symmetry MDPI

Article

An Upper Bound Asymptotically Tight for the Connectivity of the Disjointness Graph of Segments in the Plane

Aurora Espinoza-Valdez [1,†], Jesús Leaños [2,*,†], Christophe Ndjatchi [3,†] and Luis Manuel Ríos-Castro [4,†]

1 Departamento de Ciencias Computacionales, CUCEI, Universidad de Guadalajara, Guadalajara 44430, Jalisco, Mexico; aurora.espinoza@academicos.udg.mx
2 Unidad Académica de Matemáticas, Universidad Autónoma de Zacatecas, Zacatecas 98066, Mexico
3 Academia de Físico-Matemáticas, Instituto Politécnico Nacional, UPIIZ, Zacatecas 98160, Mexico; mndjatchi@ipn.mx
4 Academia de Físico-Matemáticas, Instituto Politécnico Nacional, CECYT18, Zacatecas 98160, Mexico; lriosc@ipn.mx
* Correspondence: jleanos@uaz.edu.mx
† These authors contributed equally to this work.

Abstract: Let P be a set of $n \geq 3$ points in general position in the plane. The edge disjointness graph $D(P)$ of P is the graph whose vertices are the $\binom{n}{2}$ closed straight line segments with endpoints in P, two of which are adjacent in $D(P)$ if and only if they are disjoint. In this paper we show that the connectivity of $D(P)$ is at most $\frac{7n^2}{18} + \Theta(n)$, and that this upper bound is asymptotically tight. The proof is based on the analysis of the connectivity of $D(Q_n)$, where Q_n denotes an n-point set that is almost 3-symmetric.

Keywords: disjointness graph of segments; rectilinear local crossing number; 3-symmetry; Menger's theorem; Hall's theorem

Citation: Espinoza-Valdez, A.; Leaños, J.; Ndjatchi, C.; Ríos-Castro, L.M. An Upper Bound Asymptotically Tight for the Connectivity of the Disjointness Graph of Segments in the Plane. *Symmetry* **2021**, *13*, 1050. https://doi.org/10.3390/sym13061050

Academic Editors: Walter Carballosa and Álvaro Martínez Pérez

Received: 7 May 2021
Accepted: 7 June 2021
Published: 10 June 2021

1. Introduction

We call set in general position to any finite set of points in the Euclidean plane that does not contain three collinear elements. Let P be a set of $n \geq 3$ points in general position. A segment of P is a closed straight line segment with its two endpoints being elements of P. In this paper, we shall use $\mathcal{P}$ to denote the set of all $\binom{n}{2}$ segments of P. The edge disjointness graph $D(P)$ of P is the graph whose vertex set is $\mathcal{P}$, and two elements of $\mathcal{P}$ are adjacent in $D(P)$ if and only if they are disjoint. We note that $\mathcal{P}$ naturally defines a rectilinear drawing in the plane of the complete graph K_n on n vertices. See Figure 1.

The class of edge disjointness graphs was introduced in 2005 by Araujo, Dumitrescu, Hurtado, Noy, and Urrutia [1], as a geometric version of the Kneser graphs. We recall that for $k, m \in \mathbb{Z}^+$ with $k \leq m/2$, the Kneser graph $KG(m;k)$ is defined as the graph whose vertices are all the k–subsets of $\{1, 2, \ldots, m\}$ and in which two k-subsets from an edge if and only if they are disjoint. In 1956, Kneser conjectured [2] that the chromatic number $\chi(KG(m;k))$ of $KG(m;k)$ is equal to $m - 2k + 2$. This conjecture was proved by Lovász [3] and (independently) by Bárány [4] in 1978. For more results on Kneser graphs, we refer the reader to [5–10] and the references therein.

In [1] the effort was focussed on the estimation of the chromatic number $\chi(D(P))$ of $D(P)$, and a general lower bound was established. The problem of determining the exact value of $\chi(D(P))$ remains open in general. On the other hand, there are only two families of point sets for which the exact value of $\chi(D(P))$ is known: when P is in convex position [11,12], and when P is the double chain [13]. The connectivity $\kappa(D(P))$ of $D(P)$ was studied by Leaños, Ndjatchi, and Ríos-Castro in [14], where it was shown that $\kappa(D(P)) \geq \binom{\lfloor \frac{n-2}{2} \rfloor}{2} + \binom{\lceil \frac{n-2}{2} \rceil}{2}$. We remark that in this paper we give a complementary upper bound for $\kappa(D(P))$.

Recently, Aichholzer, Kynčl, Scheucher, and Vogtenhuber [15] have established an asymptotic upper bound for the maximum size of certain independent sets of vertices of $D(P)$. In 2017 Pach, Tardos, and Tóth [16] studied the chromatic number and the clique number of $D(P)$ in the more general setting of $\mathbb{R}^d$ for $d \geq 2$, i.e., when P is a subset of $\mathbb{R}^d$. More precisely, in [16] was shown that the chromatic number of $D(P)$ is bounded by above by a polynomial function that depends on its clique number $\omega(D(P))$, and that the problem of determining any of $\chi(D(P))$ or $\omega(D(P))$ is NP-hard. Two years later, Pach and Tomon [17] have shown that if G is the disjointness graph of a set of grounded x-monotone curves in $\mathbb{R}^2$ and $\omega(G) = k$, then $\chi(G) \leq k + 1$.

Another wide research area, which is closely related to this work, is the study of the combinatorial properties of geometric graphs. We recall that a *geometric graph* is a graph whose vertex set V is a finite set of points in general positions in the plane, and the edges are straight line segments connecting some pairs of V. Clearly, the sets of segments studied in this paper correspond to a class of the geometric graph, namely the class of complete geometric graphs. See [18] for an excellent survey on geometric graphs.

Following [14], if $x, y \in P$ and $x \neq y$, then xy will be the element of $\mathcal{P}$ with endpoints x and y. Let x_1y_2 and y_1x_2 be two distinct elements of $\mathcal{P}$, and suppose that $x_1y_2 \cap y_1x_2 \neq \emptyset$. Then $x_1y_2 \cap y_1x_2$ consists precisely of one point $o \in \mathbb{R}^2$, because P is a set in general position. If o is an interior point of both x_1y_2 and y_1x_2, then we say that they cross at o, and we will refer to o as a crossing of $\mathcal{P}$. See Figure 1 (upper right).

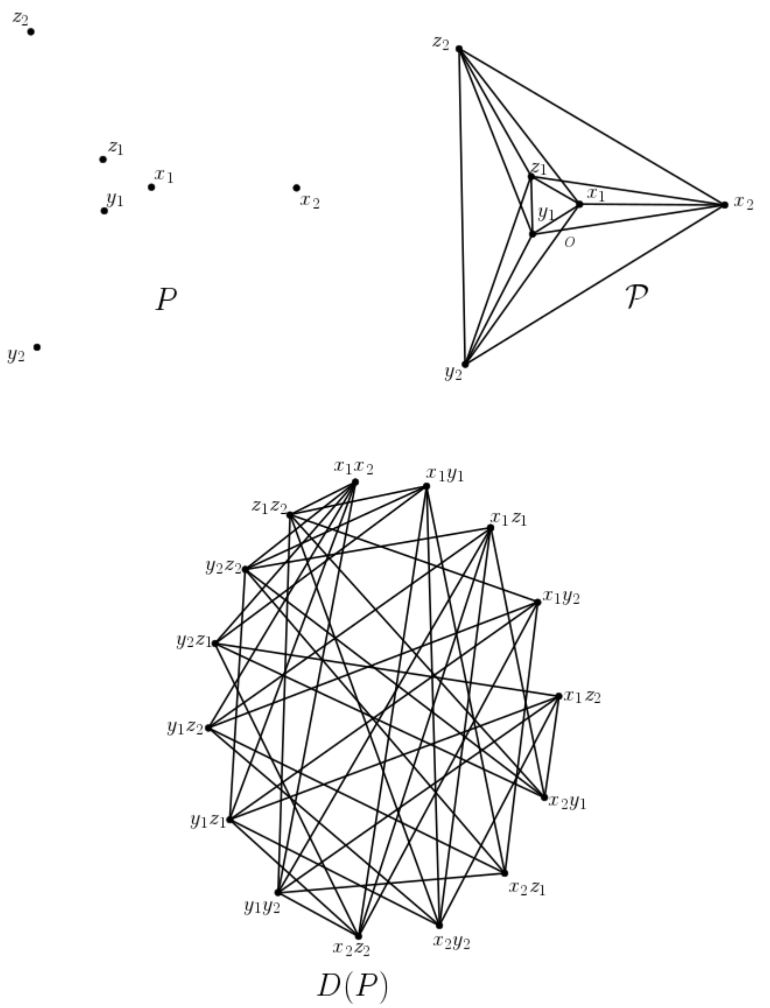

Figure 1. The point set $P = \{x_1, x_2, y_1, y_2, z_1, z_2\}$ on the upper left is a set in general position. In the upper right we have $\mathcal{P}$, which can be seen as the rectilinear drawing of K_6 induced by P. As the largest number of crossings on any segment of $\mathcal{P}$ is 1, then $\overline{\text{lcr}}(P) = 1$. The graph on the bottom part is the edge disjointness graph $D(P)$ corresponding to P.

