Article

Fixed Points of Proinov E -Contractions

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Abstract: In this paper, we consider a new type of Proinov contraction on the setting of a symmetrical abstract structure, more precisely, the metric space. Our goal is to expand on some results from the literature using admissible mappings and the concept of E -contraction. The considered examples indicate the validity of the obtained results.

Keywords: Proinov type contraction; E -contraction; alpha-admissible

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1. Introduction and Preliminaries

Fixed point theory is one of the most dynamic research topics of the last two decades. New and interesting results are obtained, following especially two directions: changing the frame (the structure of the abstract space—e.g., b -metric, delta symmetric quasi-metric or non-symmetric metric space, etc.) or changing the property of the operators.

The notion of E -contraction was introduced by Fulga and Proca [1]. Later, this concept has been improved by several authors, e.g., [2–4]. Undoubtedly, one of the most interesting, most original, most impressive fixed point theorem published in the last two decades is the result of Proinov [5]. By using certain auxiliary functions, Proinov [5] obtained interesting fixed point theorems that generalize, extend, and unify several recent fixed point results in the literature.

In this paper, we shall propose a new type of contraction, namely, Proinov type E -contraction, which combines the Proinov approach and the E -contraction setting.

First, we recall the basic results and definitions.

Definition 1. Let (X, d) be a metric space and the functions $\vartheta, \theta : (0, \infty) \rightarrow \mathbb{R}$. A mapping $\mathcal{T} : X \rightarrow X$ is said to be a Proinov type contraction if

$$\vartheta(d(\mathcal{T}x, \mathcal{T}y)) \leq \theta(d(x, y)), \quad (1)$$

for all $x, y \in X$ with $d(\mathcal{T}x, \mathcal{T}y) > 0$.

Theorem 1 ([5]). Let (X, d) be a complete metric space and $\mathcal{T} : X \rightarrow X$ be a Proinov type contraction, where the functions $\vartheta, \theta : (0, \infty) \rightarrow \mathbb{R}$ are such that the following conditions are satisfied:

- (1) ϑ is non-decreasing;
- (2) $\theta(s) < \vartheta(s)$ for any $s > 0$;
- (3) $\limsup_{s \rightarrow s_0+} \theta(s) < \vartheta(s_0+)$ for any $s_0 > 0$,

then $\mathcal{T}$ admits a unique fixed point.

Definition 2. Let (X, d) be a metric space and the functions $\vartheta, \theta : (0, \infty) \rightarrow \mathbb{R}$. A mapping $\mathcal{T} : X \rightarrow X$ is said to be a generalized Proinov type contraction if

$$\vartheta(d(Tx, Ty)) \leq \vartheta(\max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\right\}), \quad (2)$$

for all $x, y \in X$ with $d(Tx, Ty) > 0$.

Theorem 2 ([5]). Let (X, d) be a complete metric space and $T : X \rightarrow X$ be a generalized Proinov type contraction, where the functions $\vartheta, \theta : (0, \infty) \rightarrow \mathbb{R}$ are such that the following conditions are satisfied:

- (1) ϑ is non-decreasing and $\theta(s) < \vartheta(s)$ for any $s > 0$;
- (2) $\theta(s_0) < \lim_{s \rightarrow s_0} \vartheta(s)$, for any $s_0 > 0$;
- (3) $\limsup_{s \rightarrow s_0+} \theta(s) < \vartheta(s_0+)$ for any $s_0 > 0$.

Then T admits a unique fixed point.

Remark 1. Notice that in [5], the author did not put the completeness assumption of the metric space; but, he used in the proof.

Lemma 1 ([5]). Let $\{x_m\}$ be a sequence in a metric space (X, d) such that $d(x_m, x_{m+1}) \rightarrow 0$ as $m \rightarrow \infty$. If the sequence $\{x_m\}$ is not Cauchy, then there exist $\epsilon > 0$ and the subsequences $\{m_i\}$ and $\{p_i\}$ of positive integers such that

$$\begin{aligned} \lim_{i \rightarrow \infty} d(x_{m_i}, x_{p_i}) &= \lim_{i \rightarrow \infty} d(x_{m_i+1}, x_{p_i}) = \lim_{i \rightarrow \infty} d(x_{m_i}, x_{p_i+1}) \\ &= \lim_{i \rightarrow \infty} d(x_{m_i+1}, x_{p_i+1}) = \epsilon. \end{aligned} \quad (3)$$

In 2012, Karapinar–Samet proposed [6] generalized $\alpha - \psi$ -contraction. In this consideration, the role of α admissible mapping is to combine the problem of fixed point of “cyclic contractions” with the same problem in the framework of metric spaces endowed with a “partially ordered set”, see, e.g., [7–12]. Later, this notion is refined by Popescu [13]. Let (X, d) be a metric space, $T : X \rightarrow X$ be a mapping and a given mapping $\alpha : X \times X \rightarrow [0, \infty)$. We say that T is *triangular α -orbital admissible* (on short α -t.o.a.) [13] if the following two conditions are satisfied

- (o) $\alpha(x, Tx) \geq 1 \Rightarrow \alpha(Tx, T^2x) \geq 1$ for any $x \in X$;
- (t₀) $\alpha(x, y) \geq 1$ and $\alpha(y, Ty) \geq 1 \Rightarrow \alpha(x, Ty) \geq 1$ for any $x, y \in X$.

Lemma 2 ([13]). Let $\{x_m\}$ be a sequence on a non-empty set X defined by $x_m = Tx_{m-1}$ for any $m \in \mathbb{N}$, where $T : X \rightarrow X$ is an α -t.o.a. mapping. If there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$, then $\alpha(x_n, x_m) \geq 1$, for all $n, m \in \mathbb{N}$.

