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Heisenberg–Weyl Groups and Generalized Hermite Functions

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Abstract: We introduce a multi-parameter family of bases in the Hilbert space $L^2(\mathbb{R})$ that are associated to a set of Hermite functions, which also serve as a basis for $L^2(\mathbb{R})$. The Hermite functions are eigenfunctions of the Fourier transform, a property that is, in some sense, shared by these "generalized Hermite functions". The construction of these new bases is grounded on some symmetry properties of the real line under translations, dilations and reflexions as well as certain properties of the Fourier transform. We show how these generalized Hermite functions are transformed under the unitary representations of a series of groups, including the Heisenberg–Weyl group and some of their extensions.

Keywords: Hermite functions; Heisenberg–Weyl groups; group representations; Fourier transform; bases in Hilbert space $L^2(\mathbb{R})$; *rigged Hilbert spaces*

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1. Introduction

In the present paper, we study the relations between certain physical relevant low-dimensional Lie groups, in connection to affine transformations on the whole real line ($\mathbb{R}$) and their representations on the Hilbert space $L^2(\mathbb{R})$ as well as to other notions as the Hermite functions, other bases in $L^2(\mathbb{R})$ and the eigenfunctions of the Fourier transform. As a consequence of these relations, some invariance properties are disclosed.

These invariance properties come from the options between four types of freedom. These are: (i) the freedom to choose between coordinate and momentum representations and the respective bases determined by each of these representations; (ii) the freedom to choose an origin on the real line when using any of these two representations; (iii) the freedom to choose the units of length on $\mathbb{R}$; and (iv) the freedom to choose an orientation on the line. We span one-dimensional wave functions in terms of bases in either coordinate or momentum representation. The family of bases is parametrized by the set of real numbers $\mathbb{R}$, which is an homogeneous, self-similar and not oriented space, as is well known. The Fourier transform, which is an invertible correspondence between coordinate and momentum representations [1], implies some restrictions on self-similarity and orientation.

This invariance suggests a principle of relativity: Assume that two observers are located at different points of the line and that, furthermore, they use different length and/or momentum units. These observers would perceive the same physical state as exactly the same description of the reality. This means that, under these invariances, the one-dimensional physical world may be equivalently described by the coordinate x and the momentum p or, alternatively, by the coordinate $x' = kx + a$ and the momentum $p' = k^{-1}p + b$ with a, $b \in \mathbb{R}$ and $k \in \mathbb{R}^* \equiv \mathbb{R} - \{0\}$.

As with other well-known situations showing invariance properties, this type of invariance is described by a Lie group, which is usually denoted by $\widetilde{H}(1)$. This is a twofold version of the affine Heisenberg–Weyl group $\widetilde{H}_o(1)$ [2–8], since it includes the discrete symmetry associated to the reflection or Parity operator $\mathcal{P} : (x, p) \to (-x, -p)$. The Lie algebra of the affine Heisenberg–Weyl group, $\widetilde{\mathfrak{h}}(1)$ has four infinitesimal generators: D, X, P and I that correspond to dilations, position operator, momentum operator and a central operator commuting with the others, respectively. As we shall show later, the Lie group $\widetilde{H}(1)$ is isomorphic to the the central extension of the Poincaré group in 1+1 dimensions [9] enlarged with the discrete symmetry $\mathcal{P}\,\mathcal{T}$, where $\mathcal{P}$ is the parity and $\mathcal{T}$ is the time-reversal.

From now on, when we speak about symmetry or invariance on the real line, we refer to the existence of properties of spaces constructed over $\mathbb{R}$, such as $L^2(\mathbb{R})$. This includes many others depending on a unique continuous parameter.

The Hermite functions are all real and determine a basis of the (complex) space of functions $L^2(\mathbb{R})$. Self-similarity transformations do not change this property. In addition, it is rather simple to construct additional bases of $L^2(\mathbb{R})$ after some transformations on Hermite functions, for instance under the action of the group $\widetilde{H}(1)$. The results are the so-called generalized Hermite functions, to be defined later (Section 4). Contrary to the basis of Hermite functions, these bases of generalized Hermite functions are not sets of real functions as they usually have a complex phase.

As is well known, the real line $\mathbb{R}$ as one-dimensional Euclidean space is the homogeneous space $E_o(1)/\{0\}$, where $E_o(1)$ is the group of translations on the line and $\{0\}$ is the isotropy group of an arbitrary point of the line—for instance the origin. The real line supports two important continuous bases for $L^2(\mathbb{R})$: $\{|x\rangle\}_{x\in\mathbb{R}}$ and $\{|p\rangle\}_{p\in\mathbb{R}}$. Each of these bases is transformed into each other by the Fourier transform. The meaning of continuous bases will be clarified later, although it is nonetheless explained in [10].

One consequence of the homogeneity is that the continuous basis in the coordinate representation given by $\{|x\rangle\}$, where x runs out the set of real numbers, is equivalent to the continuous basis $\{|x+a\rangle\}$, where $x \xrightarrow{T_a} x + a$, for each fixed $a \in \mathbb{R}$, with $T_a \in E_o(1)$. Analogously, the continuous basis in the momentum representation, $\{|p\rangle\}$, is equivalent to the continuous basis $\{|p+b\rangle\}$, where p runs out the set of real numbers and b is an arbitrary, although fixed, real number.

If we consider the position, X and momentum, P, operators acting on their generalized eigenvectors, which are $|x\rangle$ and $|p\rangle$, respectively, we have that

$$
\begin{aligned}
X\,|x\rangle = x|x\rangle \quad &\Rightarrow \quad e^{-iXa}|x\rangle = e^{-iax}|x\rangle\,, \\
P\,|p\rangle = p|p\rangle \quad &\Rightarrow \quad e^{-iPb}|p\rangle = e^{-ibp}|p\rangle\,.
\end{aligned}
\tag{1}
$$

The Fourier transform and its inverse produce the following relations [10] :

$$
|p\rangle = \frac{1}{\sqrt{2\,\pi}} \int_{\mathbb{R}} e^{ipx}|x\rangle\,dx\,, \qquad |x\rangle = \frac{1}{\sqrt{2\,\pi}} \int_{\mathbb{R}} e^{-ipx}|p\rangle\,dp\,. \tag{2}
$$

We also have the following relations:

$$
\begin{aligned}
e^{-iXa}\,|p\rangle &= \frac{1}{\sqrt{2\,\pi}} \int_{\mathbb{R}} e^{ipx}e^{-iXa}|x\rangle\,dx = \frac{1}{\sqrt{2\,\pi}} \int_{\mathbb{R}} e^{ix(p-a)}|x\rangle\,dx = |p-a\rangle \\
e^{-iPb}\,|x\rangle &= \frac{1}{\sqrt{2\,\pi}} \int_{\mathbb{R}} e^{-ipx}e^{-iPb}|p\rangle\,dp = \frac{1}{\sqrt{2\,\pi}} \int_{\mathbb{R}} e^{-i(x+b)p}|x\rangle\,dp = |x+b\rangle.
\end{aligned}
\tag{3}
$$

The conclusion is that X and P, along with the central operator I, determine the Lie algebra for the Heisenberg–Weyl group $H(1)$. In this context, we say that the real line, meaning the space $L^2(\mathbb{R})$, supports a unitary representation of $H(1)$.

However, the group $H(1)$ does not exhaust self-similarity invariances on the real line and for our purposes is "not oriented", in the sense that it is equivalent to consider the direction on the line either from left to right or from right to left. Moreover, as commented earlier, the continuous basis $\{|x\rangle\}$ is equivalent to the continuous basis $\{|kx\rangle\}$ for each fixed $k \in \mathbb{R}^* := \mathbb{R}/\{0\}$. This suggests the use of the *dilatation operator*, D, which may be defined by the action of its exponential on the continuous basis as $e^{-idD}|x\rangle = e^{-d/2}\,|e^d x\rangle$ (d real), which defines a unique self-adjoint operator on $L^2(\mathbb{R})$. This action considers positive dilatations only as $e^d > 0$ for any real d. If $\langle x|y\rangle = \delta(x-y)$ then $\langle e^d x|e^d y\rangle = \delta(e^d(x-y)) = e^{-d}\,\delta(x-y)$. This is the reason to introduce the factor $e^{-d/2}$ in the definition of the action of e^{-idD} in $|x\rangle$, so that $\langle x|\left(e^{-idD}\right)^\dagger e^{-idD}|y\rangle = \langle x|y\rangle$.

Analogously, the continuous basis $\{|p\rangle\}$ is equivalent to the continuous basis $\{|k'p\rangle\}$, for each $k' \in \mathbb{R}^*$. Consistency with Fourier transform invariance implies that $k' = k^{-1}$. This suggests a result that shall become evident soon, which is that the algebra describing the invariance in the real line has to be $\widetilde{H}_o(1)$, i.e., the Heisenberg–Weyl group enlarged with dilatations.

Nevertheless, we need to introduce orientation invariance and negative values of k for dilatations in our picture. This is done by means of the parity operator $\mathcal{P}$, where the action of $\mathcal{P}$ on the continuous bases is given by $\mathcal{P}\,|x\rangle = |-x\rangle$ and $\mathcal{P}\,|p\rangle = |-p\rangle$. If we add this parity operator to the connected group $\widetilde{H}_o(1)$, we obtain the general group of invariance of the real line $\widetilde{H}(1)$. Then, the space $L^2(\mathbb{R})$ supports a unitary representation U of $\widetilde{H}(1)$.

This representation U can be well studied using the *generalized* Hermite functions, that we mentioned earlier. For our purposes, we need two families of bases that are constructed as follows: First, take the basis of the normalized Hermite functions $\{\psi_n(x)\}$ and add their Fourier transforms $\{\widetilde{\psi}_n(p)\}$. Then, if the unitary representation is denoted by $U(\tilde{g})$ with $\tilde{g} \in \widetilde{H}(1)$, these families are given by $\{U(\tilde{g})\psi_n(x)\}_{x\in\mathbb{R}}^{\tilde{g}\in\widetilde{H}(1)}$ and $\{U(\tilde{g})\hat{\psi}_n(p)\}_{x\in\mathbb{R}}^{\tilde{g}\in\widetilde{H}(1)}$. These two families of generalized Hermite functions are transformed into each other by the Fourier transform and its inverse, exactly as happens with the regular Hermite functions [10].

