symmetry

MDPI

Article

On Fejér Type Inequalities via (p,q)-Calculus

Nuttapong Arunrat [1], Keaitsuda Maneeruk Nakprasit [1], Kamsing Nonlaopon [1,*], Jessada Tariboon [2] and Sotiris K. Ntouyas [3,4]

1 Department of Mathematics, Faculty of Science, Khon Kaen University, Khon Kaen 40002, Thailand; nutaru@kkumail.com (N.A.); keaina@kku.ac.th (K.M.N.)
2 Department of Mathematics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Bangkok 10800, Thailand; jessada.t@sci.kmutnb.ac.th
3 Department of Mathematics, University of Ioannina, 451 10 Ioannina, Greece; sntouyas@uoi.gr
4 Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia
* Correspondence: nkamsi@kku.ac.th; Tel.: +668-6642-1582

Abstract: In this paper, we use (p,q)-integral to establish some Fejér type inequalities. In particular, we generalize and correct existing results of quantum Fejér type inequalities by using new techniques and showing some problematic parts of those results. Most of the inequalities presented in this paper are significant extensions of results which appear in existing literatures.

Keywords: Fejér type inequalities; symmetric function; (p,q)-calculus; (p,q)-derivative; (p,q)-integral

MSC: 05A30; 26D10; 26D15

Citation: Arunrat, N.; Nakprasit, K.M.; Nonlaopon, K.; Tariboon, J.; Ntouyas, S.K. On Fejér Type Inequalities via (p,q)-Calculus. *Symmetry* **2021**, *13*, 953. https://doi.org/10.3390/sym13060953

Academic Editor: Ioan Rașa

Received: 21 April 2021
Accepted: 25 May 2021
Published: 27 May 2021

1. Introduction

Quantum calculus or q-calculus, the modern name of the study of calculus without limits, has been studied since the early eighteenth century. The famous mathematician Leonhard Euler (1707–1783) established q-calculus and, in 1910, F. H. Jackson [1] determined the definite q-integral called the q-Jackson integral. Many applications of quantum calculus appear in mathematics, such as number theory, orthogonal polynomials, combinatorics, basic hypergeometric functions, and in physics, such as mechanics, relativity theory and quantum theory, see for instance [2–10] and the references therein. Furthermore, the fundamental knowledge and also the fundamental theoretical concepts of quantum calculus are covered in the book by V. Kac and P. Cheung [11].

In 2013, J. Tariboon and S. K. Ntouyas [12] defined the q-derivative and q-integral of a continuous function on finite interval along with some studying of its significant properties. In addition, they firstly extended some inequalities to q-calculus, such as Cauchy–Bunyakovsky–Schwarz, Grüss, Grüss-Čebyšev, Hermite–Hadamard, Hölder, Ostrowski and Trapezoid inequalities by applying such definitions; see [13] for more details. Based on these results, there is much research on q-calculus; see [14–20] and the references cited therein.

In recent years, many interesting quantum integral inequalities on finite interval have been considered more generally in (p,q)-calculus, which was first considered by R. Chakrabarti and R. Jagannathan [21]. In 2016, M. Tunç and E. Göv [22,23] introduced the (p,q)-derivative and (p,q)-integral on finite interval while proving some properties, and gave several inequalities of integral via (p,q)-calculus. In addition, some more results of (p,q)-calculus appear in [24–33] and the references cited therein.

The function $f:[a,b]\to\mathbb{R}$ is called convex if

$$f(\alpha x+(1-\alpha)y)\le \alpha f(x)+(1-\alpha)f(y)$$

for all $x,y\in[a,b],\alpha\in[0,1]$, and f is called concave provided that $-f$ is convex.

Let I be the interval or real numbers and $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function on I with constants $a < b$ in I. The well-known inequality which is Hermite–Hadamard inequality [34] is

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

In 2000, S. S. Dragomir et al. [35,36] proved related result to the Hermite–Hadamard inequality, as in the following.

Theorem 1. *Refs.* [35,36] *If $f : I \to \mathbb{R}$ is a twice differentiable function where $a, b \in I$ with $a < b$ and real constants m and M with $m \leq f'' \leq M$, then*

$$m\frac{(b-a)^2}{12} \leq \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx \leq M\frac{(b-a)^2}{12}, \tag{2}$$

and

$$m\frac{(b-a)^2}{24} \leq \frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right) \leq M\frac{(b-a)^2}{24}. \tag{3}$$

The Hermite–Hadamard inequality and the Hermite–Hadamard–Fejér inequalities, which are famous inequalities for convex functions, have a deep relationship to its integral mean, see [37–47] for more details and the references cited therein. A weighted generalization of inequality (1) was introduced by L. Fejé [48], as in the following.

Theorem 2. *Ref.* [48] *If $f : I \to \mathbb{R}$ is a convex function with constants $a < b$ in I, then*

$$f\left(\frac{a+b}{2}\right)\int_a^b w(x)\,dx \leq \frac{1}{b-a}\int_a^b f(x)w(x)\,dx \leq \frac{f(a)+f(b)}{2}\int_a^b w(x)\,dx,$$

where $w : [a,b] \to \mathbb{R}$ is integrable, nonnegative, and symmetric about $x = \dfrac{a+b}{2}$, that is $w(a+b-x) = w(x)$.

In [49], N. Minculete and F. C. Mitroi introduced the inequalities which become important, as follows.

Theorem 3. *Ref.* [49] *Let $f : I \to \mathbb{R}$ be a twice differentiable function with $a < b$ in I, such that $m \leq f'' \leq M$. If $\lambda \in [0,1]$, then*

$$m\frac{\lambda(1-\lambda)}{2}(b-a)^2 \leq \lambda f(a) + (1-\lambda)f(b) - f_\lambda(a,b) \leq M\frac{\lambda(1-\lambda)}{2}(b-a)^2,$$

and

$$m\frac{(1-2\lambda)^2}{8}(b-a)^2 \leq \frac{f_\lambda(a,b)+f_\lambda(b,a)}{2} - f\left(\frac{a+b}{2}\right) \leq M\frac{(1-2\lambda)^2}{8}(b-a)^2,$$

where $f_\lambda(a,b) = f(\lambda a + (1-\lambda)b)$.

Some inequalities of Hermite–Hadamard–Fejér type for differentiable functions follow from Theorem 3.

Theorem 4. *Ref.* [49] *Let $f : I \to \mathbb{R}$ be a twice differentiable function with $a < b$ in I such that $m \le f'' \le M$. If $w : [a,b] \to \mathbb{R}$ is integrable, nonnegative and symmetric about $x = \dfrac{a+b}{2}$, then*

$$\begin{aligned} \frac{m}{2}\int_a^b (t-a)(b-t)w(t)\,dt &\le \frac{f(a)+f(b)}{2}\int_a^b w(t)\,dt - \int_a^b f(t)w(t)\,dt \\ &\le \frac{M}{2}\int_a^b (t-a)(b-t)w(t)\,dt, \end{aligned} \tag{4}$$

and

$$\begin{aligned} \frac{m}{8}\int_a^b (2t-a-b)^2 w(t)\,dt &\le \int_a^b f(t)w(t)\,dt - f\left(\frac{a+b}{2}\right)\int_a^b w(t)\,dt \\ &\le \frac{M}{8}\int_a^b (2t-a-b)^2 w(t)\,dt. \end{aligned} \tag{5}$$

In q- calculus, some Fejér type inequalities for differentiable functions were established by W. Yang [50]. Moreover, Fejér type inequalities for fractional integrals were established by M. Z. Sarikaya [51].

In this paper, we propose to generalize and extend some Fejér-type inequalities in q-integral and fractional integral to (p,q)-integral. In particular, we correct existing results of quantum Fejér-type inequalities by using new techniques and showing some problematic parts of those results. The results presented here would extend some of those in existing literatures.

2. Preliminaries

In this section, we give fundamental concepts of (p,q)-calculus used in our work. We will use $I = [a,b] \subseteq \mathbb{R}$, $I^0 = (a,b)$, and p,q are constants with $0 < q < p \le 1$ throughout this paper.

Definition 1. *Refs.* [22,23] *Let $f : I \to \mathbb{R}$ be a continuous function. The (p,q)-derivative of the function f at x on $[a,b]$ is*

$${}_aD_{p,q}f(x) = \begin{cases} \dfrac{f(px+(1-p)a) - f(qx+(1-q)a)}{(p-q)(x-a)} := \dfrac{f_p(x,a) - f_q(x,a)}{(p-q)(x-a)}, & x \ne a; \\ \lim\limits_{x\to a} {}_aD_{p,q}f(x), & x = a. \end{cases}$$

A function f is called (p,q)-differentiable on I if for each $x \in I$ there exists ${}_aD_{p,q}f(x)$. If if $a = 0$ in Definition 1, then ${}_0D_{p,q}f = D_{p,q}f$, where $D_{p,q}f$ is

$$D_{p,q}f(x) = \begin{cases} \dfrac{f(px) - f(qx)}{(p-q)(x)}, & x \ne 0; \\ \lim\limits_{x\to 0} D_{p,q}f(x), & x = 0. \end{cases}$$

Furthermore, if $p = 1$, then ${}_aD_{p,q}f = {}_aD_qf$, which is the q-derivative of the function f.

Example 1. *For $x \in I$ and a natural number n, if $f(x) = (x-a)^n$, then*

$${}_aD_{p,q}f(x) = [n]_{p,q}(x-a)^{n-1},$$

where $[n]_{p,q} = \dfrac{p^n - q^n}{p-q}$.