2. Main Results

Throughout this section, we will consider that $\vartheta, \theta : (0, \infty) \rightarrow \mathbb{R}$ are two functions such that

- (m₀) $\theta(s) < \vartheta(s)$, for all $s > 0$
- and $\alpha : X \times X \rightarrow [0, +\infty)$

Definition 3. A mapping $T : X \rightarrow X$ on a complete metric space (X, d) is a $(\alpha, \vartheta, \theta)$ - $\mathcal{E}$ contraction if the inequality

$$\alpha(x, y)\vartheta(d(Tx, Ty)) \leq \theta(\mathcal{E}(x, y)), \quad (4)$$

holds for every distinct $x, y \in X$ such that $d(Tx, Ty) > 0$, where

$$\mathcal{E}(x, y) = \max\left\{d(x, y) + |d(x, Tx) - d(y, Ty)|, \frac{d(x, Ty) + d(y, Tx)}{2}\right\} \quad (5)$$

Theorem 3. Let (X, d) be a complete metric space and $T : X \rightarrow X$ be a $(\alpha, \vartheta, \theta)$ - $\mathcal{E}$ contraction such that:

- (m_1) ϑ is lower semi-continuous and non-decreasing;
- (m_2) $\limsup_{s \rightarrow s_0} \theta(s) < \vartheta(s_0)$ for any $s_0 > 0$;
- (m_3) T is α -t.o.a. and there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;
- (m_4) $\alpha(x_m, x_*) \geq 1$ for any sequence $\{x_m\}$ such that $x_m \rightarrow x_*$ and $\alpha(x_m, x_{m+1}) \geq 1$, for any $m \in \mathbb{N}$.

Then, T possesses a fixed point.

Proof. Let $\{x_m\}$ be the sequence in X defined as $x_1 = Tx_0, x_m = Tx_{m-1}$, for any $m \in \mathbb{N}$, where x_0 is an arbitrary fixed point in X such that $\alpha(x_0, Tx_0) \geq 1$. We can assume that $d(x_m, x_{m+1}) > 0$ for every $m \in \mathbb{N}$, because, on the contrary, if we can find $m_l \in \mathbb{N}$ such that $x_{m_l} = x_{m_l+1}$, due to the definition of the sequence $\{x_{m_l}\}$ we have $d(Tx_{m_l}, x_{m_l}) = d(x_{m_l+1}, x_{m_l}) = 0$, which means $Tx_{m_l} = x_{m_l}$. Therefore, letting $x = x_m$ and $y = x_{m+1}$ in (4), using (m_0) and taking Lemma 2 into account, we have

$$\begin{aligned} \vartheta(d(x_{m+1}, x_{m+2})) &= \vartheta(d(Tx_m, Tx_{m+1})) \leq \alpha(x_m, x_{m+1})\vartheta(d(Tx_m, Tx_{m+1})) \\ &\leq \theta(\mathcal{E}(x_m, x_{m+1})) < \vartheta(\mathcal{E}(x_m, x_{m+1})), \end{aligned} \quad (6)$$

where

$$\begin{aligned} \mathcal{E}(x_m, x_{m+1}) &= \max \left\{ \begin{array}{l} d(x_m, x_{m+1}) + |d(x_m, Tx_m) - d(x_{m+1}, Tx_{m+1})|, \\ \frac{d(x_m, Tx_{m+1}) + d(x_{m+1}, Tx_m)}{2} \end{array} \right\} \\ &= \max \left\{ \begin{array}{l} d(x_m, x_{m+1}) + |d(x_m, x_{m+1}) - d(x_{m+1}, x_{m+2})|, \\ \frac{d(x_m, x_{m+2}) + d(x_{m+1}, x_{m+1})}{2} \end{array} \right\} \\ &\leq \max \left\{ \begin{array}{l} d(x_m, x_{m+1}) + |d(x_m, x_{m+1}) - d(x_{m+1}, x_{m+2})|, \\ \frac{d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2})}{2} \end{array} \right\}. \end{aligned}$$

$$\text{If } \max\{d(x_m, x_{m+1}), d(x_{m+1}, x_{m+2})\} = d(x_{m+1}, x_{m+2}),$$

$$\mathcal{E}(x_m, x_{m+1}) \leq \max\{d(x_{m+1}, x_{m+2}), d(x_{m+1}, x_{m+2})\} = d(x_{m+1}, x_{m+2}) \quad (7)$$

and the inequality (6) yields

$$\vartheta(d(x_{m+1}, x_{m+2})) \leq \theta(\mathcal{E}(x_m, x_{m+1})) < \vartheta(\mathcal{E}(x_m, x_{m+1})) < \vartheta(d(x_{m+1}, x_{m+2})),$$

which is a contradiction.

$$\text{If } \max\{d(x_m, x_{m+1}), d(x_{m+1}, x_{m+2})\} = d(x_m, x_{m+1}),$$

$$\begin{aligned} \mathcal{E}(x_m, x_{m+1}) &\leq \max\{2d(x_m, x_{m+1}) - d(x_{m+1}, x_{m+2}), d(x_m, x_{m+1})\} \\ &= 2d(x_m, x_{m+1}) - d(x_{m+1}, x_{m+2}) \end{aligned} \quad (8)$$

and the inequality (6) becomes

$$\begin{aligned} \vartheta(d(x_{m+1}, x_{m+2})) &\leq \theta(\mathcal{E}(x_m, x_{m+1})) < \vartheta(\mathcal{E}(x_m, x_{m+1})) \\ &< \vartheta(2d(x_m, x_{m+1}) - d(x_{m+1}, x_{m+2})). \end{aligned} \quad (9)$$

Therefore, by (m_1) , we get

$$d(x_{m+1}, x_{m+2}) < 2d(x_m, x_{m+1}) - d(x_{m+1}, x_{m+2}) \Rightarrow d(x_{m+1}, x_{m+2}) < d(x_m, x_{m+1}),$$

so that, the sequence $\{d(x_m, x_{m+1})\}$ is a decreasing and bounded below by 0. Thus, we can find $\delta \geq 0$, such that $\lim_{m \rightarrow \infty} d(x_m, x_{m+1}) = \delta$. Thus, $\lim_{m \rightarrow \infty} \mathcal{E}(x_m, x_{m+1}) = \delta$ and we claim that, in fact, $\delta = 0$. Supposing on the contrary, that $\delta > 0$, letting $m \rightarrow \infty$ in the first part of (6), we have

$$\vartheta(\delta) = \lim_{m \rightarrow \infty} \vartheta(d(x_{m+1}, x_{m+2})) \leq \limsup_{m \rightarrow \infty} \vartheta(\mathcal{E}(x_m, x_{m+1})) \leq \limsup_{s \rightarrow \delta} \vartheta(s),$$

but this contradicts the assumption (m_2) . Consequently, $\delta = 0$, that is

$$\lim_{m \rightarrow \infty} d(x_m, x_{m+1}) = \lim_{m \rightarrow \infty} d(x_m, Tx_m) = 0. \quad (10)$$