The present article is organized as follows: In Section 2, starting from the translation groups and considering some extra symmetries for the line, we arrive at the Heisenberg–Weyl group $H(1)$. We also considered the symmetry under Fourier Transform for the Hermite functions. In Section 3, we present some general properties of the Heisenberg–Weyl (HW) group and its extension to $\widetilde{H}(1)$. This group is connected to the general symmetry on the real line. We deal with local structures, exhibited by the Lie algebra of $\widetilde{H}(1)$, which is presented in its more familiar form, which includes the parity operator.

In Section 4, we construct the unitary representations of the HW group and its generalisations defined in the previous Section. Considering the behaviour of the Hermite functions under the group $\widetilde{H}(1)$, we introduce, in Section 5, a generalization of such Hermite functions: we obtain a three-parameter family of "generalized Hermite functions" that are bases of $L^2(\mathbb{R})$. We study the properties of these generalized Hermite functions as well as their behaviour under the Fourier transform. We also construct Rigged Hilbert space structures associated with these generalized Hermite functions. We give our concluding remarks in the final Section 6.

2. From the Translation Group to the Heisenberg–Weyl Group

Let us consider the group of the translations of the real line, $E_o(1)$. It can be considered as the connected part of the isometries of the line (translations and reflexions in a point, as for instance the origin) that constitute the Euclidean group on one dimension $E(1)$.

The group $E_o(1)$ is isomorphic to the group $(\mathbb{R}, +)$. Under a translation T_a, the point x of the real line is transformed as $x \to x + a$ with $a \in \mathbb{R}$. The action of $E_o(1)$ on the space of square integrable functions defined on $\mathbb{R}$ ($L^2(\mathbb{R})$) is given by

$$(U(T_a)f)(x) = f(x-a)\,, \tag{4}$$

where we have taking into account that, if a group G acts on a space X from the left (i.e., $\forall x \in X \xrightarrow{g\in G} g\,x \in X$ such that $e\,x = x$, with e as the identity element of G, and

$g'(g\,x) = (g'\,g)\,x\,,\ \forall g, g' \in G$), then there is a representation of this group on the space of functions defined in X as

$$(U(g)f)(x) = f(g^{-1}x)\,. \tag{5}$$

Let P be the infinitesimal generator of the translation group, so that $U(T_a) = e^{-iaP}$. It is clear from Equation (5) that $P = -i\frac{d}{dx}$.

2.1. The Group $E_o(1)$ Extended by Dilations: A Matrix Realization

Let us also consider transformations, such as dilations: D_k, acting as $x \to kx$ with $k \in \mathbb{R}^*$. Thus, the composition of translations and dilations of the form $T_a \cdot D_k$ acts as

$$x \xrightarrow[k\in\mathbb{R}^*]{D_k} kx \xrightarrow[a\in\mathbb{R}]{T_a} kx + a\,. \tag{6}$$

We can represent the group spanned by both transformations as the group of matrices

$$M_{[k,a]} = \begin{pmatrix} k & a \\ 0 & 1 \end{pmatrix}, \qquad k \neq 0, b \in \mathbb{R}\,, \tag{7}$$

which acts on the real line as follows:

$$M_{[k,a]}x = \begin{pmatrix} k & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} k\,x + a \\ 1 \end{pmatrix}, \tag{8}$$

in agreement with Equation (6). Henceforth, we shall denote this group as $\widetilde{E}(1)$. It is non-connected and shows two connected components: the connected component of the unit characterized by $k > 0$ and and a second component for which $k < 0$.

2.2. The Connected Component of $\widetilde{E}(1)$: $\widetilde{E}_o(1)$

Let us start by restricting ourselves to the connected component of the unit of $\widetilde{E}(1)$ that we denote as $\widetilde{E}_o(1)$. The infinitesimal generators in the matrix representation (Equation (7)) are

$$P = \left.\frac{dM_{[k,a]}}{da}\right|_{a=0} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad D = \left.\frac{dM_{[k,a]}}{dk}\right|_{k=1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{9}$$

The commutation relation of P and D is $[D, P] = P$. We see that under exponentiation (i.e., e^{aP} and e^{kK}), we only recover $\widetilde{E}_o(1)$

$$e^{aP}\,e^{kD} = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} e^k & 0 \\ 0 & 1 \end{pmatrix} = M_{[a,e^k]}. \tag{10}$$

Let us denote by $g = (a,k) = e^{aP}\,e^{kD}$ an arbitrary element of $\widetilde{E}_o(1)$ with $a, k \in \mathbb{R}$. The group law is given by

$$g' \cdot g = (a', k')(a, k) = (a' + e^{k'}a, k' + k)\,. \tag{11}$$

Moreover, the element g can be factorized as $g = (a, 1)(0, k)$ and $g^{-1} = (-e^{-k}a, -k)$. The action of $g \in \widetilde{E}_o(1)$ on the functions $f(x)$ is given by (see Equation (5))

$$(U(a,k)f)(x) = e^{-k/2}\,f(e^{-k}(x-a))\,, \tag{12}$$

where the term $e^{-k/2}$ has been added so as to assure the unitarity of this representation [11–13]. In particular, the Hermite functions $\psi_n(x)$ are functions in $L^2(\mathbb{R})$. In addition, Hermite functions are a basis of $L^2(\mathbb{R})$. Consequently, they support the representation of $\widetilde{E}_o(1)$, so that

$$(U(a,k)\psi_n)(x) = e^{-k/2}\,\psi_n(e^{-k}(x-a))\,. \tag{13}$$

After Equation (12) ($U(a,k) = e^{-iaP}\, e^{-ikD}$), the infinitesimal generators take the explicit form

$$P = -i\frac{d}{dx}\,, \qquad D = -i\frac{1}{2}\left(x\frac{d}{dx} + \frac{d}{dx}\,x\right), \tag{14}$$

and its Lie commutator is given by $[D,P] = iP$.

2.3. The Group $\widetilde{E}(1)$

To take into account the orientation invariance of the real line or, in other words, to consider the second connected component of the group $\widetilde{E}(1)$, we must include the parity or reflexion operator around the origin $\mathcal{P}$, which acts on $\mathbb{R}$ as $x \to -x$. The infinitesimal generators P and D transform under $\mathcal{P}$ as $\mathcal{P} : (P,D) \to (-P,D)$, and the elements $g = (a,k)$ of $\widetilde{E}_o(1)$ transform under parity as $(a,k)^{\mathcal{P}} = (a^{\mathcal{P}}, k^{\mathcal{P}}) = (-a,k)$.

Each of the $\tilde{g} \in \widetilde{E}(1)$ can be parametrized by

$$\tilde{g} = (a,k,\alpha)\,, \qquad \alpha \in \mathcal{V} = \{\mathcal{I}, \mathcal{P}\} \tag{15}$$

where $\mathcal{I}$ is the identity transformation. The group law is given by

$$\tilde{g}' \cdot \tilde{g} = (a',k',\alpha')(a,k,\alpha) = (a' + e^{k'} a^{\alpha'}, k'+k, \alpha'\alpha)\,, \tag{16}$$

where, clearly,

$$a^{\alpha} = \begin{cases} a & \text{if } \alpha = \mathcal{I} \\ -a & \text{if } \alpha = \mathcal{P} \end{cases}. \tag{17}$$

Thus, $\widetilde{E}(1)$ is given by a semidirect product, such as $\widetilde{E}(1) = \widetilde{E}_o(1) \odot \mathcal{V} = (E_o(1) \odot \mathcal{V}) \odot \mathcal{D}$, where $\mathcal{D}$ is the group of dilations $\{(0,k,\mathcal{I})\}_{k\in\mathbb{R}}$. This is clear, since

$$\tilde{g} = (a,k,\alpha) = (a,k,\mathcal{I})\,(0,0,\alpha) = (a,k,\mathcal{I})\,(0,0,\alpha) = (a,0,\mathcal{I})\,(0,0,\alpha)\;(0,k,\mathcal{I})\,. \tag{18}$$

On the given representation of $\widetilde{E}(1)$, the operator $\mathcal{P}$ is realized as a linear operator, so that the representation is unitary. It has the form [13,14]

$$\begin{aligned} (U(a,k,\alpha)f)(x) &= e^{-k/2}\, f(e^{-k}(x^{\alpha} - a))\,. \\ (U(a,k,\alpha)\psi_n)(x) &= e^{-k/2}\, \psi_n(e^{-k}(x^{\alpha} - a))\,. \end{aligned} \tag{19}$$

2.4. The Heisenberg–Weyl Group $H(1)$

An important fact of the Hermite functions is that they are eigenfunctions of the Fourier transform (FT) and its inverse (IFT) [10]

$$FT\,[\psi_n(x), x, p] \;=\; i^n\,\psi_n(p)\,, \qquad IFT\,[\psi_n(p), p, x] \;=\; (-i)^n\,\psi_n(x)\,, \tag{20}$$

i.e.,

$$\begin{aligned} FT[f(x),x,p] &= \frac{1}{\sqrt{2\,\pi}} \int_{\mathbb{R}} e^{ipx}\, f(x)\, dx = \hat{f}(p)\,, \\ IFT[\hat{f}(p),p,x] &= \frac{1}{\sqrt{2\,\pi}} \int_{\mathbb{R}} e^{-ipx}\, \hat{f}(p)\, dp = f(x)\,. \end{aligned} \tag{21}$$

Henceforth, we shall use this notation.