Definition 2. *Refs.* [22,23] *Let $f : I \to \mathbb{R}$ be a continuous function. The (p,q)-integral of the function f for $x \in I$ is defined to be*

$$\int_a^x f(t)_a d_{p,q} t = (p-q)(x-a) \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} f\left(\frac{q^n}{p^{n+1}} x + \left(1 - \frac{q^n}{p^{n+1}}\right) a\right).$$

Furthermore, for $c \in (a,x)$, the (p,q)-integral is defined to be

$$\int_c^x f(t)_a d_{p,q} t = \int_a^x f(t)_a d_{p,q} t - \int_a^c f(t)_a d_{p,q} t.$$

If $\int_a^x f(t)_a d_{p,q} t$ exists for each $x \in I$,, then we say f is (p,q)-integrable on I. Observe Definition 2 reduces to the q-integral of the function f when $a = 0$ and $p = 1$.

Example 2. *Define a function $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = kx$ for $x \in I$ where $k \in \mathbb{R}$. Then*

$$\begin{aligned}
\int_a^x f(t)\ {}_a d_{p,q} t &= \int_a^x kt\ {}_a d_{p,q} t \\
&= (p-q)(x-a) \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} \left[k\left(\frac{q^n}{p^{n+1}} x + \left(1 - \frac{q^n}{p^{n+1}}\right) a\right)\right] \\
&= \frac{k(x-a)(x - a(1-p-q))}{p+q}.
\end{aligned}$$

Theorem 5. *Refs.* [22,23] *Let $f : I \to \mathbb{R}$ be a continuous function. We have*

(i) $\int_a^x {}_a D_{p,q} f(t)\ {}_a d_{p,q} t = f(x) - f(a)$;

(ii) $\int_c^x {}_a D_{p,q} f(t)\ {}_a d_{p,q} t = f(x) - f(c)$, *for $c \in (a,x)$.*

Theorem 6. *Refs.* [22,23] *If $f, g : I \to \mathbb{R}$ are two continuous functions and $\alpha \in \mathbb{R}$, then for $x \in I$,*

(i) $\int_a^x [f(t) + g(t)]\ {}_a d_{p,q} t = \int_a^x f(t)\ {}_a d_{p,q} t + \int_a^x g(t)\ {}_a d_{p,q} t$;

(ii) $\int_a^x \alpha f(t)\ {}_a d_{p,q} t = \alpha \int_a^x f(t)\ {}_a d_{p,q} t$;

(iii) $\int_a^x f(pt + (1-p)a)_a D_{p,q} g(t)\ {}_a d_{p,q} t = [(fg)(t)]_a^x - \int_a^x g(qt + (1-q)a)_a D_{p,q} f(t)\ {}_a d_{p,q} t$.

Lemma 1. *Ref.* [50] *Let $f : I \to \mathbb{R}$ be a twice differentiable function such that $m \le f'' \le M$. It follows that*

$$m \frac{\lambda(1-\lambda)}{2} (b-a)^2 \le (1-\lambda) f(a) + \lambda f(b) - f_\lambda(b,a) \le M \frac{\lambda(1-\lambda)}{2} (b-a)^2. \tag{6}$$

3. Main Results

In 2017, W. Yang [50] obtained some Fejér-type quantum integral inequalities. Unfortunately, there are many mistakes in the proofs. Many q-integrals are calculated incorrectly. Besides, the results of lemma and theorems are also wrong. Here, we will show the errors of Lemma 3 in [50].

Statement 1 (Lemma 3, [50]). *If $f : I \to \mathbb{R}$ is a twice q-differentiable function with ${}_a D_q^2 f$ q-integrable on I, then*

$$\begin{aligned}
\int_a^b (x-a)(b-x)_a D_q^2 \quad & f(x)\ {}_a d_q x \\
&= (qb - a) f_q(b,a) + (b - qa) f(a) - (1+q) \int_a^b f_{q^2}(x,a)\ {}_a d_q x,
\end{aligned} \tag{7}$$

and

$$\int_a^b \left[(x-a)^2+(x-b)^2\right]{}_aD_q^2 f(x)\ {}_ad_qx = (b-a)^2\left({}_aD_qf(b) - {}_aD_qf(a)\right) \\ -2(qb-a)f_q(b,a) - 2(b-qa)f(a) + 2(1+q)\int_a^b f_{q^2}(x,a)\ {}_ad_qx. \tag{8}$$

Example 3. *Let a function $f : [1,2] \to \mathbb{R}$ be defined by $f(x) = x$. It follows that f satisfies the conditions of Lemma 1. The left side of Equality (7) and (8) become*

$$\int_1^2 (x-1)(2-x){}_1D_q^2 x\ {}_1d_qx = 0 \quad \text{and} \quad \int_1^2 \left[(x-1)^2+(x-2)^2\right]{}_1D_q^2 f(x)\ {}_ad_qx = 0,$$

respectively. The right side of Equality (7) becomes

$$(q(2)-(1))f_q(2,1) + (2-q(1))f(1) - (1+q)\int_1^2 f_{q^2}(x,1)\ {}_1d_qx = q^2 - q, \tag{9}$$

and the right side of Equality (8) becomes

$$\begin{aligned}(2-1)^2\left({}_aD_qf(2) \quad - {}_aD_qf(1)\right) &- 2(2q-1)f_q(2,1) - 2(2-q)f(1) + 2(1+q)\int_1^2 f_{q^2}(x,1)\ {}_1d_qx \\ &= -2(2q-1)(q+1) - 2(2-q)(1) + 2(1+q)\int_1^2 f_{q^2}(x,1)\ {}_1d_qx \\ &= 2q - 2q^2.\end{aligned} \tag{10}$$

Since $q \in (0,1)$, Equalities (9) and (10) are not equal to 0. Therefore, Equality (7) and (8) are not correct.

Since Lemma 1 is used in the proof of Theorems 9 and 10 in [50], there are errors in those theorems. Now, we show that Theorem 9 in [50] is not correct.

Statement 2 (Theorem 9, [50])**.** *Let $f : I \to \mathbb{R}$ be a twice q-differentiable function with ${}_aD_q^2 f$ q-integrable on I, such that $m \le {}_aD_q^2 f \le M$. It follows that*

$$\begin{aligned}\frac{mq^2(b-a)^3}{(1+q)(1+q+q^2)} &\le (qb-a)f_q(b,a) + (b-qa)f(a) - (1+q)\int_a^b f_{q^2}(x,a)\ {}_ad_qx \\ &\le \frac{Mq^2(b-a)^3}{(1+q)(1+q+q^2)},\end{aligned} \tag{11}$$

and

$$\begin{aligned}\frac{m(1+q+q^3)(b-a)^3}{2(1+q)(1+q+q^2)} &\le (b-a)^2\left({}_aD_qf(b) - {}_aD_qf(a)\right) - 2(qb-a)f_q(b,a) \\ &\quad -2(b-qa)f(a) + 2(1+q)\int_a^b f_{q^2}(x,a)\ {}_ad_qx \\ &\le \frac{M(1+q+q^3)(b-a)^3}{2(1+q)(1+q+q^2)}.\end{aligned} \tag{12}$$

Example 4. *Let a function $f : [1,2] \to \mathbb{R}$ be defined by $f(x) = x$. Since ${}_aD_q^2 f(x) = {}_aD_q^2 x = 0$, we obtain $m \le 0$ and $M \ge 0$. It follows that f satisfies the conditions in Theorem 2 with $-1 \le {}_aD_q^2 f \le 1$. Then, we have*

$$\frac{(-1)q^2(2-1)^3}{(1+q)(1+q+q^2)} = \frac{-q^2}{(1+q)(1+q+q^2)}, \tag{13}$$

and

$$\frac{(1)q^2(2-1)^3}{(1+q)(1+q+q^2)} = \frac{q^2}{(1+q)(1+q+q^2)}. \tag{14}$$

Also,

$$(qb-a)f_q(b,a) + (b-qa)f(a) - (1+q)\int_a^b f_{q^2}(x,a)\ {}_a d_q x = q^2 - q. \tag{15}$$

As we seen, from (13) *to* (15) *and for* $q \in (0,1)$ *we write*

$$\frac{-q^2}{(1+q)(1+q+q^2)} \le q^2 - q \le \frac{q^2}{(1+q)(1+q+q^2)}.$$

For instance, choose $q = \frac{1}{2}$, *we have*

$$\frac{-\left(\frac{1}{4}\right)}{\left(1+\frac{1}{2}\right)\left(1+\frac{1}{2}+\frac{1}{4}\right)} \le \frac{1}{4} - \frac{1}{2} \le \frac{\left(\frac{1}{4}\right)}{\left(1+\frac{1}{2}\right)\left(1+\frac{1}{2}+\frac{1}{4}\right)}.$$

That is,

$$-\frac{2}{21} \le -\frac{1}{4} \le \frac{2}{21}.$$

This implies that $-\frac{2}{21} \not\le -\frac{1}{4}$. *Therefore, Inequality* (11) *is not correct. Inequality* (12) *also has the same error.*

Next, we give some inequalities of Fejér type inequalities by using (p,q)-integral. If $p = 1$, then we give the correct results of Fejér type quantum integral inequalities.