Supposing that $\{x_m\}$ is not a Cauchy sequence, from Lemma 1, we can find two subsequences $\{x_{m_i}\}$, $\{x_{p_i}\}$ of the sequence $\{x_m\}$ such that (3) hold. Thus, setting $u_i = d(x_{m_i}, x_{p_i})$, respectively $v_i = \mathcal{E}(x_{m_i}, x_{p_i})$, and taking (10), (3) into account, we have

$$\begin{aligned} \lim_{i \rightarrow \infty} u_i &= e, \\ \lim_{i \rightarrow \infty} v_i &= \lim_{i \rightarrow \infty} \mathcal{E}(x_{m_i}, x_{p_i}) = \lim_{i \rightarrow \infty} \max \left\{ \begin{array}{l} (d(x_{m_i}, x_{p_i}) + |d(x_{m_i}, Tx_{m_i}) - d(x_{p_i}, Tx_{p_i})|), \\ \frac{d(x_{m_i}, Tx_{p_i}) + d(x_{p_i}, Tx_{m_i})}{2} \end{array} \right\} \\ &= \lim_{i \rightarrow \infty} \max \left\{ \begin{array}{l} (d(x_{m_i}, x_{p_i}) + |d(x_{m_i}, x_{m_i+1}) - d(x_{p_i}, x_{p_i+1})|), \\ \frac{d(x_{m_i}, x_{p_i+1}) + d(x_{p_i}, x_{m_i+1})}{2} \end{array} \right\} \\ &= e. \end{aligned}$$

Therefore, applying (4) for $x = x_{m_i}$ and $y = x_{p_i}$, and using Lemma 2, we get

$$\vartheta(u_{i+1}) = \vartheta(d(x_{m_i+1}, x_{p_i+1})) \leq \alpha(x_{m_i}, x_{p_i}) \vartheta(d(Tx_{m_i}, Tx_{p_i})) \leq \theta(\mathcal{E}(x_{m_i}, x_{p_i})) = \theta(v_i). \quad (11)$$

Thereupon, considering the limit superior in the above inequality, we have

$$\vartheta(e) = \lim_{i \rightarrow \infty} \vartheta(u_i) \leq \limsup_{i \rightarrow \infty} \vartheta(v_i) \leq \limsup_{s \rightarrow e} \vartheta(s),$$

which contradicts (m_2) . Consequently, $\{x_m\}$ is a Cauchy sequence on a complete metric space, which guarantees that $\{x_m\}$ is a convergent sequence. Denoting by x_* the limit of this sequence, we will show that under the assumption (m_4) this point is a fixed point for the mapping T . Indeed, if not, then we have

$$0 < d(x_*, Tx_*) \leq d(x_*, Tx_m) + d(Tx_m, Tx_*). \quad (12)$$

If $d(Tx_m, Tx_*) = 0$ for infinitely many values of m , then (12) becomes $0 < d(x_*, Tx_*) \leq d(x_*, x_{m+1}) \rightarrow 0$ as $m \rightarrow \infty$, so, $d(x_*, Tx_*) = 0$.

If $d(Tx_m, Tx_*) > 0$, for any $m \in \mathbb{N}$, the relation (4) becomes

$$\vartheta(d(x_{m+1}, Tx_*)) \leq \alpha(x_m, x_*) \vartheta(d(Tx_m, Tx_*)) \leq \theta(\mathcal{E}(x_m, x_*)) < \vartheta(\mathcal{E}(x_m, x_*)). \quad (13)$$

(here, we used (m_2)), where

$$\begin{aligned} \mathcal{E}(x_m, x_*) &= \max \left\{ d(x_m, x_*) + |d(x_m, Tx_m) - d(x_*, Tx_*)|, \frac{d(x_m, Tx_*) + d(x_*, Tx_m)}{2} \right\} \\ &= \max \left\{ d(x_m, x_*) + |d(x_m, x_{m+1}) - d(x_*, Tx_*)|, \frac{d(x_m, Tx_*) + d(x_*, x_{m+1})}{2} \right\} \\ &= d(x_*, Tx_*), \end{aligned}$$

for m sufficiently large. (Here, we took into account $d(x_m, x_*) \rightarrow 0$ and $d(x_m, x_{m+1}) \rightarrow 0$ as $m \rightarrow \infty$.) Thus, letting $m \rightarrow \infty$ in (13) and keeping in mind the lower-semicontinuity of ϑ , we obtain

$$\liminf_{s \rightarrow d(Tx_*, x_*)} \vartheta(s) \leq \lim_{m \rightarrow \infty} \vartheta(d(Tx_*, Tx_m)) \leq \vartheta(d(x_*, Tx_*)) < \vartheta(d(x_*, Tx_*)) < \liminf_{s \rightarrow d(Tx_*, x_*)} \vartheta(s),$$

which is a contradiction. Then, $x_* = Tx_*$.

□

Theorem 4. Adding the condition

$$(m_5) \quad \alpha(v, u) \geq 1, \text{ for any } v, u \in \{x \in X : Tx = x\}$$

to the hypotheses of Theorem 3, one obtains uniqueness of the fixed point.

Proof. Supposing that $y_* \in X$ is such that $Ty_* = y_* \neq x_*$, by (4) we have

$$\begin{aligned} \vartheta(d(x_*, y_*)) &\leq \alpha(x_*, y_*) \vartheta(d(Tx_*, Ty_*)) \leq \vartheta(E(x_*, y_*)) < \vartheta(E(x_*, y_*)) \\ &= \vartheta(\max\{d(x_*, y_*) + |d(x_*, Tx_*) - d(y_*, Ty_*)|, \frac{d(x_*, Ty_*) + d(y_*, Tx_*)}{2}\}) \\ &= \vartheta(\max\{d(x_*, y_*) + |d(x_*, x_*) - d(y_*, y_*)|, \frac{d(x_*, y_*) + d(y_*, x_*)}{2}\}) \\ &= \vartheta(d(x_*, y_*)), \end{aligned}$$

which is a contradiction. Therefore, $x_* = y_*$.