All properties that are fulfilled by the Hermite functions $\psi_n(x)$ are also valid for their FTs $\psi_n(p)$. Hence,

$$
\begin{aligned}
\left(e^{-iPa}\,\hat{f}\right)(p) &= \frac{1}{\sqrt{2\,\pi}}\int_{\mathbb{R}} e^{ipx}\left(e^{-iPa}\,f\right)(x)\,dx = \frac{1}{\sqrt{2\,\pi}}\int_{\mathbb{R}} e^{ipx}\,f(x-a)\,dx \\
&= \frac{1}{\sqrt{2\,\pi}}\,e^{ipa}\int_{\mathbb{R}} e^{iup}\,f(u)\,du = e^{ipa}\,\hat{f}(p)\,, \\
\left(e^{-iDk}\,\hat{f}\right)(p) &= \frac{1}{\sqrt{2\,\pi}}\int_{\mathbb{R}} e^{ipx}\left(e^{-iDk}\,f\right)(x)\,dx = \frac{1}{\sqrt{2\,\pi}}\int_{\mathbb{R}} e^{ipx}\,e^{-k/2}\,f(e^{-k}x)\,dx \\
&= \frac{1}{\sqrt{2\,\pi}}\int_{\mathbb{R}} e^{k/2}\,e^{ie^{k}vp}\,f(v)\,dv = e^{k/2}\,\hat{f}(e^{k}p)\,.
\end{aligned}
\tag{22}
$$

In the above relations, we used the changes of variables given by $u = x - a$ and $v = e^{-k}x$. In addition, we need a translation operator acting on the real line in the p representation. Let us first recall some important properties of the FT, such as:

$$
x f(x) \xrightarrow{FT[\bullet,x,p]} -i\frac{d}{dp}\hat{f}(p)\,, \qquad \frac{d}{dx}f(x) \xrightarrow{FT[\bullet,x,p]} -ip\hat{f}(p)\,. \tag{23}
$$

Hence, we define a new operator X acting on the space of square integrable functions on the line as:

$$
(Xf)(x) = x\,f(x)\,, \qquad (e^{iX}f)(x) = e^{ix}\,f(x)\,, \tag{24}
$$

so that,

$$
\begin{aligned}
\left(e^{iXb}\,\hat{f}\right)(p) &= \frac{1}{\sqrt{2\,\pi}}\int_{\mathbb{R}} e^{ipx}\left(e^{iXb}\,f\right)(x)\,dx = \frac{1}{\sqrt{2\,\pi}}\int_{\mathbb{R}} e^{ipx}\,e^{ibx}\,f(x)\,dx \\
&= \frac{1}{\sqrt{2\,\pi}}\int_{\mathbb{R}} e^{ix(p+b)}\,f(x)\,dx = \hat{f}(p+b)\,.
\end{aligned}
\tag{25}
$$

Thus, X is the infinitesimal generator of translations on the p-real line.

From Equation (20) and taking into account the isomorphism between the real x-line and the real p-line, we can identify, up to a phase, the Hermite functions $\psi_n(x)$ and their FT, i.e.,

$$
\psi_n(x) \xrightarrow{TF} \hat{\psi}_n(p) = i^n\,\psi_n(p) \equiv i^n\,\psi_n(x)\,. \tag{26}
$$

Hence, we have properly determined the generators X (Equation (24)) and P (Equation (14)) acting on $L^2(\mathbb{R})$, with $\mathbb{R}$ as the x-line. From Equations (4) and (24), we note that X produces a phase and P a translation, respectively. Clearly from Equation (23), the roles of X and P interchange when $\mathbb{R}$ is the p-line. Both operators along to the central operator I determine the HW group, since they verify the Lie commutators $[X,P] = i\,I$ and $[I,\bullet] = 0\,.$

In the next section, we study the HW group as well some of its extensions in detail.

3. The Heisenberg–Weyl Group and Its Extensions

In this section, we give first a review of the HW group as well one of its extensions and their respective Lie algebras. Then, we provide the isomorphism between the extended HW group and a central extension of the Poincaré (1 + 1) group enlarged by the discrete symmetry $\mathcal{P}\,\mathcal{T}$ (parity-time inversion).

3.1. The Heisenberg–Weyl Group: A Matrix Realization

The most common commutation relation that plays a role in ordinary relativistic quantum physics, which is $[x,p] \equiv [x, -i\hbar\frac{\partial}{\partial x}] = i\hbar$, serves to define the HW group $H(1)$. This group admits a representation by real 3×3 upper uni-triangular matrices [8] given by:

$$
A[a,b,\theta] = \begin{bmatrix} 1 & a & \theta \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}, \qquad a,b,\theta \in \mathbb{R}\,. \tag{27}
$$

These matrices form a group with the usual matrix multiplication as one readily sees:

$$A[a',b',\theta'] \cdot A[a,b,\theta] \;=\; A[a'+a,b'+b,\theta'+\theta+ab'] \,. \tag{28}$$

The identity element is the identity matrix, i.e., $Id = A\big|_{a,b,\theta=0}$, and the inverse of A is given by $A^{-1}[a,b,\theta] \;= A[-a,-b,ab-\theta]$. $H(1)$ is a subgroup of the group of all upper triangular matrices 3×3, $M_3(\mathbb{R})$ [15].

3.2. The Extended Heisenberg–Weyl Group

In order to include self-similarity on the real line, one needs to look at a more general subgroup of $M_3(\mathbb{R})$, which is the set of all 3×3 matrices of the form:

$$B[a,b,\theta,k] \;=\; \begin{bmatrix} 1 & a & \theta \\ 0 & k & b \\ 0 & 0 & 1 \end{bmatrix}, \qquad a,b,\theta\in\mathbb{R},\; k\in\mathbb{R}^* \,. \tag{29}$$

The group law is also given by matrix multiplication. Hence

$$B[a',b',\theta',k'] \cdot B[a,b,\theta,k] \;=\; B[ka'+a,k'b+b',\theta'+\theta+a'b,k'k] \,. \tag{30}$$

The identity element is $Id = B\big|_{a,b,\theta=0,k=1}$ and the inverse of the element $B[a,b,\theta,k]$ is $B[-a/k,-b/k,-\theta+ab/k,1/k]$. Clearly, this group reduces to $H(1)$ if, and only if, $k=1$. In other words $H(1)$ is a subgroup of this extended HW group. Consequently, we denote the extended group as $\widetilde{H}(1)$. The group $\widetilde{H}(1)$ has two connected components: the connected component of the identity characterized for $k>0$, which is a subgroup of $\widetilde{H}(1)$, here denoted as $\widetilde{H}_o(1)$, and a second component containing the elements elements caracterized by $k<0$. This can be obtained multiplying the elements of $\widetilde{H}_o(1)$ by the "parity" matrix $\mathcal{P} = \mathrm{Diagonal}[1,-1,1]$.

3.3. The Heisenberg–Weyl Algebras

Let us go back to the group $H(1)$ of matrices of the form Equation (27). This group depends on three real parameters a, θ and b related to the generators X, I and P, respectively, of the Lie algebra $\mathfrak{h}(1)$. In addition, the Lie algebra $\widetilde{\mathfrak{h}}(1)$ contains another generator, D, which is associated with the real parameter k in the group of matrices (Equation (29)). The explicit form of these generators in this matrix representation is given by

$$\begin{aligned} \overline{X} = \frac{\partial B}{\partial a}\Big|_{Id} &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \overline{I} = \frac{\partial B}{\partial \theta}\Big|_{Id} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ \overline{P} = \frac{\partial B}{\partial c}\Big|_{Id} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \overline{D} = \frac{\partial B}{\partial k}\Big|_{Id} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{31}$$

Hereafter, we will remove the "overline" symbol of the infinitesimal generators in the matrix representation, and we will write, with same symbol, the generators in the matrix representation and in the vector field representation.

The commutation relations are

$$[X,P] \;=\; I\,, \quad [D,X] \;=\; -X\,, \quad [D,P] \;=\; P, \quad [I,\bullet\,] \;=\; 0\,. \tag{32}$$

It is noteworthy that the action of the parity matrix, $\mathcal{P} = \mathrm{Diagonal}[1,-1,1]$, on the generators is given by $\mathcal{P}\,Y\,\mathcal{P}^{-1}$ (with $Y = X,P,I,D$), so that

$$\mathcal{P}\,X\,\mathcal{P}^{-1} = -X\,, \quad \mathcal{P}\,P\,\mathcal{P}^{-1} = -P\,, \quad \mathcal{P}\,I\,\mathcal{P}^{-1} = I\,, \quad \mathcal{P}\,D\,\mathcal{P}^{-1} = D\,. \tag{33}$$

For arbitrary $\mathfrak{g} \in h(1)$, one has the commutator $[\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]] = 0$. Thus, the algebra $h(1) \equiv \langle X, P, I \rangle$ is nilpotent. On the other hand, this is not the case for $\widetilde{\mathfrak{h}}_o(1) \equiv \langle X, P, D, I \rangle$, which is not nilpotent, although solvable.

The four one-parametric subgroups of $\widetilde{\mathfrak{h}}_o(1)$, corresponding to its four independent real parameters, are constructed by direct exponentiation of the matrices in Equation (31). They are

$$e^{aX} = \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; e^{\theta I} = \begin{pmatrix} 1 & 0 & \theta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; e^{bP} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}, \; e^{dD} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^d & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

with $a, \theta, b, d \in \mathbb{R}$. Since, by exponentiation, we only obtain the elements of the connected component of the unit, which is $\widetilde{H}_o(1)$, we must have that $e^d > 0$.

The group $\widetilde{H}_o(1)$ can be factorized as a product of its four one-dimensional groups as

$$\begin{aligned} e^{\theta I}\, e^{bP}\, e^{dD}\, e^{aX} &= B[a, b, \theta, e^d]\,, \\ e^{\theta I}\, e^{bP}\, e^{aX}\, e^{dD} &= B[e^d a, b, \theta, e^d]\,, \\ e^{\theta I}\, e^{aX}\, e^{bP}\, e^{dD} &= B[e^d a, b, \theta + ab, e^d]\,, \end{aligned} \tag{34}$$

or alternatively as $e^{\theta I}\, e^{aX+bP}\, e^{dD} = B[e^d a, b, \theta + ab/2, e^d]$.

