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Fractional Weighted Ostrowski-Type Inequalities and Their Applications

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Citation: Kashuri, A.; Meftah, B.; Mohammed, P.O.; Lupaş, A.A.; Abdalla, B.; Hamed, Y.S.; Abdeljawad, T. Fractional Weighted Ostrowski-Type Inequalities and Their Applications. *Symmetry* **2021**, *13*, 968. https://doi.org/10.3390/sym13060968

Academic Editors: Wei-Shih Du and Alina Alb Lupas

Received: 22 February 2021
Accepted: 24 May 2021
Published: 29 May 2021

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Abstract: An important area in the field of applied and pure mathematics is the integral inequality. As it is known, inequalities aim to develop different mathematical methods. Nowadays, we need to seek accurate inequalities for proving the existence and uniqueness of the mathematical methods. The concept of convexity plays a strong role in the field of inequalities due to the behavior of its definition and its properties. Furthermore, there is a strong correlation between convexity and symmetry concepts. Whichever one we work on, we can apply it to the other one due the strong correlation produced between them, especially in the last few years. In this study, by using a new identity, we establish some new fractional weighted Ostrowski-type inequalities for differentiable quasi-convex functions. Further, further results for functions with a bounded first derivative are given. Finally, in order to illustrate the efficiency of our main results, some applications to special means are obtain. The obtained results generalize and refine certain known results.

Keywords: Ostrowski inequality; quasi-convexity; Riemann–Liouville integrals; Hölder inequality; power mean inequality; special means

1. Introduction

Computational and Fractional Analysis are nowadays more and more in the center of mathematics and of other related sciences either by themselves because of their rapid development, which is based on very old foundations, or because they cover a great variety of applications in the real world. In current years, fractional calculus (FC) applied in many phenomena in applied sciences, fluid mechanics, physics and other biology can be described as very effective using mathematical tools of FC. The fractional derivatives have occurred in many applied sciences equations such as reaction and diffusion processes, system identification, velocity signal analysis, relaxation of damping behaviour fabrics and creeping of polymer composites [1–4].

Definition 1 ([5]). *Let I be an interval of real numbers. A function $\varphi : I \to \mathbb{R}$ is said to be convex, if for all $x, y \in I$ and all $t \in [0,1]$, we have*

$$\varphi(tx + (1-t)y) \leq t\varphi(x) + (1-t)\varphi(y).$$

The concept of convex functions has been also generalized in diverse manners. One of them is the quasi-convex function defined as follows:

Definition 2 ([6]). *A function $\varphi : I \to \mathbb{R}$ is said to be quasi-convex, if*

$$\varphi(tx + (1-t)y) \ \leq \max\{\varphi(x), \varphi(y)\}$$

holds for all $x, y \in I$ and $t \in [0,1]$.

In 1938 Ostrowski ([7]) proved the following Ostrowski inequality:

Theorem 1 ([7]). *Let φ be a differentiable function defined on the finite interval $[a,b]$, whose derivative is integrable and bounded over $[a,b]$, i.e., $\|\varphi'\|_\infty := \sup_{x\in(a,\,b)}|\varphi'(x)| < \infty$, then*

$$\left|\varphi(x) - \frac{1}{b-a}\int_a^b \varphi(u)du\right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\|\varphi'\|_\infty. \tag{1}$$

Several generalizations, improvement, and variants of such type of inequality have been obtained, we refer readers to [8–21] and references therein.

The following notations will be used in the sequel. We denote, respectively I° the interior of I and $L[a,b]$ the set of all integrable functions on $[a,b]$.

In [22], Alomari et al. gave the following midpoint type inequalities for differentiable quasi-convex.

Theorem 2 ([22]). *Let $\varphi : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on I° such that $\varphi' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|\varphi'|$ is quasi-convex on $[a,b]$, then the following inequality holds:*

$$\left|\frac{1}{b-a}\int_a^b \varphi(u)du - \varphi\left(\frac{a+b}{2}\right)\right| \leq \frac{b-a}{8}\left(\max\left\{|\varphi'(a)|, \left|\varphi'\left(\frac{a+b}{2}\right)\right|\right\}\right.$$
$$\left. + \max\left\{\left|\varphi'\left(\frac{a+b}{2}\right)\right|, |\varphi'(b)|\right\}\right).$$

Theorem 3 ([22]). *Let $\varphi : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on I° such that $\varphi' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|\varphi'|^q$ is quasi-convex on $[a,b]$, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then the following inequality holds:*

$$\left|\frac{1}{b-a}\int_a^b \varphi(u)du - \varphi\left(\frac{a+b}{2}\right)\right| \leq \frac{b-a}{4(p+1)^{\frac{1}{p}}}\left(\left(\max\left\{|\varphi'(a)|^q, \left|\varphi'\left(\frac{a+b}{2}\right)\right|^q\right\}\right)^{\frac{1}{q}}\right.$$
$$\left. + \left(\max\left\{\left|\varphi'\left(\frac{a+b}{2}\right)\right|^q, |\varphi'(b)|^q\right\}\right)^{\frac{1}{q}}\right).$$

Theorem 4 ([22]). *Let $\varphi : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on I° such that $\varphi' \in L[a,b]$, where $a,b \in I$ with $a < b$. If $|\varphi'|^q$ is quasi-convex on $[a,b]$, $q \geq 1$, then the following inequality holds:*

$$\left|\frac{1}{b-a}\int_a^b \varphi(u)du - \varphi\left(\frac{a+b}{2}\right)\right| \leq \frac{b-a}{8}\left(\left(\max\left\{|\varphi'(a)|^q, \left|\varphi'\left(\frac{a+b}{2}\right)\right|^q\right\}\right)^{\frac{1}{q}} + \left(\max\left\{\left|\varphi'\left(\frac{a+b}{2}\right)\right|^q, |\varphi'(b)|^q\right\}\right)^{\frac{1}{q}}\right).$$

Alomari and Darus in [23] obtained the Ostrowski-type inequalities for differentiable quasi-convex functions:

Theorem 5 ([23]). *Let $\varphi : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on I° such that $\varphi' \in L[a,b]$, where $a,b \in I$ with $a < b$. If $|\varphi'|$ is quasi-convex on $[a,b]$, then the following inequality holds:*

$$\left|\frac{1}{b-a}\int_a^b \varphi(u)du - \varphi(x)\right| \leq \frac{(x-a)^2}{2(b-a)}\max\{|\varphi'(x)|, |\varphi'(a)|\} + \frac{(b-x)^2}{2(b-a)}\max\{|\varphi'(x)|, |\varphi'(b)|\}.$$

Theorem 6 ([23]). *Let $\varphi : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on I° such that $\varphi' \in L[a,b]$, where $a,b \in I$ with $a < b$. If $|\varphi'|^q$ is quasi-convex on $[a,b]$, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then the following inequality holds:*

$$\left|\frac{1}{b-a}\int_a^b \varphi(u)du - \varphi(x)\right| \leq \left(\frac{(x-a)^{p+1}}{(b-a)(p+1)}\right)^{\frac{1}{p}}\left(\max\left\{|\varphi'(x)|^q, |\varphi'(a)|^q\right\}\right)^{\frac{1}{q}} + \left(\frac{(b-x)^{p+1}}{(b-a)(p+1)}\right)^{\frac{1}{p}}\left(\max\left\{|\varphi'(x)|^q, |\varphi'(b)|^q\right\}\right)^{\frac{1}{q}}.$$

Theorem 7 ([23]). *Let $\varphi : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on I° such that $\varphi' \in L[a,b]$, where $a,b \in I$ with $a < b$. If $|\varphi'|^q$ is quasi-convex on $[a,b]$, $q \geq 1$, then the following inequality holds:*

$$\left|\frac{1}{b-a}\int_a^b \varphi(u)du - \varphi(x)\right| \leq \frac{(x-a)^2}{2(b-a)}\left(\max\left\{|\varphi'(x)|^q, |\varphi'(a)|^q\right\}\right)^{\frac{1}{q}} + \frac{(b-x)^2}{2(b-a)}\left(\max\left\{|\varphi'(x)|^q, |\varphi'(b)|^q\right\}\right)^{\frac{1}{q}}.$$

Fractional calculus has been widely studied by many researchers over the past decades. In particular, to generalize classic inequalities. Among the best known and use of these fractional integral operators we recall that of Riemann–Liouville.

Definition 3 ([24]). *Let $\varphi \in L[a,b]$. The Riemann–Liouville integrals $J^{\alpha}_{a^+}\varphi$ and $J^{\alpha}_{b^-}\varphi$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J^{\alpha}_{a^+}\varphi(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-u)^{\alpha-1}\varphi(u)du, \quad x > a,$$

$$J^{\alpha}_{b^-}\varphi(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (u-x)^{\alpha-1}\varphi(u)du, \quad b > x,$$

respectively, where $\Gamma(\alpha) = \int_0^{\infty} e^{-u}u^{\alpha-1}du$, is the Gamma function and $J^{0}_{a^+}\varphi(x) = J^{0}_{b^-}\varphi(x) = \varphi(x)$.