□

Example 1. Let the set $X = \{O, A_i, B, C\}$, $i \in \{0, 1, 2, \dots, 7\}$, where $O = (0, 0)$, $B = (0, 3)$, $C = (2, \frac{3}{2})$ and A_i with $i \in \{0, 1, \dots, 7\}$ are the corners of the regular octagon with a circumradius of $R = 4$. Let $d : X \times X \rightarrow [0, +\infty)$, where $d(U_1, U_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ for any $U_1, U_2 \in X$, $U_1 = (x_1, y_1)$, $U_2 = (x_2, y_2)$. Let, also, the mapping $T : X \rightarrow X$ be defined as

$$\begin{aligned} TO &= B, TB = C, TC = O, \\ TA_0 &= A_0 = TA_7, TA_i = A_{i+1}, \text{ for any } i \in \{1, 2, \dots, 6\}. \end{aligned}$$

First of all, we can remark that neither Theorem 1, nor Theorem 2 cannot be applied, because letting, for example, $x = A$ and $y = C$, we have

$$\begin{aligned} d(A, C) &= \frac{5}{2}, d(TA, TC) = d((4, 0), (0, 0)) = 4, d(A, TA) = d((4, 0), (4, 0)) = 0, \\ d(C, TC) &= d((2, \frac{3}{2}), (0, 0)) = \frac{5}{2}, d(A, TC) = d((4, 0), (0, 0)) = 4, \\ d(TA, C) &= d((4, 0), (2, \frac{3}{2})) = \frac{5}{2}. \end{aligned}$$

Thus, replacing in (1) and (2) we are leading to some contradictions, since

$$\vartheta(4) = \vartheta(d(TA, TC)) \leq \vartheta(d(A, C)) = \vartheta(\frac{5}{2}) < \vartheta(\frac{5}{2}),$$

and

$$\begin{aligned} \vartheta(4) &= \vartheta(d(TA, TC)) \leq \vartheta\left(\max\left\{d(A, C), d(A, TA), d(C, TC), \frac{d(A, TC) + d(C, TA)}{2}\right\}\right) \\ &= \vartheta(\frac{13}{4}) < \vartheta(\frac{13}{4}). \end{aligned}$$

Now, we consider the following functions:

$$\alpha : X \times X \rightarrow [0, +\infty), \alpha(U_1, U_2) = \begin{cases} 1, & \text{for } (U_1, U_2) \in \mathcal{A} \cup \mathcal{B} \\ \frac{d(U_1, U_2)}{2}, & \text{for } (U_1, U_2) \in \mathcal{C} \\ 0, & \text{otherwise} \end{cases},$$

$$\vartheta, \theta : (0, \infty) \rightarrow \mathbb{R}, \vartheta(s) = \frac{15}{16}s \text{ and } \theta(s) = \frac{14}{16}s,$$

where $\mathcal{A} = \{(B, O), (A, O), (O, A), (B, A), (A, B)\}$, $\mathcal{B} = \{(A_0, A_i), (A_i, A_0) | i \in \{0, 1, \dots, 5, 6\}\}$ and $\mathcal{C} = \{(A_0, A_7), (A_7, A_0)\}$. We can easily see that the assumptions (m_1) – (m_5) hold. Thus, we have to prove that the mapping T is an $(\alpha, \vartheta, \theta)$ - $\mathcal{E}$ contraction. So, consider the following cases:

- $U_1 = O, U_2 = A,$

$$d(O, A) = 4, d(TO, TA) = 5, d(A, TA) = 0, d(O, TO) = 3,$$

$$d(O, TA) = 4, d(A, TO) = 5, \mathcal{E}(O, A) = 7,$$

and

$$\alpha(O, A)\vartheta(d(TO, TA)) = \frac{15}{16} \cdot 5 < \frac{14}{16} \cdot 7 = \theta(\mathcal{E}(O, A)).$$

- $U_1 = B, U_2 = O,$

$$d(B, O) = 3, d(TB, TO) = \frac{5}{2}, d(B, TB) = \frac{5}{2}, d(O, TO) = 3,$$

$$d(B, TO) = 0, d(O, TB) = \frac{5}{2}, \mathcal{E}(B, O) = \frac{7}{2},$$

and

$$\alpha(B, O)\vartheta(d(TB, TO)) = \frac{15}{16} \cdot \frac{5}{2} < \frac{14}{16} \cdot \frac{7}{2} = \theta(\mathcal{E}(B, O)).$$

- $U_1 = B, U_2 = A,$

$$d(B, A) = 5, d(TB, TA) = \frac{5}{2}, d(B, TB) = \frac{5}{2}, d(A, TA) = 0,$$

$$d(B, TA) = 5, d(A, TB) = \frac{5}{2}, \mathcal{E}(B, A) = \frac{15}{2},$$

and

$$\alpha(B, A)\vartheta(d(TB, TA)) = \frac{15}{16} \cdot \frac{5}{2} < \frac{14}{16} \cdot \frac{15}{2} = \theta(\mathcal{E}(B, A)).$$

- $U_1 = A, U_2 = C,$

$$d(A, C) = \frac{5}{2}, d(TA, TC) = 4, d(C, TC) = \frac{5}{2}, d(A, TA) = 0,$$

$$d(C, TA) = \frac{5}{2}, d(A, TC) = 4, \mathcal{E}(A, C) = 5,$$

and

$$\alpha(A, C)\vartheta(d(TA, TC)) = \frac{15}{16} \cdot 4 < \frac{14}{16} \cdot 5 = \theta(\mathcal{E}(A, C)).$$

- $U_1 = A_0, U_2 = A_i$

A_i	$\mathcal{T}A_i$	$d(\mathcal{T}A_0, \mathcal{T}A_i)$	$\alpha(A_0, A_i)\vartheta(A_0, A_i)$	$\theta(\mathcal{E}(A_0, A_i))$
$A_0 = (4, 0)$	A_0	0	0	0
$A_1 = (2\sqrt{2}, 2\sqrt{2})$	A_2	5.656854249	5.303300859	5.357568053
$A_2 = (0, 4)$	A_3	7.39103626	6.929096494	7.628531495
$A_3 = (-2\sqrt{2}, 2\sqrt{2})$	A_4	8	7.5	9.145940754
$A_4 = (-4, 0)$	A_5	7.39103626	6.929096494	9.678784027
$A_5 = (-2\sqrt{2}, -2\sqrt{2})$	A_6	5.56854249	5.303300859	9.145940754
$A_6 = (0, -4)$	A_7	3.061467459	2.870125743	7.628531495
$A_7 = (2\sqrt{2}, -2\sqrt{2})$	A_0	0	0	5.357568053

Therefore, the mapping $\mathcal{T}$ has a unique fixed point, that is $A = (4, 0)$.