In the sequel, we shall write all $g \in \widetilde{H}_o(1)$ as a product of the four one-parametric groups using the second formula in Equation (34). This means that

$$g \equiv (\theta, b, a, d) = e^{\theta I}\, e^{bP}\, e^{iaX}\, e^{dD}\,, \qquad \theta,\, b,\, a,\, d \in \mathbb{R}\,. \tag{35}$$

With this parametrization, the group law becomes

$$g'g = (\theta', b', a', d')\,(\theta, b, a, d) = (\theta' + \theta + a'\, e^{d'} b, b' + e^{d'} b, e^{-d'} a + a', d' + d) \tag{36}$$

and the inverse element of $g = (\theta, b, a, d)$ is $g^{-1} = (-\theta + ab, -e^{-d} b, -e^d a, -d)$.

We may compute the adjoint action of the four one-parameter subgroups on the four generators of the Lie algebra $\widetilde{\mathfrak{h}}_o(1)$. The not trivial actions are

$$\begin{aligned} e^{aX} P e^{-aX} &= P + aI\,, & e^{aX} D e^{-aX} &= D + aX\,, \\ e^{bP} X e^{-bP} &= X - bI\,, & e^{bP} D e^{-bP} &= D - bP\,, \\ e^{dD} X e^{-dD} &= e^{-d} X\,, & e^{dD} P e^{-dD} &= e^d P\,. \end{aligned} \tag{37}$$

From Equations (35) and (37), we can easily compute the adjoint action of the group $\widetilde{H}_o(1)$ on its Lie algebra $\widetilde{\mathfrak{h}}_o(1)$. The result is

$$\begin{aligned} gXg^{-1} &= e^{-d} X - e^{-d} bI\,, \\ gPg^{-1} &= e^d P + e^d aI\,, \qquad\qquad g = (\theta, b, a, d) \\ gDg^{-1} &= D + a\,X - b\,P - a\,bI\,. \end{aligned} \tag{38}$$

Hence, Equation (38) show that, under the action of the elements of $\widetilde{H}_o(1)$, the position and the momentum operators are transformed as $X' = e^{-d} X - e^{-d} bI$ and $P' = e^d P + e^d aI$, respectively. Therefore, the whole group describing the invariances in the *oriented* real line should be $\widetilde{H}(1)$, as $e^{\pm d}$ is always positive, so that it does not change the orientation of X and P. However, the real line is not, properly speaking, an oriented space as can be seen equally well from left to right or from right to left. As a consequence, we have to add to $\widetilde{H}_o(1)$ a parity operator $\mathcal{P}$ acting like the parity matrix Diagonal$[1,-1,1]$ (Equation (33)). Hence,

$$\widetilde{H}(1) = \mathcal{V}_2 \otimes \widetilde{H}_o(1)\,, \tag{39}$$

where $\mathcal{V}_2$ is the group of the discrete symmetries $\{\mathcal{I},\mathcal{P}\}$.

3.4. The Extended HW Group Versus an Extension of the Poincar é (1+1) Group

The group $\widetilde{H}(1)$ is isomorphic to an extension of the Poincaré (1 + 1) group, which we denote by $\widetilde{P}(1,1)$. More especifically, it is the connected component of the identity of the extended Poincaré group in (1 + 1) dimensions [7,9]. The group $\widetilde{P}_o(1,1)$, enlarged with the symmetry $\mathcal{P}\,\mathcal{T}$, gives

$$\widetilde{P}(1,1) = \widetilde{P}_o(1,1) \cup \mathcal{P}\,\mathcal{T} \cdot P(1,1) = \mathcal{V}_2 \otimes P_o(1,1)\,. \tag{40}$$

Here, $\mathcal{V}_2$ is the group of the discrete symmetries $\{\mathcal{I},\mathcal{P}\mathcal{T}\}$. As a matter of fact, the group $\widetilde{P}_o(1,1)$ is spanned by H, P, K and C. These are the infinitesimal generators of the time-translations, space-translations, boots and the central extension, respectively. Their Lie commutators are

$$[P,H] = C, \quad [K,H] = P, \quad [K,P] = H, \quad [\bullet, C] = 0\,. \tag{41}$$

Under the discrete symmetry $\mathcal{P}\,\mathcal{T}$, the infinitesimal generators transform as

$$(H,P,K,C) \xrightarrow{\mathcal{P}\,\mathcal{T}} (-H,-P,K,C)\,. \tag{42}$$

Now, let us consider the new generators

$$X_\pm = H \pm P\,, \qquad I = 2C \tag{43}$$

together with K. Their commutation relations are

$$[X_+, X_-] = I, \quad [K, X_+] = X_+, \quad [K, X_-] = -X_-, \quad [\bullet, I] = 0\,. \tag{44}$$

From Equation (42), the behaviour of $X_\pm$ under the symmetry $\mathcal{P}\,\mathcal{T}$ is $(\mathcal{P}\,\mathcal{T})\,X_\pm\,(\mathcal{P}\,\mathcal{T})^{-1} = -X_\pm$. Hence, the identification

$$(X_+, X_-, K, I) \quad \Longleftrightarrow \quad (X, P, D, I) \tag{45}$$

along to the symmetry $(\mathcal{P}\,\mathcal{T}) \Leftrightarrow \mathcal{P}$ allows us to show the existence of an isomorphism between the Lie algebras Lie$[\widetilde{P}(1,1)]$ and Lie$[\widetilde{H}(1)]$ and their Lie groups.

4. Unitary Representations of the Heisenberg–Weyl Groups

In this section, we review the unitary representations (UR) and the unitary irreducible representations (UIR) of the different HW groups described in the previous section.

4.1. UIR of the Heisenberg–Weyl Group $H(1)$

One may consider the HW group as a central extension of the abelian group of the translations on the 2-dimensional euclidean plane. The elements of the HW group are parametrized by $g = (\theta, a, b)$ with $\theta \in \mathbb{R}$ and $a, b \in \mathbb{R}$ [7,16,17] with the multiplication law

$$\begin{aligned} g_1 \cdot g_2 &= (\theta_1, a_1, b_1)(\theta_2, a_2, b_2) \\ &= (\theta_1 + \theta_2 + \xi((a_1,b_1),(a_2,b_2)), a_1 + a_2, b_1 + b_2)\,, \end{aligned} \tag{46}$$

where the exponent ξ is

$$\xi((a_1,b_1),(a_2,b_2)) = \frac{1}{2}\,(a_1\,b_2 - a_2\,b_1)\,. \tag{47}$$

For the sake of simplicity, we write $\vec{a} = (a,b,0)$ so that, after Equation (47), we have

$$\xi(\vec{a}_1,\vec{a}_2) = \frac{1}{2}\,\vec{a}_1 \wedge \vec{a}_2\,, \qquad \vec{a}_i = (a_i,b_i,0),\ i = 1,2\,. \tag{48}$$

Note that Equation (46) comes after the more usual factorization

$$g = (\theta,a,b) := e^{\theta I}\,e^{aX+bP}\,. \tag{49}$$

The proof is very simple making use of of the Glauber formula [1,7], which states that if A and B are two operators such that $[A,[A,B]] = [B,[A,B]] = 0$, then $e^A\,e^B = e^{A+B}\,e^{\frac{1}{2}[A,B]}$ or equivalently $e^A\,e^B = e^B\,e^A\,e^{[A,B]}$. The Glauber formula is a particular case of the Baker–Campbell–Hausdorff formula [18–20]. The Glauber formula relates the different parametrizations of the group. For instance, from Equation (49) $e^{aX+bP} = e^{(\theta-\frac{1}{2}\,ab)I}\,e^{bP}\,e^{aX}$.

The UIRs of the HW group on the space of square integrable functions on the real line $L^2(\mathbb{R})$ are well known due to their applications in quantum mechanics. Here, we can distinguish two types or classes thereof:

I. The infinite-dimensional representations labeled by a real parameter $h \in \mathbb{R}^*$ given by the product of operators [7,16]

$$U_h(g) \equiv U_h(\theta,a,b) = e^{ih\theta}\,e^{ih(aX-bP)} = e^{ih(\theta-ab/2)}\,e^{ihaX}\,e^{-ihbP}\,, \tag{50}$$

for which its explicit expression acting on the functions $f(x) \in L^2(\mathbb{R})$ is given by

$$(U_h(g)f)(x) = e^{ih\theta}\,e^{iha(x-b/2)}\,f(x-b)\,. \tag{51}$$

Note that $U_{h'}$ and U_h with $h' \neq h$ are non-equivalent.

II. The one-dimensional and trivial UIR with $h = 0$, so that $(U_0(g)f)(x) = f(x)$, which is not relevant in our discussion.

Under the representations of class I, see Equation (51), the infinitesimal generators X, P, I take the form

$$(X\,f)(x) = x\,f(x)\,, \quad (P\,f)(x) = -\frac{i}{h}\,\frac{d\,f}{dx}(x)\,, \quad [X,P] = \frac{i}{h}\,I \;\Rightarrow\; I = h\,. \tag{52}$$

If $h = 1/\hbar$, we recover well-known results in quantum mechanics. We may say that the real line, by which we mean the space of square integrable functions on the real line $L^2(\mathbb{R})$, supports a UIR U_h of the HW group $H(1)$.