In so-called fractional calculus, there are not only global derivatives (for example, Riemann–Liouville and Caputo), but also local fractional derivatives (Khalil, Almeida, among others), see [25,26]. Regarding some papers dealing with fractional integral inequalities via different types of fractional integral operators, we refer readers to [27–43].

In this paper, we establish a new identity and then apply it to derive new weighted Ostrowski-type inequalities for quasi-convex functions. Further results for functions with a bounded first derivative will be given. In order to illustrate the efficiency of our main results, some applications to special means will be obtain.

2. Main Results

For brevity, we will used in the sequel $J = [a,b]$. In order to find our main results, we need to prove the following lemma.

Lemma 1. *Let $\varphi : J \to \mathbb{R}$ be differentiable function on J°, $0 \leq a < b$, and $\omega : J \to \mathbb{R}$ a continuous function. If φ', $\omega \in L(J)$, then*

$$\begin{aligned}
&J^{\alpha}_{x^-}\omega\varphi(a) + J^{\alpha}_{x^+}\omega\varphi(b) - \left[J^{\alpha}_{x^-}\omega(a) + J^{\alpha}_{x^+}\omega(b)\right]\varphi(x)\\
&= \frac{(b-x)^{\alpha+1}}{\Gamma(\alpha)}\int_0^1 p_1(\tau)\varphi'(\tau b + (1-\tau)x)d\tau - \frac{(x-a)^{\alpha+1}}{\Gamma(\alpha)}\int_0^1 p_2(\tau)\varphi'(\tau a + (1-\tau)x)d\tau,
\end{aligned}$$

where

$$p_1(\tau) := \int_{\tau}^1 (1-\sigma)^{\alpha-1}\omega(\sigma b + (1-\sigma)x)d\sigma \tag{2}$$

and

$$p_2(\tau) := \int_{\tau}^1 (1-\sigma)^{\alpha-1}\omega(\sigma a + (1-\sigma)x)d\sigma. \tag{3}$$

Proof. We integrate by parts and change the variable $(u = \sigma b + (1-\sigma)x)$ to obtain

$$\begin{aligned}
&\int_0^1 p_1(\tau)\varphi'(\tau b + (1-\tau)x)d\tau\\
&= \int_0^1 \left(\int_{\tau}^1 (1-\sigma)^{\alpha-1}\omega(\sigma b + (1-\sigma)x)d\sigma\right)\varphi'(\tau b + (1-\tau)x)d\tau\\
&= \frac{1}{b-x}\left(\int_{\tau}^1 (1-\sigma)^{\alpha-1}\omega(\sigma b + (1-\sigma)x)d\sigma\right)\varphi(\tau b + (1-\tau)x)\Big|_{\tau=0}^{\tau=1}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{b-x}\int_0^1(1-\tau)^{\alpha-1}\omega(\tau b+(1-\tau)x)\varphi(\tau b+(1-\tau)x)d\tau\\
&=-\frac{1}{b-x}\left(\int_0^1(1-\sigma)^{\alpha-1}\omega(\sigma b+(1-\sigma)x)d\sigma\right)\varphi(x)\\
&+\frac{1}{b-x}\int_0^1(1-\tau)^{\alpha-1}\omega(\tau b+(1-\tau)x)\varphi(\tau b+(1-\tau)x)d\tau\\
&=-\frac{1}{(b-x)^{\alpha+1}}\left(\int_x^b(b-u)^{\alpha-1}\omega(u)du\right)\varphi(x)\\
&+\frac{1}{(b-x)^{\alpha+1}}\int_x^b(b-u)^{\alpha-1}\omega(u)\varphi(u)du\\
&=-\frac{\Gamma(\alpha)}{(b-x)^{\alpha+1}}(J^{\alpha}_{x^+}\omega(b))\varphi(x)+\frac{\Gamma(\alpha)}{(b-x)^{\alpha+1}}(J^{\alpha}_{x^+}\omega\varphi(b)).
\end{aligned}
\tag{4}
$$

Now, multiplying (4) by $\frac{(b-x)^{\alpha+1}}{\Gamma(\alpha)}$, we get

$$
\frac{(b-x)^{\alpha+1}}{\Gamma(\alpha)}\int_0^1 p_1(\tau)\varphi'(\tau b+(1-\tau)x)d\tau=J^{\alpha}_{x^+}\omega\varphi(b)-(J^{\alpha}_{x^+}\omega(b))\varphi(x).
\tag{5}
$$

Similar work give

$$
\begin{aligned}
&\int_0^1 p_2(\tau)\varphi'(\tau a+(1-\tau)x)d\tau\\
&=\frac{\Gamma(\alpha)}{(x-a)^{\alpha+1}}(J^{\alpha}_{x^-}\omega(a))\varphi(x)-\frac{\Gamma(\alpha)}{(x-a)^{\alpha+1}}J^{\alpha}_{x^-}\omega\varphi(a).
\end{aligned}
\tag{6}
$$

Multiplying (6) by $\frac{(x-a)^{\alpha+1}}{\Gamma(\alpha)}$, we obtain

$$
\frac{(x-a)^{\alpha+1}}{\Gamma(\alpha)}\int_0^1 p_2(\tau)\varphi'(\tau a+(1-\tau)x)d\tau=(J^{\alpha}_{x^-}\omega(a))\varphi(x)-J^{\alpha}_{x^-}\omega\varphi(a).
\tag{7}
$$

By taking the difference between (5) and (7), we get

$$
\begin{aligned}
&\frac{(b-x)^{\alpha+1}}{\Gamma(\alpha)}\int_0^1 p_1(\tau)\varphi'(\tau b+(1-\tau)x)d\tau-\frac{(x-a)^{\alpha+1}}{\Gamma(\alpha)}\int_0^1 p_2(\tau)\varphi'(\tau a+(1-\tau)x)d\tau\\
&=J^{\alpha}_{x^-}\omega\varphi(a)+J^{\alpha}_{x^+}\omega\varphi(b)-\left[J^{\alpha}_{x^-}\omega(a)+J^{\alpha}_{x^+}\omega(b)\right]\varphi(x),
\end{aligned}
$$

which is the desired result. □

Theorem 8. *Consider a differentiable function $\varphi: J\to\mathbb{R}$ with $\varphi'\in L(J)$ where $0\le a<b$, and let $\omega: J\to\mathbb{R}$ be continuous function. If $|\varphi'|$ is quasi-convex, then*

$$
\begin{aligned}
&\left|J^{\alpha}_{x^-}\omega\varphi(a)+J^{\alpha}_{x^+}\omega\varphi(b)-\left[J^{\alpha}_{x^-}\omega(a)+J^{\alpha}_{x^+}\omega(b)\right]\varphi(x)\right|\\
&\le\frac{(b-x)^{\alpha+1}}{\Gamma(\alpha+2)}\max\left\{\left|\varphi'(x)\right|,\left|\varphi'(b)\right|\right\}\|\omega\|_{[x,b],\infty}\\
&+\frac{(x-a)^{\alpha+1}}{\Gamma(\alpha+2)}\max\left\{\left|\varphi'(x)\right|,\left|\varphi'(a)\right|\right\}\|\omega\|_{[a,x],\infty}.
\end{aligned}
$$

Proof. From Lemma 1, properties of modulus and quasi-convexity of $|\varphi'|$, we have

$$\begin{aligned}
&\left|J^{\alpha}_{x^-}\omega\varphi(a)+J^{\alpha}_{x^+}\omega\varphi(b)-\left[J^{\alpha}_{x^-}\omega(a)+J^{\alpha}_{x^+}\omega(b)\right]\varphi(x)\right|\\
\leq&\frac{(b-x)^{\alpha+1}}{\Gamma(\alpha)}\int_0^1|p_1(\tau)||\varphi'(\tau b+(1-\tau)x)|d\tau\\
&+\frac{(x-a)^{\alpha+1}}{\Gamma(\alpha)}\int_0^1|p_2(\tau)||\varphi'(\tau a+(1-\tau)x)|d\tau\\
\leq&\frac{(b-x)^{\alpha+1}}{\Gamma(\alpha)}\max\{|\varphi'(x)|,|\varphi'(b)|\}\int_0^1\left|\int_\tau^1(1-\sigma)^{\alpha-1}\omega(\sigma b+(1-\sigma)x)d\sigma\right|d\tau\\
&+\frac{(x-a)^{\alpha+1}}{\Gamma(\alpha)}\max\{|\varphi'(x)|,|\varphi'(a)|\}\int_0^1\left|\int_\tau^1(1-\sigma)^{\alpha-1}\omega(\sigma a+(1-\sigma)x)d\sigma\right|d\tau\\
\leq&\left(\frac{(b-x)^{\alpha+1}}{\Gamma(\alpha)}\max\{|\varphi'(x)|,|\varphi'(b)|\}\|\omega\|_{[x,b],\infty}\right.\\
&\left.+\frac{(x-a)^{\alpha+1}}{\Gamma(\alpha)}\max\{|\varphi'(x)|,|\varphi'(a)|\}\|\omega\|_{[a,x],\infty}\right)\int_0^1\left|\int_\tau^1(1-\sigma)^{\alpha-1}d\sigma\right|d\tau\\
=&\frac{(b-x)^{\alpha+1}}{\Gamma(\alpha+2)}\max\{|\varphi'(x)|,|\varphi'(b)|\}\|\omega\|_{[x,b],\infty}\\
&+\frac{(x-a)^{\alpha+1}}{\Gamma(\alpha+2)}\max\{|\varphi'(x)|,|\varphi'(a)|\}\|\omega\|_{[a,x],\infty}.
\end{aligned}$$