Corollary 1. Let (X, d) be a complete metric space and $\mathcal{T} : X \rightarrow X$ be a mapping such that

$$\vartheta(d(\mathcal{T}x, \mathcal{T}y)) \leq \theta(\mathcal{E}(x, y)), \quad (14)$$

holds for every distinct $x, y \in X$, such that $d(\mathcal{T}x, \mathcal{T}y) > 0$, where

$$\mathcal{E}(x, y) = \max \left\{ d(x, y) + |d(x, \mathcal{T}x) - d(y, \mathcal{T}y)|, \frac{d(x, \mathcal{T}y) + d(y, \mathcal{T}x)}{2} \right\} \quad (15)$$

and $\vartheta, \theta : (0, \infty) \rightarrow \mathbb{R}$. Suppose that:

- (m_0) $\theta(s) < \vartheta(s)$, for all $s > 0$;
- (m_1) ϑ lower semi-continuous and non-decreasing;
- (m_2) $\limsup_{s \rightarrow s_0} \theta(s) < \vartheta(s_0)$ for any $s_0 > 0$.

Then, $\mathcal{T}$ possesses a unique fixed point.

Proof. Let $\alpha(x, y) = 1$ in Theorem 3. $\square$

Corollary 2. On a complete metric space (X, d) let $\mathcal{T} : X \rightarrow X$ be a mapping such that

$$\vartheta(d(\mathcal{T}x, \mathcal{T}y)) \leq \kappa \vartheta(\mathcal{E}(x, y)), \quad (16)$$

holds for every distinct $x, y \in X$ with $d(\mathcal{T}x, \mathcal{T}y) > 0$, where $\mathcal{E}(x, y)$ is defined by (15), $\kappa \in [0, 1)$ and $\vartheta : (0, \infty) \rightarrow (0, \infty)$ is left-continuous and non-decreasing. Then $\mathcal{T}$ possesses a unique fixed point.

Proof. Let $\theta(s) = \kappa \vartheta(s)$ in Corollary 1. $\square$

Definition 4. A mapping $\mathcal{T} : X \rightarrow X$ on a complete metric space (X, d) is a $(\alpha, \vartheta, \theta)$ - $\mathcal{E}^2$ -contraction if the inequality

$$\alpha(x, y)\vartheta(d(\mathcal{T}^2x, \mathcal{T}^2y)) \leq \theta(\mathcal{E}^2(x, y)) \quad (17)$$

holds for every distinct $x, y \in X$, such that $d(\mathcal{T}^2x, \mathcal{T}^2y) > 0$, where

$$\mathcal{E}^2(x, y) = \max \left\{ \begin{array}{l} d(y, \mathcal{T}y) + |d(x, y) - d(x, \mathcal{T}x)|, \\ d(\mathcal{T}x, \mathcal{T}y) + |d(\mathcal{T}x, \mathcal{T}^2x) - d(\mathcal{T}y, \mathcal{T}^2y)|, \\ d(\mathcal{T}y, \mathcal{T}^2y) + |d(\mathcal{T}x, \mathcal{T}^2x) - d(y, \mathcal{T}y)| \end{array} \right\} \quad (18)$$

Theorem 5. Let (X, d) be a complete metric space and $T : X \rightarrow X$ be a $(\alpha, \vartheta, \theta)$ - $\mathcal{E}^2$ contraction such that

- (m_1) ϑ is lower semi-continuous and non-decreasing;
- (m_2) $\limsup_{s \rightarrow s_0} \theta(s) < \theta(s_0)$ for any $s_0 > 0$;
- (m_3) T is α -t.o.a. and there exists $x_0 \in X$, such that $\alpha(x_0, Tx_0) \geq 1$;
- (m_6) T^2 is continuous and $\alpha(Tv, v) \geq 1$ for any $v \in \{x \in X : T^2x = x\}$.

Then, T possesses a fixed point.

Proof. Let $x_0 \in X$ be a point such that $\alpha(x_0, Tx_0) \geq 1$ and the sequence $\{x_m\}$ in X , where $x_m = Tx_{m-1}$, for any $m \in \mathbb{N}$. Since T is α -t.o.a., by Lemma 2, we get $\alpha(x_m, x_{m+1}) \geq 1$, for any $m \in \mathbb{N}$ and then

$$\begin{aligned} \vartheta(d(x_{m+2}, x_{m+3})) &\leq \alpha(x_m, x_{m+1})\vartheta(d(T^2x_m, T^2x_{m+1})) \\ &\leq \theta(\mathcal{E}^2(x_m, x_{m+1})) < \vartheta(\mathcal{E}^2(x_m, x_{m+1})). \end{aligned} \quad (19)$$

Furthermore, the monotonicity of ϑ implies that

$$d(x_{m+2}, x_{m+3}) < \mathcal{E}^2(x_m, x_{m+1}), \quad (20)$$

where

$$\begin{aligned} \mathcal{E}^2(x_m, x_{m+1}) &= \max \left\{ \begin{array}{l} d(x_{m+1}, Tx_{m+1}) + |d(x_m, x_{m+1}) - d(x_m, Tx_m)|, \\ d(Tx_m, Tx_{m+1}) + |d(Tx_m, T^2x_m) - d(Tx_{m+1}, T^2x_{m+1})|, \\ d(Tx_{m+1}, T^2x_{m+1}) + |d(Tx_m, T^2x_m) - d(x_{m+1}, Tx_{m+1})| \end{array} \right\} \\ &= \max \left\{ \begin{array}{l} d(x_{m+1}, x_{m+2}) + |d(x_m, x_{m+1}) - d(x_m, x_{m+1})|, \\ d(x_{m+1}, x_{m+2}) + |d(x_{m+1}, x_{m+2}) - d(x_{m+2}, x_{m+3})|, \\ d(x_{m+2}, x_{m+3}) + |d(x_{m+1}, x_{m+2}) - d(x_{m+1}, x_{m+2})| \end{array} \right\} \\ &= \max \left\{ \begin{array}{l} d(x_{m+1}, x_{m+2}) + |d(x_{m+1}, x_{m+2}) - d(x_{m+2}, x_{m+3})|, \\ d(x_{m+1}, x_{m+2}), d(x_{m+2}, x_{m+3}) \end{array} \right\} \end{aligned}$$

