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Initial Value Problems of Linear Equations with the Dzhrbashyan–Nersesyan Derivative in Banach Spaces

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Abstract: Among the many different definitions of the fractional derivative, the Riemann–Liouville and Gerasimov–Caputo derivatives are most commonly used. In this paper, we consider the equations with the Dzhrbashyan–Nersesyan fractional derivative, which generalizes the Riemann–Liouville and the Gerasimov–Caputo derivatives; it is transformed into such derivatives for two sets of parameters that are, in a certain sense, symmetric. The issues of the unique solvability of initial value problems for some classes of linear inhomogeneous equations of general form with the fractional Dzhrbashyan–Nersesyan derivative in Banach spaces are investigated. An inhomogeneous equation containing a bounded operator at the fractional derivative is considered, and the solution is presented using the Mittag–Leffler functions. The result obtained made it possible to study the initial value problems for a linear inhomogeneous equation with a degenerate operator at the fractional Dzhrbashyan–Nersesyan derivative in the case of relative p -boundedness of the operator pair from the equation. Abstract results were used to study a class of initial boundary value problems for equations with the time-fractional Dzhrbashyan–Nersesyan derivative and with polynomials in a self-adjoint elliptic differential operator with respect to spatial variables.

Keywords: fractional differential equation; fractional Dzhrbashyan–Nersesyan derivative; degenerate evolution equation; initial value problem; initial boundary value problem

MSC: 34G10; 35R11; 34A08

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1. Introduction

One of the rapidly developing areas of modern mathematics is the theory of fractional differential equations and their applications [1–7] (also see the references therein). Among the many different definitions of the fractional derivative, the Riemann–Liouville [8] and Gerasimov–Caputo [8–10] derivatives are most commonly used. In this paper, we consider the equations with the Dzhrbashyan–Nersesyan fractional derivative [11], which generalizes the Riemann–Liouville and Gerasimov–Caputo derivatives; it is transformed into such derivatives for two sets of parameters that are, in a certain sense, symmetric. In this sense, the concepts of the Riemann–Liouville and Gerasimov–Caputo derivatives are symmetric. We investigate initial value problems with the Dzhrbashyan–Nersesyan fractional derivative, and the results obtained in these symmetric cases will be valid for the initial problems of equations with the Riemann–Liouville and the Gerasimov–Caputo derivatives, respectively. To begin, let us give the following definition.

Let $\{\alpha_k\}_0^n = \{\alpha_0, \alpha_1, \dots, \alpha_n\}$ be the set of real numbers satisfying the condition $0 < \alpha_k \leq 1, k = 0, 1, \dots, n, n \in \mathbb{N} \cup \{0\}$. We denote

$$D^{\sigma_0} z(t) = D_t^{\alpha_0-1} z(t), \quad (1)$$

$$D^{\sigma_k} z(t) = D_t^{\alpha_k-1} D_t^{\alpha_{k-1}} D_t^{\alpha_{k-2}} \dots D_t^{\alpha_0} z(t), \quad k = 1, 2, \dots, n. \quad (2)$$

The fractional Dzhrbashyan–Nersesyan derivative of the order σ_n associated with the sequence $\{\alpha_k\}$ is determined by the relations (1) and (2), and it includes the Riemann–Liouville ($\alpha_0 \in (0, 1)$, $\alpha_k = 1$, $k = 1, 2, \dots, n$) and the Gerasimov–Caputo ($\alpha_k = 1$, $k = 0, 1, \dots, n-1$, $\alpha_n \in (0, 1)$) fractional derivatives.

In [11], M.M. Dzhrbashyan and A.B. Nersesyan proved the existence of a unique continuous solution lying in $L_p(0, l; \mathbb{R})$ for the initial value problem

$$D^{\sigma_k} z(0) = z_k, \quad k = 0, 1, \dots, n-1 \quad (3)$$

for the equation $D^{\sigma_n} z(t) + p_0(t) D^{\sigma_{n-1}} z(t) + \dots + p_{n-1}(t) D^{\sigma_0} z(t) + p_n(t) z(t) = f(t)$ with some functions $p_k : (0, T) \rightarrow \mathbb{R}$, $k = 0, 1, \dots, n-1$, $f : (0, T) \rightarrow \mathbb{R}$. In the partial case, $p_0 \equiv p_1 \equiv \dots \equiv p_{n-1} \equiv f(t) = 0$, $p_n \equiv a \in \mathbb{R}$, the solution is presented in the form of a linear combination of the Mittag–Leffler functions.

Various differential equations with the Dzhrbashyan–Nersesyan derivative were considered in the works of A.V. Pskhu. For example, in [12], the fundamental solution of a diffusion-wave equation with the Dzhrbashyan–Nersesyan time-fractional derivative was obtained, and the unique solvability of the initial value problem $D^{\sigma_k} z(x, 0) = z_k(x)$, $k = 0, 1, \dots, n-1$, $x \in \mathbb{R}^n$ for the equation in $\mathbb{R}^n \times (0, T]$ was studied. In [13], similar issues were researched for the case of the discretely distributed Dzhrbashyan–Nersesyan time-fractional derivative.

In this paper, we study the unique solvability issues (in the classical sense) for some classes of linear equations with operator coefficients in Banach spaces. In Section 2, the formula of the Laplace transform for the fractional Dzhrbashyan–Nersesyan derivative is obtained, and the initial value problem (3) with z_k from a Banach space $\mathcal{Z}$, $k = 0, 1, \dots, n-1$, for the class of homogeneous equations $D^{\sigma_n} z(t) = Az(t)$ with a linear bounded operator in $\mathcal{Z}$ is studied; $z : \mathbb{R}_+ \rightarrow \mathcal{Z}$. Using the Laplace transform, we obtain the resolving operators' families for this equation, which are presented in the form of the Mittag–Leffler functions with an operator argument. In Section 3, the same initial value problem for the inhomogeneous equation

$$D^{\sigma_n} z(t) = Az(t) + f(t), \quad (4)$$

with a function $f \in C([0, T]; \mathcal{Z})$ is investigated.

These results are used for the proof of the unique solvability of the problem

$$D^{\sigma_k} x(0) = x_k, \quad k = 0, 1, \dots, n-1, \quad (5)$$

$$D^{\sigma_n} Lx(t) = Mx(t) + g(t). \quad (6)$$

Here, $\mathcal{X}, \mathcal{Y}$ are Banach spaces, $L \in \mathcal{L}(\mathcal{X}; \mathcal{Y})$ (linear and continuous operator from $\mathcal{X}$ into $\mathcal{Y}$), and $M \in Cl(\mathcal{X}; \mathcal{Y})$ (linear closed operator with a dense domain D_M in the space $\mathcal{X}$ and with an image in $\mathcal{Y}$). We consider the case $\ker L \neq \{0\}$; hence, Equation (5) is called a degenerate evolution equation. For this equation, we will use the condition of (L, p) -boundedness of the operator M . It allows us to reduce this equation to a system of two equations on two mutual subspaces. One of them has the form (4), and the other has a nilpotent operator at the fractional derivative. It is shown that the initial value problem

$$D^{\sigma_k} Px(0) = x_k, \quad k = 0, 1, \dots, n-1, \quad (7)$$

is more natural for the degenerate Equation (6). Here, P is a projector on one of the above-mentioned subspaces along the other subspace. A theorem of the existence and uniqueness of a classical solution of the problem in (6) and (7) is also obtained.