The proof is completed. □

Corollary 1. *Considering* $x=\frac{a+b}{2}$ *in Theorem 8, we obtain*

$$\begin{aligned}
&\left|J^{\alpha}_{\frac{a+b}{2}^-}\omega\varphi(a)+J^{\alpha}_{\frac{a+b}{2}^+}\omega\varphi(b)-\left[J^{\alpha}_{\frac{a+b}{2}^-}\omega(a)+J^{\alpha}_{\frac{a+b}{2}^+}\omega(b)\right]\varphi\left(\frac{a+b}{2}\right)\right|\\
\leq&\frac{(b-a)^{\alpha+1}}{2^{\alpha+1}\Gamma(\alpha+2)}\|\omega\|_{[a,b],\infty}\\
&\times\left(\max\left\{\left|\varphi'\left(\frac{a+b}{2}\right)\right|,|\varphi'(b)|\right\}+\max\left\{\left|\varphi'\left(\frac{a+b}{2}\right)\right|,|\varphi'(a)|\right\}\right).
\end{aligned}$$

Corollary 2. *Choosing* $\omega(u)=\frac{1}{b-a}$ *in Theorem 8, we get*

$$\begin{aligned}
&\left|J^{\alpha}_{x^-}\varphi(a)+J^{\alpha}_{x^+}\varphi(b)-\frac{(b-x)^{\alpha}+(x-a)^{\alpha}}{\Gamma(\alpha+1)}\varphi(x)\right|\\
\leq&\frac{(b-x)^{\alpha+1}}{\Gamma(\alpha+2)}\max\{|\varphi'(x)|,|\varphi'(b)|\}+\frac{(x-a)^{\alpha+1}}{\Gamma(\alpha+2)}\max\{|\varphi'(x)|,|\varphi'(a)|\}.
\end{aligned}$$

Moreover, if we take $x=\frac{a+b}{2}$, *we obtain*

$$\begin{aligned}
&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left(J^{\alpha}_{\frac{a+b}{2}^-}\varphi(a)+J^{\alpha}_{\frac{a+b}{2}^+}\varphi(b)\right)-\varphi\left(\frac{a+b}{2}\right)\right|\\
\leq&\frac{b-a}{4(\alpha+1)}\left(\max\left\{\left|\varphi'\left(\frac{a+b}{2}\right)\right|,|\varphi'(b)|\right\}+\max\left\{\left|\varphi'\left(\frac{a+b}{2}\right)\right|,|\varphi'(a)|\right\}\right).
\end{aligned}$$

Corollary 3. *Let $\alpha = 1$ in Theorem 8, then*

$$\begin{aligned}&\left|\int_a^b \omega(u)\varphi(u)du - \left(\int_a^b \omega(u)du\right)\varphi(x)\right| \\ \leq &\frac{(b-x)^2}{2}\max\{|\varphi'(x)|,|\varphi'(b)|\}\|\omega\|_{[x,b],\infty} \\ &+\frac{(x-a)^2}{2}\max\{|\varphi'(x)|,|\varphi'(a)|\}\|\omega\|_{[a,x],\infty}.\end{aligned}$$

Moreover, for $x = \frac{a+b}{2}$, we get

$$\begin{aligned}&\left|\int_a^b \omega(u)\varphi(u)du - \left(\int_a^b \omega(u)du\right)\varphi\left(\frac{a+b}{2}\right)\right| \leq \frac{(b-a)^2}{8}\|\omega\|_{[a,b],\infty} \\ &\times\left(\max\left\{\left|\varphi'\left(\frac{a+b}{2}\right)\right|,|\varphi'(b)|\right\}+\max\left\{\left|\varphi'\left(\frac{a+b}{2}\right)\right|,|\varphi'(a)|\right\}\right).\end{aligned}$$

Remark 1. *In Corollary 3, if we choose $\omega(u) = \frac{1}{b-a}$, we obtain ([23], Theorem 2). Moreover, if we take $x = \frac{a+b}{2}$, we get ([22], Theorem 7).*

Theorem 9. *Consider a differentiable function $\varphi : J \to \mathbb{R}$ with $\varphi' \in L(J)$ where $0 \leq a < b$, and let $\omega : J \to \mathbb{R}$ be continuous function. If $|\varphi'|^q$ is quasi-convex, $q > 1$ and $\frac{1}{p}+\frac{1}{q} = 1$, then*

$$\begin{aligned}&\left|J^{\alpha}_{x^-}\omega\varphi(a) + J^{\alpha}_{x^+}\omega\varphi(b) - \left[J^{\alpha}_{x^-}\omega(a) + J^{\alpha}_{x^+}\omega(b)\right]\varphi(x)\right| \\ \leq &\frac{(b-x)^{\alpha+1}}{(\alpha p+1)^{\frac{1}{p}}\Gamma(\alpha+1)}\|\omega\|_{[x,b],\infty}\left(\max\left\{|\varphi'(x)|^q,|\varphi'(b)|^q\right\}\right)^{\frac{1}{q}} \\ &+\frac{(x-a)^{\alpha+1}}{(\alpha p+1)^{\frac{1}{p}}\Gamma(\alpha+1)}\|\omega\|_{[a,x],\infty}\left(\max\left\{|\varphi'(x)|^q,|\varphi'(a)|^q\right\}\right)^{\frac{1}{q}}.\end{aligned}$$

Proof. Applying properties of modulus, Lemma 1, Hölder's inequality, and quasi-convexity of $|\varphi'|^q$, we obtain

$$\begin{aligned}&\left|J^{\alpha}_{x^-}\omega\varphi(a) + J^{\alpha}_{x^+}\omega\varphi(b) - \left[J^{\alpha}_{x^-}\omega(a) + J^{\alpha}_{x^+}\omega(b)\right]\varphi(x)\right| \\ \leq &\frac{(b-x)^{\alpha+1}}{\Gamma(\alpha)}\left(\int_0^1|p_1(\tau)|^p d\tau\right)^{\frac{1}{p}}\left(\int_0^1\left|\varphi'(\tau b+(1-\tau)x)\right|^q d\tau\right)^{\frac{1}{q}} \\ &+\frac{(x-a)^{\alpha+1}}{\Gamma(\alpha)}\left(\int_0^1|p_2(\tau)|^p d\tau\right)^{\frac{1}{p}}\left(\int_0^1\left|\varphi'(\tau a+(1-\tau)x)\right|^q d\tau\right)^{\frac{1}{q}} \\ \leq &\frac{(b-x)^{\alpha+1}}{\Gamma(\alpha)}\left(\int_0^1\left|\int_\tau^1(1-\sigma)^{\alpha-1}\omega(\sigma b+(1-\sigma)x)d\sigma\right|^p d\tau\right)^{\frac{1}{p}}\left(\max\left\{|\varphi'(x)|^q,|\varphi'(b)|^q\right\}\right)^{\frac{1}{q}} \\ &+\frac{(x-a)^{\alpha+1}}{\Gamma(\alpha)}\left(\int_0^1\left|\int_\tau^1(1-\sigma)^{\alpha-1}\omega(\sigma a+(1-\sigma)x)d\sigma\right|^p d\tau\right)^{\frac{1}{p}}\left(\max\left\{|\varphi'(x)|^q,|\varphi'(a)|^q\right\}\right)^{\frac{1}{q}} \\ \leq &\frac{(b-x)^{\alpha+1}}{\Gamma(\alpha)}\|\omega\|_{[x,b],\infty}\left(\int_0^1\left|\int_\tau^1(1-\sigma)^{\alpha-1}d\sigma\right|^p d\tau\right)^{\frac{1}{p}}\left(\max\left\{|\varphi'(x)|^q,|\varphi'(b)|^q\right\}\right)^{\frac{1}{q}}\end{aligned}$$

$$+\frac{(x-a)^{\alpha+1}}{\Gamma(\alpha)}\|\omega\|_{[a,x],\infty}\left(\int_0^1\left|\int_\tau^1(1-\sigma)^{\alpha-1}d\sigma\right|^p d\tau\right)^{\frac{1}{p}}\left(\max\left\{|\varphi'(x)|^q,|\varphi'(a)|^q\right\}\right)^{\frac{1}{q}}$$
$$=\frac{(b-x)^{\alpha+1}}{(\alpha p+1)^{\frac{1}{p}}\Gamma(\alpha+1)}\|\omega\|_{[x,b],\infty}\left(\max\left\{|\varphi'(x)|^q,|\varphi'(b)|^q\right\}\right)^{\frac{1}{q}}$$
$$+\frac{(x-a)^{\alpha+1}}{(\alpha p+1)^{\frac{1}{p}}\Gamma(\alpha+1)}\|\omega\|_{[a,x],\infty}\left(\max\left\{|\varphi'(x)|^q,|\varphi'(a)|^q\right\}\right)^{\frac{1}{q}}.$$