If for some $m \in \mathbb{N}$, $\max\{d(x_{m+1}, x_{m+2}), d(x_{m+2}, x_{m+3})\} = d(x_{m+2}, x_{m+3})$ then $\mathcal{E}^2(x_m, x_{m+1}) = d(x_{m+2}, x_{m+3})$ and (20) turns into

$$d(x_{m+2}, x_{m+3}) < d(x_{m+2}, x_{m+3}),$$

which is a contradiction. If $\max\{d(x_{m+1}, x_{m+2}), d(x_{m+2}, x_{m+3})\} = d(x_{m+1}, x_{m+2})$, then we have $\mathcal{E}^2(x_m, x_{m+1}) = 2d(x_{m+1}, x_{m+2}) - d(x_{m+2}, x_{m+3})$ and by (20) we get

$$d(x_{m+2}, x_{m+3}) < d(x_{m+1}, x_{m+2})$$

for every $m \in \mathbb{N}$. Therefore,

$$d(x_{m+2}, x_{m+3}) < d(x_{m+1}, x_{m+2}) < \dots < d(x_2, x_3) < \max\{d(x_1, x_2), d(x_0, x_1)\}.$$

Thus, the sequence $\{d(x_m, x_{m+1})\}$ is strictly decreasing and bounded below by 0. Hence, there exist $\delta \geq 0$ such that $\lim_{m \rightarrow \infty} d(x_{m+1}, x_{m+2}) = \delta$. Suppose that $\delta > 0$. Since $\lim_{m \rightarrow \infty} \mathcal{E}^2(x_m, x_{m+1}) = \lim_{m \rightarrow \infty} d(x_{m+1}, x_{m+2}) = \delta > 0$, letting the limit superior in (19), as $n \rightarrow \infty$ and taking (m_2) into account, we obtain

$$\vartheta(\delta) = \lim_{m \rightarrow \infty} \vartheta(d(x_{m+2}, x_{m+3})) \leq \limsup_{m \rightarrow \infty} \vartheta(\mathcal{E}^2(x_m, x_{m+1})) < \limsup_{s \rightarrow \delta} \vartheta(s) < \vartheta(\delta),$$

which is a contradiction. Therefore,

$$\lim_{m \rightarrow \infty} d(x_m, x_{m+1}) = 0 \quad (21)$$

and we claim that, in these conditions, the sequence $\{x_m\}$ is Cauchy. If not, from Lemma 1 we can find $\epsilon > 0$ and two subsequences $\{x_{m_i}\}$ and $\{x_{p_i}\}$ of $\{x_m\}$, such that the equalities in (3), hold. Moreover, we remark that

$$\begin{aligned} \mathcal{E}^2(x_{m_i}, x_{p_i}) &= \max \left\{ \begin{array}{l} d(x_{p_i}, Tx_{p_i}) + |d(x_{m_i}, x_{p_i}) - d(x_{m_i}, Tx_{m_i})|, \\ d(Tx_{m_i}, Tx_{p_i}) + |d(Tx_{m_i}, T^2x_{m_i}) - d(Tx_{p_i}, T^2x_{p_i})|, \\ d(Tx_{p_i}, T^2x_{p_i}) + |d(Tx_{m_i}, T^2x_{m_i}) - d(x_{p_i}, Tx_{p_i})| \end{array} \right\} \\ &= \max \left\{ \begin{array}{l} d(x_{p_i}, x_{p_i+1}) + |d(x_{m_i}, x_{p_i}) - d(x_{m_i}, x_{m_i+1})|, \\ d(x_{m_i+1}, x_{p_i+1}) + |d(x_{m_i+1}, x_{m_i+2}) - d(x_{p_i+1}, x_{p_i+2})|, \\ d(x_{p_i+1}, x_{p_i+2}) + |d(x_{m_i+1}, x_{m_i+2}) - d(x_{p_i}, x_{p_i+1})| \end{array} \right\} \end{aligned}$$

and then, using (21) we get $\lim_{i \rightarrow \infty} \mathcal{E}^2(x_{m_i}, x_{p_i}) = \epsilon$. On the other hand, letting $i \rightarrow \infty$ in the below inequality

$$\begin{aligned} d(x_{m_i+1}, x_{p_i+1}) - d(x_{m_i+1}, x_{m_i+2}) - d(x_{p_i+2}, x_{p_i+1}) &\leq d(x_{m_i+2}, x_{p_i+2}) \\ &\leq d(x_{m_i+2}, x_{m_i+1}) + d(x_{m_i+1}, x_{p_i+1}) + d(x_{p_i+1}, x_{p_i+2}). \end{aligned}$$

we find that $\lim_{i \rightarrow \infty} d(x_{m_i+2}, x_{p_i+2}) = \epsilon$. Replacing in (17), x by x_{m_i} and y by x_{p_i} , and since, by Lemma 2, $\alpha(x_{m_i}, x_{p_i}) \geq 1$, we obtain

$$\vartheta(d(x_{m_i+2}, x_{p_i+2})) \leq \vartheta(\mathcal{E}^2(x_{m_i}, x_{p_i})).$$

Thus,

$$\vartheta(\epsilon) = \lim_{i \rightarrow \infty} \vartheta(d(x_{m_i+2}, x_{p_i+2})) \leq \limsup_{i \rightarrow \infty} \vartheta(\mathcal{E}^2(x_{m_i}, x_{p_i})) \leq \limsup_{s \rightarrow \epsilon} \vartheta(s),$$

which contradicts (m_2) . Consequently, $\{x_m\}$ is a Cauchy sequence on a complete metric space, so that, there exists x_* , such that $\lim_{m \rightarrow \infty} d(x_m, x_*) = 0$. We claim that x_* is a fixed point of $\mathcal{T}$, under the assumptions (m_6) .