Abstract results for non-degenerate and degenerate equations in Banach spaces are applied to the investigation of a class of initial boundary value problems for partial differential equations with a time-fractional derivative and with polynomials in a self-adjoint elliptical differential operator with respect to spatial variables.

This article is a continuation of the previous work of the authors, who investigated equations in Banach spaces with other fractional derivatives [14–17] with applications to initial boundary value problems for partial differential equations and systems of equations.

2. Homogeneous Equation with the Dzhrbashyan–Nersesyan Fractional Derivative

Consider the fractional Dzhrbashyan–Nersesyan derivative, which is a generalization of two well-known fractional derivatives: the Riemann–Liouville and Gerasimov–Caputo [11] derivatives. Let us present their definitions.

Let $\alpha > 0$, $z : [0, T] \rightarrow \mathcal{Z}$, for some $T > 0$ and Banach space $\mathcal{Z}$. The Riemann–Liouville fractional integral of an order $\alpha > 0$ of a function z has the form

$$J_t^\alpha z(t) := \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} z(s) ds, \quad t > 0.$$

The Riemann–Liouville fractional derivative of an order $\alpha > 0$ for a function z is defined as

$${}^R D_t^\alpha z(t) := D_t^m J_t^{m-\alpha} z(t),$$

where $m-1 < \alpha \leq m \in \mathbb{N}$, and $D_t^m := \frac{d^m}{dt^m}$ is the integer-order derivative. Further, we use the notations ${}^R D_t^\alpha := D_t^\alpha$, $D_t^{-\alpha} := J_t^\alpha$ for $\alpha > 0$. The Gerasimov–Caputo fractional derivative of an order $\alpha > 0$ is defined as

$${}^C D_t^\alpha z(t) := {}^R D_t^\alpha \left(z(t) - \sum_{k=0}^{m-1} z^{(k)}(0) \frac{t^k}{k!} \right).$$

Let $\{\alpha_k\}_0^n = \{\alpha_0, \alpha_1, \dots, \alpha_n\}$ be the set of real numbers that satisfy the condition $0 < \alpha_k \leq 1$, $k = 0, 1, \dots, n \in \mathbb{N}$. We denote

$$\sigma_k := \sum_{j=0}^k \alpha_j - 1, \quad k = 0, 1, \dots, n,$$

so $-1 < \sigma_k \leq k-1$. Further, it is assumed that the condition $\sigma_n > 0$ is met everywhere. We define the Dzhrbashyan–Nersesyan fractional derivatives, which are associated with a sequence $\{\alpha_k\}_0^n$, with the relations

$$D^{\sigma_0} z(t) := D_t^{\alpha_0-1} z(t), \quad (8)$$

$$D^{\sigma_k} z(t) := D_t^{\alpha_k-1} D_t^{\alpha_{k-1}-1} D_t^{\alpha_{k-2}-1} \dots D_t^{\alpha_0-1} z(t), \quad k = 1, 2, \dots, n. \quad (9)$$

Let function $z : \mathbb{R}_+ \rightarrow \mathcal{Z}$, $\alpha > 0$, $m = \lceil \alpha \rceil$; then, the Laplace transform, which we will denote as $\widehat{z}$ —or when the expressions are too large for z , we denote it as $\text{Lap}[z]$ —has the form

$$\widehat{D_t^\alpha z}(\lambda) = \lambda^\alpha \widehat{z}(\lambda) - \sum_{k=0}^{m-1} \lambda^k D_t^{\alpha-m+k} z(0).$$

Therefore,

$$\begin{aligned} \widehat{D^{\sigma_n} z}(\lambda) &= \lambda^{\alpha_n-1} \text{Lap}[D_t^{\alpha_{n-1}-1} D_t^{\alpha_{n-2}-1} \dots D_t^{\alpha_0-1} z](\lambda) = \\ &= \lambda^{\alpha_{n-1}+\alpha_n-1} \text{Lap}[D_t^{\alpha_{n-2}-1} \dots D_t^{\alpha_0-1} z](\lambda) - \lambda^{\alpha_{n-1}} D^{\sigma_{n-1}} z(0) = \dots = \\ &= \lambda^{\alpha_1+\dots+\alpha_n-1} \widehat{D_t^{\alpha_0} z}(\lambda) - \lambda^{\alpha_n-1} D^{\sigma_{n-1}} z(0) - \lambda^{\alpha_{n-1}+\alpha_n-1} D^{\sigma_{n-2}} z(0) - \\ &\quad - \dots - \lambda^{\alpha_2+\dots+\alpha_n-1} D^{\sigma_1} z(0) = \\ &= \lambda^{\alpha_0+\dots+\alpha_n-1} \widehat{z}(\lambda) - \lambda^{\alpha_n-1} D^{\sigma_{n-1}} z(0) - \lambda^{\alpha_{n-1}+\alpha_n-1} D^{\sigma_{n-2}} z(0) - \\ &\quad - \dots - \lambda^{\alpha_1+\dots+\alpha_n-1} D^{\sigma_0} z(0) = \end{aligned}$$

$$\begin{aligned}
&= \lambda^{\sigma_n} \widehat{z}(\lambda) - \lambda^{\sigma_n - \sigma_{n-1} - 1} D^{\sigma_{n-1}} z(0) - \lambda^{\sigma_n - \sigma_{n-2} - 1} D^{\sigma_{n-2}} z(0) - \dots - \\
&\quad - \lambda^{\sigma_n - \sigma_0 - 1} D^{\sigma_0} z(0) = \lambda^{\sigma_n} \widehat{z}(\lambda) - \sum_{k=0}^{n-1} \lambda^{\sigma_n - \sigma_k - 1} D^{\sigma_k} z(0).
\end{aligned} \quad (10)$$

Let $\mathcal{L}(\mathcal{Z})$ be the Banach space of all linear bounded operators on $\mathcal{Z}$, $A \in \mathcal{L}(\mathcal{Z})$, and let D^{σ_n} be the Dzhrbashyan–Nersesyan fractional derivative, which is defined by a set of numbers $\{\alpha_k\}_0^n = \{\alpha_0, \alpha_1, \dots, \alpha_n\}$, $0 < \alpha_k \leq 1$, $k = 0, 1, \dots, n \in \mathbb{N}$ using Formulas (8) and (9). It is required that the inequality $\sigma_n > 0$ is satisfied. Consider the equation

$$D^{\sigma_n} z(t) = Az(t), \quad t > 0, \quad (11)$$

with the initial conditions

$$D^{\sigma_k} z(0) = z_k, \quad k = 0, 1, \dots, n-1. \quad (12)$$

A function $z \in C(\mathbb{R}_+; \mathcal{Z})$ is called a solution to problem (11), (12), if $D_t^{\sigma_k} z \in C(\overline{\mathbb{R}_+}; \mathcal{Z})$, $k = 0, 1, \dots, n-1$, $D_t^{\sigma_n} z \in C(\mathbb{R}_+; \mathcal{Z})$, equality (11) is fulfilled for all $t \in \mathbb{R}_+$, and conditions (12) are true. Here, $\overline{\mathbb{R}_+} := \mathbb{R}_+ \cup \{0\}$.