The proof is completed. □

Corollary 4. *Let $x=\frac{a+b}{2}$ in Theorem 9, then*

$$\left|J^{\alpha}_{\frac{a+b}{2}^-}\omega\varphi(a)+J^{\alpha}_{\frac{a+b}{2}^+}\omega\varphi(b)-\left[J^{\alpha}_{\frac{a+b}{2}^-}\omega(a)+J^{\alpha}_{\frac{a+b}{2}^+}\omega(b)\right]\varphi\left(\frac{a+b}{2}\right)\right|$$
$$\leq\frac{(b-a)^{\alpha+1}}{2^{\alpha+1}(\alpha p+1)^{\frac{1}{p}}\Gamma(\alpha+1)}\|\omega\|_{[a,b],\infty}\left(\left(\max\left\{\left|\varphi'\left(\frac{a+b}{2}\right)\right|^q,|\varphi'(b)|^q\right\}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\max\left\{\left|\varphi'\left(\frac{a+b}{2}\right)\right|^q,|\varphi'(a)|^q\right\}\right)^{\frac{1}{q}}\right).$$

Corollary 5. *Choosing $\omega(u)=\frac{1}{b-a}$ in Theorem 9, we get*

$$\left|J^{\alpha}_{x^-}\varphi(a)+J^{\alpha}_{x^+}\varphi(b)-\frac{(b-x)^{\alpha}+(x-a)^{\alpha}}{\Gamma(\alpha+1)}\varphi(x)\right|$$
$$\leq\frac{(b-x)^{\alpha+1}}{(\alpha p+1)^{\frac{1}{p}}\Gamma(\alpha+1)}\left(\max\left\{|\varphi'(x)|^q,|\varphi'(b)|^q\right\}\right)^{\frac{1}{q}}$$
$$+\frac{(x-a)^{\alpha+1}}{(\alpha p+1)^{\frac{1}{p}}\Gamma(\alpha+1)}\left(\max\left\{|\varphi'(x)|^q,|\varphi'(a)|^q\right\}\right)^{\frac{1}{q}}.$$

Moreover, if we take $x=\frac{a+b}{2}$, we obtain

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left(J^{\alpha}_{\frac{a+b}{2}^-}\varphi(a)+J^{\alpha}_{\frac{a+b}{2}^+}\varphi(b)\right)-\varphi\left(\frac{a+b}{2}\right)\right|$$
$$\leq\frac{b-a}{4(\alpha p+1)^{\frac{1}{p}}}\left(\left(\max\left\{\left|\varphi'\left(\frac{a+b}{2}\right)\right|^q,|\varphi'(b)|^q\right\}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\max\left\{\left|\varphi'\left(\frac{a+b}{2}\right)\right|^q,|\varphi'(a)|^q\right\}\right)^{\frac{1}{q}}\right).$$

Corollary 6. *For $\alpha=1$ in Theorem 9, we have*

$$\left|\int_a^b\omega(u)\varphi(u)du-\left(\int_a^b\omega(u)du\right)\varphi(x)\right|$$
$$\leq\frac{(b-x)^2}{(p+1)^{\frac{1}{p}}}\|\omega\|_{[x,b],\infty}\left(\max\left\{|\varphi'(x)|^q,|\varphi'(b)|^q\right\}\right)^{\frac{1}{q}}$$
$$+\frac{(x-a)^2}{(p+1)^{\frac{1}{p}}}\|\omega\|_{[a,x],\infty}\left(\max\left\{|\varphi'(x)|^q,|\varphi'(a)|^q\right\}\right)^{\frac{1}{q}}.$$

Moreover, considering $x = \frac{a+b}{2}$, *we get*

$$\begin{aligned}
&\left|\int_a^b \omega(u)\varphi(u)du - \left(\int_a^b \omega(u)du\right)\varphi\left(\frac{a+b}{2}\right)\right| \\
\leq &\frac{(b-a)^2}{4(p+1)^{\frac{1}{p}}}\|\omega\|_{[a,b],\infty}\left(\left(\max\left\{\left|\varphi'\left(\frac{a+b}{2}\right)\right|^q, |\varphi'(b)|^q\right\}\right)^{\frac{1}{q}}\right. \\
&\left. + \left(\max\left\{\left|\varphi'\left(\frac{a+b}{2}\right)\right|^q, |\varphi'(a)|^q\right\}\right)^{\frac{1}{q}}\right).
\end{aligned}$$

Corollary 7. *In Corollary 6, if we choose* $\omega(u) = \frac{1}{b-a}$, *we obtain*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b \varphi(u)du - \varphi(x)\right| \\
\leq &\left(\frac{(b-x)^2}{(b-a)(p+1)^{\frac{1}{p}}}\left(\max\left\{|\varphi'(x)|^q, |\varphi'(b)|^q\right\}\right)^{\frac{1}{q}}\right. \\
&\left. + \frac{(x-a)^2}{(b-a)(p+1)^{\frac{1}{p}}}\left(\max\left\{|\varphi'(x)|^q, |\varphi'(a)|^q\right\}\right)^{\frac{1}{q}}\right).
\end{aligned}$$

For $x = \frac{a+b}{2}$, *we get ([22], Theorem 8).*

Theorem 10. *Consider a differentiable function* $\varphi : J \to \mathbb{R}$ *with* $\varphi' \in L(J)$ *where* $0 \leq a < b$, *and let* $\omega : J \to \mathbb{R}$ *be continuous function. If* $|\varphi'|^q$ *is quasi-convex, where* $q \geq 1$, *then*

$$\begin{aligned}
&|J^{\alpha}_{x^-}\omega\varphi(a) + J^{\alpha}_{x^+}\omega\varphi(b) - [J^{\alpha}_{x^-}\omega(a) + J^{\alpha}_{x^+}\omega(b)]\varphi(x)| \\
\leq &\frac{(b-x)^{\alpha+1}}{\Gamma(\alpha+2)}\|\omega\|_{[x,b],\infty}\left(\max\left\{|\varphi'(x)|^q, |\varphi'(b)|^q\right\}\right)^{\frac{1}{q}} \\
&+ \frac{(x-a)^{\alpha+1}}{\Gamma(\alpha+2)}\|\omega\|_{[a,x],\infty}\left(\max\left\{|\varphi'(x)|^q, |\varphi'(a)|^q\right\}\right)^{\frac{1}{q}}.
\end{aligned}$$

Proof. By properties of modulus, applying Lemma 1, power mean inequality and quasi-convexity of $|\varphi'|^q$, we obtain

$$\begin{aligned}
&|J^{\alpha}_{x^-}\omega\varphi(a) + J^{\alpha}_{x^+}\omega\varphi(b) - [J^{\alpha}_{x^-}\omega(a) + J^{\alpha}_{x^+}\omega(b)]\varphi(x)| \\
\leq &\frac{(b-x)^{\alpha+1}}{\Gamma(\alpha)}\left(\int_0^1 |p_1(\tau)|d\tau\right)^{1-\frac{1}{q}}\left(\int_0^1 |p_1(\tau)||\varphi'(\tau b + (1-\tau)x)|^q d\tau\right)^{\frac{1}{q}} \\
&+ \frac{(x-a)^{\alpha+1}}{\Gamma(\alpha)}\left(\int_0^1 |p_2(\tau)|d\tau\right)^{1-\frac{1}{q}}\left(\int_0^1 |p_2(\tau)||\varphi'(\tau a + (1-\tau)x)|^q d\tau\right)^{\frac{1}{q}} \\
\leq &\frac{(b-x)^{\alpha+1}}{\Gamma(\alpha)}\left(\int_0^1 |p_1(\tau)|d\tau\right)^{1-\frac{1}{q}}\left(\max\left\{|\varphi'(x)|^q, |\varphi'(b)|^q\right\}\int_0^1 |p_1(\tau)|d\tau\right)^{\frac{1}{q}} \\
&+ \frac{(x-a)^{\alpha+1}}{\Gamma(\alpha)}\left(\int_0^1 |p_2(\tau)|d\tau\right)^{1-\frac{1}{q}}\left(\max\left\{|\varphi'(x)|^q, |\varphi'(a)|^q\right\}\int_0^1 |p_2(\tau)|d\tau\right)^{\frac{1}{q}} \\
= &\frac{(b-x)^{\alpha+1}}{\Gamma(\alpha)}\left(\int_0^1 |p_1(\tau)|d\tau\right)\left(\max\left\{|\varphi'(x)|^q, |\varphi'(b)|^q\right\}\right)^{\frac{1}{q}}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{(x-a)^{\alpha+1}}{\Gamma(\alpha)}\left(\int_0^1|p_2(\tau)|d\tau\right)\left(\max\left\{|\varphi'(x)|^q,|\varphi'(a)|^q\right\}\right)^{\frac{1}{q}}\\
\leq&\frac{(b-x)^{\alpha+1}}{\Gamma(\alpha)}\|\omega\|_{[x,b],\infty}\left(\int_0^1\left(\int_\tau^1(1-\sigma)^{\alpha-1}d\sigma\right)d\tau\right)\left(\max\left\{|\varphi'(x)|^q,|\varphi'(b)|^q\right\}\right)^{\frac{1}{q}}\\
&+\frac{(x-a)^{\alpha+1}}{\Gamma(\alpha)}\|\omega\|_{[a,x],\infty}\left(\int_0^1\left(\int_\tau^1(1-\sigma)^{\alpha-1}d\sigma\right)d\tau\right)\left(\max\left\{|\varphi'(x)|^q,|\varphi'(a)|^q\right\}\right)^{\frac{1}{q}}\\
=&\frac{(b-x)^{\alpha+1}}{\Gamma(\alpha+2)}\|\omega\|_{[x,b],\infty}\left(\max\left\{|\varphi'(x)|^q,|\varphi'(b)|^q\right\}\right)^{\frac{1}{q}}\\
&+\frac{(x-a)^{\alpha+1}}{\Gamma(\alpha+2)}\|\omega\|_{[a,x],\infty}\left(\max\left\{|\varphi(x)|^q,|\varphi'(a)|^q\right\}\right)^{\frac{1}{q}}.
\end{aligned}
$$

