

Parts and Composites of Quantum Systems

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Abstract: We consider three types of entities for quantum measurements. In order of generality, these types are observables, instruments and measurement models. If α and β are entities, we define what it means for α to be a part of β . This relationship is essentially equivalent to α being a function of β and in this case β can be employed to measure α . We then use the concept to define the coexistence of entities and study its properties. A crucial role is played by a map $\hat{\alpha}$ which takes an entity of a certain type to one of a lower type. For example, if $\mathcal{I}$ is an instrument, then $\hat{\mathcal{I}}$ is the unique observable measured by $\mathcal{I}$. Composite systems are discussed next. These are constructed by taking the tensor product of the Hilbert spaces of the systems being combined. Composites of the three types of measurements and their parts are studied. Reductions in types to their local components are discussed. We also consider sequential products of measurements. Specific examples of Lüders, Kraus and trivial instruments are used to illustrate various concepts. We only consider finite-dimensional systems in this article. Finally, we mention the role of symmetry representations for groups using quantum channels.

Keywords: quantum instruments; quantum measurements; quantum observables; quantum parts



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1. Introduction

Two important operations on quantum systems are the formations of parts and composites. In a rough sense, these operations are opposites to each other. The parts of a measurement α are smaller components of α in the sense that they can be simultaneously measured by α . A composite system is a combination of two or more other systems. This combination is formed using the tensor product $H = H_1 \otimes H_2$, where H_1 and H_2 are the Hilbert spaces describing two subsystems. The composite system contains more information than the individual systems because H describes how H_1 and H_2 interact. We can reduce measurements on H to simpler ones on H_1 and H_2 but information is lost in the process.

Section 2 presents the basic definitions that are needed in the sequel. Three types of quantum measurements are considered. In order of generality, these types are observables, instruments and measurement models. At the basic level is an observable A which is a measurement whose outcome probabilities $\text{tr}(\rho A_x)$ are determined by the state ρ of the system. At the next level is an instrument $\mathcal{I}$. We think of $\mathcal{I}$ as an apparatus that can be employed to measure an observable $\hat{\mathcal{I}}$. Although $\hat{\mathcal{I}}$ is unique, there are many instruments that can be used to measure an observable. Moreover, $\mathcal{I}$ gives more information than $\hat{\mathcal{I}}$ because, depending on the outcome x , $\mathcal{I}$ updates the input state ρ to the output state $\mathcal{I}_x(\rho)/\text{tr}(\rho \hat{\mathcal{I}}_x)$. At the highest level is a measurement model $\mathcal{M}$ that measures a unique instrument $\hat{\mathcal{M}}$. Again, there are many measurement models that measure an instrument and $\mathcal{M}$ contains more detailed information. For conciseness, we call these types of measurements *entities*. We should mention that all the quantum systems in this article are assumed to be finite-dimensional.

Section 3 considers system parts. If α and β are entities, we define what it means for α to be a *part* of β and when this is the case, we write $\alpha \rightarrow \beta$. If $\alpha \rightarrow \beta$ and $\beta \rightarrow \alpha$, we say that α and β are *equivalent*. We show that $\alpha \rightarrow \beta$ implies $\hat{\alpha} \rightarrow \hat{\beta}$ and that $\rightarrow$ is a partial order to within equivalence. The relation $\alpha \rightarrow \beta$ is the same as α being a function of β or $\hat{\beta}$ and in this case, β can be employed to measure α . We then use this concept to define

the coexistence of entities and study its properties. We show that joint measurability is equivalent to coexistence. We then introduce the sequential products of observables and use this concept to illustrate parts of entities.

Section 4 discusses composite systems. These are constructed by taking the tensor product $H = H_1 \otimes H_2$, where H_1, H_2 are the Hilbert spaces of the systems being combined. Composites of the three types of measurements and parts of these composites are studied. Reductions of types into their local components are discussed. Specific examples of Lüders, Kraus and trivial instruments are employed to illustrate various concepts.