Let a solution of (11) have the Laplace transform; then, Equation (11) implies that

$$\lambda^{\sigma_n} \widehat{z}(\lambda) - \sum_{k=0}^{n-1} \lambda^{\sigma_n - \sigma_k - 1} D^{\sigma_k} z(0) = A \widehat{z}(\lambda). \quad (13)$$

For a fixed value $l \in \{0, 1, \dots, n-1\}$, consider the problem

$$D^{\sigma_l} z(0) = z_l, \quad D^{\sigma_k} z(0) = 0, \quad k \in \{0, 1, \dots, n-1\} \setminus \{l\}. \quad (14)$$

for Equation (11). If its solution has the Laplace transform, then the equality (13) for it has the form

$$\lambda^{\sigma_n} \widehat{z}(\lambda) - \lambda^{\sigma_n - \sigma_l - 1} z_l = A \widehat{z}(\lambda).$$

From here, we have

$$\begin{aligned}
\widehat{z}(\lambda) &= \lambda^{\sigma_n - \sigma_l - 1} (\lambda^{\sigma_n} I - A)^{-1} z_l, \\
z(t) &= \frac{1}{2\pi i} \int_{\gamma} \lambda^{\sigma_n - \sigma_l - 1} (\lambda^{\sigma_n} I - A)^{-1} e^{\lambda t} d\lambda z_l,
\end{aligned}$$

where $\gamma = \{\lambda = re^{-i\pi} \in \mathbb{C} : r \in (\infty, a]\} \cup \{\lambda = ae^{i\varphi} \in \mathbb{C} : \varphi \in (-\pi, \pi)\} \cup \{\lambda = re^{i\pi} \in \mathbb{C} : r \in [a; \infty)\}$ with $a > \|A\|_{\mathcal{L}(\mathcal{Z})}^{1/\sigma_n}$.

So, we define the operators for $k = 0, 1, \dots, n-1$:

$$Z_k(t) = \frac{1}{2\pi i} \int_{\gamma} \lambda^{\sigma_n - \sigma_k - 1} (\lambda^{\sigma_n} I - A)^{-1} e^{\lambda t} d\lambda, \quad t > 0.$$

Note that due to the boundedness of the operator A ,

$$\begin{aligned}
Z_k(t) &= \sum_{j=0}^{\infty} \frac{A^j}{2\pi i} \int_{\gamma} \lambda^{-\sigma_k - 1 - j\sigma_n} e^{\lambda t} d\lambda = \sum_{j=0}^{\infty} \frac{t^{j\sigma_n + \sigma_k} A^j}{2\pi i} \int_{\gamma_t} \frac{e^v dv}{v^{j\sigma_n + \sigma_k + 1}} = \\
&= \sum_{j=0}^{\infty} \frac{t^{j\sigma_n + \sigma_k} A^j}{\Gamma(j\sigma_n + \sigma_k + 1)} = t^{\sigma_k} E_{\sigma_n, \sigma_k + 1}(t^{\sigma_n} A).
\end{aligned}$$

The Mittag–Leffler function is used here:

$$E_{\alpha,\beta}(Z) = \sum_{j=0}^{\infty} \frac{Z^j}{\Gamma(\alpha j + \beta)}, \quad Z \in \mathcal{L}(\mathcal{Z}).$$

Lemma 1. Let $A \in \mathcal{L}(\mathcal{Z})$, $z_l \in \mathcal{Z}$ for $l \in \{0, \dots, n-1\}$, $\sigma_n > 0$, $\alpha_n + \sigma_l > 0$. Then, function $Z_l(t) = t^{\sigma_l} E_{\sigma_n, \sigma_l+1}(t^{\sigma_n} A)$ is the unique solution to the problem in (11) and (14).

Proof. We have

$$D^{\sigma_0} t^{\sigma_l} E_{\sigma_n, \sigma_l+1}(t^{\sigma_n} A) = D_t^{\alpha_0-1} \sum_{j=0}^{\infty} \frac{t^{j\sigma_n+\sigma_l} A^j}{\Gamma(j\sigma_n + \sigma_l + 1)} = \sum_{j=0}^{\infty} \frac{t^{j\sigma_n+\sigma_l-\sigma_0} A^j}{\Gamma(j\sigma_n + \sigma_l - \sigma_0 + 1)},$$

$\sigma_l - \sigma_0 > 0$ for $l > 0$, so $D^{\sigma_0} Z_l(0) = 0$, $D^{\sigma_0} Z_0(0) = I$. For $k \in \{0, 1, \dots, l\}$,

$$D^{\sigma_k} Z_l(t) = D_t^{\alpha_k-1} D_t^{\alpha_{k-1}-1} D_t^{\alpha_{k-2}-1} \dots D_t^{\alpha_0} Z_l(t) = \sum_{j=0}^{\infty} \frac{t^{j\sigma_n+\sigma_l-\sigma_k} A^j}{\Gamma(j\sigma_n + \sigma_l - \sigma_k + 1)},$$

at $k < l$ $\sigma_l - \sigma_k = \alpha_{k+1} + \alpha_{k+2} + \dots + \alpha_l > 0$; hence, $D^{\sigma_k} Z_l(0) = 0$, $D^{\sigma_l} Z_l(0) = I$.

Further,

$$\begin{aligned} D^{\sigma_{l+1}} Z_l(t) &= D_t^{\alpha_{l+1}-1} D_t^{\alpha_l} D_t^{\alpha_{l-1}-1} \dots D_t^{\alpha_0} Z_l(t) = D_t^{\alpha_{l+1}-1} D_t^{\alpha_l} \sum_{j=0}^{\infty} \frac{t^{j\sigma_n+\alpha_l-1} A^j}{\Gamma(j\sigma_n + \alpha_l)} = \\ &= D_t^{\alpha_{l+1}-1} \sum_{j=1}^{\infty} \frac{t^{j\sigma_n-1} A^j}{\Gamma(j\sigma_n)} = \sum_{j=1}^{\infty} \frac{t^{j\sigma_n+\sigma_l-\sigma_{l+1}} A^j}{\Gamma(j\sigma_n + \sigma_l - \sigma_{l+1} + 1)}, \end{aligned}$$

for $k \in \{l+1, l+2, \dots, n-1\}$

$$D^{\sigma_k} Z_l(t) = \sum_{j=1}^{\infty} \frac{t^{j\sigma_n+\sigma_l-\sigma_k} A^j}{\Gamma(j\sigma_n + \sigma_l - \sigma_k + 1)}.$$

For $l \in \{0, 1, \dots, n-2\}$, $k \in \{l+1, l+2, \dots, n-1\}$, we have

$$\sigma_n + \sigma_l - \sigma_k \geq \sigma_n + \sigma_l - \sigma_{n-1} = \alpha_n + \sigma_l > 0.$$

Therefore, $D^{\sigma_k} Z_l(0) = 0$.