Theorem 7. *Let* $f : I \to \mathbb{R}$ *be a twice* (p,q)*-differentiable function such that* $m \le f'' \le M$. *It follows that*

$$\begin{aligned} \frac{mpq^2(b-a)^2}{(p+q)(p^2+pq+q^2)} &\le [f(a)+f(b)] - \frac{1}{p(b-a)} \int_a^{pb+(1-p)a} [f(x)+f(a+b-x)]\ {}_a d_{p,q}x \\ &\le \frac{Mpq^2(b-a)^2}{(p+q)(p^2+pq+q^2)}, \end{aligned} \tag{16}$$

and

$$\begin{aligned} &\frac{m(b-a)^2}{4} \left[1 - \frac{4p}{p+q} + \frac{4p^2}{p^2+pq+q^2}\right] \\ &\le \frac{1}{p(b-a)} \int_a^{pb+(1-p)a} [f(x)+f(a+b-x)]\ {}_a d_{p,q}x - 2f\left(\frac{a+b}{2}\right) \\ &\le \frac{M(b-a)^2}{4}\left[1 - \frac{4p}{p+q} + \frac{4p^2}{p^2+pq+q^2}\right]. \end{aligned} \tag{17}$$

Proof. Taking (p,q)-integral for Inequality (6) with respect to λ over $[0,p]$ yields

$$\begin{aligned} \frac{m(b-a)^2}{2} \int_0^p \lambda(1-\lambda)\ d_{p,q}\lambda &\le f(a)\int_0^p (1-\lambda)\ d_{p,q}\lambda + f(b)\int_0^p \lambda\ d_{p,q}\lambda - \int_0^p f_\lambda(b,a)\ d_{p,q}\lambda \\ &\le \frac{M(b-a)^2}{2} \int_0^p \lambda(1-\lambda)\ d_{p,q}\lambda. \end{aligned} \tag{18}$$

Using direct computation and variable changing in (18), we have

$$\begin{aligned}\frac{mp^2q^2(b-a)^2}{2(p+q)(p^2+pq+q^2)} &\leq \frac{pqf(a)}{p+q}+\frac{p^2f(b)}{p+q}-\frac{1}{b-a}\int_a^{pb+(1-p)a} f(x)\ {}_a d_{p,q}x \\ &\leq \frac{Mp^2q^2(b-a)^2}{2(p+q)(p^2+pq+q^2)}.\end{aligned} \tag{19}$$

Similarly, using (p,q)-integration on the first inequality of Theorem 3 with respect to λ over $[0,p]$, we obtain

$$\begin{aligned}\frac{m(b-a)^2}{2}\int_0^p \lambda(1-\lambda)\ d_{p,q}\lambda &\leq f(a)\int_0^p \lambda\ d_{p,q}\lambda + f(b)\int_0^p (1-\lambda)\ d_{p,q}\lambda - \int_0^p f_\lambda(a,b)\ d_{p,q}\lambda \\ &\leq \frac{M(b-a)^2}{2}\int_0^p \lambda(1-\lambda)\ d_{p,q}\lambda.\end{aligned} \tag{20}$$

Using direct computation and variable changing in (20), we obtain

$$\begin{aligned}\frac{mp^2q^2(b-a)^2}{2(p+q)(p^2+pq+q^2)} &\leq \frac{p^2f(a)}{p+q}+\frac{pqf(b)}{p+q}-\frac{1}{b-a}\int_a^{pb+(1-p)a} f(a+b-x)\ {}_a d_{p,q}x \\ &\leq \frac{Mp^2q^2(b-a)^2}{2(p+q)(p^2+pq+q^2)}.\end{aligned} \tag{21}$$

Inequality (16) comes from (19) and (21).

Next, using (p,q)-integration on the second inequality of Theorem 3 with respect to λ over $[0,p]$, we obtain

$$\begin{aligned}m\frac{(b-a)^2}{8}\int_0^p (1-2\lambda)^2\ d_{p,q}\lambda &\leq \int_0^p \frac{f_\lambda(a,b)+f_\lambda(b,a)}{2}\ d_{p,q}\lambda - \int_0^p f\left(\frac{a+b}{2}\right)\ d_{p,q}\lambda \\ &\leq M\frac{(b-a)^2}{8}\int_0^p (1-2\lambda)^2\ d_{p,q}\lambda.\end{aligned}$$

Changing the variable, we have

$$\begin{aligned}&\frac{m(b-a)^2}{8}\left[p-\frac{4p^2}{p+q}+\frac{4p^3}{p^2+pq+q^2}\right] \\ &\quad\leq \frac{1}{2(b-a)}\int_a^{pb+(1-p)a}[f(x)+f(a+b-x)]\ {}_a d_{p,q}x - pf\left(\frac{a+b}{2}\right) \\ &\quad\leq \frac{M(b-a)^2}{8}\left[p-\frac{4p^2}{p+q}+\frac{4p^3}{p^2+pq+q^2}\right],\end{aligned}$$

which implies Inequality (17). This completes the proof of theorem. □

Remark 1. *(i)* *If $p=1$, then Theorem 7 reduces to Theorem 7 in [50].*
(ii) *If $p=1$ and $q\to 1$, then Inequality (16) reduces to (2), and Inequality (17) reduces (3).*

Theorem 8. *Let $f: I \to \mathbb{R}$ be a twice (p,q)-differentiable function such that $m \leq f'' \leq M$. If $w: I\to\mathbb{R}$ is (p,q)-integrable on I, nonnegative and symmetric about $x=(a+b)/2$, then*

$$\begin{aligned}m\int_a^{pb+(1-p)a} &(x-a)(b-x)w(x)\ {}_a d_{p,q}x \\ &\leq [f(a)+f(b)]\int_a^{pb+(1-p)a} w(x)\ {}_a d_{p,q}x - \int_a^{pb+(1-p)a}[f(x)+f(a+b-x)]w(x)\ {}_a d_{p,q}x \\ &\leq M\int_a^{pb+(1-p)a}(x-a)(b-x)w(x)\ {}_a d_{p,q}x,\end{aligned} \tag{22}$$

and

$$\begin{aligned}\frac{m}{4}\int_a^{pb+(1-p)a} & (2x-a-b)^2 w(x)\ {}_a d_{p,q}x \\ & \leq \int_a^{pb+(1-p)a}[f(x)+f(a+b-x)]w(x)\ {}_a d_{p,q}x - 2f\left(\frac{a+b}{2}\right)\int_a^{pb+(1-p)a} w(x)\ {}_a d_{p,q}x \\ & \leq \frac{M}{4}\int_a^{pb+(1-p)a}(2x-a-b)^2 w(x)\ {}_a d_{p,q}x.\end{aligned} \tag{23}$$

Proof. Multiplying Inequality (6) by $w_\lambda(b,a)$, we get

$$\begin{aligned} m\frac{\lambda(1-\lambda)}{2}(b-a)^2 w_\lambda(b,a) & \leq [(1-\lambda)f(a)+\lambda f(b)-f_\lambda(b,a)]w_\lambda(b,a) \\ & \leq M\frac{\lambda(1-\lambda)}{2}(b-a)^2 w_\lambda(b,a).\end{aligned} \tag{24}$$

Taking (p,q)-integral for Inequality (24) with respect to λ over $[0,p]$ yields

$$\begin{aligned}\frac{m(b-a)^2}{2}\int_0^p \lambda(1-\lambda)w_\lambda(b,a)\ d_{p,q}\lambda & \leq \int_0^p[(1-\lambda)f(a)+\lambda f(b)-f_\lambda(b,a)]w_\lambda(b,a)\ d_{p,q}\lambda \\ & \leq \frac{M(b-a)^2}{2}\int_0^p \lambda(1-\lambda)w_\lambda(b,a)\ d_{p,q}\lambda.\end{aligned} \tag{25}$$

Using directly computation and variable changing in (25), we obtain

$$\begin{aligned}\frac{m}{2}\int_a^{pb+(1-p)a} & \frac{(x-a)(b-x)w(x)}{b-a}\ {}_a d_{p,q}x \\ & \leq f(a)\int_a^{pb+(1-p)a}\frac{(b-x)w(x)}{(b-a)^2}\ {}_a d_{p,q}x + f(b)\int_a^{pb+(1-p)a}\frac{(x-a)w(x)}{(b-a)^2}\ {}_a d_{p,q}x \\ & \quad -\int_a^{pb+(1-p)a}\frac{f(x)w(x)}{b-a}\ {}_a d_{p,q}x \\ & \leq \frac{M}{2}\int_a^{pb+(1-p)a}\frac{(x-a)(b-x)w(x)}{b-a}\ {}_a d_{p,q}x.\end{aligned} \tag{26}$$

Similarly, multiplying the first inequality of Theorem 3 by $w_\lambda(b,a)$, and subsequently take (p,q)-integral on the obtained inequality with respect to λ over $[0,p]$ yield

$$\begin{aligned}\frac{m(b-a)^2}{2}\int_0^p & \lambda(1-\lambda)w_\lambda(b,a)\ d_{p,q}\lambda \\ & \leq f(a)\int_0^p \lambda w_\lambda(b,a)\ d_{p,q}\lambda + f(b)\int_0^p(1-\lambda)w_\lambda(b,a)\ d_{p,q}\lambda - \int_0^p f_\lambda(a,b)w_\lambda(b,a)\ d_{p,q}\lambda \\ & \leq \frac{M(b-a)^2}{2}\int_0^p \lambda(1-\lambda)w_\lambda(b,a)\ d_{p,q}\lambda.\end{aligned} \tag{27}$$

From (27), we change the variable and apply the symmetry of $w(x)$, it follows that

$$\begin{aligned}\frac{m}{2} & \int_a^{pb+(1-p)a}\frac{(x-a)(b-x)w(x)}{b-a}\ {}_a d_{p,q}x \\ & \leq f(a)\int_a^{pb+(1-p)a}\frac{(x-a)w(x)}{(b-a)^2}\ {}_a d_{p,q}x + f(b)\int_a^{pb+(1-p)a}\frac{(b-x)w(x)}{(b-a)^2}\ {}_a d_{p,q}x \\ & \quad -\int_a^{pb+(1-p)a}\frac{f(a+b-x)w(x)}{b-a}\ {}_a d_{p,q}x \\ & \leq \frac{M}{2}\int_a^{pb+(1-p)a}\frac{(x-a)(b-x)w(x)}{b-a}\ {}_a d_{p,q}x.\end{aligned} \tag{28}$$

Then, we obtain Inequality (22) from (26) and (28).