2. Basic Definitions

This section discusses the basic concepts and definitions that are needed in the sequel. Since these ideas are well developed in the literature [1–5], we shall proceed quickly and leave details and motivation to the reader's discretion. In this article, we shall only consider finite-dimensional complex Hilbert spaces H . Let $\mathcal{L}(H)$ be the set of linear operators on H . For $S, T \in \mathcal{L}(H)$, we write $S \leq T$ if $\langle \phi, S\phi \rangle \leq \langle \phi, T\phi \rangle$ for all $\phi \in H$. We define the set of effects by

$$\mathcal{E}(H) = \{a \in \mathcal{L}(H) : 0 \leq a \leq 1\}$$

where $0, 1$ are the zero and identity operators, respectively. Effects correspond to yes–no measurements and when the result of the measurement a is yes, we say that a occurs. The complement of $a \in \mathcal{E}(H)$ is $a' = 1 - a$ and a' occurs if and only if a does not occur. A one-dimensional projection $P_\phi = |\phi\rangle\langle\phi|$, where $\|\phi\| = 1$ is an effect called an *atom*. We call $\rho \in \mathcal{E}(H)$ a *partial state* if $\text{tr}(\rho) \leq 1$ and ρ is a *state* if $\text{tr}(\rho) = 1$. We denote the set of partial states by $\mathcal{S}_p(H)$ and the set of states by $\mathcal{S}(H)$. If $\rho \in \mathcal{S}(H)$, $a \in \mathcal{E}(H)$, we call $\mathcal{P}_\rho(a) = \text{tr}(\rho a)$ the *probability that a occurs in the state ρ* [1–3]. For $a, b \in \mathcal{E}(H)$, their *sequential product* is the effect $a \circ b = a^{1/2} b a^{1/2}$, where $a^{1/2}$ is the unique square root of a [6–8]. We interpret $a \circ b$ as the effect that results from first measuring a and then measuring b . We also call $a \circ b$ the effect b *conditioned on* the effect a and write $(b | a) = a \circ b$.

Let Ω_A be a finite set. A (finite) *observable with outcome-space Ω_A* is a subset:

$$A = \{A_x : x \in \Omega_A\} \subseteq \mathcal{E}(H)$$

satisfying $\sum_{x \in \Omega_A} A_x = 1$. We denote the set of observables on H by $\mathcal{O}(H)$. If $B = \{B_y : y \in \Omega_B\}$ is another observable, we define the *sequential product* $A \circ B \in \mathcal{O}(H)$ [8–10] to be the observable with outcome-space $\Omega_A \times \Omega_B$ given by

$$A \circ B = \{A_x \circ B_y : (x, y) \in \Omega_A \times \Omega_B\}$$

We also define the observable B *conditioned by A* as

$$(B | A) = \{(B | A)_y : y \in \Omega_B\} \subseteq \mathcal{E}(H)$$

where $(B | A)_y = \sum_{x \in \Omega_A} (A_x \circ B_y)$. If $A \in \mathcal{O}(H)$, we define the *effect-valued measure* (or POVM) $X \rightarrow A_X$ from 2^{Ω_A} to $\mathcal{E}(H)$ by $A_X = \sum_{x \in X} A_x$ and we also call $X \mapsto A_X$ an *observable* [2,3,8]. Moreover, we have the observables:

$$(A \circ B)_\Delta = \sum_{(x,y) \in \Delta} (A_x \circ B_y)$$

and:

$$(B | A)_Y = \sum_{x \in \Omega_A} (A_x \circ B_Y)$$

If $\rho \in \mathcal{S}(H)$ and $A \in \mathcal{O}(H)$, the probability that A has an outcome in $X \subseteq \Omega_A$ when the system is in state ρ is $\mathcal{P}_\rho(A_X) = \text{tr}(\rho A_X)$. Notice that $X \mapsto \mathcal{P}_\rho(A_X)$ is a probability measure on Ω_A . We call:

$$\mathcal{P}_\rho(A_X \text{ then } B_Y) = \text{tr}[\rho(A \circ B)_{X \times Y}]$$

the joint probability of A_X then B_Y [8–10].

