ with respect to the evolution parameters $t_k \in \mathbb{R}$ for all $k \in \mathbb{Z}_+$.

It is worthy to observe now that in the case of the Casimir functionals $h^{(k)} := \frac{1}{k+1}(l^k|l), k \in \mathbb{Z}_+$, the flows (24) can be equivalently rewritten as the Hamiltonian systems

$$\partial \tilde{l}(x;p)/\partial t_k = \quad [\widetilde{l_+^k}(x;p), \tilde{l}(x;p)] \tag{25}$$

on $\mathcal{G}^*$ for all $k \in \mathbb{Z}_+$, where, by definition, $\tilde{l}(x;p) := \frac{\partial l}{\partial p}\frac{\partial}{\partial x} - \frac{\partial l}{\partial x}\frac{\partial}{\partial p} \in \Gamma(T_{(x,p)}(T^*(\mathbb{S}^1)))$ at point $(x;p) \in T^*(\mathbb{S}^1)$. Using the Lie bracket (16), Equation (25) can be rewritten as the Hamiltonian flows on the cotangent space $T^*(\mathbb{S}^1)$

$$\partial l(x;p)/\partial t_k = \{H_k(x,p), l(x;p)\}, \tag{26}$$

where, by definitions, $H_k(x;p) = l_+^k(x;p)$ for any $k \in \mathbb{N}, (x;p) \in T^*(\mathbb{S}^1)$.

Remark 1. *It is worth also to remark here that we can pose the following vector field isospectral problem*

$$\tilde{l}(x;p)\psi(x;p|z) = z\,\psi(x;p|z), \tag{27}$$

where $\psi(\cdot;z) \in C^\infty(T^*(\mathbb{S}^1);\mathbb{C})$ is the eigenfunction corresponding to an eigenvalue $z \in \mathbb{C}$, which is a priori invariant with respect to all vector fields (25). The latter naturally allows to apply to (27) the modified inverse scattering transform technique developed in [40] and describe many classes of symbols $l \in \mathcal{G}$, generating important dispersion-less heavenly type [41] dynamical systems, important for applications in modern mathematical physics.

As the point variables $(x;p) \in T^*(\mathbb{S}^1)$ are constant parameters for the evolution flows (25) on analytic at $p = \infty$ element $l \in \mathcal{G}^*$, one can put, by definition, $l(x;p) = z \in \mathbb{C}$ and resolve the functional equation $l(x;p) = z$ with respect to the symbol parameter $p \in \mathbb{R}$, obtaining the following expression:

$$p := \xi(x;z) = z - u - \sum_{j \in \mathbb{N}} \xi_j(x) z^{-j} \tag{28}$$

with coefficients $\xi_j \in C^\infty(\mathbb{S}^1;\mathbb{R}), j \in \mathbb{N}$, characterized by the following lemma.

Lemma 1. *The element $\xi \in C^\infty(\mathbb{S}^1 \times \mathbb{R};\mathbb{C})$ satisfies the following hierarchy of compatible evolution equations*

$$\frac{\partial}{\partial t_k}\xi(x;z) = \frac{\partial \mathcal{H}_k(x;z)}{\partial x}, \tag{29}$$

where the elements $\mathcal{H}_k(x;z) := l_+^k(x;\xi(x;z)), k \in \mathbb{N}$, are determined, using the following simple algebraic expressions:

$$\mathcal{H}_k(x;z) := H_k(x;\xi(x;z)), \tag{30}$$

which hold jointly with compatibility relationships

$$\frac{\partial \mathcal{H}_s(x;z)}{\partial t_k} = \frac{\partial \mathcal{H}_k(x;z)}{\partial t_s} \tag{31}$$

for all $k, s \in \mathbb{N}$.