Finally,

$$D^{\sigma_n} Z_l(t) = \sum_{j=1}^{\infty} \frac{t^{j\sigma_n+\sigma_l-\sigma_n} A^j}{\Gamma(j\sigma_n + \sigma_l - \sigma_n + 1)} = A \sum_{j=0}^{\infty} \frac{t^{j\sigma_n+\sigma_l} A^j}{\Gamma(j\sigma_n + \sigma_l + 1)} = A Z_l(t).$$

We will prove the uniqueness of the solution. Suppose that $z_1(t)$ and $z_2(t)$ are two solutions of the problem in (11) and (14). Let us fix some $T > 0$; then, $y(t) = z_1(t) - z_2(t)$ is a solution of the problem $D^{\sigma_k} y(0) = 0$, $k = 0, 1, \dots, n$, for Equation (11) on the interval $(0, T)$. We define the function $y(t)$ as zero on $[T, +\infty)$. Such a function is bounded and is also a solution to this problem for Equation (11) for $t > 0$, except it may be a point $t = T$. After acting with the Laplace transform on both parts of the equality $D_t^{\sigma_n} y(t) = A y(t)$, we get $\lambda^{\sigma_n} \hat{y}(\lambda) = A \hat{y}(\lambda)$. Therefore, $(\lambda^{\sigma_n} - A) \hat{y}(\lambda) \equiv 0$. If $|\lambda| > \|A\|_{\mathcal{L}(\mathcal{Z})}^{1/\sigma_n}$; then, $\hat{y}(\lambda) = 0$. Consequently, $z_1(t) - z_2(t) = y(t) \equiv 0$ for all $t \in (0, T)$. Because $T > 0$ can be chosen at a large enough value, then $z_1(t) = z_2(t)$ for all $t > 0$. $\square$

Theorem 1. Let $A \in \mathcal{L}(\mathcal{Z})$, $z_k \in \mathcal{Z}$, $k = 0, 1, \dots, n-1$, $0 < \alpha_k \leq 1$, $k = 0, 1, \dots, n$, $\sigma_n > 0$, $\alpha_0 + \alpha_n > 1$. Then, the function

$$z(t) = \sum_{k=0}^{n-1} t^{\sigma_k} E_{\sigma_n, \sigma_k+1}(t^{\sigma_n} A) z_k$$

is a unique solution of the problem in (11) and (12).

Proof. For any $l \in \{0, 1, \dots, n-1\}$, we have

$$\sigma_l + \alpha_n \geq \sigma_0 + \alpha_n = \alpha_0 + \alpha_n - 1 > 0.$$

Therefore, Lemma 1 is valid for all l . From the linearity of the problem in (11) and (12), we get what we need. $\square$

Remark 1. The result for $\mathcal{Z} = \mathbb{R}$ was obtained in [11].

3. Inhomogeneous Equation

Consider the inhomogeneous equation

$$D^{\sigma_n} z(t) = Az(t) + f(t), \quad t \in (0, T], \quad (15)$$

for some $f \in C([0, T]; \mathcal{Z})$. A function $z \in C((0, T]; \mathcal{Z})$ is called a solution of the problem in (12) and (15) if $D_t^{\sigma_k} z \in C([0, T]; \mathcal{Z})$, $k = 0, 1, \dots, n-1$, $D_t^{\sigma_n} z \in C((0, T]; \mathcal{Z})$, equality (15) is satisfied for all $t \in (0, T]$, and conditions (12) are true.

Assuming the convergence of the corresponding integrals, we denote

$$\begin{aligned} Z(t) &= \frac{1}{2\pi i} \int_{\gamma} (\lambda^{\sigma_n} I - A)^{-1} e^{\lambda t} d\lambda = \sum_{j=0}^{\infty} \frac{A^j}{2\pi i} \int_{\gamma} \lambda^{-(j+1)\sigma_n} e^{\lambda t} d\lambda = \\ &= \sum_{j=0}^{\infty} \frac{t^{(j+1)\sigma_n-1} A^j}{2\pi i} \int_{\gamma_t} \frac{e^{\nu} d\nu}{\nu^{(j+1)\sigma_n}} = \sum_{j=0}^{\infty} \frac{t^{(j+1)\sigma_n-1} A^j}{\Gamma((j+1)\sigma_n)} = t^{\sigma_n-1} E_{\sigma_n, \sigma_n}(t^{\sigma_n} A), \end{aligned}$$

for $k = 0, 1, \dots, n-1$, and

$$\begin{aligned} Z_{\sigma_k}(t) &= \frac{1}{2\pi i} \int_{\gamma} \lambda^{\sigma_k} (\lambda^{\sigma_n} I - A)^{-1} e^{\lambda t} d\lambda = \sum_{j=0}^{\infty} \frac{A^j}{2\pi i} \int_{\gamma} \lambda^{\sigma_k-(j+1)\sigma_n} e^{\lambda t} d\lambda = \\ &= \sum_{j=0}^{\infty} \frac{t^{(j+1)\sigma_n-\sigma_k-1} A^j}{2\pi i} \int_{\gamma_t} \frac{e^{\nu} d\nu}{\nu^{(j+1)\sigma_n-\sigma_k}} = \sum_{j=0}^{\infty} \frac{t^{(j+1)\sigma_n-\sigma_k-1} A^j}{\Gamma((j+1)\sigma_n-\sigma_k)} = t^{\sigma_n-\sigma_k-1} E_{\sigma_n, \sigma_n-\sigma_k}(t^{\sigma_n} A). \end{aligned}$$

We note that $\sigma_n - \sigma_k > 0$, and by assumption, $\sigma_n > 0$; hence, as $t \rightarrow 0+$,

$$Z(t) \sim \frac{t^{\sigma_n-1}}{\Gamma(\sigma_n)}, \quad Z_{\sigma_k}(t) \sim \frac{t^{\sigma_n-\sigma_k-1}}{\Gamma(\sigma_n-\sigma_k)}, \quad k = 0, 1, \dots, n-1. \quad (16)$$

Lemma 2. Let $A \in \mathcal{L}(\mathcal{Z})$, $0 < \alpha_k \leq 1$, $k = 0, 1, \dots, n$, $\sigma_n > 0$, $\alpha_0 + \alpha_n > 1$, $f \in C([0, T]; \mathcal{Z})$. Then, the function

$$z_f(t) = \int_0^t (t-s)^{\sigma_n-1} E_{\sigma_n, \sigma_n}((t-s)^{\sigma_n} A) f(s) ds$$

is a unique solution for the problem

$$D^{\sigma_k} z(0) = 0, \quad k = 0, 1, \dots, n-1, \quad (17)$$

for Equation (15).