Next, multiplying the second inequality of Theorem 3 by $w_\lambda(b,a)$, and subsequently taking (p,q)-integral on the obtained inequality with respect to λ over $[0,p]$ yields

$$\begin{aligned} m\frac{(b-a)^2}{8} & \int_0^p(1-2\lambda)^2 w_\lambda(b,a)\ d_{p,q}\lambda \\ & \leq \frac{1}{2}\left[\int_0^p f_\lambda(a,b)w_\lambda(b,a)\ d_{p,q}\lambda + \int_0^p f_\lambda(b,a)w_\lambda(b,a)\ d_{p,q}\lambda\right] - f\left(\frac{a+b}{2}\right)\int_0^p w_\lambda(b,a)\ d_{p,q}\lambda \\ & \leq M\frac{(b-a)^2}{8}\int_0^p(1-2\lambda)^2 w_\lambda(b,a)\ d_{p,q}\lambda.\end{aligned} \tag{29}$$

By using the change of the variable of (29), we get

$$\begin{aligned}
&\frac{m}{8}\int_a^{pb+(1-p)a}\frac{(2x-a-b)^2w(x)}{b-a}\ {}_ad_{p,q}x\\
&\le\frac{1}{2(b-a)}\left[\int_a^{pb+(1-p)a}f(x)w(x)\ {}_ad_{p,q}x+\int_a^{pb+(1-p)a}f(a+b-x)w(x)\ {}_ad_{p,q}x\right]\\
&\quad-f\left(\frac{a+b}{2}\right)\int_a^{pb+(1-p)a}\frac{w(x)}{b-a}\ {}_ad_{p,q}x\\
&\le\frac{M}{8}\int_a^{pb+(1-p)a}\frac{(2x-a-b)^2w(x)}{b-a}\ {}_ad_{p,q}x,
\end{aligned}$$

which implies Inequality (23). The proof of the theorem is complete. □

Remark 2. *(i) If $p=1$, then Theorem 8 reduces to Theorem 8 in [50].*
(ii) If $p=1$ and $q\to1$, then Inequality (22) reduce to (4) and Inequility (23) reduce to (5).

Lemma 2. *If $f:I\to\mathbb{R}$ is a twice (p,q)-differentiable function with ${}_aD^2_{p,q}f$ (p,q)-integrable on I, then*

$$\begin{aligned}
&\int_a^{pb+(1-p)a}(x-a)(b-x)\,{}_aD^2_{p,q}f(x)\ {}_ad_{p,q}x\\
&=\frac{q}{p^2}(b-a)f_{pq}(b,a)+\frac{1}{p}(b-a)f(a)-\left(\frac{p+q}{p^3}\right)\int_a^{pb+(1-p)a}f_{q^2}(x,a)\ {}_ad_{p,q}x,
\end{aligned}\tag{30}$$

and

$$\begin{aligned}
&\int_a^{pb+(1-p)a}\left[(x-a)^2+(x-b)^2\right]{}_aD^2_{p,q}f(x)\ {}_ad_{p,q}x\\
&=(b-a)^2\Big({}_aD_{p,q}f_p(b,a)-{}_aD_{p,q}f(a)\Big)-\frac{2q}{p^2}(b-a)f_{pq}(b,a)-\frac{2}{p}(b-a)f(a)\\
&\quad+2\left(\frac{p+q}{p^3}\right)\int_a^{pb+(1-p)a}f_{q^2}(x,a)\ {}_ad_{p,q}x.
\end{aligned}\tag{31}$$

Proof. Using (p,q)-integration by parts yields

$$\begin{aligned}
&\int_a^{pb+(1-p)a}(x-a)(b-x)\,{}_aD^2_{p,q}f(x)\ {}_ad_{p,q}x\\
&=\left[\frac{1}{p^2}(x-a)(pb+(1-p)a-x)\,{}_aD_{p,q}f(x)\right]_a^{pb+(1-p)a}\\
&\quad-\int_a^{pb+(1-p)a}{}_aD_{p,q}f(qx+(1-q)a)\,{}_aD_{p,q}\left(\frac{1}{p^2}(x-a)(pb+(1-p)a-x)\right)\ {}_ad_{p,q}x\\
&=-\frac{1}{p^2}\int_a^{pb+(1-p)a}[pb+qa-(p+q)x]\,{}_aD_{p,q}f_q(x,a)\ {}_ad_{p,q}x\\
&=-\frac{1}{p^2}\left\{\left[\left(pb+qa-(p+q)\left(\frac{x-(1-p)a}{p}\right)\right)f_q(x,a)\right]_a^{pb+(1-p)a}\right.\\
&\quad\left.-\int_a^{pb+(1-p)a}f_q(qx+(1-q)a,a)\,{}_aD_{p,q}\left(pb+qa-(p+q)\left(\frac{x-(1-p)a}{p}\right)\right)\ {}_ad_{p,q}x\right\}\\
&=-\frac{1}{p^2}\left\{\left[q(a-b)f_{pq}(b,a)-p(b-a)f(a)\right]+\left(\frac{p+q}{p}\right)\int_a^{pb+(1-p)a}f_{q^2}(x,a)\ {}_ad_{p,q}x\right\}\\
&=\frac{q}{p^2}(b-a)f_{pq}(b,a)+\frac{1}{p}(b-a)f(a)-\left(\frac{p+q}{p^3}\right)\int_a^{pb+(1-p)a}f_{q^2}(x,a)\ {}_ad_{p,q}x,
\end{aligned}$$

which is Inequality (30).

Next, we prove Inequality (31). Using (p,q)-integration by parts, we obtain

$$\begin{aligned}
\int_a^{pb+(1-p)a} & (x-a)^2 {}_aD^2_{p,q}f(x)\ {}_ad_{p,q}x \\
&= \left[\left(\frac{x-a}{p}\right)^2 {}_aD_{p,q}f(x)\right]_a^{pb+(1-p)a} - \int_a^{pb+(1-p)a} {}_aD_{p,q}f(qx+(1-q)a)\,{}_aD_{p,q}\left(\frac{x-a}{p}\right)^2 {}_ad_{p,q}x \\
&= (b-a)^2 {}_aD_{p,q}f_p(b,a) - \frac{1}{p^2}\int_a^{pb+(1-p)a}(p+q)(x-a)\,{}_aD_{p,q}f_q(x,a)\ {}_ad_{p,q}x \\
&= (b-a)^2 {}_aD_{p,q}f_p(b,a) - \frac{p+q}{p^2}\int_a^{pb+(1-p)a}(x-a)\,{}_aD_{p,q}f_q(x,a)\ {}_ad_{p,q}x \\
&= (b-a)^2 {}_aD_{p,q}f_p(b,a) - \frac{p+q}{p^2}\left\{\left[\left(\frac{x-a}{p}\right)f_q(x,a)\right]_a^{pb+(1-p)a}\right. \\
&\quad \left. - \int_a^{pb+(1-p)a} f_q(qx+(1-q)a,a)\,{}_aD_{p,q}\left(\frac{x-a}{p}\right)\ {}_ad_{p,q}x\right\} \\
&= (b-a)^2 {}_aD_{p,q}f_p(b,a) - \frac{p+q}{p^2}\left[(b-a)f_{pq}(b,a) - \frac{1}{p}\int_a^{pb+(1-p)a} f_{q^2}(x,a)\ {}_ad_{p,q}x\right] \\
&= (b-a)^2 {}_aD_{p,q}f_p(b,a) - \frac{(p+q)(b-a)}{p^2}f_{pq}(b,a) + \left(\frac{p+q}{p^3}\right)\int_a^{pb+(1-p)a} f_{q^2}(x,a)\ {}_ad_{p,q}x,
\end{aligned} \tag{32}$$

and

$$\begin{aligned}
\int_a^{pb+(1-p)a} & (x-b)^2 {}_aD^2_{p,q}f(x)\ {}_ad_{p,q}x \\
&= \left[\left(\frac{x-(pb+(1-p)a)}{p}\right)^2 {}_aD_{p,q}f(x)\right]_a^{pb+(1-p)a} \\
&\quad - \int_a^{pb+(1-p)a} {}_aD_{p,q}f(qx+(1-q)a)\,{}_aD_{p,q}\left(\frac{x-(pb+(1-p)a)}{p}\right)^2 {}_ad_{p,q}x \\
&= -(b-a)^2 {}_aD_{p,q}f(a) - \frac{1}{p^2}\int_a^{pb+(1-p)a}[(p+q)(x-a)-2p(b-a)]\,{}_aD_{p,q}f_q(x,a)\ {}_ad_{p,q}x
\end{aligned}$$

$$\begin{aligned}
&= -(b-a)^2 {}_aD_{p,q}f(a) - \frac{1}{p^2}\left\{\left[\left(\left(\frac{p+q}{p}\right)(x-a)-2p(b-a)\right)f_q(x,a)\right]_a^{pb+(1-p)a}\right. \\
&\quad \left. - \int_a^{pb+(1-p)a} f_{q^2}(x,a)\,{}_aD_{p,q}\left[\left(\frac{p+q}{p}\right)(x-a)-2p(b-a)\right]\ {}_ad_{p,q}x\right\} \\
&= -(b-a)^2 {}_aD_{p,q}f(a) - \frac{(p+q)(b-a)}{p^2}f_{pq}(b,a) + \frac{2}{p}(b-a)f_{pq}(b,a) - \frac{2}{p}(b-a)f(a) \\
&\quad + \left(\frac{p+q}{p^3}\right)\int_a^{pb+(1-p)a} f_{q^2}(x,a)\ {}_ad_{p,q}x.
\end{aligned} \tag{33}$$