4.2. UIR of the Heisenberg–Weyl Group with Dilations $\widetilde{H}_o(1)$

As mentioned in Section 1, the group $H(1)$ does not exhaust self-similarity invariances on the real line, which, for our purposes, should be considered as "non-oriented". By non-orientation, we refer to the equivalence of both directions from left to right or from right to left. The Lie algebra describing the invariance on the real line is $\tilde{\mathfrak{h}}(1)$, and its generators fulfil the commutation relations (Equation (32)). Then, we take into account the realization of the infinitesimal generators of the HW group (Equation (52)) and Section 2.2 (in particular expression (14)). With all these ingredients, we obtain the following expression for the infinitesimal generator D:

$$(D\,f)(x) = -\frac{i}{2h}\left(x\frac{d}{dx} + \frac{d}{dx}\,x\right)f(x) = -\frac{i}{2h}\cdot\left(2x\frac{df(x)}{dx} + f(x)\right) \tag{53}$$

Hence, $\left(e^{-ihdD}f\right)(x) = e^{-d/2}\, f(e^{-d}x)$. Another interesting fact is that this group has two Casimir elements: I (central charge) and $\mathcal{C} = X\,P - I\,D$ (the quadratic Casimir). The eigenvalues of these central elements $(h,\mathcal{C}) \in \mathbb{R}^2$ label the UIRs of $\widetilde{H}_o(1)$. For the sake of our purposes, the suitable UIRs of $\widetilde{H}_o(1)$ $(h \neq 0, \mathcal{C})$ are given by

$$(U_{h,\mathcal{C}}(\hat{g})\, f)(x) = e^{-d/2}\, e^{ih(\theta+\mathcal{C})}\, e^{iha(x-b/2)}\, f(e^{-d}(x-b))\,, \tag{54}$$

where, according to Equation (49) $\hat{g} = (g,d) = (\theta,a,b,d) = e^{\theta I}\, e^{aX+bP}\, e^{dD}$ with $g \in H(1)$ and $d \in \mathbb{R}$. Now, the group law is given by

$$\hat{g}_1\, \hat{g}_2 = (\theta_1 + \theta_2 + \frac{1}{2}\,\xi((a_1,b_1),(e^{d_1}a_2, e^{-d_1}b_2)), a_1 + e^{d_1}a_2, b_1 + e^{-d_1}b_2, d_1 + d_2)\,, \tag{55}$$

where we have considered Equation (46). The inverse of the element $\hat{g} = (\theta,a,b,d)$ is the element

$$\hat{g}^{-1} = (-\theta, -e^{-d}a, -e^{d}b, -d)\,. \tag{56}$$

Following the notation used in Equation (47), we can rewrite the exponent ξ of Equation (55) as

$$\xi(\hat{g}_1\, \hat{g}_2) = \xi((a_1,b_1),(e^{d_1}a_2, e^{-d_1}b_2)) = \xi\left(\vec{a}_1, \vec{a}_2^{\,d_1}\right), \qquad \vec{a}^{\,d} = (e^{d}a, e^{-d}b)\,. \tag{57}$$

The factor systems [21] $\omega^{\widetilde{H}_o(1)} = e^{ih\xi}$ of the group $\widetilde{H}_o(1)$ are

$$\omega^{\widetilde{H}_o(1)}(\hat{g}_1\, \hat{g}_2) = e^{ih\xi\left(\vec{a}_1, \vec{a}_2^{\,d_1}\right)}\,. \tag{58}$$

In [9], the UIRs of the Poincaré (1 + 1) group are constructed. Taking into account the relationship between this group and $\widetilde{H}_o(1)$, which was discussed in Section 3.4, it is straightforward to rewrite these representations in terms of our results for $\widetilde{H}_o(1)$.

4.3. UR of the Extended Heisenberg–Weyl Group $\widetilde{H}(1)$

The invariance under orientation, or invariance under the change $x \leftrightarrow -x$ suggests the need for the use of the parity operator, $\mathcal{P}$. The connected group $\widetilde{H}_o(1)$ plus the parity operator provide the general group of invariance of the real line as a semidirect product of the group of the discrete symmetries $\mathcal{V}_2 = \{\mathcal{I},\mathcal{P}\}$, where $\mathcal{I}$ is the identity operator and the affine HW group (Equation (39)). This semidirect group is

$$\widetilde{H}(1) = \mathcal{V}_2 \odot \widetilde{H}_o(1)\,. \tag{59}$$

The action of the parity into $\widetilde{H}_o(1)$ is given by $(\theta,a,b,d) \xrightarrow{\mathcal{P}} (\theta,-a,-b,d)$. The elements of the group $\widetilde{H}(1)$ can be written as $\tilde{g} = (\hat{g},\alpha)$ where $\hat{g} = (\theta,a,b,d) \in \widetilde{H}_o(1)$ and $\alpha \in \mathcal{V}_2$. The law group of $\widetilde{H}(1)$ is given by

$$\tilde{g}_1 \cdot \tilde{g}_2 = (\hat{g}_1,\alpha_1)(\hat{g}_2,\alpha_2) = (\hat{g}_1 \cdot \hat{g}_2^{\alpha_1}, \alpha_1\alpha_2)\,, \tag{60}$$

where $\hat{g}^{\alpha} = \hat{g}$ if $\alpha = \mathcal{I}$ and $\hat{g}^{\mathcal{P}} = (\theta,-a,-b,d)$ if $\alpha = \mathcal{P}$. From Equations (56) and (60), the inverse of $\tilde{g}$ is

$$\tilde{g}^{-1} = (\hat{g},\alpha)^{-1} = (\left(\hat{g}^{-1}\right)^{\alpha}, \alpha) = (-\theta, -e^{-d}a^{\alpha}, -e^{d}b^{\alpha}, -d, \alpha)\,. \tag{61}$$

Before constructing the representations of $\widetilde{H}(1)$, we display some well-known facts about representations of non-connected groups.

Let us consider a non-connected Lie group G, a subgroup $H \subset G$ of index 1 or 2 in G and a realization of G on the group of linear and antilinear operators in a Hilbert space such that $U(g)$ be linear or antilinear if $g \in H$ or $g \in G - H$. Hence, the action of

$U(g)$ on a function $f(x)$ would be $(U(g)f)(x) = \eta(g,x)\, f^g(g^{-1}x)$ such that $f^g(x) = f(x)$ or $f^g(x) = f(x)^*$ if $g \in H$ or $g \in G - H$, respectively, with $\eta : G \times X \to U(1)$. Moreover, from the relation $U(g')U(g) = \omega(g',g)U(g'g)$, with $\omega : G \times G \to U(1)$, we find that $\eta(g', gx)\, \eta(g,x)^{g'} = \omega(g',g)\, \eta(g'gx)$. The factor system ω verifies the two-cocycle condition

$$\omega(g_1,g_2)\,\omega(g_1g_2,g_3) = \omega(g_2,g_3)^{g_1}\,\omega(g_1,g_2g_3) \tag{62}$$

together with $\omega(e,e) = \omega(e,g) = \omega(g,e) = 1$, $\forall g \in G$, with e as the identity element of G. The action of G on $U(1)$, denoted by ${}^*H$, is defined by $\beta^g = \beta$ if $g \in H$ and $\beta^g = \beta^*$ if $g \in G - H$ with $\beta \in U(1)$. The set of two-cocycles is denoted by $\mathbf{Z}^2_{*H}(G, U(1))$. The two-coboundaries are those two-cocycles ω verifying

$$\omega_1(g_1,g_2) = \lambda(g_1)\,\lambda(g_2)\,\lambda(g_1,g_2)^{-1}\,. \tag{63}$$

The set of classes of equivalence of two-cocycles modulo two-coboundaries determines the second cohomology group of G: $\mathbf{H}^2(G,U(1)) = \mathbf{Z}^2_{*H}(G,U(1))/\mathbf{B}^2_{*H}(G,U(1))$ [22].

Let G be a semidirect product $G = G_o \odot V$, where G_o is the connected component of the identity and $V = \pi_o(G)$ is the group of the connected components, with the action $g \in G \xrightarrow{\alpha \in V} g^\alpha \in G$. In this case, the restrictions to G_o and V of the action of G on $U(1)$ give the actions of G_o and V on $U(1)$ (denoted by ${}^*H\big|_{G_o}$ and ${}^*H\big|_V$ respectively). In this case, ${}^*H\big|_{G_o}$ is trivial. Then, for each $[\omega] \in \mathbf{H}^2_{*H}(G,U(1))$, we can find a factor system ω, which is an element of $\mathbf{Z}^2_{*H}(G,U(1))$ given by

$$\omega^G(g_1,\alpha_1; g_2,\alpha_2) = \omega^{G_o}(g_1, g_2^{\alpha_1})\,\omega^V(\alpha_1,\alpha_2)\,\Lambda(g_2,\alpha_1)\,, \tag{64}$$

where $\omega^{G_o} \in \mathbf{Z}^2_{{}^*H|_{G_o}}(G_o, U(1))$, $\omega^V \in \mathbf{Z}^2_{{}^*H|_V}(V,U(1))$ and $\Lambda : G_o \times V \to U(1)$ verifying

$$\omega^{G_o}(g_1^\alpha, g_2^\alpha) = \omega^{G_o}(g_1,g_2)^{{}^*H|_V(\alpha)}\,\Lambda(g_1g_2,\alpha)(\Lambda(g_1,\alpha)\,\Lambda(g_2,\alpha))^{-1}\,, \tag{65}$$

$$\Lambda(g,\alpha_1\alpha_2) = \Lambda(g^{\alpha_2},\alpha_1)\,(\Lambda(g,\alpha_2))^{{}^*H|_V(\alpha_1)}\,. \tag{66}$$

For more details, see [14] and the references therein.

Returning to the parity operator $\mathcal{P}$, we may select two choices for its representation $U(\mathcal{P})$, either by a linear or by an antilinear operator. In the sequel, we analyse both possibilities [14].

(I) If we look at $\mathcal{P}$ as a linear operator, then $H = \widetilde{H}(1)$. The factor systems of $\widetilde{H}(1)$ can be written as

$$\omega^{\widetilde{H}(1)}(\hat{g}_1,\alpha_1,\hat{g}_2,\alpha_2) = \omega^{\widetilde{H}_o(1)}(\hat{g}_1,\hat{g}_2^{\alpha_1})\,\omega^{\mathcal{V}_2}(\alpha_1,\alpha_2)\,\Lambda(\hat{g}_2,\alpha_1)\,, \tag{67}$$

where the factors $\omega^{\widetilde{H}_o(1)}$, $\omega^{\mathcal{V}_2}$, and $\Lambda : \widetilde{H}_o(1) \times \mathcal{V}_2 \to U(1)$ fulfil the equations

$$\omega^{\widetilde{H}_o(1)}(\hat{g}_1^\alpha,\hat{g}_2^\alpha) = \omega^{\widetilde{H}_o(1)}(\hat{g}_1,\hat{g}_2)\,\Lambda(\hat{g}_1\hat{g}_2,\alpha)(\Lambda(\hat{g}_1,\alpha)\,\Lambda(\hat{g}_2,\alpha))^{-1}\,, \tag{68}$$

$$\Lambda(\hat{g},\alpha_1\alpha_2) = \Lambda(\hat{g}^{\alpha_2},\alpha_1)\,\Lambda(\hat{g},\alpha_2)\,. \tag{69}$$

where we have taken into account that the action ${}^*H\big|_{\mathcal{V}_2}(\mathcal{P})$ is here trivial after the linearity of $\mathcal{P}$.