Proof. We have

$$\|z_f(t)\|_{\mathcal{Z}} \leq \max_{s \in [0, T]} \|E_{\sigma_n, \sigma_n}(s^{\sigma_n} A)\|_{\mathcal{L}(\mathcal{Z})} \max_{s \in [0, T]} \|f(s)\|_{\mathcal{Z}} \frac{t^{\sigma_n}}{\sigma_n},$$

so $z_f(0) = 0$. For $\alpha_0 \in (0, 1)$,

$$\|D^{\sigma_0} z_f(t)\|_{\mathcal{Z}} = \left\| \frac{1}{\Gamma(1 - \alpha_0)} \int_0^t (t - s)^{-\alpha_0} z_f(s) ds \right\|_{\mathcal{Z}} \leq \frac{\max_{s \in [0, T]} \|z_f(s)\|_{\mathcal{Z}}}{\Gamma(1 - \alpha_0)} \frac{t^{1 - \alpha_0}}{1 - \alpha_0}.$$

Therefore, $D^{\sigma_0} z_f(0) = 0$.

The Laplace transform is

$$\begin{aligned} \widehat{Z}(\mu) &= \frac{1}{2\pi i} \int_{\gamma} (\lambda^{\sigma_n} I - A)^{-1} \int_0^{\infty} e^{(\lambda - \mu)t} dt d\lambda = \\ &= \frac{1}{2\pi i} \int_{\gamma} (\lambda^{\sigma_n} I - A)^{-1} \frac{d\lambda}{\mu - \lambda} = (\mu^{\sigma_n} I - A)^{-1}, \end{aligned}$$

because

$$\left\| \frac{1}{\mu - \lambda} (\lambda^{\sigma_n} I - A)^{-1} \right\|_{\mathcal{Z}} \leq \frac{C}{|\lambda|^{1 + \sigma_n}}.$$

We define f with zero outside the segment $[0, T]$. We have $z_f = Z * f$; consequently,

$$\widehat{z}_f(\mu) = \widehat{Z}(\mu) \widehat{f}(\mu) = (\mu^{\sigma_n} I - A)^{-1} \widehat{f}(\mu),$$

$$\widehat{D^{\sigma_0} z_f}(\mu) = \mu^{\sigma_0} (\mu^{\sigma_n} I - A)^{-1} \widehat{f}(\mu), \quad D^{\sigma_0} z_f(t) = \int_0^t Z_{\sigma_0}(t - s) f(s) ds,$$

$$\widehat{D^{\sigma_1} z_f}(\mu) = \mu^{\sigma_1} (\mu^{\sigma_n} I - A)^{-1} \widehat{f}(\mu), \quad D^{\sigma_1} z_f(t) = \int_0^t Z_{\sigma_1}(t - s) f(s) ds$$

due to (10). Then, for $k = 0, 1$, by virtue of (16),

$$\|D^{\sigma_k} z_f(t)\| \leq C_k \max_{s \in [0, T]} \|f(s)\|_{\mathcal{Z}} \int_0^t (t - s)^{\sigma_n - \sigma_k - 1} ds = \frac{C_k t^{\sigma_n - \sigma_k}}{\sigma_n - \sigma_k} \max_{s \in [0, T]} \|f(s)\|_{\mathcal{Z}}.$$

Consequently, $D^{\sigma_k} z_f(0) = 0$, and

$$\widehat{D^{\sigma_2} z_f}(\mu) = \mu^{\sigma_2} (\mu^{\sigma_n} I - A)^{-1} \widehat{f}(\mu), \quad D^{\sigma_2} z_f(t) = \int_0^t Z_{\sigma_2}(t - s) f(s) ds.$$

Continuing these arguments, we get

$$D^{\sigma_k} z_f(t) = \int_0^t Z_{\sigma_k}(t - s) f(s) ds, \quad k = 0, 1, \dots, n,$$

$$D^{\sigma_k} z_f(0) = 0, \quad k = 0, 1, \dots, n - 1.$$

Therefore, conditions (17) are valid.

Due to the boundedness of the operator A ,

$$\widehat{Az}_f(\mu) = A\widehat{z}_f(\mu) = A(\mu^{\sigma_n}I - A)^{-1}\widehat{f}(\mu) = \mu^{\sigma_n}(\mu^{\sigma_n}I - A)^{-1}\widehat{f}(\mu) - \widehat{f}(\mu),$$

so

$$Az_f(t) = \int_0^t Z_{\sigma_n}(t-s)f(s)ds - f(t) = D^{\sigma_n}z_f(t) - f(t)$$

for all $t > 0$. Thus, equality (15) is satisfied for the function z_f .

The uniqueness of the solution can be proved in the same way as for the homogeneous equation above. $\square$

From Theorem 1 and Lemma 2, we immediately get the following result.

Theorem 2. Let $A \in \mathcal{L}(\mathcal{Z})$, $z_k \in \mathcal{Z}$, $k = 0, 1, \dots, n-1$, $0 < \alpha_k \leq 1$, $k = 0, 1, \dots, n$, $\sigma_n > 0$, $\alpha_0 + \alpha_n > 1$, $f \in C([0, T]; \mathcal{Z})$. Then, function

$$z(t) = \sum_{k=0}^{n-1} t^{\sigma_k} E_{\sigma_n, \sigma_k+1}(t^{\sigma_n} A) z_k + \int_0^t (t-s)^{\sigma_n-1} E_{\sigma_n, \sigma_n}((t-s)^{\sigma_n} A) f(s) ds$$

is a unique solution of the problem (12) in (15).

4. Degenerate Equation

Let $L \in \mathcal{L}(\mathcal{X}; \mathcal{Y})$ and $M \in \mathcal{Cl}(\mathcal{X}; \mathcal{Y})$; D_M is a domain of an operator M . We define the L -resolvent set $\rho^L(M) = \{\mu \in \mathbb{C} : (\mu L - M)^{-1} \in \mathcal{L}(\mathcal{Y}; \mathcal{X})\}$ of an operator M and denote $R_\mu^L(M) := (\mu L - M)^{-1}L$, $L_\mu^L := L(\mu L - M)^{-1}$.

An operator M is called (L, σ) -bounded if

$$\exists a > 0 \quad \forall \mu \in \mathbb{C} \quad (|\mu| > a) \Rightarrow (\mu \in \rho^L(M)).$$

Lemma 3. ([18], pp. 89, 90). Let an operator M be (L, σ) -bounded; $\gamma = \{\mu \in \mathbb{C} : |\mu| = r > a\}$. Then, operators

$$P = \frac{1}{2\pi i} \int_\gamma R_\mu^L(M) d\mu \in \mathcal{L}(\mathcal{X}), \quad Q = \frac{1}{2\pi i} \int_\gamma L_\mu^L(M) d\mu \in \mathcal{L}(\mathcal{Y})$$

are projections.