Proof. Making use of the Equation (25), one can easily calculate for any $k \in \mathbb{N}$ the evolution equations

$$\frac{\partial}{\partial t_k}\left(\frac{1}{\xi(x;z)-p}\right) := \left\{H_k(x;p), \left(\frac{1}{\xi(x;z)-p}\right)\right\},$$

giving rise to the following expressions

$$\begin{aligned}\frac{\partial \xi(x;z)}{\partial t_k} &= \frac{\partial H_k(x;p)}{\partial x} + \left.\frac{\partial H_k(x;p)}{\partial p}\right|_{p=\xi(x;z)} \frac{\partial \xi(x;z)}{\partial x} = \\ &= dH_k(x;\xi(x;z)/dx := \frac{\partial \mathcal{H}_k(x;z)}{\partial x},\end{aligned} \tag{32}$$

which hold for all $k \in \mathbb{N}$ and all $z \in \mathbb{R}$. The compatibility relationships are obvious, following from the commuting to each other flows (29). □

Consider now the functional identity

$$\frac{1}{\xi(x;z)-p} = \sum_{k\in\mathbb{N}} \frac{z^{-k}}{k}\frac{\partial}{\partial p}H_k(x;p), \tag{33}$$

which is satisfied as $z \to \infty$, owing to the following residuum calculation:

$$\begin{aligned}&\frac{1}{2\pi i}\oint \frac{z^{k-1}dz}{\xi(x;z)-p} = \frac{1}{2\pi i}\oint \frac{z^{k-1}dz}{[z-l(x;p)]\frac{[\xi(x;z)-p]}{[z-l(x;p)]}} = \\ &= \left.\frac{l(x;p)^{k-1}}{\partial\xi(x;z)/\partial z}\right|_{z\to\infty} = l(x;p)^{k-1}\partial l(x;p)/\partial p\Big|_{z\to\infty} = \frac{1}{k}\frac{\partial}{\partial p}H_k(x;p)|_+,\end{aligned} \tag{34}$$

which holds for any $k \in \mathbb{N}$. Consider now Hamiltonian functions $H_k : T^*(\mathbb{S}^1) \to \mathbb{R}, k \in \mathbb{N}$, and consider the related canonical Hamiltonian vector fields on the cotangent space $T^*(\mathbb{R})$:

$$\frac{\partial x}{\partial t_k} = \frac{\partial H_k(x;p)}{\partial p}, \qquad \frac{\partial p}{\partial t_k} = -\frac{\partial H_k(x;p)}{\partial x} \tag{35}$$

with respect to a point $(x,p) \in T^*(\mathbb{S}^1)$ subject to the evolution parameter $t_k \in \mathbb{R}, k \in \mathbb{N}$. Taking into account the evolution flows (35) and the fact that $\partial/\partial t_1 = \partial/\partial x$, the identity (33) can be rewritten as

$$\frac{1}{\xi(x;z)-p} = \sum_{k\in\mathbb{N}} \frac{z^{-k}}{k}\frac{\partial x}{\partial t_k} = D(z)x(t),$$

from which and the relationships (31) one ensues the functional representation

$$\xi(x;z) = z - \frac{\partial \mathcal{F}(t)}{\partial x} - D(z)\frac{\partial \mathcal{F}(t)}{\partial x} \tag{36}$$

for some smooth function $\mathcal{F} : M \to \mathbb{R}$. Based now on Lemma 1 and relationships (33), (34) one can state now the following proposition.

Proposition 2. *Let $F : M \to \mathbb{R}$ be a potential function on the Frobenius manifold M, defined by means of the set of asymptotic relationship*

$$D(y)F(t) + D(y)D(z)F(t) = -\ln(1-z/y) - \sum_{k\in\mathbb{N}} \frac{y^{-k}}{k}\mathcal{H}_k(x;z) \tag{37}$$

where, by definition, the operator $D(\alpha) = \sum_{k\in\mathbb{N}} \frac{\alpha^{-k}}{k}\frac{\partial}{\partial t_k}, \alpha \in \mathbb{R}$, is the well known vertex operator. Then the element (28) satisfies the asymptotic representation (36) for all $x \in \mathbb{S}^1$ as $z \to \infty$.