Let $H = (V(H), E(H))$ be a (non-empty) simple connected graph. As usual, if $u, v \in V(H)$, then the distance between u and v in H will be denoted by $d_H(u, v)$, and we write uv to mean that u and v are adjacent in H. We note that the uv notation is similar to that used to denote the straight line segment xy defined by the points $x, y \in P$. However, none of these notations should be a source of confusion, because the former objects are vertices of a graph, and the latter are points in the plane.

The neighborhood of v in H is the set $\{u \in V(H) : uv \in E(H)\}$ and is denoted by $N_H(v)$. If $S \subseteq V(H)$, then $N_H(S) := \cup_{v \in S} N_H(v)$. The *degree* $\deg_H(v)$ of v is the number $|N_H(v)|$. The number $\delta(H) := \min\{\deg_H(v) : v \in V(H)\}$ is the minimum degree of H. A $u - v$ path of H is a path of H having an endpoint in u and the other endpoint in v. Similarly, if U is a subgraph of H, then $H \setminus U$ is the subgraph of H that results by removing U from H.

We recall that if k is a nonnegative integer, then H is *k–connected* if $|V(H)| > k$ and $H \setminus W$ is connected for every set $W \subset V(H)$ with $|W| < k$. The connectivity $\kappa(H)$ of H is greatest integer k such that H is k-connected. See Figure 2.

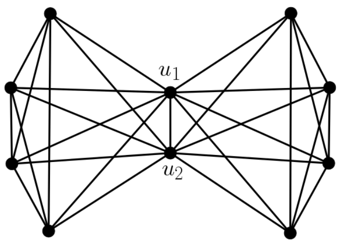

Figure 2. This graph is 2–connected, but not 3–connected. Indeed, if we remove any vertex, what remains is still connected. On the other hand, note that $U = \{u_1, u_2\}$ is vertex cut of order 2, and hence U is a vertex cut of minimum order.

Our aim in this paper is to show the following result.

Theorem 1. *Let P be a set of $n \geq 3$ points in general positions in the plane. Then $\kappa(D(P)) \leq \frac{7}{18}n^2 + \Theta(n)$, and this bound is asymptotically tight.*

The rest of the paper is organized as follows. The validity of inequality in Theorem 1 is proved in Section 2, by making strong use of the main result of Ábrego and Fernández-Merchant in [19]. In Section 3 we briefly explain the strategy to prove Theorem 1 and present the key ingredients of our proof. Finally, in Section 4, we show that the upper bound in Theorem 1 is asymptotically tight, by proving that $\kappa(D(Q_n)) = \frac{7}{18}n^2 - \Theta(n)$ for certain infinite family $\{Q_n\}_{n=3}^{\infty}$ of point sets in general position with certain symmetry property.

2. The Rectilinear Local Crossing Number of K_n and the Validity of Inequality in Theorem 1

In what follows n is an integer with $n \geq 3$, and P is an n-point set in general position. Our aim in this section is to show that $\kappa(D(P)) \leq \frac{7}{18}n^2 + \Theta(n)$. In order to do that, we need to establish a straightforward, yet essential, relationship between the minimum degree $\delta(D(P))$ of the graph $D(P)$ and the rectilinear local crossing number of the drawing $\mathcal{P}$ of K_n induced by P.

We recall that the rectilinear local crossing number of P denoted by $\overline{\text{lcr}}(P)$, is the largest number of crossings on any element of $\mathcal{P}$, and that the rectilinear local crossing number of K_n denoted by $\overline{\text{lcr}}(K_n)$, is the minimum of $\overline{\text{lcr}}(P)$ taken over all n-point sets P in general position. See Figure 1 (upper right).

Proposition 2. *The minimum degree $\delta(D(P))$ of the graph $D(P)$ is equal to $\binom{n-2}{2} - \overline{\text{lcr}}(P)$.*

Proof. Let $e = xy$ be an element of $\mathcal{P}$, and let $\mathcal{E}$ be the subset of $\mathcal{P}$ consisting of all segments that cross e. From the definitions of e and $\mathcal{E}$ it follows that an element of $\mathcal{P} \setminus \{e\}$ is adjacent to e in $D(P)$ if and only if has both endpoints in $P \setminus \{x, y\}$ and does not belong to $\mathcal{E}$. Since $|P| = n$, then the degree of e in $D(P)$ is exactly $\binom{n-2}{2} - |\mathcal{E}|$. The last fact and $|\mathcal{E}| \leq \overline{\text{lcr}}(P)$ imply $\delta(D(P)) \geq \binom{n-2}{2} - \overline{\text{lcr}}(P)$. On the other hand, from the definition of $\overline{\text{lcr}}(P)$ we know that $\mathcal{P}$ contains a segment, say g, that is crossed by exactly $\overline{\text{lcr}}(P)$ elements of $\mathcal{P} \setminus \{g\}$, and hence the degree of g in $D(P)$ is $\binom{n-2}{2} - \overline{\text{lcr}}(P)$, as required. □

The following result was proved in [19] and is the key ingredient in the proof of Corollary 4.

Theorem 3 (Theorem 1 [19])**.** *If n is a positive integer, then*

$$\overline{\text{lcr}}(K_n) = \begin{cases} \frac{1}{9}(n-3)^2 & if \quad n \equiv 0 \pmod 3, \\ \frac{1}{9}(n-1)(n-4) & if \quad n \equiv 1 \pmod 3, \\ \frac{1}{9}(n-2)^2 - \lfloor \frac{n-2}{6} \rfloor & if \quad n \equiv 2 \pmod 3, n \notin \{8, 14\}. \end{cases} \tag{1}$$

In addition, $\overline{\text{lcr}}(K_8) = 4$ and $\overline{\text{lcr}}(K_{14}) = 15$.

Corollary 4. *Let P be an n-point set in general position, and let $n \geq 3$. Then $\kappa(D(P)) \leq \frac{7}{18}n^2 + \Theta(n)$.*

Proof. It is well-known that $\kappa(D(P)) \leq \delta(D(P))$. Thus, it suffices to show that $\delta(D(P)) \leq \frac{7}{18}n^2 + \Theta(n)$. A trivial manipulation of Equation (1) allow us to see that $\overline{\text{lcr}}(K_n) = \frac{1}{9}n^2 - \Theta(n)$. On the other hand, from Proposition 2 and the fact that $\overline{\text{lcr}}(K_n) \leq \overline{\text{lcr}}(P)$, it follows that $\delta(D(P)) \leq \binom{n-2}{2} - \overline{\text{lcr}}(K_n) = \binom{n-2}{2} - \left(\frac{1}{9}n^2 - \Theta(n)\right) = \frac{7}{18}n^2 + \Theta(n)$, as required. □

3. The Key Ingredients of the Proof of Theorem 1

Our strategy to prove that the upper bound in Theorem 1 is asymptotically tight is as follows. First, for any integer n with $n \geq 3$, we define a certain n-point set in a general position, which we denote by Q_n. The family $\{Q_n\}_{n=3}^{\infty}$ was originally defined by Lara, Rubio-Montiel and Zaragoza in [20], where it was shown that $\overline{\text{lcr}}(K_n) = \frac{1}{9}n^2 - \Theta(n)$. More recently, Ábrego and Fernández-Merchant [19] showed that $\overline{\text{lcr}}(K_n) = \overline{\text{lcr}}(Q_n)$ for any $n \not\equiv 2 \pmod 3$. Then, we will give some notation and basic facts related to the connectivity of $D(Q_n)$, which will allow us to simplify the remaining part of the proof of Theorem 1. Finally, in Section 4.2, we will show that $\kappa(D(Q_n)) = \frac{7}{18}n^2 - \Theta(n)$.

We now recall a couple of classical results in graph theory which are fundamental in our proof.

Theorem 5 (Hall's theorem)**.** *Let H be a bipartite graph with bipartition $\{A, B\}$, and let C be an element of $\{A, B\}$ of minimum cardinality. Then H contains a matching of size $|C|$ if and only if $|N_H(S)| \geq |S|$ for any $S \subseteq C$.*

Theorem 6 (Menger's theorem)**.** *A graph is k-connected if and only if it contains k pairwise internally disjoint paths between any two distinct vertices.*

It is straightforward to check that the graph $D(P)$ is connected for any n-point set P in general position with $n \geq 5$. In view of this, the following consequence of Menger's theorem will be useful.