In this case, we take the factors (Equation (58)) for $\widetilde{H}_o(1)$. Then,

$$\omega^{\widetilde{H}_o(1)}(\hat{g}_1^\alpha,\hat{g}_2^\alpha) = \omega^{\widetilde{H}_o(1)}(\hat{g}_1,\hat{g}_2)\,, \qquad \alpha \in \{\mathcal{I},\mathcal{P}\}\,. \tag{70}$$

Hence, Λ, is two-coboundary (Equation (63)), and therefore we may dismiss it. The factor $\omega^{\mathcal{V}_2}(\alpha_1,\alpha_2)$ is easily shown to be trivial in this case. It is straightforward that $\omega^{\mathcal{V}_2}(\mathcal{P},\mathcal{P}) = m \in U(1)$, while all the others : $\omega(\mathcal{I},\mathcal{I}) = \omega(\mathcal{I},\mathcal{P}) = \omega(\mathcal{P},\mathcal{I}) = 1$. Now, from Equation (63), we can write

$$\omega_1(\mathcal{P},\mathcal{P}) = m = \lambda(\mathcal{P})\,\lambda(\mathcal{P})\,\lambda(\mathcal{P}^2)^{-1} = \lambda(\mathcal{P})^2 \;\;\Rightarrow\;\; \lambda(\mathcal{P}) = m^{1/2}\,, \tag{71}$$

since $\lambda(\mathcal{I}) = 1$. Thus, the UIRs are given by

$$(U_{h,\mathcal{C}}(\hat{g},\alpha)f)(x) = e^{-d/2}\,e^{ih(\theta+\mathcal{C})}\,e^{iha(x-b/2)^\alpha}\,f(e^{-d}(x-b)^\alpha)\,. \tag{72}$$

(II) According to the second option, $\mathcal{P}$ is an antilinear operator. Then, $H = \widetilde{H}_o(1)$. The factors for $\widetilde{H}(1)$ satisfy the relation (67). $\omega^{\widetilde{H}_o(1)}$, $\omega^{\mathcal{V}_2}$ and Λ verify the Equations (65) and (66). Since $\mathcal{P}$ is an antilinear operator, the action ${}^*H\big|_{\mathcal{V}_2}(\mathcal{P})$ is the complex conjugation. Hence,

$$\omega^{\widetilde{H}_o(1)}(\hat{g}_1^\alpha,\hat{g}_2^\alpha) \;=\; \omega^{\widetilde{H}_o(1)}(\hat{g}_1,\hat{g}_2)^{{}^*H|_{\mathcal{V}_2}(\alpha)}\,\Lambda(\hat{g}_1\hat{g}_2,\alpha)(\Lambda(\hat{g}_1,\alpha)\,\Lambda(\hat{g}_2,\alpha))^{-1}\,, \tag{73}$$

$$\Lambda(\hat{g},\alpha_1\alpha_2) \;=\; \Lambda(\hat{g}^{\alpha_2},\alpha_1)\,\Lambda(\hat{g},\alpha_2)^{{}^*H|_{\mathcal{V}_2}(\alpha)}\,. \tag{74}$$

From Equations (58) and (70), we conclude that Eqaution (73) have no solutions for Λ unless $h = 0$. Moreover, $\omega^{\mathcal{V}_2}$ is non trivial now and $\omega_m^{\mathcal{V}_2}(\mathcal{P},\mathcal{P}) = m$ with $m = \pm 1$. Then, Λ becomes trivial. Its factor system is now

$$\omega^{\widetilde{H}(1)}(\hat{g}_1^\alpha,\hat{g}_2^\alpha) = \omega_m^{\mathcal{V}_2}(\alpha_1,\alpha_2)\,. \tag{75}$$

We, thus, obtained a semi-unitary representation of the whole group such that its restriction to the connected component is a realization with $h = 0$. We have now

$$(U_{0,\mathcal{C}}(\hat{g},\alpha)f)(x) = \Delta(\alpha)\,f(x^\alpha)\,, \tag{76}$$

where $\Delta(\mathcal{I}) =$ Identity and $\Delta(\mathcal{P}) = \mathbf{K}$, the conjugation operator.

In the following, we shall focus our attention in the representations of class I, i.e., in the unitary representations (72) or (81) as they are the only non-trivial.

4.4. Unitary Representations of $\widetilde{H}(1)$ and Fourier Transform

The above unitary representations can be translated to functions $\hat{f}(p)$ via the Fourier transform. Thus, for the representation (72), we have

$$\begin{aligned} \left(U_{h,\mathcal{C}}(\tilde{g})\hat{f}\right)(p) &= \int_{\mathbb{R}} e^{ihpx}\,(U_{h,\mathcal{C}}(\tilde{g})f)(x)\,dx \\ &= e^{d/2}\,e^{ih(\theta+\mathcal{C})}\,e^{ihpb}\,\hat{f}(e^d(p+a)^\alpha)\,. \end{aligned} \tag{77}$$

For the representation (76), we have

$$\begin{aligned} \left(U_{0,\mathcal{C}}(\hat{g},\alpha)\hat{f}\right)(p) &= \int_{\mathbb{R}} e^{\varepsilon_\alpha ihpx}\,(U_{0,\mathcal{C}}(\hat{g},\alpha)f)(x)\,dx \\ &= \int_{\mathbb{R}} e^{\varepsilon_\alpha ihpx}\,\Delta(\alpha)\,f(x^\alpha)\,dx = \Delta(\alpha)\int_{\mathbb{R}} e^{ihp^\alpha x^\alpha}\,f(x^\alpha)\,dx \\ &= \Delta(\alpha)\,f(p^\alpha)\,, \end{aligned} \tag{78}$$

where $\varepsilon_\alpha = \mathrm{sign}(\Delta(\alpha)i)$.

5. A Generalization of the Hermite Functions

The most used orthonormal basis for the Hilbert space $L^2(\mathbb{R})$ is the basis of the normalized Hermite functions, $\{\psi_n(x)\}$, defined as [23,24]

$$\psi_n(x) := \frac{e^{-x^2/2}}{\sqrt{2^n n! \sqrt{\pi}}}\, H_n(x)\,, \quad H_n(x) := (-1)^n\, e^{x^2} \frac{d^n}{dx^n} e^{-x^2}\,, \quad n = 0, 1, 2, \ldots\,, \tag{79}$$

where the $H_n(x)$ are the so-called the (physicists) Hermite polynomials [10,25]. We recall the following well known relations of orthognormality an completeness, respectively, that assure that the normalized Hermite functions are a basis for $L^2(\mathbb{R})$:

$$\int_{-\infty}^{+\infty} \psi_n(x)\, \psi_{n'}(x)^*\, dx = \delta_{nn'}\,, \quad \sum_{n=0}^{\infty} \psi_n(x)\, \psi_n(y)^* = \delta(x-y)\,. \tag{80}$$

The basis of Hermite functions (Equation (79)) has two interesting properties: (i) despite the complex character of the functions in the Hilbert space $L^2(\mathbb{R})$, all Hermite functions are real and (ii) they are eigenfunctions of the FT and also of the IFT (Equation (20)) [10].

We can restrict the UIR of $\widetilde{H}(1)$ (Equation (72)) to those elements $\tilde{g} = (\hat{g}, \alpha)$ with $\theta = 0$, recall that $\tilde{g} = (\theta, a, b, d, \alpha)$. Let us denote $\tilde{g}_0 = (0, a, b, d, \alpha)$, and take $\mathcal{C} = 0$. The action of $\tilde{g}_0$ on the Hermite functions is given by

$$(U_{h,0}(\tilde{g}_0)\psi_n)(x) = e^{-d/2}\, e^{iha(x-b/2)^\alpha}\, \psi_n(e^{-d}(x-b)^\alpha)\,. \tag{81}$$

Note that the joint action of Parity (Equation (17)) and dilatation becomes

$$e^{-d}\, x^\alpha = \begin{cases} e^{-d}\, x = kx & \text{with } k > 0 \quad \text{if } \alpha = \mathcal{I} \\ -e^{-d}\, x = kx & \text{with } k < 0 \quad \text{if } \alpha = \mathcal{P} \end{cases} \tag{82}$$

Since $U_{h,0}$ is a UIR, it preserves the orthonormality and the completeness relations (Equation (80)) for the transformed Hermite functions $(U_{h,0}(\tilde{g}_0)\psi_n)(x)$. If we split the completeness relation for the $(U_{h,0}(\tilde{g}_0)\psi_n)(x)$ into its real and imaginary parts, we arrive at the following pair of equations, both together equivalent

$$\begin{aligned} \sum_{n=0}^{\infty} \cos[ha(x-y)]\, \psi_n(kx+b)\, \psi_n(ky+b) &= \delta(x-y)\,, \\ \sum_{n=0}^{\infty} \sin[a(x-y)]\, \psi_n(kx+b)\, \psi_n(ky+b) &= 0\,. \end{aligned} \tag{83}$$

In the sequel, we shall introduce a generalization of the Hermite functions and study some of their properties.