Proof. The functional identity (37) easily reduces to the set of asymptotic expressions

$$\mathcal{H}_k(x;z) = z^k - \partial F/\partial t_k - D(z)\partial F/\partial t_k \tag{38}$$

for all $k \in \mathbb{N}$ as $z \to \infty$. Simultaneously one can observe that the expression (29) and (30) reduce to the representation (36), proving the proposition. □

This proposition is useful for constructing Frobenius manifolds, naturally related with some *generating function* $\mathcal{F} : M \to \mathbb{R}$, satisfying the relationship (36). As an example, we suggest the following element

$$l(x;p) = p + u(x) + \ln\left(1 + \frac{v(x)}{p}\right) \in \mathcal{G}^*, \tag{39}$$

where $u, v \in C^\infty(\mathbb{S}^1;\mathbb{R})$ are some functional parameters. The corresponding Casimir functions $h^{(t_1)} := (l|l)/2, h^{(t_2)} := (l^2|l)/3$ and $h^{(t_3)} := (l^3|l)/4, h^{(t_4)} := (l^4|l)/5$, etc., generate the following Hamiltonian flows on $\mathcal{G}^* \simeq \mathcal{G}$:

$$\partial l/\partial x = [l_+, l], \partial l/\partial y = [l_+^2, l], \quad \partial l/\partial t = [l_+^3, l], \partial l/\partial s = [l_+^4, l] \tag{40}$$

with respect to the evolution parameters $x = t_1 \in \mathbb{R}, t_2, t_3 \in \mathbb{R}$, etc., where, for instance,

$$\begin{aligned} l_+^2 \ &: \ = H_2(x;p) = p^2 + 2pu \in \mathcal{G}_+, \\ l_+^3 \ &: \ = H_3(x;p) = p^3 + 3p^2u + 3pu^2 + 3pv \in \mathcal{G}_+ \end{aligned} \tag{41}$$

and so on. The above commutator expressions with respect to the evolution parameters t_1, t_2 and $t_3 \in \mathbb{R}$ reduce to the next commuting to each other non-linear Monge type evolution systems

$$u_{t_1} = u_x, \quad v_{t_1} = v_x, \quad u_{t_2} = -(u^2 + 2v)_x, v_{t_2} = (v^2 - 2uv)_x, \tag{42}$$

and

$$u_{t_3} = (\frac{3}{2}v^2 - 6uv - u^3)_x, v_{t_3} = (-v^3 - 3u^2v + 3uv^2 - 3v^2)_x, \tag{43}$$

being also compatible dispersion-less Hamiltonian flows on the corresponding functional phase. Moreover, the evolution systems (42) and (43) are equivalent to the Lax-Sato vector field commutator representation (22), where

$$\begin{aligned} \nabla h_+^{(t_1)}(\tilde{l}) \ &= \ (p+u)\frac{\partial}{\partial x} - u_x p\frac{\partial}{\partial p}, \\ \nabla h_+^{(t_1)}(\tilde{l}) \ &= \ (p^2 + 2up + 2v + u^2)\frac{\partial}{\partial x} - (u_x p^2 + v_x p + 2uu_x p)\frac{\partial}{\partial p}. \end{aligned} \tag{44}$$

The vector fields (44), being considered as elements of the Lie algebra $\tilde{\mathcal{G}} \simeq diff(\mathbb{S}^1 \times \mathbb{C})$ of holomorphic with respect to the variable $p \in \mathbb{C}$ vector fields on $\mathbb{S}^1 \times \mathbb{C}$, naturally splits into the direct sum of two sub-algebras $\tilde{\mathcal{G}} = \tilde{\mathcal{G}}_+ \oplus \tilde{\mathcal{G}}_-$, holomorphic in the parameter $p \in \mathbb{C}$ inside $\mathbb{D}_+^1(0)$ of the unit circle $\mathbb{D}_+^1(0) \subset \mathbb{C}$ and outside $\mathbb{D}_-^1(0)$ of this disk, respectively, appear to be generated by the corresponding Casimir functionals on the adjoint space $\tilde{\mathcal{G}}^* \simeq \Omega^1(\mathbb{S}^1 \times \mathbb{C})$ at some root element $\tilde{l} \in \tilde{\mathcal{G}}^*$ subject to the following canonical non-degenerate bi-linear form on $\tilde{\mathcal{G}}^* \times \tilde{\mathcal{G}}$:

$$(\tilde{l}|\tilde{a}) := \int_0^{2\pi} res_p \langle l|a\rangle dx, \tag{45}$$

where we put, by definition, $\tilde{l} := \langle l|d\mathrm{x}\rangle, \tilde{a} := \langle a|\partial/\partial\mathrm{x}\rangle$,x$:= (p;x) \in \mathbb{C} \times \mathbb{S}^1$. Based on the definition of Casimir functionals, one easily enough obtains that this root element equals

$$\begin{aligned}\tilde{l} &= (u_x p^2 + (v+u^2)_x p)dx + (p^2 + 2up + v + u^2)dp = \\ &= d\left(\frac{1}{3}p^3 + up^2 + (v+u^2)p\right),\end{aligned} \tag{46}$$

being a complete derivative of the scalar element $\tilde{\eta} = \frac{1}{3}p^3 + up^2 + (v+u^2)p \in \Omega^0(\mathbb{S}^1 \times \mathbb{C}), \tilde{l} = d\tilde{\eta}$, for all $(p;x) \in \mathbb{C} \times \mathbb{S}^1$. Moreover, the system of evolution equations (42) and (43) becomes equivalent to the following co-adjoint flows

$$\partial\tilde{l}/\partial y = -ad^*_{\nabla h_+^{(t_2)}(\tilde{l})}\tilde{l}, \partial\tilde{l}/\partial t = -ad^*_{\nabla h_+^{(t_3)}(\tilde{l})}\tilde{l} \tag{47}$$

on the adjoint space $\tilde{\mathcal{G}}^*$, generated by the corresponding Casimir functionals $h^{(t_2)}, h^{(t_3)} \in I(\tilde{\mathcal{G}}^*)$ and satisfying the determining relationships $ad^*_{\nabla h^{(t_2)}(\tilde{l})}\tilde{l} = 0, ad^*_{\nabla h^{(t_3)}(\tilde{l})}\tilde{l} = 0$. As now the basic Lie algebra $\tilde{\mathcal{G}} \simeq diff(\mathbb{S}^1 \times \mathbb{C})$ of holomorphic vector fields on $\mathbb{S}^1 \times \mathbb{C}$ is not, evidently, metrized, the flows (47) on $\tilde{\mathcal{G}}^*$ do not possess the standard Lax type commutator representation.

Taking into account the expressions (36) and (39), one can formulate the following proposition.

Proposition 3. *Let a function $F : M \to \mathbb{R}$ be defined by the following differential relationships*

$$\begin{aligned}\frac{\partial^2 F(t_1,t_2,t_3)}{\partial t_1 \partial t_2} &= v, \frac{\partial^2 F(t_1,t_2,t_3)}{\partial t_1 \partial t_3} = v(2u - v), \\ \frac{\partial^2 F(t_1,t_2,t_3)}{\partial t_1 \partial t_3} &= 2v[v^2 + 3v - 3u(u-v)],\end{aligned} \tag{48}$$

where the pair of functions $(u,v) \in C^\infty(M;\mathbb{R}^2)$ satisfies the evolution flows (42) and (43). Then it is a potential function of the Frobenius manifold M, describing the related Frobenius manifold algebraic structures.

This result makes it possible to describe a wide variety of Frobenius manifold potential functions in terns of solutions to these Monge type Hamiltonian systems (42) and (43).

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