Set $\mathcal{X}^0 = \ker P$, $\mathcal{X}^1 = \operatorname{im} P$, $\mathcal{Y}^0 = \ker Q$, $\mathcal{Y}^1 = \operatorname{im} Q$. We denote by $L_k(M_k)$ the restriction of the operator $L(M)$ on $\mathcal{X}^k$ ($D_{M_k} = D_M \cap \mathcal{X}^k$), $k = 0, 1$.

Theorem 3. ([18], pp. 90, 91). Let an operator M be (L, σ) -bounded. Then,

- (i) $M_1 \in \mathcal{L}(\mathcal{X}^1; \mathcal{Y}^1)$, $M_0 \in \mathcal{Cl}(\mathcal{X}^0; \mathcal{Y}^0)$, $L_k \in \mathcal{L}(\mathcal{X}^k; \mathcal{Y}^k)$, $k = 0, 1$;
- (ii) there exist operators $M_0^{-1} \in \mathcal{L}(\mathcal{Y}^0; \mathcal{X}^0)$, $L_1^{-1} \in \mathcal{L}(\mathcal{Y}^1; \mathcal{X}^1)$.

We denote $G := M_0^{-1}L_0$. For $p \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, the operator M is called (L, p) -bounded if it is (L, σ) -bounded; $G^p \neq 0$, $G^{p+1} = 0$.

Consider the initial problem

$$D^{\sigma_k} x(0) = x_k, \quad k = 0, 1, \dots, n-1, \quad (18)$$

for a linear inhomogeneous fractional-order equation

$$D^{\sigma_n} Lx(t) = Mx(t) + g(t), \quad (19)$$

in which, as before, D^{σ_n} is the Dzhrbashyan–Nersesyan fractional derivative, which is defined by a set of numbers $\{\alpha_0, \alpha_1, \dots, \alpha_n\}$, $0 < \alpha_k \leq 1$, $k = 0, 1, \dots, n$, $g \in C([0, T]; \mathcal{Y})$.

A solution to the problem in (18) is (19) is called a function $x : (0, T] \rightarrow D_M$, for which $Mx \in C((0, T]; \mathcal{Y})$, $D^{\sigma_k} x \in C([0, T]; \mathcal{X})$, $k = 0, 1, \dots, n-1$, $D^{\sigma_n} Lx \in C((0, T]; \mathcal{X})$, the equality (19) is valid for all $t \in (0, T]$, and conditions (18) are true.

Lemma 4. Let $H \in \mathcal{L}(\mathcal{X})$ be a nilpotent operator with a power $p \in \mathbb{N}_0$, $h : [0, T] \rightarrow \mathcal{X}$, such that $(D^{\sigma_n} H)^l h \in C((0, T]; \mathcal{X})$ at $l = 0, 1, \dots, p$, $D^{\sigma_k} (D^{\sigma_n} H)^l h \in C([0, T]; \mathcal{X})$ for $k = 0, 1, \dots, n-1$, $l = 0, 1, \dots, p$. Then, there exists a unique solution to the equation

$$D^{\sigma_n} Hx(t) = x(t) + h(t). \quad (20)$$

It has the form

$$x(t) = - \sum_{l=0}^p (D^{\sigma_n} H)^l h(t). \quad (21)$$

Proof. Let $z = z(t)$ be a solution of Equation (20). We act with the operator H on both parts of (20) and get the equality $HD^{\sigma_n} Hz(t) = Hz(t) + Hh(t)$. Due to the theorem's conditions, there exists a fractional derivative D^{σ_n} for the the right-hand side of this equality, as well as for its left-hand side. Acting with the operator D^{σ_n} on both parts of this equality, we will have

$$(D^{\sigma_n} H)^2 z = D^{\sigma_n} Hz + D_t^{\sigma_n} Hh = z + h + D^{\sigma_n} Hh.$$

At the p -th step, sequentially continuing this reasoning, we obtain the equality

$$(D^{\sigma_n} H)^{p+1} z = z + \sum_{l=0}^p (D^{\sigma_n} H)^l h.$$

By virtue of the continuity and nilpotency of the operator H , we have

$$(D^{\sigma_n} H)^{p+1} z = (D^{\sigma_n})^{p+1} H^{p+1} z \equiv 0.$$

Hence, equality (21) for is true the function z . This equality implies the existence of a solution to Equation (20) (it is checked by substituting this function into the equation) and its uniqueness. Indeed, the difference of two solutions corresponds to a solution of Equation (20) with the function $h \equiv 0$. According to Formula (21), its solution is identically equal to zero. The lemma has been proved. $\square$

Theorem 4. Let an operator M be (L, p) -bounded, $0 < \alpha_k \leq 1$, $k = 0, 1, \dots, n$, $\sigma_n > 0$, $\alpha_0 + \alpha_n > 1$, $g \in C([0, T]; \mathcal{Y})$, $(D^{\sigma_n} G)^l M_0^{-1} (I - Q)g \in C((0, T]; \mathcal{X})$, $l = 0, 1, \dots, p$, $D^{\sigma_k} (D^{\sigma_n} G)^l M_0^{-1} (I - Q)g \in C([0, T]; \mathcal{X})$ for $k = 0, 1, \dots, n-1$, $l = 0, 1, \dots, p$, and let $x_k \in \mathcal{X}$ satisfy the conditions

$$(I - P)x_k = -D^{\sigma_k} \sum_{l=0}^p (D^{\sigma_n} G)^l M_0^{-1} (I - Q)g(t)|_{t=0}, \quad k = 0, 1, \dots, n-1. \quad (22)$$

Then, there exists a unique solution to the problem (18) in (19); it has the form

$$\begin{aligned} x(t) = & \sum_{k=0}^{n-1} t^{\sigma_k} E_{\sigma_n, \sigma_k+1} (t^{\sigma_n} L_1^{-1} M) P x_k + \int_0^t (t-s)^{\sigma_n-1} E_{\sigma_n, \sigma_n} ((t-s)^{\sigma_n} L_1^{-1} M) L_1^{-1} Q g(s) ds - \\ & - \sum_{l=0}^p (D^{\sigma_n} G)^l M_0^{-1} (I - Q)g(t). \end{aligned} \quad (23)$$

Proof. Acting on (19) with the operator $L_1^{-1}Q \in \mathcal{L}(\mathcal{Y}^1; \mathcal{X}^1)$, we get the equation

$$D^{\sigma_n}v(t) = L_1^{-1}Mv(t) + L_1^{-1}Qg(t), \quad (24)$$

where $v(t) = Px(t)$. Indeed, $L_1^{-1}QD^{\sigma_n}Lx(t) = D^{\sigma_n}L_1^{-1}QLx(t) = D^{\sigma_n}L_1^{-1}L_1Px(t) = D^{\sigma_n}v(t) = L_1^{-1}Q(Mx(t) + g(t)) = L_1^{-1}MPx(t) + L_1^{-1}Qg(t) = L_1^{-1}Mv(t) + L_1^{-1}Qg(t)$. In this case, the equality $MP = QM$ is used (see Lemma 3 and Theorem 3).