5.1. Generalized Hermite Functions

Let us define a three-parameter family of square integrable functions based on the Hermite functions as follows:

$$\chi_n(x, k, a, b) := \sqrt{|k|}\; e^{-iax}\, \psi_n(kx+b)\,, \qquad a, b \in \mathbb{R};, k \neq 0 \in \mathbb{R}^*\,. \tag{84}$$

These also verify the orthonormality and completeness relations (Equation (80)) as the Hermite functions as the reader can easily verify. This shows that, for fixed a, b and $k \neq 0$, the functions $\chi_n(x, k, a, b)$, $n = 0, 1, 2, \ldots$, form a basis for $L^2(\mathbb{R})$. Thus, we have constructed a family of bases for this Hilbert space, whose elements under the Laplace transform and its inverse become

$$\begin{aligned} FT[\chi_n(x,k,a,b),x,p] &= i^n \chi_n(p,k^{-1},b,-a)\,, \\ IFT[\chi_n(p,k,a,b),p,x] &= (-i)^n \chi_n(x,k^{-1},-b,a)\,. \end{aligned} \tag{85}$$

Thus, the generalized Hermite functions are not eigenvectors of FT (IFT) contrarily to the Hermite functions (Equation (20)). On the other hand, if

$$k = k^{-1},\ a = b,\ b = -a \qquad \Longrightarrow \qquad k = \pm 1,\ a = 0,\ b = 0, \tag{86}$$

the corresponding generalized Hermite functions are eigenvalues of FT (IFT). This only happens for the standard Hermite functions.

Note that while the Hermite functions are real, the generalized Hermite functions are not real and they are only real for the particular choice $a = 0$, where the three-parameter family of bases becomes restricted to a two-parameter family.

Finally, we may disregard translational invariance and consider self-similarity and invalid orientation only. Then, the three-parameter family of bases Equation (84) reduces to a one-parameter family, depending only on $k \in \mathbb{R}^*$. This is

$$\{\chi_n(x,k)\}^{n\in\mathbb{N}}_{\in\mathbb{R}^*} \equiv \{\chi_n(x,k,0,0)\}^{n\in\mathbb{N}}_{\in\mathbb{R}^*} \equiv \{\sqrt{|k|}\ \psi_n(kx)\}^{n\in\mathbb{N}}_{\in\mathbb{R}^*}\,. \tag{87}$$

We shall discuss the importance of these bases in the sequel.

5.2. $\widetilde{P}(1,1)$ *and the "Classical" Real Line*

In Section 3, we extended the group $H(1)$ to include non-commutativity and self-similarity. Thus, we arrived to $\widetilde{H}(1)$, which is isomorphic to an extension of the Poincaré group in 1+1 dimensions, $\widetilde{P}(1,1)$, see Section 3.4. Nevertheless, it is always possible to start from symmetries of "classical physics" given by $P_o(1,1)$, which is the connected component of the Poincaré group in $(1+1)$ dimensions to arrive again at $\widetilde{P}(1+1)$ using the central extension and the $\mathcal{PT}$ symmetry as a tool.

In order to implement this programme, we start with the algebra Lie$(P_o(1+1)) = \mathcal{P}_o(1+1)$ with the basis $\{H, P, K\}$ [9]. Here, H and P are the infinitesimal generators of the time and space translations, respectively, and K is the infinitesimal generator of the Lorentz transformations. Their commutation relations are

$$[H,P] = 0, \quad [H,K] = P, \quad [P,K] = H\,. \tag{88}$$

The action of an arbitrary element $(a^0, a^1, \Lambda(\eta)) \in P_o(1+1)$ on the space-time is given by

$$(a,b,\Lambda(\eta))\mathbf{x} \equiv \begin{pmatrix} \cosh\eta & \sinh\eta \\ \sinh\eta & \cosh\eta \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \end{pmatrix} + \begin{pmatrix} a^0 \\ a^1 \end{pmatrix}, \tag{89}$$

where $\mathbf{x} = (x^0, x^1)^T$. Using the relations (43) and (45), we obtain a new basis $\{X, P, K\}$, such that $[X, P] = 0$. These new basis elements are related to to the light-cone coordinates:

$$x_\pm = x^0 \pm x^1 \quad \Leftrightarrow \quad x^0 = \frac{x_+ + x_-}{2}\,,\ x^1 = \frac{x_+ - x_-}{2}\,. \tag{90}$$

The commutator $[X, P] = 0$ justifies the label of "classicality" for the symmetry with group of invariance $P_o(1,1)$. As previously remarked, the group $P(1,1)$ is the result of the addition of the operator $\mathcal{PT}$ to $P_o(1,1)$. The action of each $g = (a,b,d,\alpha) \in P(1,1)$ on any square integrable function in the coordinate and the momentum representation is ($x_+ = x$, $x_- = p$), respectively, according to Equations (81) and (82):

$$\begin{aligned} U(g)\, f(x) &= |k|^{-1/2}\, f(k^{-1}(x-b)) \\ U(g)\, f(p) &= |k|^{1/2}\, f(k(p+a)) \end{aligned} \qquad k = \left(e^{d}\right)^{\alpha} \in \mathbb{R}^* \,. \tag{91}$$

Now, let us consider self-similarity and parity transformations on the line, performing the operations $x \Longrightarrow kx$ and $p \Longrightarrow k^{-1}p$, along the symmetries induced by these transformations. The translation invariance introduced in quantum physics by the non-commutativity is not relevant here. For $k \neq 0$ and real, Equation (87) yields to

$$\chi_n(x,k) = \sqrt{|k|}\, \frac{e^{-k^2x^2/2}}{\sqrt{2^n n! \sqrt{\pi}}}\, H_n(kx) \,. \tag{92}$$

One readily obtains that, for any $k \in \mathbb{R}^*$, these functions verify orthogonality an completeness relations, such as the Hermite functions (Equation (80)). This shows that $\{\chi_n(x,k)\}$ is a one-parameter family of orthonormal bases for $L^2(\mathbb{R})$. Under FT and IFT these bases become

$$FT[\chi_n(x,k),x,y] = i^n \chi_n(y,k^{-1})\,, \qquad IFT[\chi_n(p,k^{-1}),y,x] = (-i)^n \chi_n(x,k)\,. \tag{93}$$

The functions belonging to the family of basis $\{\chi_n(x,k)\}$ are all real for all $k \in \mathbb{R}^*$, a property also shared by the basis of Hermite functions $\{\psi_n(x)\}$. This means that both sets of bases are equally appropriate for the Hilbert space $L^2(\mathbb{R})$, no matter if this is a Hilbert space over either the complex or the real field. This property is, in general, false if we choose $\{\chi_n(x,k,a,b)\}$ as a basis, which, for most values of the parameters, is solely a basis for $L^2(\mathbb{R})$ as a Hilbert space over the complex field.

On the other hand, all the bases $\{\psi_n(x)\}$, $\{\chi_n(x,k,a,b)\}$ and $\{\chi_n(x,k)\}$ have a similar behaviour under the Fourier transform and its inverse, so that all serve as bases in the momentum representations (Equations (20), (85) and (93)).

5.3. Generalized Hermite Polynomials

Some comments on the functions $\{\chi_n(x,k)\}$ are in order here. For each value of $n = 0,1,2,\ldots$, these functions include the factor $H_n(kx)$, which is the n-th Hermite polynomial (Equation (79)) with a dilation on its argument. The Rodrigues formula for $H_n(kx)$ follows straightforwardly from Equation (79) and gives

$$H_n(kx) = (-1)^n\, e^{k^2x^2}\, \frac{d^n}{k^n dx^n} e^{-k^2x^2} = \left(2kx - \frac{1}{k}\frac{d}{dx}\right)^n * 1\,, \tag{94}$$

with the generating function

$$e^{2kxt-t^2} = \sum_{n=0}^{\infty} H_n(kx)\frac{t^n}{n!}\,. \tag{95}$$

Other relevant formulas or recurrence relations of the Hermite polynomials $H_n(x)$ are straightforwardly obtained from $H_n(kx)$. As for instance, the differential equation for $H_n(kx)$, which is

$$H_n''(kx) - 2k^2xH_n'(kx) + 2k^2nH_n(kx) = 0\,. \tag{96}$$

5.4. The Set of Functions $\{\chi_n(x,k)\}$ as Basis for Representations of the HW Algebra $H(1)$

As already mentioned, $\{\psi_n(x)\} \equiv \{\chi_n(x,1)\}$ is a basis for representations of the HW algebra $h(1)$ [26], which are supported on $L^2(\mathbb{R})$. In addition, following previous experiences with the use of ladder operators, we may also here construct a set of operators, $\{H, A_+, A_-\}$, for $h(1)$ such that the basis functions $\{\chi_n(x,k)\}$ are eigenfunctions of H and are transformed into each other using $A_\pm$ as ladder operators. The explicit form of these operators for $h(1)$ is

$$H := \frac{1}{2}(k^2X^2 + k^{-1}P^2)\,, \qquad A_\pm := \frac{k}{\sqrt{2}}\,x \mp \frac{1}{\sqrt{2}\,k}\,\frac{d}{dx}\,. \tag{97}$$

They fulfil the following commutation relations in $h(1)$:

$$[H, A_\pm] = \pm A_\pm\,, \qquad [A_+, A_-] = -1\,. \tag{98}$$

It is quite simple to show that the operators $A_\pm$ act as ladder operators with respect to the family of bases $\{\chi_n(x,k)\}$:

$$A_+\,\chi_n(x,k) = \sqrt{n+1}\,\chi_{n+1}(x,k)\,, \quad A_-\,\chi_n(x,k) = \sqrt{n}\,\chi_{n-1}(x,k)\,. \tag{99}$$

Then, we may define the number operator $N := A_+A_-$ so that, from Equation (99), we have

$$N\chi_n(x,k) = n\,\chi_n(x,k)\,, \tag{100}$$

as we may have expected. Note that $H = N + 1/2$ and that relations (98) and (99) are independent on k. This representation of $h(1)$ has the zero operator as a Casimir [26,27]:

$$\left[H - \frac{1}{2}\{A_+, A_-\}\right]\chi_n(x,k) = 0\,. \tag{101}$$

This relation may be extended to the common domain of the operators $\{H, A_+, A_-\}$. This domain is dense in $L^2(\mathbb{R})$ since it contains the Schwartz space. We also may write the Casimir in terms of the basis $\{X, P, H\}$. Needless to say that, in this explicit realization (Equation (97)), the Casimir is also zero, i.e.,

$$\left[H - \frac{1}{2}(k^2X^2 + k^{-2}P^2)\right]\chi_n(x,k) = 0\,. \tag{102}$$

Observe that the formal expression for the Casimir depends now on k. This is also the case of the kinetic energy operator, which, on each member of the basis $\{\chi_n(x,k)\}$, acts as

$$\frac{P^2}{2}\,\chi_n(x,k) = k^2\left[(N + 1/2) - \frac{k^2X^2}{2}\right]\chi_n(x,k)\,. \tag{103}$$

Note that the right hand side of Equation (103) goes to the free particle of zero energy in the limit $k \to 0$. This exhibits a limiting connection between the harmonic oscillator and the free particle within the context of quantum mechanics.