Adding (32) and (33), we obtain

$$\begin{aligned}
\int_a^{pb+(1-p)a} & \left[(x-a)^2+(x-b)^2\right]{}_aD^2_{p,q}f(x)\ {}_ad_{p,q}x \\
&= (b-a)^2\left({}_aD_{p,q}f_p(b,a) - {}_aD_{p,q}f(a)\right) - \frac{2q}{p^2}(b-a)f_{pq}(b,a) - \frac{2}{p}(b-a)f(a) \\
&\quad + 2\left(\frac{p+q}{p^3}\right)\int_a^{pb+(1-p)a} f_{q^2}(x,a)\ {}_ad_{p,q}x,
\end{aligned}$$

which is Inequality (31). Thus the proof is completed. □

Taking $p=1$ in Lemma 2 yields the correct result of Statement 1.

Corollary 1. *If $f: I \to \mathbb{R}$ is a twice q-differentiable function where ${}_aD^2_q f$ q-integrable on I, then*

$$\begin{aligned}\int_a^b (x-a)(b-x)_a D_q^2 \quad & f(x)\ {}_a d_q x \\ & = q(b-a) f_q(b,a) + (b-a) f(a) - (1+q) \int_a^b f_{q^2}(x,a)\ {}_a d_q x,\end{aligned} \tag{34}$$

and

$$\begin{aligned}\int_a^b \left[(x-a)^2 + (x-b)^2\right]_a D_q^2 f(x)\ {}_a d_q x = (b-a)^2 \Big({}_a D_q f(b) - {}_a D_q f(a) \Big) \\ -2q(b-a) f_q(b,a) - 2(b-a) f(a) + 2(1+q) \int_a^b f_{q^2}(x,a)\ {}_a d_q x.\end{aligned} \tag{35}$$

Remark 3. *From Example 3, the left side of Equality (34) and (35) become*

$$\int_1^2 (x-1)(2-x)_1 D_q^2 x\ {}_1 d_q x = 0 \quad \textit{and} \quad \int_1^2 \left[(x-1)^2 + (x-2)^2\right]_1 D_q^2 f(x)\ {}_a d_q x = 0,$$

respectively. The right side of Equality (34) becomes

$$\begin{aligned} q(2-1) f_q(2,1) + (2-1) f(1) - (1+q) \int_1^2 f_{q^2}(x,1)\ {}_1 d_q x &= q(q+1) + 1 - (1+q) \int_1^2 f_{q^2}(x,1)\ {}_1 d_q x \\ &= q^2 + q + 1 - (1+q) \left[q^2 \left(\frac{1}{1+q} \right) + 1 \right] \\ &= 0, \end{aligned}$$

and the right side of Equality (35) becomes

$$\begin{aligned} (2-1)^2 \Big({}_1 D_q f(2) - {}_1 D_q f(1) \Big) &- 2q(2-1) f_q(2,1) - 2(2-1) f(1) + 2(1+q) \int_1^2 f_{q^2}(x,1)\ {}_1 d_q x \\ &= -2q(q+1) - 2(1) + 2(1+q) \left[q^2 \left(\frac{1}{1+q} \right) + 1 \right] \\ &= 0, \end{aligned}$$

which shows the result appearing in Corollary 1.

Theorem 9. *Let $f : I \to \mathbb{R}$ be a twice (p,q)-differentiable function where ${}_a D_{p,q}^2 f$ (p,q)-integrable on I with $m \le {}_a D_{p,q}^2 f \le M$. It follows that*

$$\begin{aligned} \frac{m p^3 q^2 (b-a)^3}{(p+q)(p^2+pq+q^2)} \quad & \le \frac{q}{p}(b-a) f_{pq}(b,a) + (b-a) f(a) - \left(\frac{p+q}{p^2} \right) \int_a^{pb+(1-p)a} f_{q^2}(x,a)\ {}_a d_{p,q} x \\ & \le \frac{M p^3 q^2 (b-a)^3}{(p+q)(p^2+pq+q^2)}, \end{aligned} \tag{36}$$

and

$$\begin{aligned} \frac{m(p^4 + 2p^3 q + pq^3)(b-a)^3}{(p+q)(p^2+pq+q^2)} \quad & \le (b-a)^2 \Big({}_a D_{p,q} f_p(b,a) - {}_a D_{p,q} f(a) \Big) - \frac{2q}{p^2}(b-a) f_q(b,a) \\ & \quad -\frac{2}{p}(b-a) f(a) + 2 \left(\frac{p+q}{p^3} \right) \int_a^{pb+(1-p)a} f_{q^2}(x,a)\ {}_a d_{p,q} x \\ & \le \frac{M(p^4 + 2p^3 q + pq^3)(b-a)^3}{(p+q)(p^2+pq+q^2)}. \end{aligned} \tag{37}$$

Proof. Since $m \le {}_a D_{p,q}^2 f \le M$, it follows that

$$m(x-a)(b-x) \le (x-a)(b-x)_a D_{p,q}^2 f(x) \le M(x-a)(b-x), \quad \forall x \in I. \tag{38}$$

Take (p,q)-integral for Inequality (38) with respect to x from a to $pb+(1-p)a$, we obtain

$$\begin{aligned} m\int_a^{pb+(1-p)a}(x-a)(b-x)\ {}_ad_{p,q}x &\leq \int_a^{pb+(1-p)a}(x-a)(b-x){}_aD^2_{p,q}f(x)\ {}_ad_{p,q}x \\ &\leq M\int_a^{pb+(1-p)a}(x-a)(b-x)\ {}_ad_{p,q}x. \end{aligned} \tag{39}$$

Applying Inequality (30) in Lemma 2 and

$$\int_a^{pb+(1-p)a}(x-a)(b-x)\ {}_ad_{p,q}x = \frac{p^2q^2(b-a)^3}{(p+q)(p^2+pq+q^2)}$$

into (39), we get

$$\begin{aligned} \frac{mp^2q^2(b-a)^3}{(p+q)(p^2+pq+q^2)} &\leq \frac{q}{p^2}(b-a)f_{pq}(b,a)+\frac{1}{p}(b-a)f(a)-\left(\frac{p+q}{p^3}\right)\int_a^{pb+(1-p)a} f_{q^2}(x,a)\ {}_ad_{p,q}x \\ &\leq \frac{Mp^2q^2(b-a)^3}{(p+q)(p^2+pq+q^2)}, \end{aligned}$$

which implies Inequality (36). From $m \leq {}_aD^2_{p,q}f \leq M$, we have

$$m(x-a)^2 \leq (x-a)^2{}_aD^2_{p,q}f(x) \leq M(x-a)^2, \tag{40}$$

and

$$m(x-b)^2 \leq (x-b)^2{}_aD^2_{p,q}f(x) \leq M(x-b)^2, \tag{41}$$

for all $x \in I$. Taking (p,q)-integral on (40) and (41) with respect to x from a to $pb+(1-p)a$, we obtain

$$m\int_a^{pb+(1-p)a}(x-a)^2\ {}_ad_{p,q}x \leq \int_a^{pb+(1-p)a}(x-a)^2{}_aD^2_{p,q}f(x)\ {}_ad_{p,q}x \leq M\int_a^{pb+(1-p)a}(x-a)^2\ {}_ad_{p,q}x, \tag{42}$$

and

$$m\int_a^{pb+(1-p)a}(x-b)^2\ {}_ad_{p,q}x \leq \int_a^{pb+(1-p)a}(x-b)^2{}_aD^2_{p,q}f(x)\ {}_ad_{p,q}x \leq M\int_a^{pb+(1-p)a}(x-b)^2\ {}_ad_{p,q}x, \tag{43}$$

respectively. By directly computation, we obtain

$$\int_a^{pb+(1-p)a}(x-a)^2\ {}_ad_{p,q}x = \frac{p^3(b-a)^3}{p^2+pq+q^2}, \tag{44}$$

and

$$\int_a^{pb+(1-p)a}(x-b)^2\ {}_ad_{p,q}x = \frac{(p^3q+pq^3)(b-a)^3}{(p+q)(p^2+pq+q^2)}. \tag{45}$$

Substituting (44) into (42), we get

$$\frac{mp^3(b-a)^3}{p^2+pq+q^2} \leq \int_a^{pb+(1-p)a}(x-a)^2{}_aD^2_{p,q}f(x)\ {}_ad_{p,q}x \leq \frac{Mp^3(b-a)^3}{p^2+pq+q^2}. \tag{46}$$

And substituting (45) into (43), we get

$$\frac{m(p^3q+pq^3)(b-a)^3}{(p+q)(p^2+pq+q^2)} \leq \int_a^{pb+(1-p)a}(x-b)^2{}_aD^2_{p,q}f(x)\ {}_ad_{p,q}x \leq \frac{M(p^3q+pq^3)(b-a)^3}{(p+q)(p^2+pq+q^2)}. \tag{47}$$

Adding (46) and (47), we obtain

$$\begin{aligned} \frac{m(p^4+2p^3q+pq^3)(b-a)^3}{(p+q)(p^2+pq+q^2)} &\leq \int_a^{pb+(1-p)a}\left[(x-a)^2+(x-b)^2\right]{}_aD^2_{p,q}f(x)\ {}_ad_{p,q}x \\ &\leq \frac{M(p^4+2p^3q+pq^3)(b-a)^3}{(p+q)(p^2+pq+q^2)}. \end{aligned} \tag{48}$$

Substituting Equality (31) into (48), we get Inequality (37). This completes the proof. □

Taking $p = 1$ in Theorem 9 yields the correct result of Statement 2.