Corollary 7. *Let H be a connected graph. Then H is k-connected if and only if H has k pairwise internally disjoint $a - b$ paths, for any two vertices a and b of H such that $d_H(a, b) = 2$.*

Proof. The forward implication follows directly from Menger's theorem. Conversely, let U be a vertex cut of H of minimum order. Let H_1 and H_2 be two distinct components of $H \setminus U$, and let $u \in U$. Since U is a minimum cut, then u has at least a neighbor v_i in H_i, for $i = 1, 2$. Then $d_H(v_1, v_2) = 2$. By hypothesis, H has k pairwise internally disjoint $v_1 - v_2$ paths. Since each of these k paths intersects U, then we have that $|U| \geq k$, as required. □

Another ingredient that plays a central role in this work is the property of 3-symmetry of point sets in the plane, which is a recurrent concept in crossing number theory. A subset X in $\mathbb{R}^2$ is called 3-symmetric if X contains a subset X_1 such that $X = X_1 \cup \rho(X_1) \cup \rho^2(X_1)$, where ρ is a $2\pi/3$ clockwise rotation around a suitable point in the plane. The relationship between the concept of 3-symmetry and several variants of crossing number have been investigated by a number of authors [19–22]. If X is finite and $|X| \not\equiv 0 \pmod 3$, then we say that X is almost 3-symmetric if X contains a subset X' with at most two elements such that $X \setminus X'$ is 3-symmetric. As we shall see in the next section, the main part of the proof of Theorem 1 is based on the estimation of the connectivity of $D(Q_n)$, where Q_n is an almost 3-symmetric set with n points.

4. The Upper Bound in Theorem 1 Is Asymptotically Tight

For the rest of the paper, n is an integer with $n \geq 3$. We begin by introducing the family of point sets that we use in the proof of our main result, and some notation.

4.1. The Family $\{Q_n\}_{n=3}^{\infty}$ and Its Properties

Following [19], let C_0 be the arc of the circumference passing through the points $(1,0)$, $(3,0)$, and $(2,\epsilon)$, where $\epsilon \in \mathbb{R}^+$ is close to zero. Let C_1 (resp. C_2) be the $2\pi/3$ (resp. $4\pi/3$) counterclockwise rotation of C_0 around the origin $O := (0,0)$. We choose ϵ small enough so that any straight line passing through two distinct points of C_0 separates C_1 from C_2. See Figure 3. Since $Y := C_0 \cup C_1 \cup C_2$ is a 3-symmetric set, we can choose an almost 3-symmetric subset Q_n of Y with exactly n points. For $i \in \{0,1,2\}$, let $n_i := |Q_n \cap Y|$. Then, $n_0 + n_1 + n_2 = n$, and $|n_i - n_j| \leq 1$ for $i, j \in \{0,1,2\}$.

For the rest of the paper, we use $\mathcal{Q}_n$ to denote the set of all $\binom{n}{2}$ segments of Q_n, and $G_n := D(Q_n)$. Similarly, if $a, b \in \mathcal{Q}_n$, then $\eta(Q_n; a, b)$ will denote the maximum number of pairwise internally-disjoint $a - b$ paths in G_n.

Remark 8. *Let a, b be vertices of G_n such that $d_{G_n}(a,b) = 2$. By Corollary 7 and Menger's theorem, in order to show the last assertion of Theorem 1 it is enough to show that $\eta(Q_n; a, b) = \frac{7}{18}n^2 - \Theta(n)$.*

In view of the previous remark, for the rest of the work, we can assume that a and b are two fixed vertices of G_n such that $d_{G_n}(a,b) = 2$. Then a and b are not adjacent in G_n, and hence $a \cap b \neq \emptyset$. This inequality and the fact that Q_n is a set in general position imply that $a \cap b$ consists precisely of one point of $\mathbb{R}^2$, which will be denoted by o. Then either a and b cross at o or o is common endpoint of them.

We note that if $e \in \mathcal{Q}_n$, then there is a unique $i \in \{0,1,2\}$ such that e has an endpoint in C_i and the other in $C_i \cup C_{i+2}$, where addition is taken mod 3. We will say that such an i is the type of e. In particular, note that in any of the three cases of Figure 3, a and b are of type 0 and 1, respectively.

Clearly, there are only two possibilities for a and b with respect to their types: they have the same type, or they have different types. If a and b have the same type, then, by rotating Q_n around O (if necessary) but not the labels C_0, C_1, and C_2, we may assume that a and b are both of type 0. Analogously, if a and b have different types, then, by rotating Q_n (if necessary) we can assume that 0 and 1 are the types of a and b.

It is not hard to check that if a and b are of type 0 and 1, respectively, then the three configurations illustrated in Figure 3 are the only possibilities to put the endpoints of a and b on the arcs C_0, C_1, and C_2. Similarly, if a and b are both of type 0, then ten configurations illustrated in Figure 4 are the only possibilities to put the endpoints of a and b on the arcs C_0 and C_2. Then, from now on, we can assume without any loss of generality that a and b are placed in $\mathcal{Q}_n$ according to some case of Figures 3 and 4. We abuse notation and we shall use C_i to refer to the subset of points of Q_n that lies on the arc C_i. In particular, we will assume that $|C_i| = n_i$. For distinct $i, j \in \{0,1,2\}$, we let $C_{i,j} := C_i \cup C_j$.

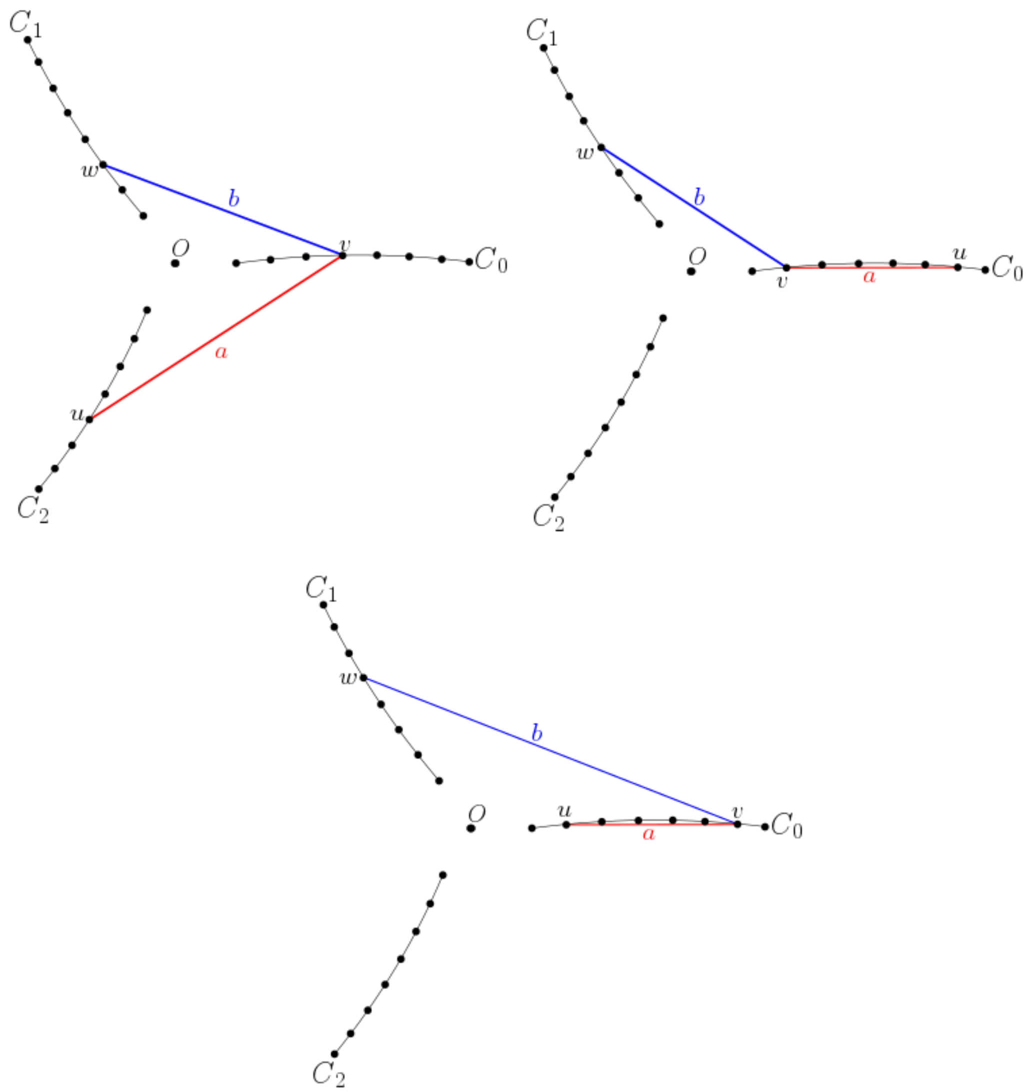

Figure 3. The underlying point set in these three drawings is Q_{24}. Note that $n_i = 8$ for each $i = 0, 1, 2$. The difference between these configurations is the way in which the endpoints of $a = uv$ and $b = vw$ are located on C_0, C_1, and C_2.