Now, since $\mathcal{T}^2$ is continuous, we have

$$\lim_{m \rightarrow \infty} d(x_m, \mathcal{T}^2x_*) = \lim_{m \rightarrow \infty} d(\mathcal{T}^2x_{m-2}, \mathcal{T}^2x_*) = 0, \quad (22)$$

and then, we derive that $\mathcal{T}^2x_* = x_*$. By reductio ad absurdum, we assume that $\mathcal{T}x_* \neq x_*$. Thus, by (17), and keeping in mind (m_1) and (m_6) , we have

$$\begin{aligned}
\vartheta(d(\mathcal{T}x_*, x_*)) &= \vartheta(d(\mathcal{T}^2(\mathcal{T}x_*), \mathcal{T}^2x_*)) \leq \alpha(\mathcal{T}x_*, x_*)\vartheta(d(\mathcal{T}^2(\mathcal{T}x_*), \mathcal{T}^2x_*)) \\
&\leq \vartheta(\mathcal{E}^2(\mathcal{T}x_*, x_*)) < \vartheta(\mathcal{E}^2(\mathcal{T}x_*, x_*)) \\
&= \vartheta\left(\max\left\{\begin{array}{l} d(x_*, \mathcal{T}x_*) + |d(\mathcal{T}x_*, x_*) - d(\mathcal{T}x_*, \mathcal{T}(\mathcal{T}x_*))|, \\ d(\mathcal{T}x_*, \mathcal{T}(\mathcal{T}x_*)) + |d(\mathcal{T}(\mathcal{T}x_*), \mathcal{T}^2(\mathcal{T}x_*)) - d(\mathcal{T}x_*, \mathcal{T}^2x_*)|, \\ d(\mathcal{T}x_*, \mathcal{T}^2x_*) + |d(\mathcal{T}(\mathcal{T}x_*), \mathcal{T}^2(\mathcal{T}x_*)) - d(x_*, \mathcal{T}x_*)| \end{array}\right\}\right) \\
&= \vartheta\left(\max\left\{\begin{array}{l} d(x_*, \mathcal{T}x_*) + |d(\mathcal{T}x_*, x_*) - d(\mathcal{T}x_*, x_*)|, \\ d(\mathcal{T}x_*, x_*) + |d(x_*, \mathcal{T}x_*) - d(\mathcal{T}x_*, x_*)|, \\ d(\mathcal{T}x_*, x_*) + |d(x_*, \mathcal{T}x_*) - d(x_*, \mathcal{T}x_*)| \end{array}\right\}\right) \\
&= \vartheta(d(x_*, \mathcal{T}x_*)),
\end{aligned}$$

which is a contradiction. Therefore, $d(x_*, \mathcal{T}x_*) = 0$, so that, x_* is a fixed point of $\mathcal{T}$.
□

Theorem 6. Adding the condition

(m_5) $\alpha(v, u) \geq 1$, for any $v, u \in \{x \in X : \mathcal{T}x = x\}$

to the hypotheses of Theorem 5, one obtains uniqueness of the fixed point.

Proof. In order to prove the uniqueness of the fixed point, we suppose, by reductio ad absurdum, that there exists $x_*, y_* \in \{x \in X : \mathcal{T}x = x\}$ such that $x_* \neq y_*$. Then, by (17) and (m_5) we obtain

$$\vartheta(d(x_*, y_*)) \leq \alpha(x_*, y_*)\vartheta(d(\mathcal{T}^2x_*, \mathcal{T}^2y_*)) \leq \vartheta(\mathcal{E}^2(x_*, y_*)) < \vartheta(\mathcal{E}^2(x_*, y_*)) = \vartheta(d(x_*, y_*)),$$

which is a contradiction. Therefore, the mapping $\mathcal{T}$ possesses a unique fixed point. □

Remark 2. In case of the mapping $\mathcal{T}$ is continuous, we get the same result without the assumption (m_6) .

Example 2. Let the set $X = [0, 5]$ be endowed with the usual metric, $d : X \times X \rightarrow [0, +\infty)$,

$$d(x, y) = |x - y| \text{ and a mapping } \mathcal{T} : X \rightarrow X \text{ defined as } \mathcal{T}x = \begin{cases} 0, & \text{if } x \in [0, 1] \\ 1, & \text{if } x \in (1, 2] \\ \frac{x-2}{2}, & \text{if } x \in (2, 4] \\ \frac{x}{2}, & \text{if } x \in (4, 5] \end{cases}.$$

Assume that $\vartheta, \theta : (0, \infty) \rightarrow \mathbb{R}$ are two functions such that ϑ is non-decreasing and $\theta(s) < \vartheta(s)$, for any $s > 0$. First of all, we remark that, taking $x = 4/6$ and $y = 7/6$, we have

$$d(\frac{4}{6}, \frac{7}{6}) = \frac{3}{6}, d(\mathcal{T}\frac{4}{6}, \mathcal{T}\frac{7}{6}) = d(0, 1) = 1, d(\frac{7}{6}, \mathcal{T}\frac{7}{6}) = d(\frac{7}{6}, 1) = \frac{1}{6},$$

$$d(\frac{4}{6}, \mathcal{T}\frac{4}{6}) = d(\frac{4}{6}, 0) = \frac{4}{6}, d(\frac{7}{6}, \mathcal{T}\frac{4}{6}) = d(\frac{7}{6}, 0) = \frac{7}{6},$$

$$d(\frac{4}{6}, \mathcal{T}\frac{7}{6}) = d(\frac{4}{6}, 1) = \frac{2}{6}.$$

Thus,

•

$$\vartheta(1) = \vartheta\left(d(\mathcal{T}\frac{4}{6}, \mathcal{T}\frac{7}{6})\right) \leq \theta\left(d(\frac{4}{6}, \frac{7}{6})\right) = \theta\left(\frac{1}{2}\right) < \vartheta\left(\frac{1}{2}\right),$$

which is a contradiction, so, Theorem 1 can not be applied.

•

$$\begin{aligned}\vartheta(1) &= \vartheta(d(\mathcal{T}\frac{4}{6}, \mathcal{T}\frac{7}{6})) \leq \theta\left(\max\left\{d(\frac{4}{6}, \frac{7}{6}), d(\frac{7}{6}, d(\frac{4}{6}, \mathcal{T}\frac{4}{6}), \mathcal{T}\frac{7}{6}), \frac{d(\frac{7}{6}, \mathcal{T}\frac{4}{6}) + d(\frac{4}{6}, \mathcal{T}\frac{7}{6})}{2}\right\}\right) \\ &= \theta\left(\max\left\{\frac{1}{2}, \frac{2}{3}, \frac{1}{6}, \frac{\frac{7}{6} + \frac{2}{6}}{2}\right\}\right) = \theta(\frac{3}{4}) < \vartheta(\frac{3}{4}),\end{aligned}$$

which is a contradiction. Therefore, Theorem 2 can not be used.