The proof is completed. □

Corollary 8. *For $x=\frac{a+b}{2}$ in Theorem 10, we get*

$$
\begin{aligned}
&\left|J^{\alpha}_{\frac{a+b}{2}-}\omega\varphi(a)+J^{\alpha}_{\frac{a+b}{2}+}\omega\varphi(b)-\left[J^{\alpha}_{\frac{a+b}{2}-}\omega(a)+J^{\alpha}_{\frac{a+b}{2}+}\omega(b)\right]\varphi\left(\frac{a+b}{2}\right)\right|\\
\leq&\frac{(b-a)^{\alpha+1}}{2^{\alpha+1}\Gamma(\alpha+2)}\|\omega\|_{[a,b],\infty}\left(\left(\max\left\{\left|\varphi'\left(\frac{a+b}{2}\right)\right|^q,|\varphi'(b)|^q\right\}\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\max\left\{\left|\varphi'\left(\frac{a+b}{2}\right)\right|^q,|\varphi(a)|^q\right\}\right)^{\frac{1}{q}}\right).
\end{aligned}
$$

Corollary 9. *Choosing $\omega(u)=\frac{1}{b-a}$ in Theorem 10, we have*

$$
\begin{aligned}
&\left|J^{\alpha}_{x-}\varphi(a)+J^{\alpha}_{x+}\varphi(b)-\frac{(b-x)^{\alpha}+(x-a)^{\alpha}}{\Gamma(\alpha+1)}\varphi(x)\right|\\
\leq&\frac{(b-x)^{\alpha+1}}{\Gamma(\alpha+2)}\left(\max\left\{|\varphi'(x)|^q,|\varphi'(b)|^q\right\}\right)^{\frac{1}{q}}\\
&+\frac{(x-a)^{\alpha+1}}{\Gamma(\alpha+2)}\left(\max\left\{|\varphi'(x)|^q,|\varphi'(a)|^q\right\}\right)^{\frac{1}{q}}.
\end{aligned}
$$

Moreover, if we take $x=\frac{a+b}{2}$, we obtain

$$
\begin{aligned}
&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left(J^{\alpha}_{\frac{a+b}{2}-}\varphi(a)+J^{\alpha}_{\frac{a+b}{2}+}\varphi(b)\right)-\varphi\left(\frac{a+b}{2}\right)\right|\leq\frac{b-a}{4(\alpha+1)}\\
&\times\left(\left(\max\left\{\left|\varphi'\left(\frac{a+b}{2}\right)\right|^q,|\varphi'(b)|^q\right\}\right)^{\frac{1}{q}}+\left(\max\left\{\left|\varphi'\left(\frac{a+b}{2}\right)\right|^q,|\varphi'(a)|^q\right\}\right)^{\frac{1}{q}}\right).
\end{aligned}
$$

Corollary 10. *Considering $\alpha=1$ in Theorem 10, we get*

$$
\begin{aligned}
&\left|\int_a^b\omega(u)\varphi(u)du-\left(\int_a^b\omega(u)du\right)\varphi(x)\right|\\
\leq&\frac{(b-x)^2}{2}\|\omega\|_{[x,b],\infty}\left(\max\left\{|\varphi'(x)|^q,|\varphi'(b)|^q\right\}\right)^{\frac{1}{q}}\\
&+\frac{(x-a)^2}{2}\|\omega\|_{[a,x],\infty}\left(\max\left\{|\varphi'(x)|^q,|\varphi'(a)|^q\right\}\right)^{\frac{1}{q}}.
\end{aligned}
$$

Moreover, for $x = \frac{a+b}{2}$, *we have*

$$\begin{aligned}&\left|\int_a^b \omega(u)\varphi(u)du - \left(\int_a^b \omega(u)du\right)\varphi\left(\frac{a+b}{2}\right)\right|\\ \leq &\frac{(b-a)^2}{8}\|\omega\|_{[a,b],\infty}\left(\left(\max\left\{\left|\varphi'\left(\frac{a+b}{2}\right)\right|^q, |\varphi'(b)|^q\right\}\right)^{\frac{1}{q}}\right.\\ &\left.+\left(\max\left\{\left|\varphi'\left(\frac{a+b}{2}\right)\right|^q, |\varphi'(a)|^q\right\}\right)^{\frac{1}{q}}\right).\end{aligned}$$

Remark 2. *Choosing* $\omega(u) = \frac{1}{b-a}$ *in Corollary 10, we obtain ([23], Theorem 4). Moreover, for* $x = \frac{a+b}{2}$, *we get ([22], Theorem 9).*

3. Further Results

In this section, we will prove the following results.

Theorem 11. *Consider a differentiable function* $\varphi : J \to \mathbb{R}$ *with* $\varphi' \in L(J)$, $0 \leq a < b$, *and let* $\omega : J \to \mathbb{R}$ *be continuous function. If there exist constants* $m < M$ *such that* $-\infty < m \leq \varphi'(x) \leq M < +\infty$ *for all* $x \in J$, *then*

$$|\Lambda^\alpha(a,x,b,\omega,\varphi)| \leq \frac{(M-m)(b-x)^{\alpha+1}}{2\Gamma(\alpha+2)}\|\omega\|_{[x,b],\infty} + \frac{(M-m)(x-a)^{\alpha+1}}{2\Gamma(\alpha+2)}\|\omega\|_{[a,x],\infty},$$

where

$$\begin{aligned}\Lambda^\alpha(a,x,b,\omega,\varphi) :=& J^\alpha_{x^-}\omega\varphi(a) + J^\alpha_{x^+}\omega\varphi(b) - \left[J^\alpha_{x^-}\omega(a) + J^\alpha_{x^+}\omega(b)\right]\varphi(x)\\ &- \frac{(M+m)\left[(b-x)^{\alpha+1} - (x-a)^{\alpha+1}\right]}{2\Gamma(\alpha)}\int_0^1 (p_1(\tau)+p_2(\tau))d\tau, \qquad (8)\end{aligned}$$

and p_1, p_2 *are defined as in* (2) *and* (3), *respectively.*

Proof. Applying Lemma 1, we get

$$\begin{aligned}&J^\alpha_{x^-}\omega\varphi(a) + J^\alpha_{x^+}\omega\varphi(b) - \left[J^\alpha_{x^-}\omega(a) + J^\alpha_{x^+}\omega(b)\right]\varphi(x)\\ =&\frac{(b-x)^{\alpha+1}}{\Gamma(\alpha)}\int_0^1 p_1(\tau)\varphi'(\tau b + (1-\tau)x)d\tau - \frac{(x-a)^{\alpha+1}}{\Gamma(\alpha)}\int_0^1 p_2(\tau)\varphi'(\tau a + (1-\tau)x)d\tau\\ =&\frac{(b-x)^{\alpha+1}}{\Gamma(\alpha)}\int_0^1 p_1(\tau)\left[\varphi'(\tau b + (1-\tau)x) - \frac{M+m}{2} + \frac{M+m}{2}\right]d\tau\\ &- \frac{(x-a)^{\alpha+1}}{\Gamma(\alpha)}\int_0^1 p_2(\tau)\left[\varphi'(\tau a + (1-\tau)x) - \frac{M+m}{2} + \frac{M+m}{2}\right]d\tau\\ =&\frac{(b-x)^{\alpha+1}}{\Gamma(\alpha)}\int_0^1 p_1(\tau)\left(\varphi'(\tau b + (1-\tau)x) - \frac{M+m}{2}\right)d\tau\\ &+ \frac{(M+m)(b-x)^{\alpha+1}}{2\Gamma(\alpha)}\int_0^1 p_1(\tau)d\tau\end{aligned}$$

$$- \frac{(x-a)^{\alpha+1}}{\Gamma(\alpha)} \int_0^1 p_2(\tau)\left(\varphi'(\tau a + (1-\tau)x) - \frac{M+m}{2}\right)d\tau$$
$$- \frac{(M+m)(x-a)^{\alpha+1}}{2\Gamma(\alpha)} \int_0^1 p_2(\tau)d\tau. \tag{9}$$