We now split the set of neighbours of a and b into three sets as follows.

$$\mathcal{A} := \{e \in \mathcal{Q}_n | e \cap a = \emptyset \text{ and } e \cap b \neq \emptyset\},$$

$$\mathcal{B} := \{e \in \mathcal{Q}_n | e \cap b = \emptyset \text{ and } e \cap a \neq \emptyset\},$$

$$\mathcal{D} := \{e \in \mathcal{Q}_n | e \cap b = \emptyset \text{ and } e \cap a = \emptyset\}.$$

Clearly, $\mathcal{A}, \mathcal{B}, \mathcal{D}$, and $\{a, b\}$ are pairwise disjoint. We also note that $N_{G_n}(a) = \mathcal{A} \cup \mathcal{D}, N_{G_n}(b) = \mathcal{B} \cup \mathcal{D}$, and hence

$$\eta(Q_n; a, b) \leq |\mathcal{D}| + \min\{|\mathcal{A}|, |\mathcal{B}|\}. \tag{2}$$

Keeping in mind the facts and the terminology given in this subsection, we now proceed to show the last assertion of our main result.

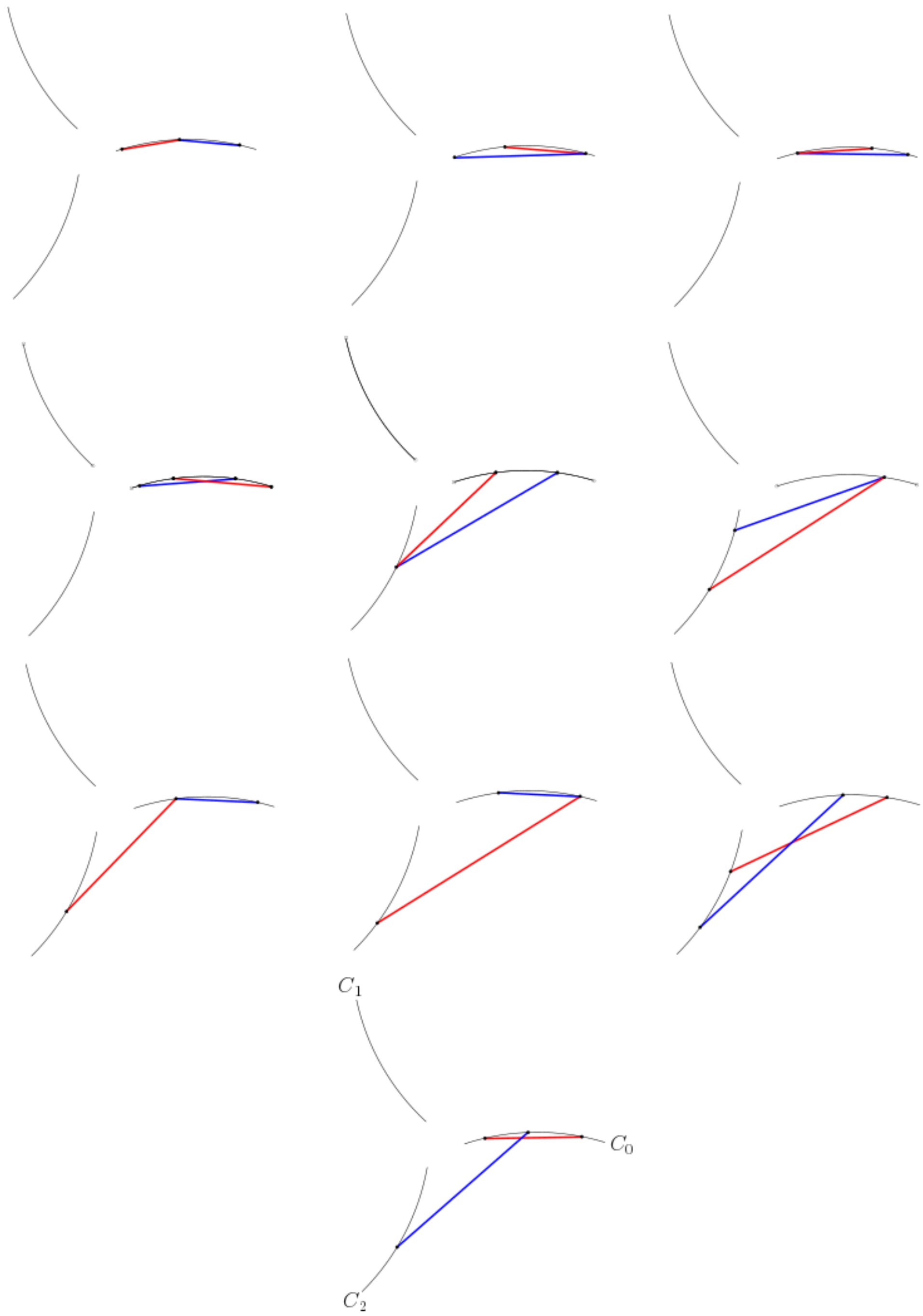

Figure 4. Except for rotations, these ten configurations together with those three in Figure 3, are the only possibilities to put the endpoints of a and b on C_0, C_1, and C_2. The arcs here are not as flat as they should be; we have drawn them in this way for illustrative purposes only.

4.2. Constructing the Required $a - b$ Paths of G_n

As we have mentioned in Remark 8, all we need to finish the proof of Theorem 1 is to show the following lemma.

Lemma 9. *If a and b are as in Remark 8, then $\eta(Q_n; a, b) = \frac{7}{18}n^2 - \Theta(n)$.*

We devote this section to show Lemma 9, and hence to complete the proof of Theorem 1. Our proof is constructive and uses two types of constructions, depending on whether a and b are located in $\mathcal{Q}_n$ according to some case of Figure 3 or Figure 4. In the interest of readability, we start by presenting several technical issues needed to define both types of constructions.

Since the equality in Lemma 9 is asymptotic, in what remains of this section, we assume that $n \geq 15$. Note that this assumption guarantees that each of C_0, C_1, and C_2 has at least one point that does not belong to a or b. This will be used in Section 4.2.2.

4.2.1. Suppose That a and b Have Distinct Types

Then, a and b are located in $\mathcal{Q}_n$ according to some case of Figure 3, and so $a = uv, b = vw, v \in C_0, w \in C_1$, and $u \in C_{0,2} \setminus \{v\}$. Let H be the bipartite subgraph of G_n with bipartition $\{\mathcal{A}, \mathcal{B}\}$ such that $f \in \mathcal{A}$ is adjacent to $g \in \mathcal{B}$ in H if and only if f and g are adjacent in G_n. Thus, H is an induced bipartite subgraph of G_n. If X and Y are nonempty subsets of Q_n, we will denote the set of all segments of $\mathcal{Q}_n$ that have an endpoint in X and the other in Y by $X * Y$.

A simple inspection of the several cases in which two elements of $\mathcal{Q}_n$ can cross each other yields the following assertion.

Observation 10. *If $f, g \in \mathcal{Q}_n$ cross each other, then f and g are of the same type.*

A key fact that will allow us to construct the first class of paths is that H satisfies the Hall's condition. We formalize this idea as follows.

Proposition 11. *Let $\mathcal{C}$ be an element of $\{\mathcal{A}, \mathcal{B}\}$ of minimum cardinality. Then $|N_H(\mathcal{S})| \geq |\mathcal{S}|$ for any $\mathcal{S} \subseteq \mathcal{C}$.*

Proof. We break the proof into two cases, depending on whether $|\mathcal{A}| \leq |\mathcal{B}|$ or $|\mathcal{A}| > |\mathcal{B}|$. We remark that Observation 10 is often used in this proof without explicit mention.

(A) Suppose that $|\mathcal{A}| \leq |\mathcal{B}|$, and let $\mathcal{S} \subseteq \mathcal{A}$. We can assume that $\mathcal{S} \neq \emptyset$ and that $\mathcal{B} \setminus N_H(\mathcal{S}) \neq \emptyset$, as otherwise we are done. Let e^* be a fixed segment of $\mathcal{B} \setminus N_H(\mathcal{S})$, and assume that $e^* = pq$.