• Since

$$\mathcal{E}(4/6, 7/6) = \max\left\{\frac{3}{6} + \left|\frac{4}{6} - \frac{1}{6}\right|, \frac{\frac{2}{6} + \frac{7}{6}}{2}\right\} = 1,$$

the inequality (4) becomes

$$\vartheta(1) \leq \alpha(4/6, 7/6)\vartheta(d(\mathcal{T}(4/6), \mathcal{T}(7/6))) \leq \theta(\mathcal{E}(4/6, 7/6)) < \vartheta(\mathcal{E}(4/6, 7/6)) = \vartheta(1),$$

which is again a contradiction (for any function $\alpha : X \times X \rightarrow [0, +\infty]$). Thus, Theorem 3 can not be used.

However, we have $\mathcal{T}^2x = \begin{cases} 0, & \text{if } x \in [0, 4] \\ \frac{x-4}{4}, & \text{if } x \in (4, 5] \end{cases}$, so that, $\mathcal{T}^2$ is continuous. Let $\alpha : X \times X \rightarrow [0, +\infty]$ be the function defined as follows:

$$\alpha(x, y) = \begin{cases} x^2 + y^2 + 1, & \text{if } x, y \in [0, 4] \\ 1, & \text{if } x, y \in (4, 5] \\ 2, & \text{if } x \in (4, 5], y \in [0, 1] \\ 0, & \text{otherwise.} \end{cases}$$

In this case, it is easy to check that the assumptions (m_3) and (m_6) hold. Let $\vartheta, \theta : (0, \infty) \rightarrow \mathbb{R}$ be two functions such that (m_1) and (m_2) are satisfied, for example $\vartheta(s) = \frac{s}{2}$ and $\theta(s) = \frac{s}{3}$. We show that, the mapping $\mathcal{T}$ is a $(\alpha, \vartheta, \theta)$ - $\mathcal{E}^2$ contraction.

1. If $x \in (4, 5]$ and $y \in [0, 1]$, since

$$d(\mathcal{T}x, \mathcal{T}y) + |d(\mathcal{T}x, \mathcal{T}^2x) - d(\mathcal{T}y, \mathcal{T}^2y)| = d(\frac{x}{2}, 0) + d(\frac{x}{2}, \frac{x-4}{4}) = \frac{3x+4}{4} \leq \mathcal{E}^2(x, y),$$

$$d(\mathcal{T}^2x, \mathcal{T}^2y) = d(\frac{x-4}{4}, 0) = \frac{x-4}{4},$$

we have

$$\alpha(x, y) \frac{d(\mathcal{T}^2x, \mathcal{T}^2y)}{2} = 2 \frac{x-4}{8} \leq \frac{3x+4}{12} \leq \frac{\mathcal{E}^2(x, y)}{3} = \theta(\mathcal{E}^2(x, y)).$$

2. If $x, y \in (4, 5]$, since

$$\begin{aligned}d(\mathcal{T}x, \mathcal{T}y) + |d(\mathcal{T}x, \mathcal{T}^2x) - d(\mathcal{T}y, \mathcal{T}^2y)| &= \frac{|x-y|}{2} + \left|\frac{x+4}{4} - \frac{y+4}{4}\right| \\ &= \frac{3|x-y|}{2} \leq \mathcal{E}^2(x, y),\end{aligned}$$

$$d(\mathcal{T}^2x, \mathcal{T}^2y) = d(\frac{x-4}{4}, \frac{y-4}{4}) = \frac{|x-y|}{4},$$

we get

$$\alpha(x, y) \vartheta(d(\mathcal{T}^2x, \mathcal{T}^2y)) = \frac{|x-y|}{8} \leq \frac{3|x-y|}{6} \leq \frac{\mathcal{E}^2(x, y)}{3} = \theta(\mathcal{E}^2(x, y)).$$

Therefore, by Theorem 5, the mapping $\mathcal{T}$ has a unique fixed point, this being $x = 0$.

Corollary 3. Let (X, d) be a complete metric space and $T : X \rightarrow X$ be a mapping, such that

$$\vartheta(d(T^2x, T^2y)) \leq \theta(\mathcal{E}^2(x, y)), \quad (23)$$

holds for every distinct $x, y \in X$ with $d(T^2x, T^2y) > 0$, where

$$\mathcal{E}^2(x, y) = \max \left\{ \begin{array}{l} d(y, Ty) + |d(x, y) - d(x, Tx)|, \\ d(Tx, Ty) + |d(Tx, T^2x) - d(Ty, T^2y)|, \\ d(Ty, T^2y) + |d(Tx, T^2x) - d(y, Ty)| \end{array} \right\}, \quad (24)$$

and $\vartheta, \theta : (0, \infty) \rightarrow \mathbb{R}$. Suppose that:

- (m_0) $\theta(s) < \vartheta(s)$, for all $s > 0$;
- (m_1) ϑ is lower semi-continuous and non-decreasing;
- (m_2) $\limsup_{s \rightarrow s_0} \theta(s) < \vartheta(s_0)$ for any $s_0 > 0$;
- (m_6) T^2 is continuous.

Then, T possesses a unique fixed point.

Proof. Let $\alpha(x, y) = 1$ in Theorem 5. $\square$

Corollary 4. On a complete metric space (X, d) let $T : X \rightarrow X$ be a mapping, such that

$$\vartheta(d(T^2x, T^2y)) \leq \kappa \vartheta(\mathcal{E}^2(x, y)), \quad (25)$$

holds for every distinct $x, y \in X$ with $d(T^2x, T^2y) > 0$, where

$$\mathcal{E}^2(x, y) = \max \left\{ \begin{array}{l} d(y, Ty) + |d(x, y) - d(x, Tx)|, \\ d(Tx, Ty) + |d(Tx, T^2x) - d(Ty, T^2y)|, \\ d(Ty, T^2y) + |d(Tx, T^2x) - d(y, Ty)| \end{array} \right\} \quad (26)$$

κ in $(0, 1)$ and $\vartheta : (0, \infty) \rightarrow (0, \infty)$ is left-continuous and non-decreasing. If T^2 is continuous, then T possesses a unique fixed point.

Proof. Let $\theta(s) = \kappa \vartheta(s)$ in Corollary 3. $\square$

Remark 3. We mention that a series of known results can be found through a convenient choices of functions α , ϑ and θ . For example, by choosing $\alpha(x, y) = 1$, $\vartheta(s) = s$ and $\theta(s) = \beta(s)s$, where $\beta : (0, \infty) \rightarrow (0, 1)$ is such that $\limsup_{s \rightarrow e} \beta(s) < 1$ for any $e > 0$, Theorem 3 reduces to the main Theorem in [1].

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