An operation is a completely positive map $\mathcal{A}: \mathcal{S}_p(H) \rightarrow \mathcal{S}_p(H)$ [1–3]. Any operation has a Kraus decomposition:

$$\mathcal{A}(\rho) = \sum_{i=1}^n S_i \rho S_i^*$$

where $S_i \in \mathcal{L}(H)$ with $\sum_{i=1}^n S_i^* S_i \leq 1$. An operation $\mathcal{A}$ is a channel if $\mathcal{A}(\rho) \in \mathcal{S}(H)$ for all

$\rho \in \mathcal{S}(H)$. In this case, $\sum_{i=1}^n S_i^* S_i = 1$ and we denote the set of channels on H by $\mathcal{C}(H)$.

Notice that if $a \in \mathcal{E}(H)$, then $\rho \mapsto (\rho | a) = a \circ \rho$ is an operation and if $A \in \mathcal{O}(H)$, then $\rho \mapsto (\rho | A) = \sum_{x \in \Omega_A} (A_x \circ \rho)$ is a channel. The simplest example of a channel has the

form $\mathcal{A}_U(\rho) = U \rho U^*$ where U is a unitary operator on H . Letting $\mathcal{U}(H)$ be the group of unitary operators on H , we have that $\mathcal{A}_I = I$ and $\mathcal{A}_{U_1 U_2} = \mathcal{A}_{U_1} \mathcal{A}_{U_2}$ for all $U_1, U_2 \in \mathcal{U}(H)$. In particular, if we have a symmetry group $G \subseteq \mathcal{U}(H)$ for the system, then $U \mapsto \mathcal{A}_U, U \in G$, gives a symmetry representation of G . For a finite set $\Omega_{\mathcal{I}}$, a (finite) instrument with outcome-space $\Omega_{\mathcal{I}}$ is a set of operations $\mathcal{I} = \{\mathcal{I}_x: x \in \Omega_{\mathcal{I}}\}$ satisfying $\mathcal{C}_{\mathcal{I}} = \sum_{x \in \Omega_{\mathcal{I}}} \mathcal{I}_x \in \mathcal{C}(H)$ [1–3,11].

Defining $\mathcal{I}_X = \sum_{x \in X} \mathcal{I}_x$ for $X \subseteq \Omega_{\mathcal{I}}$, we see that $X \mapsto \mathcal{I}_X$ is an operation-valued measure on H that we also call an instrument. We denote the set of instruments on H by $\text{In}(H)$. We say that $\mathcal{I} \in \text{In}(H)$ measures $A \in \mathcal{O}(H)$ if $\Omega_A = \Omega_{\mathcal{I}}$ and:

$$\mathcal{P}_\rho(A_X) = \text{tr}[\mathcal{I}_X(\rho)] \quad (1)$$

for every $\rho \in \mathcal{S}(H)$, $X \subseteq \Omega_A$. There is a unique $A \in \mathcal{O}(H)$ that $\mathcal{I}$ measures and we write $A = \widehat{\mathcal{I}}$ [1,2,11]. For $\mathcal{I}, \mathcal{J} \in \text{In}(H)$, we define the product instrument with outcome space $\Omega_{\mathcal{I}} \times \Omega_{\mathcal{J}}$ by

$$(\mathcal{I} \circ \mathcal{J})_{(x,y)}(\rho) = \mathcal{J}_y[\mathcal{I}_x(\rho)]$$

for every $\rho \in \mathcal{S}(H)$. We also define the conditioned instrument with outcome-space $\Omega_{\mathcal{J}}$ by

$$(\mathcal{J} | \mathcal{I})_y = \sum_{x \in \Omega_{\mathcal{I}}} (\mathcal{I} \circ \mathcal{J})_{(x,y)} = \mathcal{J}_y[\mathcal{C}_{\mathcal{I}}(\rho)]$$

We conclude that:

$$(\mathcal{I} \circ \mathcal{J})_{\Delta}(\rho) = \sum_{(x,y) \in \Delta} \mathcal{J}_y(\mathcal{I}_x(\rho))$$

for all $\Delta \subseteq \Omega_{\mathcal{I}} \times \Omega_{\mathcal{J}}$ and:

$$(\mathcal{J} | \mathcal{I})_Y(\rho) = \sum_{y \in Y} \mathcal{J}_y(\mathcal{C}_{\mathcal{I}}(\rho))$$

for all $Y \subseteq \Omega_{\mathcal{J}}$ [8–10].