The next facts follow easily from the choice of e^* and $\mathcal{S}$. (F1) e^* intersects each element of $\mathcal{S} \cup \{a\}$, (F2) b intersects each element of $\mathcal{S}$, and (F3) each element of $\mathcal{S}$ has at least one endpoint in C_1.

(A1) Suppose that $p \in C_0$ and $q \in C_2$. Then (F1) and (F3) imply that $\mathcal{S}$ is a subset of $\mathcal{A}' := \mathcal{X} \cup \{qw\}$, where $\mathcal{X}$ denotes the set of all segments that intersect b and are in $\{p\} * C_1$. Note that if $\mathcal{S} = \{qw\}$, then $\{u\} * (C_0 \setminus \{u, v\}) \subseteq N_H(\mathcal{S})$, and so $|N_H(\mathcal{S})| \geq n_0 - 2 \geq 1 = |\mathcal{S}|$. Thus, we may assume that $\mathcal{X} \neq \emptyset$. This implies that if $qw \in \mathcal{S}$ (respectively, $qw \notin \mathcal{S}$), then $\mathcal{B}' := \{u\} * (C_{0,2} \setminus \{q, u, v\})$ (respectively, $\mathcal{B}' := \{u\} * (C_{0,2} \setminus \{p, u, v\})$) is a subset of $N_H(\mathcal{S})$. Since $|\mathcal{A}'| \leq n_1 + 1, |\mathcal{B}'| \geq n_0 + n_2 - 3$ and $n_0 + n_2 \geq n_1 + 4$, we have $|N_H(\mathcal{S})| \geq |\mathcal{S}|$.

(A2) Suppose that $p, q \in C_0$. Then (F1) and (F3) imply that $\mathcal{S}$ is a subset of the set $\mathcal{A}'$, which consists of all segments that intersect b and are in $\{p, q\} * C_1$. Then, $|\mathcal{S}| \leq |\mathcal{A}'| \leq 2n_1$. Let $e \in \mathcal{S}$, and assume without loss of generality that $p \in e$. Since $e \cap a = \emptyset$, then $p \notin \{u, v\}$ and $\mathcal{B}' := \{u\} * (C_{0,2} \setminus \{u, v, p\})$ is a subset of $N_H(e)$. Then, $|N_H(\mathcal{S})| \geq |\mathcal{B}'| = n_0 + n_2 - 3$, and we can assume that $|\mathcal{A}'| > n_0 + n_2 - 3$, as otherwise we are done. This implies that there exists $g \in \mathcal{S}$ with $q \in g$.

Since $u \in C_2$ implies $|\mathcal{A}'| \leq n_1 + 1 < n_0 + n_2 - 3$, we can assume that $u \in C_0$. Let I_0 be the set of points of C_0 that are between u and v. From (F1) we know that at most one of p or q is in I_0. If $z \in \{p, q\}$ is a point of I_0, then $\mathcal{B}' \cup (\{z\} * C_2)$ is a subset of $N_H(\{e, g\})$. Then $|N_H(\mathcal{S})| \geq (n_0 + n_2 - 3) + n_2 \geq 2n_1 \geq |\mathcal{S}|$, as required. Then, neither p nor q is in I_0. Since $e^* \cap a \neq \emptyset$, we must have $u = q$. This implies that $\mathcal{S} \subseteq \{p\} * C_1$, and so $|N_H(\mathcal{S})| \geq n_0 + n_2 - 3 \geq n_1 \geq |\mathcal{S}|$.

(A3) Suppose that $p, q \in C_2$. Since e^* is of type 2 and $e^* \cap a \neq \emptyset$, then u must be a common endpoint of e^* and a, by Observation 10. Without loss of generality, suppose that $p = u$.

From (F1) and (F3) we know that $\mathcal{S}$ is a subset of $\{w\} * (Y \cup \{q\})$, where Y denotes the set of all points of C_2 that lie between u and q. Note that if u and q are consecutive points of C_2, then $Y = \emptyset$. Let e be the segment of $\mathcal{S}$ whose endpoint $q' \in Y \cup \{q\}$ is closest to q. Let Y' be the set of all points of C_2 that lie between u and q'. Then $|Y'| + 1 \geq |\mathcal{S}|$, and $\mathcal{B}' :=$

$\{u\} * (Y' \cup C_0 \setminus \{v\})$ is a subset of $N_H(e)$. Since $|\mathcal{S}| \leq |Y'| + 1$ and $|\mathcal{B}'| = (n_0 - 1) + |Y'|$, then $n_0 \geq 5$ implies $|N_H(\mathcal{S})| \geq |\mathcal{S}|$.

(A4) Suppose that e^* has an endpoint in C_1. We assume without loss of generality that $p \in C_1$. Clearly, $p \neq w$. Since $e^* \cap a \neq \emptyset$, then $q = u$, and so $q \in C_{0,2} \setminus \{v\}$.

(A4.1) Suppose that $q \in C_0 \setminus \{v\}$. Then $e^*, b \in C_1 * \{u, v\}$, and $e^* \cap b = \emptyset$. Since $u, v \in C_0$, then $\{w\} * C_2 \subseteq \mathcal{A} \setminus \mathcal{S}$ due to (F1). Let $\mathcal{A}' := \mathcal{A} \setminus (\{w\} * C_2)$, and let $\mathcal{B}'$ be the set that results by removing any segment of $\{u\} * (C_1 \setminus \{p\})$ from $\mathcal{B}$. Then, $|\mathcal{B}'| \geq |\mathcal{A}'|$ because $|\mathcal{B}| \geq |\mathcal{A}|$. We note that (F1) implies $\mathcal{S} \subseteq \mathcal{A}'$. If $\mathcal{S}$ contains a segment e with both endpoints in C_1, then (F1) implies $\mathcal{B}' \subseteq N_H(e)$, and hence $|N_H(\mathcal{S})| \geq |\mathcal{B}'| \geq |\mathcal{A}'| \geq |\mathcal{S}|$, as required. Thus, we may assume that $\mathcal{S} \cap (C_1 * C_1) = \emptyset$, and so $\mathcal{S} \subseteq C_0 * C_1$.

Let I_0 be the set of points of C_0 that are between u and v (if u and v are contiguous, $I_0 = \emptyset$), and let $J_0 := C_0 \setminus (I_0 \cup \{v\})$. By (F1) and (F2) we know that each segment of $\mathcal{S}$ intersects both e^* and b. Since b and e^* are disjoint, no segment of $\mathcal{S}$ can have an endpoint in $I_0 \cup \{v\}$, and, on the other hand, each segment of $\mathcal{B}'$ must have at least one endpoint in $I_0 \cup \{u\}$. Note that if J_0 has at least two points being incident with segments of $\mathcal{S}$, then each segment of $\mathcal{B}'$ is disjoint from at least one of these points of J_0, and hence $\mathcal{B}' \subseteq N_H(\mathcal{S})$, as required.

Thus, we can assume that all segments of $\mathcal{S}$ are incident with one point of J_0, say z. Then, $\mathcal{S} \subseteq \{z\} * (C_1 \setminus \{p\})$ or $\mathcal{S} \subseteq \{z\} * (C_1 \setminus \{w\})$, and so $|\mathcal{S}| \leq n_1 - 1 \leq n_2$. On the other hand, note that $\{u\} * C_2 \subseteq N_H(\mathcal{S})$, and so $|N_H(\mathcal{S})| \geq n_2$, as required.

(A4.2) Suppose now that $q \in C_2$. Then (F1), (F2), and (F3) imply that each segment of $\mathcal{S}$ must be incident with at least one of p or w. Then, $\mathcal{S}$ is the disjoint union $\mathcal{S}_p$ and $\mathcal{S}_w$, where $\mathcal{S}_p$ (respectively, $\mathcal{S}_w$) denotes the subset of $\mathcal{S}$ contained in $\{p\} * (C_{0,1} \setminus \{p, v\})$ (respectively, $\{w\} * (C_2 \setminus \{u\})$). Then, $|\mathcal{S}| = |\mathcal{S}_p| + |\mathcal{S}_w| \leq (n_0 + n_1 - 2) + (n_2 - 1)$.

Let $\mathcal{B}_u^1 := \{u\} * (C_1 \setminus \{p, w\})$ and $\mathcal{B}_u^{0,2} := \{u\} * (C_{0,2} \setminus \{u, v\})$. Clearly, $\mathcal{B}_u^1$ and $\mathcal{B}_u^{0,2}$ are disjoint sets. Moreover, by (A1) and (A3), we may assume that $\mathcal{B}_u^{0,2} \subset N_H(\mathcal{S})$.