From (9), we have

$$\Lambda^{\alpha}(a,x,b,\omega,\varphi) = \frac{(b-x)^{\alpha+1}}{\Gamma(\alpha)} \int_0^1 p_1(\tau)\left(\varphi'(\tau b + (1-\tau)x) - \frac{M+m}{2}\right)d\tau$$
$$- \frac{(x-a)^{\alpha+1}}{\Gamma(\alpha)} \int_0^1 p_2(\tau)\left(\varphi'(\tau a + (1-\tau)x) - \frac{M+m}{2}\right)d\tau, \tag{10}$$

where $\Lambda^{\alpha}(a,x,b,\omega,\varphi)$ is defined as in (8). Applying absolute value on both sides of (10) and using the fact that $m \le \varphi'(x) \le M$ for all $x \in J$, we obtain

$$\begin{aligned}
&|\Lambda^{\alpha}(a,x,b,\omega,\varphi)| \\
&\le \frac{(b-x)^{\alpha+1}}{\Gamma(\alpha)} \int_0^1 |p_1(\tau)|\left|\varphi'(\tau b + (1-\tau)x) - \frac{M+m}{2}\right|d\tau \\
&\quad + \frac{(x-a)^{\alpha+1}}{\Gamma(\alpha)} \int_0^1 |p_2(\tau)|\left|\varphi'(\tau a + (1-\tau)x) - \frac{M+m}{2}\right|d\tau \\
&\le \frac{(M-m)(b-x)^{\alpha+1}}{2\Gamma(\alpha)} \int_0^1 \left|\int_\tau^1 (1-\sigma)^{\alpha-1}\omega(\sigma b + (1-\sigma)x)d\sigma\right|d\tau \\
&\quad + \frac{(M-m)(x-a)^{\alpha+1}}{2\Gamma(\alpha)} \int_0^1 \left|\int_\tau^1 (1-\sigma)^{\alpha-1}\omega(\sigma a + (1-\sigma)x)d\sigma\right|d\tau \\
&\le \frac{(M-m)(b-x)^{\alpha+1}}{2\Gamma(\alpha+2)}\|\omega\|_{[x,b],\infty} + \frac{(M-m)(x-a)^{\alpha+1}}{2\Gamma(\alpha+2)}\|\omega\|_{[a,x],\infty}.
\end{aligned}$$

The proof is completed. □

Corollary 11. *For $x = \frac{a+b}{2}$ in Theorem 11, we have*

$$\begin{aligned}
&\left|J^{\alpha}_{\frac{a+b}{2}-}\omega\varphi(a) + J^{\alpha}_{\frac{a+b}{2}+}\omega\varphi(b) - \left[J^{\alpha}_{\frac{a+b}{2}-}\omega(a) + J^{\alpha}_{\frac{a+b}{2}+}\omega(b)\right]\varphi\left(\frac{a+b}{2}\right)\right| \\
&\le \frac{(M-m)(b-a)^{\alpha+1}}{2^{\alpha+1}\Gamma(\alpha+2)}\|\omega\|_{[a,b],\infty}.
\end{aligned}$$

Corollary 12. *Putting $\omega(u) = \frac{1}{b-a}$ in Theorem 11, we get*

$$\left|J^{\alpha}_{x^-}\varphi(a) + J^{\alpha}_{x^+}\varphi(b) - \frac{(b-x)^{\alpha} + (x-a)^{\alpha}}{\Gamma(\alpha+1)}\varphi(x) - \frac{(M+m)\left[(b-x)^{\alpha+1} - (x-a)^{\alpha+1}\right]}{2\Gamma(\alpha+2)}\right|$$
$$\le \frac{(M-m)\left[(b-x)^{\alpha+1} + (x-a)^{\alpha+1}\right]}{2\Gamma(\alpha+2)}.$$

Moreover, if we take $x = \frac{a+b}{2}$, *we obtain*

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left(J^{\alpha}_{\frac{a+b}{2}-}\varphi(a)+J^{\alpha}_{\frac{a+b}{2}+}\varphi(b)\right)-\varphi\left(\frac{a+b}{2}\right)\right|\leq\frac{(M-m)(b-a)}{4(\alpha+1)}.$$

Corollary 13. *Choosing* $\alpha = 1$ *in Theorem 11, we have*

$$\begin{aligned}&\left|\int_a^b\omega(u)\varphi(u)du-\left(\int_a^b\omega(u)du\right)\varphi(x)\right.\\&\left.-\frac{(M+m)\left[(b-x)^2-(x-a)^2\right]}{2}\left(\int_\tau^1(\omega(\sigma a+(1-\sigma)x)+\omega(\sigma b+(1-\sigma)x))d\sigma\right)d\tau\right|\\\leq&\frac{(M-m)}{4}\left((b-x)^2\|\omega\|_{[x,b],\infty}+(x-a)^2\|\omega\|_{[a,x],\infty}\right).\end{aligned}$$

Moreover, for $x = \frac{a+b}{2}$, *we get*

$$\left|\int_a^b\omega(u)\varphi(u)du-\left(\int_a^b\omega(u)du\right)\varphi\left(\frac{a+b}{2}\right)\right|\leq\frac{(M-m)(b-a)^2}{8}\|\omega\|_{[a,b],\infty}.$$

Corollary 14. *Let* $\omega(u) = \frac{1}{b-a}$ *in Corollary 13, then*

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b\varphi(u)du-\varphi(x)-\frac{(M+m)\left[(b-x)^2-(x-a)^2\right]}{4(b-a)}\right|\\\leq&\frac{\left[(b-x)^2+(x-a)^2\right](M-m)}{4(b-a)}.\end{aligned}$$

For $x = \frac{a+b}{2}$, *then*

$$\left|\frac{1}{b-a}\int_a^b\varphi(u)du-\varphi\left(\frac{a+b}{2}\right)\right|\leq\frac{(b-a)(M-m)}{8}.$$

Before giving our next result, we recall that a function $\varphi : J \to \mathbb{R}$ is r-H-Hölder (Hölder condition, see [44]), if

$$|\varphi(x)-\varphi(y)|\leq H|x-y|^r$$

holds for all $x, y \in J$, for some $H > 0$ and $r \in (0, 1]$.

Theorem 12. *Consider a differentiable function* $\varphi : J \to \mathbb{R}$ *with* $\varphi' \in L(J)$, $0 \leq a < b$, *and let* $\omega : J \to \mathbb{R}$ *be continuous function. If* φ' *satisfies Hölder condition for some* $H > 0$ *and* $r \in (0, 1]$, *then*

$$|\Xi^{\alpha}(a,x,b,\omega,\varphi)|\leq H\left(\frac{(b-x)^{\alpha+r+1}}{(\alpha+r+1)\Gamma(\alpha+1)}\|\omega\|_{[x,b],\infty}+\frac{(x-a)^{\alpha+r+1}}{(\alpha+r+1)\Gamma(\alpha+1)}\|\omega\|_{[a,x],\infty}\right),$$

where

$$\Xi^{\alpha}(a,x,b,\omega,\varphi) := J^{\alpha}_{x^-}\omega\varphi(a) + J^{\alpha}_{x^+}\omega\varphi(b) - \left[J^{\alpha}_{x^-}\omega(a) + J^{\alpha}_{x^+}\omega(b)\right]\varphi(x) - \frac{\varphi'(b)(b-x)^{\alpha+1} - \varphi'(a)(x-a)^{\alpha+1}}{2\Gamma(\alpha)}\int_0^1 (p_1(\tau)+p_2(\tau))d\tau, \tag{11}$$

and p_1, p_2 are defined as in (2) and (3), respectively.