Corollary 2. *If $f : I \to \mathbb{R}$ is a twice q-differentiable function with ${}_aD_q^2 f$ q-integrable on I such that $m \leq {}_aD_q^2 f \leq M$, then*

$$\begin{aligned}\frac{mq^2(b-a)^3}{(1+q)(1+q+q^2)} &\leq q(b-a)f_q(b,a) + (b-a)f(a) - (1+q)\int_a^b f_{q^2}(x,a)\ {}_ad_qx \\ &\leq \frac{Mq^2(b-a)^3}{(1+q)(1+q+q^2)},\end{aligned}$$

and

$$\begin{aligned}\frac{m(1+2q+q^3)(b-a)^3}{(1+q)(1+q+q^2)} &\leq (b-a)^2\big({}_aD_qf(b) -{}_a D_qf(a)\big) - 2q(b-a)f_q(b,a) \\ &\quad - 2(b-a)f(a) + 2(1+q)\int_a^b f_{q^2}(x,a)\ {}_ad_qx \\ &\leq \frac{M(1+2q+q^3)(b-a)^3}{(1+q)(1+q+q^2)}.\end{aligned}$$

Remark 4. *From Example 4, f satisfies the conditions of Corollary 2 with $-1 \leq {}_aD_q^2 f \leq 1$. Then we have*

$$\frac{(-1)q^2(2-1)^3}{(1+q)(1+q+q^2)} = \frac{-q^2}{(1+q)(1+q+q^2)}, \tag{49}$$

and

$$\frac{(1)q^2(2-1)^3}{(1+q)(1+q+q^2)} = \frac{q^2}{(1+q)(1+q+q^2)}. \tag{50}$$

Also,

$$q(b-a)f_q(b,a) + (b-a)f(a) - (1+q)\int_a^b f_{q^2}(x,a)\ {}_ad_qx = 0. \tag{51}$$

As we seen, from (49) to (51) and for $q \in (0,1)$ we write

$$\frac{-q^2}{(1+q)(1+q+q^2)} \leq 0 \leq \frac{q^2}{(1+q)(1+q+q^2)}.$$

For instance, choose $q = \dfrac{1}{2}$, we have

$$\frac{-\left(\frac{1}{4}\right)}{\left(1+\frac{1}{2}\right)\left(1+\frac{1}{2}+\frac{1}{4}\right)} \leq 0 \leq \frac{\left(\frac{1}{4}\right)}{\left(1+\frac{1}{2}\right)\left(1+\frac{1}{2}+\frac{1}{4}\right)}.$$

That is,

$$-\frac{2}{21} \leq 0 \leq \frac{2}{21},$$

which shows the result described in Corollary 2.

Theorem 10. *If $f : I \to \mathbb{R}$ is a twice (p,q)-differentiable function with ${}_aD^2_{p,q}f$ (p,q)-integrable on I such that $m \le {}_aD^2_{p,q}f \le M$, then*

$$\begin{aligned}&\Bigg| \left(\frac{q}{p^2}(b-a)f_{pq}(b,a) + \frac{1}{p}(b-a)f(a) - \left(\frac{p+q}{p^3}\right)\int_a^{pb+(1-p)a} f_{q^2}(x,a)\ {}_ad_{p,q}x\right) \\ &\quad - \frac{pq^2(b-a)^2\left({}_aD_{p,q}f_p(b,a) - {}_aD_{p,q}f(a)\right)}{(p+q)(p^2+pq+q^2)}\Bigg| \le \frac{p(b-a)^3}{16}(M-m).\end{aligned} \tag{52}$$

Proof. We observe that

$$\sup_{x\in[a,pb+(1-p)a]}(x-a)(b-x) = p(1-p)(b-a)^2 \le \frac{(b-a)^2}{4}, \quad \text{for } 0 < p < \frac{1}{2},$$
$$\sup_{x\in[a,pb+(1-p)a]}(x-a)(b-x) = \frac{(b-a)^2}{4}, \quad \text{for } \frac{1}{2} \le p \le 1,$$

and

$$\inf_{x\in[a,pb+(1-p)a]}(x-a)(b-x) = 0.$$

Consequently,

$$0 \le (x-a)(b-x) \le \frac{(b-a)^2}{4}, \quad \text{for } x \in [a, pb+(1-p)a].$$

Substituting $b, f(x)$, and $g(x)$ in Theorem 9 in [23] by $pb+(1-p)a, (x-a)(b-x)$, and ${}_aD^2_{p,q}f(x)$, respectively, we obtain

$$\begin{aligned}&\Bigg|\frac{1}{(pb+(1-p)a-a)}\int_a^{pb+(1-p)a}(x-a)(b-x){}_aD^2_{p,q}f(x)\ {}_ad_{p,q}x \\ &- \frac{1}{(pb+(1-p)a-a)^2}\left(\int_a^{pb+(1-p)a}(x-a)(b-x)\ {}_ad_{p,q}x\right)\left(\int_a^{pb+(1-p)a}{}_aD^2_{p,q}f(x)\ {}_ad_{p,q}x\right)\Bigg| \\ &\le \frac{1}{4}\left(\frac{(b-a)^2}{4}\right)(M-m).\end{aligned}$$

By Equality (31) in Lemma 2, we obtain

$$\begin{aligned}&\Bigg|\frac{1}{p(b-a)}\left(\frac{q}{p^2}(b-a)f_{pq}(b,a) + \frac{1}{p}(b-a)f(a) - \left(\frac{p+q}{p^3}\right)\int_a^{pb+(1-p)a} f_{q^2}(x,a)\ {}_ad_{p,q}x\right) \\ &\quad - \frac{q^2(b-a)\left({}_aD_{p,q}f_p(b,a) - {}_aD_{p,q}f(a)\right)}{(p+q)(p^2+pq+q^2)}\Bigg| \le \frac{(b-a)^2}{16}(M-m),\end{aligned}$$

which implies Inequality (52). This completes the proof. □

Taking $p = 1$ in Theorem 10 yields the correct result of Theorem 10 in [50].

Corollary 3. *If $f : I \to \mathbb{R}$ is a twice q-differentiable function with ${}_aD^2_q f$ q-integrable on I and $m \le {}_aD^2_q f \le M$, then*

$$\begin{aligned}&\Bigg|\left(q(b-a)f_q(b,a) + (b-a)f(a) - (1+q)\int_a^b f_{q^2}(x,a)\ {}_ad_qx\right) \\ &\quad - \frac{q^2(b-a)^2\left({}_aD_qf(b) - {}_aD_qf(a)\right)}{(1+q)(1+q+q^2)}\Bigg| \le \frac{(b-a)^3}{16}(M-m).\end{aligned}$$

Remark 5. *If $p = 1$ and $q \to 1$, then Theorem 10 reduces to the result obtained in [36].*

Lemma 3. *Let $\phi, \varphi : I \to \mathbb{R}$ be two continuous and (p,q)-differentiable functions on I^0. If ${}_aD_{p,q}\varphi(x) \neq 0$ on I^0 and $m \leq {}_aD_{p,q}\phi(x)/{}_aD_{p,q}\varphi(x) \leq M$ on I^0, then*

$$\begin{aligned} m &\left(p(b-a)\int_a^{pb+(1-p)a}\varphi^2(x)\ {}_ad_{p,q}x - \left(\int_a^{pb+(1-p)a}\varphi(x)\ {}_ad_{p,q}x\right)^2\right) \\ &\leq p(b-a)\int_a^{pb+(1-p)a}\phi(x)\varphi(x)\ {}_ad_{p,q}x - \left(\int_a^{pb+(1-p)a}\phi(x)\ {}_ad_{p,q}x\right)\left(\int_a^{pb+(1-p)a}\varphi(x)\ {}_ad_{p,q}x\right) \\ &\leq M\left(p(b-a)\int_a^{pb+(1-p)a}\varphi^2(x)\ {}_ad_{p,q}x - \left(\int_a^{pb+(1-p)a}\varphi(x)\ {}_ad_{p,q}x\right)^2\right). \end{aligned} \tag{53}$$

Proof. If ${}_aD_{p,q}\varphi(x) > 0$, then $\varphi(x)$ is an increasing function with

$$m\,{}_aD_{p,q}\varphi(x) \leq {}_aD_{p,q}\phi(x) \leq M\,{}_aD_{p,q}\varphi(x), \tag{54}$$

for all $x \in I^0$. For $a \leq x \leq y \leq b$, taking (p,q)-integral for Inequality (54) from x to y with respect to x yields

$$m(\varphi(y) - \varphi(x)) \leq \phi(y) - \phi(x) \leq M(\varphi(y) - \varphi(x)).$$

Multiplying the inequality above by $\varphi(y) - \varphi(x) \geq 0$, we have

$$m(\varphi(y) - \varphi(x))^2 \leq (\phi(y) - \phi(x))(\varphi(y) - \varphi(x)) \leq M(\varphi(y) - \varphi(x))^2. \tag{55}$$