If we use the operator $M_0^{-1}(I - Q) \in \mathcal{L}(\mathcal{Y}^0; \mathcal{X}^0)$ in the same way, then we get the equation

$$D^{\sigma_n}Gw(t) = w(t) + M_0^{-1}(I - Q)g(t), \quad (25)$$

$w(t) = (I - P)x(t)$. Here, we use the equalities $MP_0 = M(I - P) = (I - Q)M = Q_0M$.

Equations (24) and (25) are endowed with the initial conditions

$$D^{\sigma_k}v(0) = Px_k, \quad k = 0, 1, \dots, n-1, \quad (26)$$

$$D^{\sigma_k}w(0) = (I - P)x_k, \quad k = 0, 1, \dots, n-1. \quad (27)$$

By Theorem 2 and with $\mathcal{Z} = \mathcal{X}^1$, $A = L_1^{-1}M_1 \in \mathcal{L}(\mathcal{X}^1)$ (see Theorem 3), $f(t) = L_1^{-1}Qg(t)$, $z_k = Px_k$, $k = 0, 1, n-1$, the problem in (24) and (26) has a unique solution, and it has the form

$$v(t) = \sum_{k=0}^{n-1} t^{\sigma_k} E_{\sigma_n, \sigma_k+1}(t^{\sigma_n} L_1^{-1}M) Px_k + \int_0^t (t-s)^{\sigma_n-1} E_{\sigma_n, \sigma_n}((t-s)^{\sigma_n} L_1^{-1}M) L_1^{-1}Qg(s) ds.$$

By virtue of Lemma 4, if conditions (22) are fulfilled, the problem in (25) and (27) has a unique solution:

$$w(t) = - \sum_{l=0}^p (D^{\sigma_n}G)^l M_0^{-1}(I - Q)g(t).$$

In this case, the following conditions are used: $D^{\sigma_k}(D^{\sigma_n}G)^l M_0^{-1}(I - Q)g \in C([0, T]; \mathcal{X})$ for $k = 0, \dots, n-1$, $l = 0, 1, \dots, p$. $\square$

To avoid the need to satisfy the approval conditions (22), consider the problem

$$D^{\sigma_k}Px(0) = x_k, \quad k = 0, 1, \dots, n-1, \quad (28)$$

for Equation (19). Its solution is called a function $x : (0, T] \rightarrow D_M$, for which $x \in C((0, T]; D_M)$, $D^{\sigma_k}Px \in C([0, T]; \mathcal{X})$, $k = 0, 1, \dots, n-1$, $D^{\sigma_n}Lx \in C((0, T]; \mathcal{X})$, equality (19) are fulfilled for all $t \in (0, T]$, and conditions (28) are valid.

Remark 2. It is not difficult to make sure that, for $p = 0$, the initial conditions (28) are equivalent to the conditions

$$D^{\sigma_k}Lx(0) = y_k, \quad k = 0, 1, \dots, n-1, \quad (29)$$

where $y_k = Lx_k$, or $x_k = L_1^{-1}y_k$, $k = 0, 1, \dots, n-1$.

The existence and uniqueness theorem for the problem in (19) and (28) is proved similarly with help of a reduction to the system in (24) and (25) with initial conditions (26) and without conditions (27).

Theorem 5. Let an operator M be (L, p) -bounded, $0 < \alpha_k \leq 1$, $k = 0, 1, \dots, n$, $\sigma_n > 0$, $\alpha_0 + \alpha_n > 1$, $g \in C([0, T]; \mathcal{Y})$, $(D^{\sigma_n}G)^l M_0^{-1}(I - Q)g \in C((0, T]; \mathcal{X})$, $l = 0, 1, \dots, p$, $x_k \in \mathcal{X}^1$, $k = 0, 1, \dots, n-1$. Then, there exists a unique solution to the problem in (19) and (28), and it has the form of (23).

5. Application to a Class of Initial Boundary Value Problems

Let $P_q(\lambda) = \sum_{j=0}^q c_j \lambda^j$, $Q_q(\lambda) = \sum_{j=0}^q d_j \lambda^j$, $c_j, d_j \in \mathbb{C}$, $j = 0, 1, \dots, q \in \mathbb{N}_0$, $c_q \neq 0$, $\Omega \subset \mathbb{R}^d$ be a bounded region with a smooth boundary $\partial\Omega$,

$$(\Lambda u)(s) := \sum_{|q| \leq 2r} a_q(s) \frac{\partial^{|q|} u(s)}{\partial s_1^{q_1} \partial s_2^{q_2} \dots \partial s_d^{q_d}}, \quad a_q \in C^\infty(\overline{\Omega}),$$

$$(B_l u)(s) := \sum_{|q| \leq r_l} b_{lq}(s) \frac{\partial^{|q|} u(s)}{\partial s_1^{q_1} \partial s_2^{q_2} \dots \partial s_d^{q_d}}, \quad b_{lq} \in C^\infty(\partial\Omega), \quad l = 1, 2, \dots, r,$$

$q = (q_1, q_2, \dots, q_d) \in \mathbb{N}_0^d$, $|q| = q_1 + \dots + q_d$, and let the operator pencil $\Lambda, B_1, B_2, \dots, B_r$ be regularly elliptical [19]. Let an operator $\Lambda_1 \in \mathcal{CL}(L_2(\Omega))$ with the domain

$$D_{\Lambda_1} = H_{\{B_l\}}^{2r}(\Omega) := \{v \in H^{2r}(\Omega) : B_l v(s) = 0, \quad l = 1, 2, \dots, r, \quad s \in \partial\Omega\}$$

act as $\Lambda_1 u := \Lambda u$. Assume that Λ_1 is a self-adjoint operator; then, the spectrum $\sigma(\Lambda_1)$ of the operator Λ_1 is real and discrete, with finite multiplicity [19]. In addition, the spectrum $\sigma(\Lambda_1)$ is bounded from the right and does not contain zero; $\{\varphi_k : k \in \mathbb{N}\}$ is orthonormal in the $L_2(\Omega)$ system of eigenfunctions of the operator Λ_1 , which is numbered in the non-increasing order of the corresponding eigenvalues $\{\lambda_k : k \in \mathbb{N}\}$, taking their multiplicity into account.