Proof. Applying Lemma 1, we have

$$\begin{aligned}
&J^{\alpha}_{x^-}\omega\varphi(a) + J^{\alpha}_{x^+}\omega\varphi(b) - \left[J^{\alpha}_{x^-}\omega(a) + J^{\alpha}_{x^+}\omega(b)\right]\varphi(x)\\
=&\frac{(b-x)^{\alpha+1}}{\Gamma(\alpha)}\int_0^1 p_1(\tau)\varphi'(\tau b+(1-\tau)x)d\tau - \frac{(x-a)^{\alpha+1}}{\Gamma(\alpha)}\int_0^1 p_2(\tau)\varphi'(\tau a+(1-\tau)x)d\tau\\
=&\frac{(b-x)^{\alpha+1}}{\Gamma(\alpha)}\int_0^1 p_1(\tau)\left[\varphi'(\tau b+(1-\tau)x) - \varphi'(b) + \varphi'(b)\right]d\tau\\
&-\frac{(x-a)^{\alpha+1}}{\Gamma(\alpha)}\int_0^1 p_2(\tau)\left[\varphi'(\tau a+(1-\tau)x) - \varphi'(a) + \varphi'(a)\right]d\tau\\
=&\frac{(b-x)^{\alpha+1}}{\Gamma(\alpha)}\int_0^1 p_1(\tau)\left(\varphi'(\tau b+(1-\tau)x) - \varphi'(b)\right)d\tau\\
&+\frac{\varphi'(b)(b-x)^{\alpha+1}}{2\Gamma(\alpha)}\int_0^1 p_1(\tau)d\tau\\
&-\frac{(x-a)^{\alpha+1}}{\Gamma(\alpha)}\int_0^1 p_2(\tau)\left(\varphi'(\tau a+(1-\tau)x) - \varphi'(a)\right)d\tau\\
&-\frac{\varphi'(a)(x-a)^{\alpha+1}}{2\Gamma(\alpha)}\int_0^1 p_2(\tau)d\tau.
\end{aligned} \tag{12}$$

From (12), we get

$$\begin{aligned}
\Xi^{\alpha}(a,x,b,\omega,\varphi) =& \frac{(b-x)^{\alpha+1}}{\Gamma(\alpha)}\int_0^1 p_1(\tau)\left(\varphi'(\tau b+(1-\tau)x) - \varphi'(b)\right)d\tau\\
&-\frac{(x-a)^{\alpha+1}}{\Gamma(\alpha)}\int_0^1 p_2(\tau)\left(\varphi'(\tau a+(1-\tau)x) - \varphi'(a)\right)d\tau,
\end{aligned} \tag{13}$$

where $\Xi^{\alpha}(a,x,b,\omega,\varphi)$ is defined as in (11). Applying absolute value on both sides of (13) and r-H-Hölder property of φ', we obtain

$$\begin{aligned}
&|\Xi^{\alpha}(a,x,b,\omega,\varphi)|\\
\leq&\frac{(b-x)^{\alpha+1}}{\Gamma(\alpha)}\int_0^1 |p_1(\tau)|\left|\varphi'(\tau b+(1-\tau)x) - \varphi'(b)\right|d\tau\\
&+\frac{(x-a)^{\alpha+1}}{\Gamma(\alpha)}\int_0^1 |p_2(\tau)|\left|\varphi'(\tau a+(1-\tau)x) - \varphi'(a)\right|d\tau
\end{aligned}$$

$$\leq H\left(\frac{(b-x)^{\alpha+r+1}}{\Gamma(\alpha+1)}\|\omega\|_{[x,b],\infty}+\frac{(x-a)^{\alpha+r+1}}{\Gamma(\alpha+1)}\|\omega\|_{[a,x],\infty}\right)\int_0^1(1-\tau)^{\alpha+r}d\tau$$
$$=H\left(\frac{(b-x)^{\alpha+r+1}}{(\alpha+r+1)\Gamma(\alpha+1)}\|\omega\|_{[x,b],\infty}+\frac{(x-a)^{\alpha+r+1}}{(\alpha+r+1)\Gamma(\alpha+1)}\|\omega\|_{[a,x],\infty}\right).$$

The proof is completed. □

Corollary 15. *Letting* $x=\frac{a+b}{2}$ *in Theorem 12, we have*

$$\left|J^{\alpha}_{\frac{a+b}{2}-}\omega\varphi(a)+J^{\alpha}_{\frac{a+b}{2}+}\omega\varphi(b)-\left[J^{\alpha}_{\frac{a+b}{2}-}\omega(a)+J^{\alpha}_{\frac{a+b}{2}+}\omega(b)\right]\varphi\left(\frac{a+b}{2}\right)\right|$$
$$\leq H\frac{(b-a)^{\alpha+r+1}}{2^{\alpha+r}(\alpha+r+1)\Gamma(\alpha+1)}\|\omega\|_{[a,b],\infty}$$
$$+\frac{(b-a)^{\alpha+1}(\varphi'(b)-\varphi'(a))}{2^{\alpha+2}\Gamma(\alpha)}\int_0^1(p_1(\tau)+p_2(\tau))d\tau.$$

Corollary 16. *Choosing* $\omega(u)=\frac{1}{b-a}$ *in Theorem 12, we get*

$$\left|J^{\alpha}_{x-}\varphi(a)+J^{\alpha}_{x+}\varphi(b)-\frac{(b-x)^{\alpha}+(x-a)^{\alpha}}{\Gamma(\alpha+1)}\varphi(x)\right|$$
$$\leq H\frac{(b-x)^{\alpha+r+1}+(x-a)^{\alpha+r+1}}{(\alpha+r+1)\Gamma(\alpha+1)}+\frac{\varphi'(b)(b-x)^{\alpha+1}-\varphi'(a)(x-a)^{\alpha+1}}{2\Gamma(\alpha+2)}.$$

Moreover, if we take $x=\frac{a+b}{2}$, *we obtain*

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left(J^{\alpha}_{\frac{a+b}{2}-}\varphi(a)+J^{\alpha}_{\frac{a+b}{2}+}\varphi(b)\right)-\varphi\left(\frac{a+b}{2}\right)\right|$$
$$\leq H\frac{(b-a)^{r+1}}{2^{1+r}(\alpha+r+1)}+\frac{(b-a)(\varphi'(b)-\varphi'(a))}{8(\alpha+1)}.$$

Corollary 17. *For* $\alpha=1$ *in Theorem 12, we have*

$$\left|\int_a^b\omega(u)\varphi(u)du-\left(\int_a^b\omega(u)du\right)\varphi(x)\right|$$
$$\leq H\frac{(b-x)^{r+2}\|\omega\|_{[x,b],\infty}+(x-a)^{r+2}\|\omega\|_{[a,x],\infty}}{r+2}$$
$$+\frac{\varphi'(b)(b-x)^2-\varphi'(a)(x-a)^2}{2}$$
$$\times\int_0^1\left(\int_\tau^1(\omega(\sigma a+(1-\sigma)x)+\omega(\sigma b+(1-\sigma)x))d\sigma\right)d\tau.$$

Moreover, for $x = \frac{a+b}{2}$, *we get*

$$\begin{aligned}&\left|\int_a^b \omega(u)\varphi(u)du - \left(\int_a^b \omega(u)du\right)\varphi\left(\frac{a+b}{2}\right)\right| \\ \leq & H\frac{(b-a)^{r+2}}{2^{r+1}(r+2)}\|\omega\|_{[a,b],\infty} + \frac{\varphi'(b)(b-a)^2 - \varphi'(a)(b-a)^2}{8} \\ & \times \int_0^1 \left(\int_\tau^1 \left(\omega\left(\sigma a + (1-\sigma)\frac{a+b}{2}\right) + \omega\left(\sigma b + (1-\sigma)\frac{a+b}{2}\right)\right)d\sigma\right)d\tau.\end{aligned}$$

Corollary 18. *In Corollary 17, if we choose* $\omega(u) = \frac{1}{b-a}$, *we obtain*

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b \varphi(u)du - \varphi(x)\right| \\ \leq & H\frac{(b-x)^{r+2} + (x-a)^{r+2}}{(r+2)(b-a)} + \frac{\varphi'(b)(b-x)^2 - \varphi'(a)(x-a)^2}{4(b-a)}.\end{aligned}$$

Moreover, if we take $x = \frac{a+b}{2}$, *we have*

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b \varphi(u)du - \varphi\left(\frac{a+b}{2}\right)\right| \\ \leq & H\frac{(b-a)^{r+2}}{2^{r+1}(r+2)} + \frac{(\varphi'(b) - \varphi'(a))(b-a)}{16}.\end{aligned}$$

Corollary 19. *Let* φ' *be a L–Lipschitzian function* ($H = L$ *and* $r = 1$) *in Theorem 12, then*

$$|\Xi^\alpha(a,x,b,\omega,\varphi)| \leq L\left(\frac{(b-x)^{\alpha+2}}{(\alpha+2)\Gamma(\alpha+1)}\|\omega\|_{[x,b],\infty} + \frac{(x-a)^{\alpha+2}}{(\alpha+2)\Gamma(\alpha+1)}\|\omega\|_{[a,x],\infty}\right),$$

where Ξ^α *is defined by* (11).