Similarly, if ${}_aD_{p,q}\varphi(x) < 0$, then we also obtain (55). Taking (p,q)-integral for Inequality (55) from a to $pb - (1-p)a$ with respect to x and y, we have

$$\begin{aligned} m\int_a^{pb+(1-p)a} &\int_a^{pb+(1-p)a}(\varphi(y) - \varphi(x))^2\ {}_ad_{p,q}x\ {}_ad_{p,q}y \\ &\leq \int_a^{pb+(1-p)a}\int_a^{pb+(1-p)a}(\phi(y) - \phi(x))(\varphi(y) - \varphi(x))\ {}_ad_{p,q}x\ {}_ad_{p,q}y \\ &\leq M\int_a^{pb+(1-p)a}\int_a^{pb+(1-p)a}(\varphi(y) - \varphi(x))^2\ {}_ad_{p,q}x\ {}_ad_{p,q}y. \end{aligned} \tag{56}$$

A direct calculation yields

$$\begin{aligned} &\int_a^{pb+(1-p)a}\int_a^{pb+(1-p)a}(\varphi(y) - \varphi(x))^2\ {}_ad_{p,q}x\ {}_ad_{p,q}y \\ &\quad = 2\left[p(b-a)\int_a^{pb+(1-p)a}\varphi^2(x)\ {}_ad_{p,q}x - \left(\int_a^{pb+(1-p)a}\varphi(x)\ {}_ad_{p,q}x\right)^2\right], \end{aligned} \tag{57}$$

and

$$\begin{aligned} &\int_a^{pb+(1-p)a}\int_a^{pb+(1-p)a}(\phi(y) - \phi(x))(\varphi(y) - \varphi(x))\ {}_ad_{p,q}x\ {}_ad_{p,q}y \\ &\quad = 2\left[p(b-a)\int_a^{pb+(1-p)a}\phi(x)\varphi(x)\ {}_ad_{p,q}x - \left(\int_a^{pb+(1-p)a}\phi(x)\ {}_ad_{p,q}x\right)\left(\int_a^{pb+(1-p)a}\varphi(x)\ {}_ad_{p,q}x\right)\right]. \end{aligned} \tag{58}$$

Substituting (57) and (58) into (56) , we obtain

$$\begin{aligned} m&\left(p(b-a)\int_a^{pb+(1-p)a}\varphi^2(x)\ {}_ad_{p,q}x - \left(\int_a^{pb+(1-p)a}\varphi(x)\ {}_ad_{p,q}x\right)^2\right) \\ &\leq p(b-a)\int_a^{pb+(1-p)a}\phi(x)\varphi(x)\ {}_ad_{p,q}x - \left(\int_a^{pb+(1-p)a}\phi(x)\ {}_ad_{p,q}x\right)\left(\int_a^{pb+(1-p)a}\varphi(x)\ {}_ad_{p,q}x\right) \\ &\leq M\left(p(b-a)\int_a^{pb+(1-p)a}\varphi^2(x)\ {}_ad_{p,q}x - \left(\int_a^{pb+(1-p)a}\varphi(x)\ {}_ad_{p,q}x\right)^2\right), \end{aligned}$$

which is Inequality (53). This completes the proof. □

Theorem 11. *Let $f : I \to \mathbb{R}$ be a twice (p,q)-differentiable function with ${}_aD^2_{p,q}f$ (p,q)-integrable on I such that $m \le {}_aD^2_{p,q}f \le M$. It follows that*

$$\begin{aligned}\frac{mp^5q(b-a)^3}{(p+q)^2(p^2+pq+q^2)} &\le \frac{p(b-a)(qf_p(b,a)+pf(a))}{p+q} - \int_a^{pb+(1-p)a} f_q(x,a)\ {}_ad_{p,q}x \\ &\le \frac{Mp^5q(b-a)^3}{(p+q)^2(p^2+pq+q^2)}.\end{aligned} \tag{59}$$

Proof. Let

$$\phi(x) = {}_aD_{p,q}f(x) \quad \text{and} \quad \varphi(x) = x - \left(\frac{a+b}{2}\right).$$

Then $m \le {}_aD_{p,q}\phi(x)/{}_aD_{p,q}\varphi(x) \le M$ on I^0. Lemma 3 yields

$$\begin{aligned} m &\left(p(b-a)\int_a^{pb+(1-p)a}\left(x-\left(\frac{a+b}{2}\right)\right)^2 {}_ad_{p,q}x - \left(\int_a^{pb-(1-p)a}\left(x-\left(\frac{a+b}{2}\right)\right) {}_ad_{p,q}x\right)^2\right) \\ &\le p(b-a)\int_a^{pb+(1-p)a}\left(x-\left(\frac{a+b}{2}\right)\right){}_aD_{p,q}f(x)\ {}_ad_{p,q}x \\ &\quad -\left(\int_a^{pb+(1-p)a}\left(x-\left(\frac{a+b}{2}\right)\right) {}_ad_{p,q}x\right)\left(\int_a^{pb+(1-p)a} {}_aD_{p,q}f(x)\ {}_ad_{p,q}x\right) \\ &\le M\left(p(b-a)\int_a^{pb+(1-p)a}\left(x-\left(\frac{a+b}{2}\right)\right)^2 {}_ad_{p,q}x - \left(\int_a^{pb+(1-p)a}\left(x-\left(\frac{a+b}{2}\right)\right) {}_ad_{p,q}x\right)^2\right).\end{aligned} \tag{60}$$

A direct calculation shows that

$$\int_a^{pb+(1-p)a}\left(x-\left(\frac{a+b}{2}\right)\right) {}_ad_{p,q}x = \frac{(p^2-pq)(b-a)^2}{2(p+q)}, \tag{61}$$

$$\int_a^{pb+(1-p)a}\left(x-\left(\frac{a+b}{2}\right)\right)^2 {}_ad_{p,q}x = \frac{(p^4-2p^2q^2+2p^3q+pq^3)(b-a)^3}{4(p+q)(p^2+pq+q^2)}, \tag{62}$$

and

$$\begin{aligned}\int_a^{pb+(1-p)a} &\left(x-\left(\frac{a+b}{2}\right)\right){}_aD_{p,q}f(x)\ {}_ad_{p,q}x \\ &= \frac{(b-a)}{2}f_p(b,a) + \frac{b-a}{2}f(a) - \frac{1}{p}\int_a^{pb-(1-p)a} f_q(x,a)\ {}_ad_{p,q}x.\end{aligned} \tag{63}$$

Substituting (61)–(63) into (60), we obtain

$$\begin{aligned}\frac{mp^5q(b-a)^4}{(p+q)^2(p^2+pq+q^2)} &\le \frac{p(b-a)^2(qf_p(b,a)+pf(a))}{p+q} - (b-a)\int_a^{pb+(1-p)a} f_q(x,a)\ {}_ad_{p,q}x \\ &\le \frac{Mp^5q(b-a)^4}{(p+q)^2(p^2+pq+q^2)}.\end{aligned}$$

which implies Inequality (59). The proof is complete. □

Remark 6. *If $p = 1$, then Theorem 11 reduces to the result obtained in [50].*

4. Conclusions

We have established some inequalities of Fejér-type inequalities by using (p,q)-integral, such as the trapezoid-like inequalities, the midpoint-like inequalities, the Fejér-like inequalities. In particular, we generalized and corrected existing results of quantum Fejér-type inequalities by using new techniques and showing some problematic parts of those results. Our work improves the results of Fejér-type quantum integral inequalities. By taking $q \to 1$ and $p = 1$, our results give classical inequalities. The (p,q)-integral inequalities deduced in the present research are very general and helpful in error estimations involved in various approximation processes. With these contributions, we hope that these techniques and ideas established in this article will inspire the interest of readers in exploring the field of (p,q)-integral inequalities.