5.5. Representations on a Rigged Hilbert Space

Thus far, we have discussed representations of some Lie algebras as operators on the Hilbert space $L^2(\mathbb{R})$. These operators, although self-adjoint, are unbounded. It would be interesting to represent these algebras of operators as *continuous* operators on some topological vector space. The formalism of *rigged Hilbert spaces* (RHS), or Gelfand triplets, is very suitable in achieving this goal. A rigged Hilbert space is a triplet of spaces [28] $\Phi \subset \mathcal{H} \subset \Phi^\times$, such that $\mathcal{H}$ is a complex separable infinite dimensional Hilbert space.

The locally convex space Φ is endowed with a strictly finer topology than the inherited by Φ from $\mathcal{H}$, so that the canonical injection $\Phi \longmapsto \mathcal{H}$ is continuous. Finally, the space of all continuous *antilinear* functionals on Φ is $\Phi^\times$, which is the *antidual* space of Φ. It may have any topology compatible with the dual pair $\{\Phi, \Phi^\times\}$, i.e., weak, strong or MacKey. We usually choose this antiduality instead of duality for notational convenience [29,30]. See also [10,31–34].

The simplest example for Φ is the Schwartz space $\mathcal{S}$ of all complex indefinitely differentiable functions on the real line, such that they and their derivatives go to zero at infinity faster than the inverse of any polynomial. A good discussion on the Schwartz space may be found in [35]. The Schwartz space contains all the basis $\{\chi_n(x,k,a,b)\}$ and $\mathcal{S} \subset L^2(\mathbb{R}) \subset \mathcal{S}^\times$ is a RHS. In the sequel, we shall see why this RHS is suitable for our purposes. We should note first that, if A is a symmetric (Hermitian) continuous operator [35] on $\mathcal{S}$, then it may be extended to a continuous operator on $\mathcal{S}^\times$ by using the *duality formula* $\langle A\varphi|F\rangle = \langle\varphi|AF\rangle$ for all $\varphi \in \mathcal{S}$ and $F \in \mathcal{S}^\times$, and $\langle\varphi|F\rangle$ is the action of $F \in \mathcal{S}^\times$ on $\varphi \in \mathcal{S}$.

The usual Frèchet topology on $\mathcal{S}$ is given by a countable set of norms. There are several countable families of norms given the same topology on $\mathcal{S}$, although the most convenient for our purposes in the following [35]: A square integrable function $f(x) \in L^2(\mathbb{R})$ with $f(x) = \sum_{n=0}^\infty a_n\,\psi_n(x)$ is in $\mathcal{S}$ if, and only if,

$$\sum_{n=0}^\infty |a_n|^2\,(n+1)^{2r} < \infty\,, \qquad r = 0,1,2,\ldots\,. \tag{104}$$

Then, for any $f \equiv f(x) \in \mathcal{S}$, we define the following countable family of norms, $p_r(f)$, as:

$$p_r(f) := \sqrt{\sum_{n=0}^\infty |a_n|^2\,(n+1)^{2r}}\,, \qquad r = 0,1,2,\ldots\,. \tag{105}$$

For $r = 0$, we have the Hilbert space norm, and thus the canonical injection $i : \mathcal{S} \longmapsto L^2(\mathbb{R})$ is continuous.

What happens if we use the other families of bases such as $\{\chi_n(x,k)\}$ or $\{\chi_n(x,k,a,b)\}$? Note that for fixed real numbers a, b and $k \neq 0$, we have

$$\begin{aligned} f(x) &= \sum_{n=0}^\infty b_n\,\chi_n(x,k,a,b) = \sum_{n=0}^\infty b_n\,\sqrt{k}\,e^{-iax}\,\psi_n(kx+b) \\ &= \sum_{n=0}^\infty b_n\,\sqrt{k}\,e^{-i(y/k-b/k)}\,\psi_n(y)\,, \end{aligned} \tag{106}$$

so that for all $r = 0,1,2,\ldots$,

$$p_r^2(f) = k\sum_{n=0}^\infty |b_n|^2\,(n+1)^{2r}\,, \tag{107}$$

and hence $|a_n|^2 = k\,|b_n|^2$, $n = 0,1,2,\ldots$, for k fixed. This is the same for the span of $f(x)$ in terms of the family of basis $\{\chi_n(x,k)\}$.

With these ideas in mind, it is trivial to prove that the operators $A_\pm$, H and N, defined in Equations (97)–(99), are continuous operators on $\mathcal{S}$ and, therefore, continuously extensible to $\mathcal{S}^\times$. This comes from the following result [35]:

Theorem. Let Φ be a locally convex space for which the topology is defined by the family of seminorms $\{p_i(\cdot)\}_{i\in I}$. A *linear* operator $A : \Phi \longmapsto \Phi$ is continuous on Φ if, and only if, for each seminorm p_j of the previous family, there exist a positive constant $K > 0$ and k fixed seminorms of the same collection $p_{n_1}, p_{n_2}, \ldots, p_{n_k}$ such that *for all* $\varphi \in \Phi$, we have

$$p_i(\varphi) \leq K\{p_{n_1}(\varphi) + p_{n_2}(\varphi) + \cdots + p_{n_k}(\varphi)\}\,. \tag{108}$$

The constant K, the seminorms $p_{n_1}, p_{n_2}, \ldots, p_{n_k}$ and its number k may depend on p_j.

Proof. In order to prove our claim, let us first show that, for any $f(x) \in \mathcal{S}$, $A_\pm f(x) \in \mathcal{S}$, and the same property is true for H and N. Take,

$$[A_+ f](x) = \sum_{n=0}^\infty a_n\,\sqrt{n+1}\,\chi_{n+1}(x,k)\,, \tag{109}$$

so that for any norm, p_r, in Equation (105), one has for $r = 0, 1, 2, \ldots$

$$\begin{aligned} p_r(A_+ f) &= \sqrt{k} \sqrt{\sum_{k=0}^{\infty} |a_n|^2 \, (n+1) \, (n+1)^{2r}} \leq \sqrt{k} \sqrt{\sum_{k=0}^{\infty} |a_n|^2 \, (n+1)^{2(r+1)}} \\ &\leq \sqrt{k} \, p_{r+1}(f) \, . \end{aligned} \tag{110}$$

This proves both that $A_+ f \in \mathcal{S}$ for any $f \in \mathcal{S}$ and that, according to the previous Theorem, A_+ is continuous on $\mathcal{S}$. Similar proofs can be used for A_-, H and N. Since,

$$X = \frac{1}{\sqrt{2}\, k} \, (A_+ + A_-) \, , \qquad P = \frac{ik}{\sqrt{2}} \, (A_- - A_+) \, , \tag{111}$$

it comes that X and P are also continuous operators on $\mathcal{S}$. The same property holds for the parity operator $\mathbb{P}$. All these operators are continuously extensible to $\mathcal{S}^{\times}$.

6. Concluding Remarks

We studied how invariance properties on the real line under geometric transformations, such as translations, dilations and inversions, can be represented as unitary mappings on $L^2(\mathbb{R})$. This representation transforms the basis of Hermite functions to a new basis of functions, which generalizes the notion of Hermite functions. In the process, we arrive at the Euclidean group on the line $E(1)$.

The properties of the Fourier transform and, in particular, transforming coordinates into momenta and vice versa, $\mathrm{FT}[f(x), x, p] = \widehat{f}(p)$, forced us to introduce an enlarged group adding a new generator to extend the Heisenberg–Weyl group $H(1)$ to the group $\widetilde{H}(1)$. This group is isomorphic to the central extension of the Poincaré group in (1 + 1) dimensions enlarged with the $\mathcal{P}\mathcal{T}$ transformation. Analogously, $\widetilde{H}(1)$ is isomorphic to the central extension group of isometries of the two dimensional space $\mathbb{R}^2$ with signature $(+,-)$. This extension is denoted as $\widetilde{P}(1,1)$ or also $\widetilde{E}(1,1)$.

One representation of the infinitesimal generators of $\widetilde{E}(1,1)$ as operators on $L^2(\mathbb{R})$ is explicitly given by $X = x$, $P = -(\mathbf{i}/h)\, \partial_x$, $D = -\frac{i}{2h}(x\partial_x + \partial_x\, x)$, $I = h$. While X and P algebraically express the connection between the configuration and momenta representation described analytically by the Fourier transform, the dilatation operator is given to obtain the factor $e^{\mp d/2}$. This factor is necessary in order to normalize the representation (54) and (77). Finally, if we choose, for h, the value $1/\hbar$, we recover all the well-known results of quantum mechanics.

We introduced a generalization of the Hermite functions, which are quite appropriate to our discussion due to their behaviour under transformations by the group $\widetilde{H}(1)$. These newly generalized Hermite functions also provide a three-parameter family of bases of $L^2(\mathbb{R})$. However, these generalized Hermite functions are not eigenvectors of the Fourier transform on $L^2(\mathbb{R})$, regardless of if the Fourier transform maps the orthonormal basis into orthonormal basis. We may say that, from this point of view, the usual Hermite functions are those with better properties among all types of generalized Hermite functions.

Let us also mention that the generalized Hermite functions are discrete bases in a rigged Hilbert space on which the generators of $H(1)$ or $\widetilde{H}(1)$ are continuous. Finally, since the generalized Hermite functions belong to all spaces $L^p(\mathbb{R})$ with $p \geq 1$ and, in particular, to $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$, they are useful in the decomposition of wavelets in signal analysis by making use of the Gabor transform [36–38]. This could be the subject of future research.

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