Corollary 20. *For* $x = \frac{a+b}{2}$ *in Corollary 19, we get*

$$\begin{aligned}&\left|J^\alpha_{\frac{a+b}{2}-}\omega\varphi(a) + J^\alpha_{\frac{a+b}{2}+}\omega\varphi(b) - \left[J^\alpha_{\frac{a+b}{2}-}\omega(a) + J^\alpha_{\frac{a+b}{2}+}\omega(b)\right]\varphi\left(\frac{a+b}{2}\right)\right| \\ \leq & L\frac{(b-a)^{\alpha+2}}{2^{\alpha+1}(\alpha+2)\Gamma(\alpha+1)}\|\omega\|_{[a,b],\infty} \\ & + \frac{(b-a)^{\alpha+1}(\varphi'(b) - \varphi'(a))}{2^{\alpha+2}\Gamma(\alpha)}\int_0^1 (p_1(\tau) + p_2(\tau))d\tau.\end{aligned}$$

Corollary 21. *In Corollary 19, if we choose* $\omega(u) = \frac{1}{b-a}$, *we obtain*

$$\begin{aligned}&\left|J^\alpha_{x^-}\varphi(a) + J^\alpha_{x^+}\varphi(b) - \frac{(b-x)^\alpha + (x-a)^\alpha}{\Gamma(\alpha+1)}\varphi(x)\right| \\ \leq & L\frac{(b-x)^{\alpha+2} + (x-a)^{\alpha+2}}{(\alpha+2)\Gamma(\alpha+1)} + \frac{\varphi'(b)(b-x)^{\alpha+1} - \varphi'(a)(x-a)^{\alpha+1}}{2\Gamma(\alpha+2)}.\end{aligned}$$

Moreover, if we take $x = \frac{a+b}{2}$, *we have*

$$\begin{aligned}&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left(J^{\alpha}_{\frac{a+b}{2}-}\varphi(a)+J^{\alpha}_{\frac{a+b}{2}+}\varphi(b)\right)-\varphi\left(\frac{a+b}{2}\right)\right|\\ \leq& L\frac{(b-a)^2}{4(\alpha+2)}+\frac{(b-a)(\varphi'(b)-\varphi'(a))}{8(\alpha+1)}.\end{aligned}$$

Corollary 22. *Choosing* $\alpha = 1$ *in Corollary 19, we get*

$$\begin{aligned}&\left|\int_a^b\omega(u)\varphi(u)du-\left(\int_a^b\omega(u)du\right)\varphi(x)\right|\\ \leq& L\frac{(b-x)^3\|\omega\|_{[x,b],\infty}+(x-a)^3\|\omega\|_{[a,x],\infty}}{3}\\ &+\frac{\varphi'(b)(b-x)^2-\varphi'(a)(x-a)^2}{2}\\ &\times\int_0^1\left(\int_\tau^1(\omega(\sigma a+(1-\sigma)x)+\omega(\sigma b+(1-\sigma)x))d\sigma\right)d\tau.\end{aligned}$$

Moreover, for $x = \frac{a+b}{2}$, *we obtain*

$$\begin{aligned}&\left|\int_a^b\omega(u)\varphi(u)du-\left(\int_a^b\omega(u)du\right)\varphi\left(\frac{a+b}{2}\right)\right|\\ \leq& L\frac{(b-a)^3}{12}\|\omega\|_{[a,b],\infty}+\frac{\varphi'(b)(b-a)^2-\varphi'(a)(b-a)^2}{8}\\ &\times\int_0^1\left(\int_\tau^1\left(\omega\left(\sigma a+(1-\sigma)\frac{a+b}{2}\right)+\omega\left(\sigma b+(1-\sigma)\frac{a+b}{2}\right)\right)d\sigma\right)d\tau.\end{aligned}$$

Corollary 23. *For* $\omega(u) = \frac{1}{b-a}$ *in Corollary 22, we have*

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b\varphi(u)du-\varphi(x)\right|\\ \leq& L\frac{(b-x)^3+(x-a)^3}{3(b-a)}+\frac{\varphi'(b)(b-x)^2-\varphi'(a)(x-a)^2}{4(b-a)}.\end{aligned}$$

Moreover, for $x = \frac{a+b}{2}$, *we get*

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b\varphi(u)du-\varphi\left(\frac{a+b}{2}\right)\right|\\ \leq& L\frac{(b-a)^3}{12}+\frac{(\varphi'(b)-\varphi'(a))(b-a)}{16}.\end{aligned}$$

4. Applications

We shall consider the following special means for different positive real numbers a and b, where $a < b$:

- Arithmetic mean: $\mathcal{A}(a,b) = \frac{a+b}{2}$.
- p–Logarithmic mean: $\mathcal{L}_p(a,b) = \left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}}$, $\quad p \in \mathbb{Z}\setminus\{0,-1\}$.

Let recall from [23] the following quasi-convex functions, $\varphi(u) = u^n$, $n \in \mathbb{N}$, $n \geq 2$ and $\varphi(u) = \frac{1}{u}$ for all $u > 0$, respectively.

Using Section 2, we are in position to prove the following results regarding above special means.

Proposition 1. *Let $a, b \in \mathbb{R}$, $0 < a < b$, $n \in \mathbb{N}$ and $n \geq 2$, then*

$$\left|\mathcal{L}_{n+1}^{n+1}(a,b) - \mathcal{A}^{n+1}(a,b)\right| \leq \frac{nb(b-a)}{4}\mathcal{A}\Big(\mathcal{A}^{n-1}(a,b), b^{n-1}\Big).$$

Proof. Taking $\varphi(u) = u^n$, $u > 0$ for $x = \frac{a+b}{2}$ and $\omega(u) = u$ in Corollary 3, we get the desired result. □

Proposition 2. *Let $a, b \in \mathbb{R}$, $0 < a < b$, $n \in \mathbb{N}$ and $n \geq 2$, then*

$$\left|\mathcal{L}_{n-1}^{n-1}(a,b) - \frac{\mathcal{L}_n^n(a,b)}{\mathcal{A}(a,b)}\right| \leq \frac{b^n(b-a)}{4}\mathcal{A}\left(\frac{1}{a^2}, \frac{1}{\mathcal{A}^2(a,b)}\right).$$

Proof. Choosing $\varphi(u) = \frac{1}{u}$, $u > 0$ for $x = \frac{a+b}{2}$ and $\omega(u) = u^n$ in Corollary 3, we obtain the desired result. □

Proposition 3. *Let $a, b \in \mathbb{R}$, $0 < a < b$, $p > 1$, $n \in \mathbb{N}$ and $n \geq 2$, then*

$$\left|\mathcal{L}_{n+1}^{n+1}(a,b) - \mathcal{A}^{n+1}(a,b)\right| \leq \frac{nb(b-a)}{2(p+1)^{\frac{1}{p}}}\mathcal{A}\Big(\mathcal{A}^{n-1}(a,b), b^{n-1}\Big).$$

Proof. Taking $\varphi(u) = u^n$, $u > 0$ for $x = \frac{a+b}{2}$ and $\omega(u) = u$ in Corollary 6, we get the desired result. □

Proposition 4. *Let $a, b \in \mathbb{R}$, $0 < a < b$, $p > 1$, $n \in \mathbb{N}$ and $n \geq 2$, then*

$$\left|\mathcal{L}_{n-1}^{n-1}(a,b) - \frac{\mathcal{L}_n^n(a,b)}{\mathcal{A}(a,b)}\right| \leq \frac{b^n(b-a)}{2(p+1)^{\frac{1}{p}}}\mathcal{A}\left(\frac{1}{a^2}, \frac{1}{\mathcal{A}^2(a,b)}\right).$$

Proof. Choosing $\varphi(u) = \frac{1}{u}$, $u > 0$ for $x = \frac{a+b}{2}$ and $\omega(u) = u^n$ in Corollary 6, we obtain the desired result. □

5. Conclusions

The main results and future research of the article can be summarized as follows:

- A new identity regarding fractional weighted Ostrowski-type is established.
- New fractional weighted Ostrowski-type inequalities for quasi-convex functions using the above identity are deduced.
- Several further results for function with a bounded first derivative are given.
- Some applications to special means are obtained.
- The efficiency of our results is shown.
- As future research, from our results, interested reader can find several new interesting inequalities from many areas of pure and applied sciences. Moreover, they can derive (using our technique) applications to special means for different quasi-convex functions.

Author Contributions: Conceptualization, A.K., B.M., P.O.M., T.A.; methodology, B.M., P.O.M., A.A.L., Y.S.H.; software, P.O.M., Y.S.H., A.K.; validation, P.O.M., Y.S.H., B.A.; formal analysis, P.O.M., Y.S.H., T.A.; investigation, P.O.M.; resources, P.O.M., A.K., A.A.L.; data curation, P.O.M., Y.S.H.; writing—original draft preparation, B.M., Y.S.H.; writing—review and editing, Y.S.H., P.O.M., B.A.; visualization, Y.S.H.; supervision, P.O.M., Y.S.H., T.A., A.K. All authors have read and agreed to the final version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: This work was supported by the Taif University Researchers Supporting Project (No. TURSP-2020/155), Taif University, Taif, Saudi Arabia, and the authors Bahaaeldin Abdalla and Thabet Abdeljawad would like to thank Prince Sultan University for funding this work through research group Nonlinear Analysis Methods in Applied Mathematics (NAMAM) group number RG-DES-2017-01-17.

Conflicts of Interest: The authors declare no conflict of interest.

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