Suppose first that $\mathcal{S}_p$ has no segments in $\{p\} * (C_0 \setminus \{v\})$. Then $\mathcal{S}_p \subseteq \{p\} * (C_1 \setminus \{p\})$, and so $|\mathcal{S}| = |\mathcal{S}_p| + |\mathcal{S}_w| \leq (n_1 - 1) + (n_2 - 1)$. We may assume that $n_0 + n_2 - 2 = |\mathcal{B}_u^{0,2}| \leq |N_H(\mathcal{S})| < |\mathcal{S}_p| + |\mathcal{S}_w| \leq n_1 + n_2 - 2$, as otherwise we are done. Since $n_0 \geq n_1 - 1$, then we must have that $n_0 = n_1 - 1, \mathcal{S}_p = \{p\} * (C_1 \setminus \{p\})$, and $\mathcal{S}_w = \{w\} * (C_2 \setminus \{u\})$. From $\mathcal{S}_p = \{p\} * (C_1 \setminus \{p\})$ it follows that $pw \in \mathcal{S}$, and hence $\mathcal{B}_u^1 \subset N_H(pw) \subset N_H(\mathcal{S})$. Since $\mathcal{B}_u^1$ and $\mathcal{B}_u^{0,2}$ are pairwise disjoint, $|N_H(\mathcal{S})| \geq (n_0 + n_2 - 2) + (n_1 - 2) > n_1 + n_2 - 2 \geq |\mathcal{S}|$, as required.

Suppose now that $\mathcal{S}_p$ has a segment e in $\{p\} * (C_0 \setminus \{v\})$. Then $\mathcal{B}_u^1 \subset N_H(e) \subset N_H(\mathcal{S})$, and hence $|N_H(\mathcal{S})| \geq |\mathcal{B}_u^{0,2}| + |\mathcal{B}_u^1| = (n_0 + n_2 - 2) + (n_1 - 2)$. Thus, we may assume that $|\mathcal{S}_p| = n_0 + n_1 - 2$ and $|\mathcal{S}_w| = n_2 - 1$, as otherwise we are done. Then, $\mathcal{S}_p = \{p\} * (C_{0,1} \setminus \{p, v\})$ and $\mathcal{S}_w = \{w\} * (C_2 \setminus \{u\})$.

From $\mathcal{S}_p = \{p\} * (C_{0,1} \setminus \{p, v\})$ and (F2) it is not hard to see that vp must be the segment of $C_0 * C_1$ with either the smallest or the largest length. Let $\mathcal{B}' := (C_0 \setminus \{v\}) * (C_2 \setminus \{u\})$. If vp is the segment of $C_0 * C_1$ with the smallest (respectively, largest) length, then $\mathcal{S}_w = \{w\} * (C_2 \setminus \{u\})$ and (F1) imply that pu must have the largest (respectively, smallest) length among all segments in $\{p\} * C_2$, and hence $\mathcal{B}' \subset N_H(\mathcal{S})$. Since $\mathcal{B}_u^1, B_u^{0,2}$, and $\mathcal{B}'$ are pairwise disjoint and $|\mathcal{B}'| \geq (n_0 - 1)(n_2 - 1) \geq 4$, then $|N_H(\mathcal{S})| \geq (n_0 + n_2 - 2) + (n_1 - 2) + 4 > |\mathcal{S}|$, as required.

(B) Suppose that $|\mathcal{B}| < |\mathcal{A}|$, and let $\mathcal{S} \subseteq \mathcal{B}$. We can assume that $\mathcal{S} \neq \emptyset$ and that $\mathcal{A} \setminus N_H(\mathcal{S}) \neq \emptyset$, as otherwise we are done. Let $d : \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ denote the ordinary euclidean distance in $\mathbb{R}^2$. For $i \in \{0, 1, 2\}$ and $x \in C_i$, we let $C_i^{>x} := \{y \mid y \in C_i \text{ and } d(O, y) > d(O, x)\}$. The set $C_i^{<x}$ is defined analogously. It is not hard to see that $\mathcal{A} \subseteq \mathcal{F}$, where

$$\mathcal{F} := (\{w\} * (Q_n \setminus \{u, v, w\})) \cup (C_0^{<v} * C_1^{>w}) \cup (C_0^{>v} * C_1^{<w}) \cup (C_1^{<w} * C_1^{>w}).$$

Note that $\mathcal{F}_w := \{w\} * (Q_n \setminus \{u, v, w\})$ is a subset of $\mathcal{A}$. Since no segment of $\mathcal{B}$ intersects every segment of $\mathcal{F}_w$, then at least one segment of $\mathcal{F}_w$ is in $N_H(\mathcal{S})$, and so $|N_H(\mathcal{S})| \geq 1$. Thus $|\mathcal{S}| \geq 2$, as otherwise we are done.

(B1) Suppose that $\mathcal{S}$ has two distinct segments of type 0, say e_1 and e_2, so that $e_1 \cap e_2 \cap C_0 = \emptyset$. Then $\mathcal{A} \subseteq N_H(\{e_1, e_2\}) \subseteq N_H(\mathcal{S})$, unless $e_1 \cap e_2 = y \in C_2$. Since the former case contradicts $\mathcal{A} \setminus N_H(\mathcal{S}) \neq \emptyset$, we assume that $e_1 \cap e_2 = y \in C_2$. Then $\mathcal{A} \setminus \{yw\} \subseteq N_H(\{e_1, e_2\})$, and hence $|N_H(\mathcal{S})| \geq |N_H(\{e_1, e_2\})| \geq |\mathcal{A}| - 1 \geq |\mathcal{B}| \geq |\mathcal{S}|$, as required.

(B2) Suppose now that each segment of $\mathcal{S}$ is incident with exactly one point $x \in C_0$. Clearly, $x \neq v$. If $x \neq u$, then $\mathcal{S} \subseteq \{x\} * (C_{0,2} \setminus \{x, v\})$, and hence $|\mathcal{S}| \leq n_0 + n_2 - 2$. Since $|\mathcal{S}| \geq 2$, then $C_{0,2} \setminus \{x, v\}$ has at least two points, say z_1 and z_2, such that $xz_1, xz_2 \in \mathcal{S}$. We note that $\mathcal{F}_w \setminus \{wx\} \subseteq N_H(\{xz_1, xz_2\}) \subseteq N_H(\mathcal{S})$, and so $|N_H(\mathcal{S})| \geq |\mathcal{F}_w| - 1 \geq n_0 + n_1 + n_2 - 4$. Since $n_1 \geq 3$, then $|N_H(\mathcal{S})| \geq n_0 + n_2 - 1 \geq |\mathcal{S}|$.

Suppose now that $u = x$. Then $\mathcal{S} \subseteq \{u\} * (\mathcal{Q}_n \setminus \{u, v, w\})$, and so $|\mathcal{S}| \leq n_0 + n_1 + n_2 - 3$. If $\mathcal{S}$ has at least one segment in $\{u\} * C_2$, then $|\mathcal{S}| \geq 2$ implies that $\mathcal{F}_w \subseteq N_H(\mathcal{S})$, and so $|N_H(\mathcal{S})| \geq |\mathcal{F}_w| \geq n_0 + n_1 + n_2 - 3 \geq |\mathcal{S}|$. Then, we may assume that $\mathcal{S}$ has no segments in $\{u\} * C_2$, and hence $\{w\} * C_2 \subseteq N_H(\mathcal{S})$.

Let W_1 be the subset of all points in C_1 that are incident with a segment of $\mathcal{S}$. Then $\mathcal{S} \subseteq \{u\} * (W_1 \cup C_0 \setminus \{u, v\})$, and so $|\mathcal{S}| \leq |W_1| + n_0 - 2$. If $W_1 = \emptyset$, then $|\mathcal{S}| \leq n_0 - 2 \leq n_2 = |\{w\} * C_2| \leq |N_H(\mathcal{S})|$, as required. Then we can assume that $W_1 \neq \emptyset$. Let w' be the point in W_1 that is farthest from w. Since any segment in $\{w\} * (W_1 \setminus \{w'\})$ is disjoint from uw', then $\mathcal{F}'_w := \{w\} * (C_2 \cup (W_1 \setminus \{w'\}))$ is a subset of $N_H(\mathcal{S})$, and hence $|N_H(\mathcal{S})| \geq |\mathcal{F}'_w| \geq |W_1| - 1 + n_2 \geq |W_1| + n_0 - 2 \geq |\mathcal{S}|$.