Consider the initial boundary value problem

$$D_t^{\sigma_k} u(s, 0) = u_k(s), \quad k = 0, 1, \dots, n-1, \quad s \in \Omega, \quad (30)$$

$$B_l \Lambda^k u(s, t) = 0, \quad k = 0, 1, \dots, q-1, \quad l = 1, 2, \dots, r, \quad (s, t) \in \partial\Omega \times (0, T], \quad (31)$$

$$D_t^{\sigma_n} P_q(\Lambda) u(s, t) = Q_q(\Lambda) u(s, t) + h(s, t), \quad (s, t) \in \Omega \times (0, T], \quad (32)$$

where $D_t^{\sigma_k}$ are the Dzhrbashyan–Nersesyan fractional derivatives with respect to the variable t , corresponding to the set $\{\alpha_k\}_{k=0}^n$, $\alpha_k \in (0, 1]$, $k = 0, 1, \dots, n$, $h : \Omega \times [0, T] \rightarrow \mathbb{R}$. Take

$$\mathcal{X} = \{v \in H^{2rq}(\Omega) : B_l \Lambda^k v(s) = 0, \quad k = 0, 1, \dots, q-1, \quad l = 1, 2, \dots, r, \quad s \in \partial\Omega\},$$

$$\mathcal{Y} = L_2(\Omega), \quad L = P_q(\Lambda), \quad M = Q_q(\Lambda) \in \mathcal{L}(\mathcal{X}; \mathcal{Y}).$$

Let $P_q(\lambda_k) \neq 0$ for all $k \in \mathbb{N}$; then, there exists an inverse operator $L^{-1} \in \mathcal{L}(\mathcal{Y}; \mathcal{X})$, and the problem in (30)–(32) is representable as the problem in (12) and (15), where $\mathcal{Z} = \mathcal{X}$, $A = L^{-1}M \in \mathcal{L}(\mathcal{Z})$, $z_k = u_k(\cdot)$, $k = 0, 1, \dots, n-1$, $f(t) = L^{-1}h(\cdot, t)$. By Theorem 2, for $\sigma_n > 0$, $\alpha_0 + \alpha_n > 1$, there exists a unique solution to problem (30)–(32) for any $u_k \in \mathcal{X}$, $k = 0, 1, \dots, n-1$, and $h \in C([0, T]; L_2(\Omega))$ (in this case, $L^{-1}h \in C([0, T]; \mathcal{X})$).

Example 1. Take $q = 2$, $P_2(\lambda) = \lambda^2$, $Q_2(\lambda) = a_0 + a_1 \lambda$, $d = 1$, $\Omega = (0, \pi)$, $r = 1$, $\Lambda u = \frac{\partial^2 u}{\partial s^2}$, $B_1 = I$. Then, the problem in (30)–(32) has the form

$$D_t^{\sigma_n} \frac{\partial^4 u}{\partial s^4}(s, t) = a_0 u(s, t) + a_1 \frac{\partial^2 u}{\partial s^2}(s, t) + h(s, t), \quad (s, t) \in (0, \pi) \times (0, T],$$

$$u(0, t) = u(\pi, t) = \frac{\partial^2 u}{\partial s^2}(0, t) = \frac{\partial^2 u}{\partial s^2}(\pi, t) = 0, \quad t \in (0, T],$$

$$D_t^{\sigma_k} u(s, 0) = u_k(s), \quad k = 0, 1, \dots, n-1, \quad s \in (0, \pi).$$

Now, consider the degenerate case. Suppose that $P_\varrho(\lambda_k) = 0$ for some $k \in \mathbb{N}$. If the polynomials P_ϱ and Q_ϱ have no common roots on the set $\{\lambda_k\}$, the operator M is $(L, 0)$ -bounded (see [20]), and the projectors have the form

$$P = \sum_{P_\varrho(\lambda_k) \neq 0} \langle \cdot, \varphi_k \rangle \varphi_k, \quad Q = \sum_{P_\varrho(\lambda_k) \neq 0} \langle \cdot, \varphi_k \rangle \varphi_k,$$

where $\langle \cdot, \varphi_k \rangle$ is the inner product in $L_2(\Omega)$. Considering Remark 2, the initial conditions can be given in the form

$$D_t^{\sigma_k} P_\varrho(\Lambda) u(s, 0) = y_k(s), \quad k = 0, 1, \dots, n-1, \quad s \in \Omega. \quad (33)$$

Then, the problem in (31)–(33) is represented as (19) and (29) with the spaces $\mathcal{X}, \mathcal{Y}$ and the operators L and M selected above. Theorem 5 implies the unique solvability of the problem in (31)–(33) if $\sigma_n > 0$, $\alpha_0 + \alpha_n > 1$, $h \in C([0, T]; L_2(\Omega))$, and $y_k \in L_2(\Omega)$, $k = 0, 1, \dots, n-1$, such that $\langle y_k, \varphi_l \rangle = 0$ for all $l \in \mathbb{N}$, for which $P_\varrho(\lambda_l) = 0$ (in other words, $y_k \in \mathcal{Y}^1$, $k = 0, 1, \dots, n-1$).

Example 2. Let $\varrho = 2$, $P_2(\lambda) \equiv \lambda(\lambda + 9)$, $Q_2(\lambda) = 1 + \lambda$, $d = 1$, $\Omega = (0, \pi)$, $r = 1$, $\Lambda u = \frac{\partial^2 u}{\partial s^2}$, $B_1 = I$. Then, the degenerate problem in (31)–(33) has the form

$$D_t^{\sigma_n} \left(\frac{\partial^4 u}{\partial s^4} + 9 \frac{\partial^2 u}{\partial s^2} \right) (s, t) = \left(u + \frac{\partial^2 u}{\partial s^2} \right) (s, t), \quad (s, t) \in (0, \pi) \times (0, T],$$

$$u(0, t) = u(\pi, t) = \frac{\partial^2 u}{\partial s^2}(0, t) = \frac{\partial^2 u}{\partial s^2}(\pi, t) = 0, \quad t \in (0, T],$$

$$D_t^{\sigma_k} \left(\frac{\partial^4 u}{\partial s^4} + 9 \frac{\partial^2 u}{\partial s^2} \right) (s, 0) = y_k(s), \quad k = 0, 1, \dots, n-1, \quad s \in (0, \pi).$$

Here, $P_2(0) = P_2(-9) = 0$, $0 \notin \sigma(\Lambda_1)$, $-9 = -3^2 \in \sigma(\Lambda_1)$; therefore, $\mathcal{X}^0 = \mathcal{Y}^0 = \text{span}\{\sin 3s\}$, $\mathcal{X}^1$, and $\mathcal{Y}^1$ are closures of $\text{span}\{\sin ks : k \in \mathbb{N} \setminus \{3\}\}$ in $H^4(0, \pi)$ and $L_2(0, \pi)$, respectively. Thus, the conditions

$$\langle y_k, \sin 3s \rangle = \int_0^\pi y_k(s) \sin 3s ds = 0, \quad k = 0, 1, \dots, n-1,$$

must be satisfied for the solvability of this initial boundary value problem.

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