Author Contributions: Conceptualization, J.T. and S.K.N.; investigation, N.A., K.M.N. and K.N.; methodology, K.N.; validation, N.A., K.M.N., K.N., J.T. and S.K.N.; visualization, K.M.N., K.N., J.T. and S.K.N.; writing—original draft, N.A. and K.N.; writing—review and editing, K.N. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The first and second authors have received a scholarship under the Research Fund for Supporting Lecturer to Admit High Potential Student to Study and Research on His Expert Program Year 2018 from the Graduate School, Khon Kaen University, Thailand (Grant no. 612JT217). We would like to express our thanks to the anonymous reviewers for their suggestions and comments.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Jackson, F.H. On a q-definite integrals. *Quart. J. Pure Appl. Math.* **1910**, *41*, 193–203.
2. Jackson, F.H. q-difference equations. *Am. J. Math.* **1910**, *32*, 305–314. [CrossRef]
3. Aslam, M.; Awan, M.U.; Noor, K.I. Quantum Ostrowski inequalities for q-differentiable convex function. *J. Math. Inequal.* **2016**, *10*, 1013–1018.
4. Aral, A.; Gupta, V.; Agarwal, R.P. *Applications of q-Calculus in Operator Theory*; Springer Science + Business Media: New York, NY, USA, 2013.
5. Gauchman, H. Integral inequalities in q-calculus. *J. Comput. Appl. Math.* **2002**, *47*, 281–300. [CrossRef]
6. Ahmad, B. Boundary-value problems for nonlinear third-order q-difference equations. *Electron. J. Differ. Equ.* **2011**, *94*, 1–7. [CrossRef]
7. Ahmad, B.; Alsaedi, A.; Ntouyas, S.K. A study of second-order q-difference equations with boundary conditions. *Adv. Differ. Equ.* **2012**, *2012*, 35. [CrossRef]
8. Ahmad, B.; Ntouyas, S.K.; Purnaras, I.K. Existence results for nonlinear q-difference equations with nonlocal boundary conditions. *Commun. Appl. Nonlinear Anal.* **2012**, *19*, 59–72.
9. Ahmad, B.; Nieto, J.J. On nonlocal boundary value problems of nonlinear q-difference equation. *Adv. Differ. Equ.* **2012**, *2012*, 81. [CrossRef]
10. Bukweli-Kyemba, J.D.; Hounkonnou, M.N. Quantum deformed algebra: coherent states and special functions. *arXiv* **2013**, arXiv:1301.0116v1.
11. Kac, V.; Cheung, P. *Quantum Calculus*; Springer: New York, NY, USA, 2002.
12. Tariboon, J.; Ntouyas, S.K. Quantum calculus on finite intervals and applications to impulsive difference equations. *Adv. Differ. Equ.* **2013**, *2013*, 282. [CrossRef]
13. Tariboon, J.; Ntouyas, S.K. Quantum integral inequalities on finite intervals. *J. Inequal. Appl.* **2014**, *2014*, 121. [CrossRef]
14. Kunt, M.; Latif, M.A.; Iscan, I.; Dragomir, S.S. Quantum Hermite-Hadamard type inequality and some estimates of quantum midpoint type inequalities for double integrals. *Sigma J. Eng. Nat. Sci.* **2019**, *37*, 207–223.
15. Bermudo, S.; Korus, P.; Valdes, J.E.N. On q-Hermite-Hadamard inequalities for general convex functions. *Acta Math. Hung.* **2020**, *162*, 364–374. [CrossRef]
16. Jhanthanam, S.; Tariboon, J.; Ntouyas, S.K.; Nonlaopon, K. On q-Hermite-Hadamard inequalities for differentiable convex functions. *Mathematics* **2019**, *7*, 632. [CrossRef]
17. Noor, M.A.; Noor, K.I.; Awan, M.U. Some quantum estimates for Hermite-Hadamard inequalities. *Appl. Math. Comput.* **2015**, *251*, 675–679. [CrossRef]
18. Prabseang, J.; Nonlaopon, K.; Tariboon, J. Quantum Hermite-Hadamard inequalities for double integral and q-differentiable convex functions. *J. Math. Inequal.* **2019**, *13*, 675–686. [CrossRef]
19. Sudsutad, W.; Ntouyas, S.K.; Tariboon, J. Quantum integral inequalities for convex functions. *J. Math. Inequal.* **2015**, *9*, 781–793. [CrossRef]
20. Prabseang, J.; Nonlaopon, K.; Ntouyas, S.K. On refinement of quantum Hermite-Hadamard inequalities for convex functions. *J. Math. Inequal.* **2020**, *14*, 875–885. [CrossRef]
21. Chakrabarti, R.; Jagannathan, R. A (p,q)-oscillator realization of two-parameter quantum algebras. *J. Phys. A Math. Gen.* **1991**, *24*, L711–L718. [CrossRef]
22. Tunç, M.; Göv, E. Some integral inequalities via (p,q)-calculus on finite intervals. *RGMIA Res. Rep. Coll* **2016**, *19*, 1–12.
23. Tunç, M.; Göv, E. (p,q)-integral inequalities. *RGMIA Res. Rep. Coll* **2016**, *19*, 1–13.
24. Prabseang, J.; Nonlaopon, K.; Tariboon, J. (p,q)-Hermite-Hadamard inequalities for double integral and (p,q)-differentiable convex functions. *Axioms* **2019**, *8*, 68. [CrossRef]

25. Kalsoom, H.; Amer, M.; Junjua, M.D.; Hassain, S.; Shahxadi, G. (p, q)-estimates of Hermite-Hadamard-type inequalities for coordinated convex and quasi convex function. *Mathematics* **2019**, *7*, 683. [CrossRef]
26. Hounkonnou, M.N.; Désiré, J.; Kyemba, B.R. (p, q)-calculus: Differentiation and integration. *SUT J. Math.* **2013**, *49*, 145–167.
27. Sadjang, P.N. On the fundamental theorem of (p, q)-calculus and some (p, q)-Taylor formulas. *Results Math.* **2018**, *73*, 39. [CrossRef]
28. Chu, Y.M.; Awan, M.U.; Talib, S.; Noor, M.A.; Noor, K.I. New post quantum analogues of Ostrowski-type inequalities using new definitions of left-right (p, q)-derivatives and definite integrals. *Adv. Differ. Equ.* **2020**, *2020*, 634. [CrossRef]
29. Kalsoom, H.; Rashid, S.; Tdrees, M.; Safdar, F.; Akram, S.; Baleanu, D.; Chu, Y.M. Post quantum inequalities of Hermite-Hadamard-type associated with co-ordinated higher-order generalized strongly pre-index and quasi-pre-index mappings. *Symmetry* **2020**, *12*, 443. [CrossRef]
30. Kunt, M.; Iscan, I.; Alp, N.; Sarikaya, M.Z. (p, q)-Hermite-Hadamard and (p, q)-estimates for midpoint type inequalities via convex and quasi-convex functions. *RACSAM* **2018**, *112*, 969–992. [CrossRef]
31. Ali, M.A.; Budak, H.; Kalsoom, H.; Chu, Y.M. Post-quantum Hermite-Hadamard inequalities involving newly defined (p, q)-integral. *Authorea* **2020**. [CrossRef]
32. Thongjob, S.; Nonlaopon, K.; Ntouyas, S.K. Some (p, q)-Hardy type inequalities for (p, q)-integrable functions. *AIMS Math.* **2020**, *6*, 77–89. [CrossRef]
33. Wannalookkhee, F.S.; Nonlaopon, K.; Tariboon, J.; Ntouyas, S.K. On Hermite-Hadamard type inequalities for coordinated convex functions via (p, q)-calculus. *Mathematics* **2021**, *9*, 698. [CrossRef]
34. Pečarić, J.; Proschan, F.; Tong, Y.L. *Convex Functions, Partial Ordering and Statistical Applications*; Academic Press: New York, NY, USA, 1991.
35. Dragomir, S.S.; Cerone, P.; Sofo, A. Some remarks on the midpoint rule in numerical integration. *Studia Univ. Babes. Bolyai. Math.* **2000**, *XLV*, 63–74.
36. Dragomir, S.S.; Cerone, P.; Sofo, A. Some remarks on the trapezoid rule in numerical integration. *Indian J. Pure Appl. Math.* **2000**, *31*, 475–494.
37. Sarikaya, M.Z. On new Hermite Hadamard Fejér type integral inequalities. *Studia Univ. Babes. Bolyai. Math.* **2012**, *57*, 377–386.
38. Sarikaya, M.Z.; Budak, H. Generalized Ostrowski type inequalities for local fractional integrals. *RGMIA Res. Rep. Collect.* **2015**, *62*, 1–11. [CrossRef]
39. Sarikaya, M.Z.; Erden, S.; Budak, H. Some generalized Ostrowski type inequalities involving local fractional integrals and applications. *Adv. Inequal. Appl.* **2016**, *2016*, 1–6.
40. Sarikaya, M.Z.; Budak, H. On generalized Hermite-Hadamard inequality for generalized convex function. *RGMIA Res. Rep. Collect.* **2015**, *64*, 1–15.
41. Sarikaya, M.Z.; Erden, S.; Budak, H. Some integral inequalities for local fractional integrals. *RGMIA Res. Rep. Collect.* **2015**, *65*, 1–12. [CrossRef]
42. Sarikaya, M.Z.; Budak, H.; Erden, S. On new inequalities of Simpson's type for generalized convex functions. *RGMIA Res. Rep. Collect.* **2015**, *66*, 1–13.
43. Tseng, K.L.; Yang, G.S.; Hsu, K.C. Some inequalities for differentiable mappings and applications to Fejér inequality and weighted trapozidal formula. *Taiwan. J. Math.* **2011**, *15*, 1737–1747. [CrossRef]
44. Wang, C.L.; Wang, X.H. On an extension of Hadamard inequality for convex functions. *Chin. Ann. Math.* **1982**, *3*, 567–570.
45. Wu, S.H. On the weighted generalization of the Hermite-Hadamard inequality and its applications. *Rocky Mt. J. Math.* **2009**, *39*, 1741–1749. [CrossRef]
46. Xi, B.Y.; Qi, F. Some Hermite-Hadamard type inequalities for differentiable convex functions and applications. *Hacet. J. Math. Stat.* **2013**, *42*, 243–257.
47. Xi, B.Y.; Qi, F. Hermite-Hadamard type inequalities for functions whose derivatives are of convexities. *Nonlinear Funct. Anal. Appl.* **2013**, *18*, 163–176.
48. Fejér, L. Űber die Fourierreihen. *II. Math. Naturwiss. Anz Ungar. Akad. Wiss.***1906**, *24*, 369–390. (In Hungarian)
49. Minculete, N.; Mitroi, F.C. Fejér-type inequalities. *Aust. J. Math. Anal. Appl.* **2012**, *9*, 1–8.
50. Yang, W. Some new Fejér type inequalities via quantum calculus on finite intervals. *Sci. Asia* **2017**, *2017*, 123–134. [CrossRef]
51. Sarikaya, M.Z. On Fejér type inequalities via fractional integrals. *J. Interdiscip. Math.* **2018**, *21*, 143–155. [CrossRef]