(B3) Suppose that $u \in C_2$. Let $\mathcal{S}'$ be the set of all segments of $\mathcal{S}$ that are not incident with u. Suppose first that $\mathcal{S}' = \emptyset$. Then (B1) implies that $\mathcal{S}$ has at most one segment in $\{u\} * C_0$. If there exists $x \in C_0$ such that $ux \in \mathcal{S}$, then $|\mathcal{S}| \geq 2$ implies that $\mathcal{A} \subseteq N_H(\mathcal{S})$, and hence $|N_H(\mathcal{S})| \geq |\mathcal{A}| > |\mathcal{B}| \geq |\mathcal{S}|$. Thus, we may assume that $\mathcal{S} \subseteq \{u\} * (C_{1,2} \setminus \{u, w\})$. For $k = 1, 2$, let W_k be the subset of all points in $C_k \setminus \{u, w\}$ that are incident with a segment of $\mathcal{S}$. Then $\mathcal{S} = \{u\} * (W_1 \cup W_2)$, and so $|\mathcal{S}| = |W_1| + |W_2|$. If $W_k \neq \emptyset$, then we use w_k to denote the point in W_k that is closest to w if $k = 1$ (resp. u if $k = 2$). Since any segment in $\{w\} * (W_k \setminus \{w_k\})$ is disjoint from at least one of uw_1 or uw_2, then $\mathcal{F}'_w := \{w\} * ((C_0 \cup W_1 \cup W_2) \setminus \{v, w_1, w_2\})$ is a subset of $N_H(\mathcal{S})$, and hence $|N_H(\mathcal{S})| \geq |\mathcal{F}'_w| \geq n_0 + |W_1| + |W_2| - 3 \geq |W_1| + |W_2| \geq |\mathcal{S}|$.

Suppose now that $\mathcal{S}' \neq \emptyset$. Then each segment of $\mathcal{S}'$ crosses a, and has at least one endpoint in $C_0 \setminus \{v\}$. From (B2) it follows that there is $x \in C_0 \setminus \{v\}$ such that $\mathcal{S}' \subseteq \{x\} * C_2$. This fact and (B2) imply the existence of $uy \in \mathcal{S}$ with $y \in C_{1,2} \setminus \{u, w\}$. Note that if $|\mathcal{S}'| > 1$, then $\mathcal{A} \setminus \{xy\} \subseteq N_H(\mathcal{S}' \cup \{uy\}) \subseteq N_H(\mathcal{S})$. On the other hand, if $\mathcal{S}' = \{xz\}$ for some $z \in C_2 \setminus \{u\}$, then $\mathcal{A} \setminus \{wz\} \subseteq N_H(\{xz, uy\}) \subseteq N_H(\mathcal{S})$. In any case, we have $|N_H(\mathcal{S})| \geq |\mathcal{A}| - 1 \geq |\mathcal{B}| \geq |\mathcal{S}|$.

(B4) Suppose that $u \in C_0$. Let $\mathcal{S}_u$ be the set of all segments of $\mathcal{S}$ in $\{u\} * (C_1 \setminus \{w\})$, and let $\mathcal{S}' = \mathcal{S} \setminus \mathcal{S}_u$. Then, any element in $\mathcal{S}'$ is a segment of type 0. We may assume that $\mathcal{S}_u \neq \emptyset$, as otherwise we are in (B1) or (B2) due to $|\mathcal{S}| \geq 2$. Similarly, if $\mathcal{S}' = \emptyset$, then $|\mathcal{S}| = |\mathcal{S}_u| \leq n_1 - 1$. From $\mathcal{S}_u \neq \emptyset$ we have that $\{w\} * C_2 \subseteq N_H(\mathcal{S})$, and so $|N_H(\mathcal{S})| \geq n_2 \geq n_1 - 1 \geq |\mathcal{S}|$. Thus, we also assume that $\mathcal{S}' \neq \emptyset$.

From (B1) and $\mathcal{S}' \neq \emptyset$ we know that there exists a point $x \in C_0 \setminus \{v\}$ that is incident with any segment of $\mathcal{S}'$. Since $\{w\} * C_2 \subseteq N_H(\mathcal{S}_u)$ and $\{w\} * (C_0 \cup C_1 \setminus \{u, v, x, w\}) \subseteq N_H(\mathcal{S}')$, then $|N_H(\mathcal{S})| \geq n_0 + n_1 + n_2 - 4$. Seeking a contradiction, suppose that $|\mathcal{S}| \geq n_0 + n_1 + n_2 - 3$. Since $|\mathcal{S}_u| \leq n_1 - 1$, then $|\mathcal{S}'| \geq n_0 + n_2 - 2$. From $\mathcal{S}' \subseteq \{x\} * C_{0,2} \setminus \{x, v\}$ and $n_0 \geq 3$ it follows that $\mathcal{S}'$ has a segment $e = xy$ with $y \in C_0 \setminus \{x, v\}$. Since e together with any other $e' \in \mathcal{S}' \setminus \{e\}$ satisfy the conditions of (B1), then we must have that $\mathcal{S}' = \{e\}$, contradicting $|\mathcal{S}'| \geq n_0 + n_2 - 2$. $\square$

4.2.2. Suppose That a and b Have The Same Type

Then, a and b are located in $\mathcal{Q}_n$ according to some case of Figure 4. For $i = 0, 1, 2$ let E_i be the set of points in C_i that are endpoints of at least one of a and b, and let $C'_i := C_i \setminus E_i$. Then, $3 \leq |E_0| + |E_1| + |E_2| \leq 4$. From a simple inspection of the ten cases in Figure 4 we can see that $|E_0| \in \{1,2,3,4\}$, $|E_1| = 0$, and $|E_2| \in \{0,1,2\}$. Thus, for $i = 0,1,2$ the following holds:

$$\lfloor n/3 \rfloor + 1 \geq |C_i| \geq |C_i'| \geq \lfloor n/3 \rfloor - 4 \geq 1. \tag{3}$$

Let G_n' be the subgraph of G_n induced by $C_{0,2}$, i.e., $G_n' := D(C_{0,2})$. Similarly, let

$$\mathcal{D}_1 := (C_1 * C_1) \cup (C_1 * C_0') \cup (C_1 * C_2').$$

It is easy to check that $\mathcal{D}_1$ and G_n' satisfy the following properties: (i) the three sets forming $\mathcal{D}_1$ are pairwise disjoint, (ii) no segment of $\mathcal{D}_1$ belongs to G_n', (iii) $\mathcal{D}_1 \subseteq \mathcal{D}$, and (iv) a and b are vertices of G_n'.

By applying the main result of [14] to G_n' we have that the connectivity of G_n' is at least

$$\kappa_n' := \binom{\lfloor \frac{m-2}{2} \rfloor}{2} + \binom{\lceil \frac{m-2}{2} \rceil}{2}, \tag{4}$$

where $m = |C_{0,2}|$.

We are finally ready to prove Lemma 9.

Proof of Lemma 9. We analyze two cases separately, depending on whether a and b are located in $\mathcal{Q}_n$ according to some case of Figure 3 or Figure 4. □

Case 1. Suppose that a and b are located in $\mathcal{Q}_n$ according to some case of Figure 3, and let $H \subseteq G_n$ be as in Section 4.2.1. From Proposition 11 and Hall's theorem it follows that H has a matching M of size $m_1 := \min\{|\mathcal{A}|, |\mathcal{B}|\}$. Suppose that $M = \{a_k b_k \mid a_k \in \mathcal{A}, b_k \in \mathcal{B}, \text{ and } k = 1,2,\ldots,m_1\}$. Then

$$\mathbb{L} := \{a a_k b_k b \mid a_k b_k \in M\},$$

is a collection $a - b$ paths of G_n of length 3. Furthermore, the paths in $\mathbb{L}$ are pairwise internally disjoint, because M is a matching of H. On the other hand, note that

$$\mathbb{L}' := \{adb \mid d \in \mathcal{D}\},$$

is also a collection of pairwise internally disjoint $a - b$ paths of G_n of length 2. Since $\mathcal{D} \cap (\mathcal{A} \cup \mathcal{B}) = \emptyset$, then the paths in $\mathbb{L} \cup \mathbb{L}'$ are pairwise internally disjoint. The existence of such $m_1 + |\mathcal{D}|$ paths, and (2) imply

$$\eta(Q_n; a, b) = |\mathcal{D}| + m_1 = |\mathcal{D}| + \min\{|\mathcal{A}|, |\mathcal{B}|\} = \min\{\deg_{G_n}(a), \deg_{G_n}(b)\}. \tag{5}$$

Combining Proposition 2 and Theorem 3 we obtain that $\delta(G_n) = \frac{7}{18}n^2 + \Theta(n)$. Since $\min\{\deg_{G_n}(a), \deg_{G_n}(b)\} \geq \delta(G_n)$, then (2) and (5) imply $\eta(Q_n; a, b) = \frac{7}{18}n^2 + \Theta(n) = \frac{7}{18}n^2 - \Theta(n)$, as required.

Case 2. Suppose now that a and b are located in $\mathcal{Q}_n$ according to some case of Figure 4, and let $G_n', C_i', \mathcal{D}_1, \kappa_n', m$, and Properties (i)–(iv) as in Section 4.2.2.

Since $2\lfloor \frac{n}{3} \rfloor \leq m \leq 2\lfloor \frac{n}{3} \rfloor + 2$, a straightforward manipulation on (4) gives that $k_n' = \frac{2}{18}n^2 - \Theta(n)$. This last equality, (iv), and Menger's theorem imply that G_n' has collection $\mathbb{T}$ of at least $\frac{2}{18}n^2 - \Theta(n)$ pairwise internally disjoint $a - b$ paths. On the other hand, from (iii) it follows that

$$\mathbb{T}' := \{adb \mid d \in \mathcal{D}_1\},$$

is a collection of $|\mathcal{D}_1|$ pairwise internally disjoint $a - b$ paths of G_n of length 2. Moreover, by (ii) we have that $\mathbb{T} \cup \mathbb{T}'$ is also a collection of pairwise internally disjoint $a - b$ paths of G_n. From (3), (i), and the definition of $\mathbb{T}'$, it is not hard to see that

$$|\mathbb{T}'| = |\mathcal{D}_1| = \binom{|C_1|}{2} + |C_1||C_0'| + |C_1||C_2'| \geq \frac{5}{18}n^2 - \Theta(n).$$

Thus, $\eta(Q_n; a, b) \geq |\mathbb{T} \cup \mathbb{T}'| \geq \frac{7}{18}n^2 - \Theta(n)$. This last and (2) imply $\eta(Q_n; a, b) = \frac{7}{18}n^2 - \Theta(n)$, as required. $\square$

5. Concluding Remarks

Let P be a set of $n \geq 3$ points in general position in the plane. We have observed that the minimum degree $\delta(D(P))$ of the disjointness graph of segments defined by P can be expressed in terms of n and the rectilinear local crossing number of the rectilinear drawing $\mathcal{P}$ of K_n induced by P. From this observation and the exact value of $\overline{\text{lcr}}(K_n)$ provided by Theorem 1 in [19], it is easy to see that $\binom{n}{2} - \overline{\text{lcr}}(K_n)$ is an upper bound for $\delta(D(P))$, which is tight for each n. Since the connectivity $\kappa(H)$ of a graph H is upper bounded by its minimum degree $\delta(H)$, then $\binom{n}{2} - \overline{\text{lcr}}(K_n)$ is also a general upper bound for $\kappa(D(P))$.

On the other hand, the main goal in this work is to estimate the connectivity of $D(Q_n)$, where $\{Q_n\}_{n=3}^{\infty}$ is one of the families of point sets best understood from the point of view of rectilinear crossing number. In particular, it is known that Q_n is almost 3-symmetric and that $\overline{\text{lcr}}(K_n) = \overline{\text{lcr}}(\mathcal{Q}_n)$ for each $n \not\equiv 2 \pmod 3$. The basic idea behind our approach is to show that $\delta(D(Q_n)) - \kappa(D(Q_n))$ cannot be too large. In fact, we strongly believe that $\delta(D(Q_n)) = \kappa(D(Q_n))$ holds, and that this equality can be verified by analogous arguments to those used in Section 4.2.1. We finally remark that Theorem 1 is an asymptotic solution for an open question posed in [14].

Author Contributions: The authors contributed equally to this work. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: We thank two anonymous referees for careful reading and improvements to the presentation.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Araujo, G.; Dumitrescu, A.; Hurtado, F.; Noy, M.; Urrutia, J. On the chromatic number of some geometric type Kneser graphs. *Comput. Geom.* **2005**, *32*, 59–69. [CrossRef]
2. Kneser, M., Aufgabe 360 , Jahresbericht der Deutschen Mathematiker-Vereinigung. *Abteilung* **1955**, *58*, 27.
3. Lovász, L. Kneser's conjecture, chromatic number, and homotopy. *J. Combin. Theory Ser. A* **1978**, *25*, 319–324. [CrossRef]
4. Bárány, I. A short proof of Kneser's conjecture. *J. Combin. Theory Ser. A* **1978**, *25*, 325–326. [CrossRef]
5. Chen, Y.-C. Kneser graphs are Hamiltonian for $n \geq 3k$. *J. Combin. Theory Ser. B* **2000**, *80*, 69–79. [CrossRef]
6. Ekinci, G.B.; Gauci, J.B. The super-connectivity of Kneser graphs. *Discuss. Math. Graph Theory* **2019**, *39*, 5–11. [CrossRef]
7. Albertson, M.O.; Boutin, D.L. Using determining sets to distinguish Kneser graphs. *Electron. J. Combin.* **2007**, *14*, 20. [CrossRef]
8. Matoušěk, J. Using the Borsuk-Ulam theorem. In *Lectures on Topological Methods in Combinatorics and Geometry*; Anders, B., Günter, M.Z., Eds.; Springer: Berlin, Germany, 2003; ISBN 3-540-00362-2.
9. Valencia-Pabon, M.; Vera, J.-C. On the diameter of Kneser graphs. *Discret. Math.* **2005**, *305*, 383–385. [CrossRef]
10. Reinfeld, P. Chromatic Polynomials and the Spectrum of the Kneser Graph. Tech. rep., London School of Economics. LSE-CDAM-2000-02. 2000. Available online: http://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.27.4627 (accessed on 7 June 2021).
11. Jonsson, J. The Exact Chromatic Number of the Convex Segment Disjointness Graph. 2011. Available online: https://people.kth.se/~jakobj/doc/preprints/chrom.pdf (accessed on 7 June 2021).
12. Fabila-Monroy, R.; Wood, D.R. The chromatic number of the convex segment disjointness graph. In *Computational Geometry, Volume 7579 of Lecture Notes in Computer Science*; Springer: Cham, Switzerland, 2011; pp. 79–84.
13. Fabila-Monroy, R.; Hidalgo-Toscano, C.; Leaños, J.; Lomelí-Haro, M. The Chromatic Number of the Disjointness Graph of the Double Chain. *Discret. Math. Theor. Comput. Sci.* **2020**, *22*, 11.
14. Leaños, J.; Ndjatchi, C.; Ríos-Castro, L.R. On the connectivity of the disjointness graph of segments of point sets in general position in the plane. *arXiv* **2007**, arXiv:2007.15127v1.

15. Aichholzer, O.; Kynčl, J.; Scheucher, M.; Vogtenhuber, B. On 4–Crossing-Families in Point Sets and an Asymptotic Upper Bound. In Proceedings of the 37th European Workshop on Computational Geometry (EuroCG 2021), Saint Petersburg, Russia, 7–9 April 2021; pp. 1–8.
16. Pach, J.; Tardos, G.; Tóth, G. Disjointness graphs of segments. In *33th International Symposium on Computational Geometry (SoCG 2017) of Leibniz International Proceedings in Informatics (LIPIcs)*; Katz 59; Boris, A., Matthew, J., Eds.; Leibniz-Zentrum für Informatik: Wadern, Germany, 2017; Volume 77, pp. 1–15.
17. Pach, J.; Tomon, I. On the Chromatic Number of Disjointness Graphs of Curves. In *35th International Symposium on Computational Geometry (SoCG 2019)*; Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik: Wadern, Germany, 2019.
18. Pach, J. Geometric Graph Theory, Surveys in Combinatorics, BOOK_CHAP. 1999; pp. 167–200. Available online: https://www.cambridge.org/core/books/surveys-in-combinatorics-1999/geometric-graph-theory/E57E1273C7FAA8FD47B1B28F0E539C90 (accessed on 7 June 2021).
19. Ábrego, B.M.; Fernández-Merchant, S. The rectilinear local crossing number of K_n. *J. Combinat. Ser. A* **2017**, *151*, 131–145. [CrossRef]
20. Lara, D.; Rubio-Montiel, C.; Zaragoza, F. Grundy and pseudo–Grundy indices for complete graphs. In Proceedings of the Abstracts of the XVI Spanish Meeting on Computational Geometry, Barcelona, Spain, 1–3 July 2015.
21. Ábrego, B.M.; Cetina, M.; Fernández-Merchant, S.; Leaños, J.; Salazar, G. 3-symmetric and 3–decomposable geometric drawings of K_n. *Discret. Appl. Math.* **2010**, *158*, 1240–1258. [CrossRef]
22. Ábrego, B.M.; Cetina, M.; Fernández-Merchant, S.; Leaños, J.; Salazar, G. On $\leq k$-Edges, Crossings, and Halving Lines of Geometric Drawings of K_n. *Discret. Comput. Geom.* **2012**, *48*, 192–215. [CrossRef]