A finite measurement model (MM) is a 5-tuple $\mathcal{M} = (H, K, \eta, \nu, F)$ where H, K are finite-dimensional Hilbert spaces called the base and probe systems, respectively, $\eta \in \mathcal{S}(K)$ is an initial probe state, $\nu \in \mathcal{C}(H \otimes K)$ is a channel describing the measurement interaction between the base and probe systems, and $F \in \mathcal{O}(K)$ is the probe (or meter) observable [1,2,12]. We say that $\mathcal{M}$ measures the model instrument $\widehat{\mathcal{M}} \in \text{In}(H)$ where $\widehat{\mathcal{M}}$ is the unique instrument satisfying:

$$\widehat{\mathcal{M}}_X(\rho) = \text{tr}_K[\nu(\rho \otimes \eta)(I \otimes F_X)] \quad (2)$$

for all $\rho \in \mathcal{S}(H)$, $X \subseteq \Omega_F$. In (2), tr_K is the partial trace over K [2,3]. We also say that $\mathcal{M}$ measures the model observable $\mathcal{M}^{\wedge\wedge}$.

We thus have three levels of abstraction. At the basic level is an observable A which is a measurement whose outcome probabilities $\text{tr}(\rho A_x)$ are determined by the state ρ of the system. At the next level is an instrument $\mathcal{I}$. We think of $\mathcal{I}$ as an apparatus that can be employed to measure an observable $\widehat{\mathcal{I}}$. Although $\widehat{\mathcal{I}}$ is unique, there are many instruments that can be used to measure an observable. Moreover, $\mathcal{I}$ gives more information than $\widehat{\mathcal{I}}$ because, depending on the outcome x (or event X), $\mathcal{I}$ updates the input state ρ to the output partial state $\mathcal{I}_x(\rho)$ (or $\mathcal{I}_X(\rho)$). At the highest level is a measurement model $\mathcal{M}$ that measures a unique model instrument $\widehat{\mathcal{M}}$ and a unique model observable $\mathcal{M}^{\wedge\wedge}$. Again, there are many MMs that measure any instrument or observable and $\mathcal{M}$ contains more detailed information on how the measurement is performed.

3. System Parts

We begin by discussing parts of systems at the three levels considered in Section 2. We then show how parts can be used to define coexistence at these levels and even between levels. We also show that coexistence is equivalent to simultaneous measurability.

An element at one of the three levels discussed in Section 2 is called an *entity*. The three levels are said to be the *types* 1, 2 and 3, respectively. The concept of an entity being part of another entity was originally introduced in [12,13]. If $A, B \in \mathcal{O}(H)$, we say that A is *part of* B (and write $A \rightarrow B$) if there exists a surjection $f: \Omega_B \rightarrow \Omega_A$ such that $A_x = B_{f^{-1}(x)}$ for all $x \in \Omega_A$. We then write $A = f(B)$. It follows that $A_X = B_{f^{-1}(X)}$ for all $X \in \Omega_A$ and that:

$$A_X = \sum \{B_y: f(y) \in X\} \quad (3)$$

If $\mathcal{I}, \mathcal{J} \in \text{In}(H)$, we say that $\mathcal{I}$ is *part of* $\mathcal{J}$ (and write $\mathcal{I} \rightarrow \mathcal{J}$) if there exists a surjection $f: \Omega_{\mathcal{J}} \rightarrow \Omega_{\mathcal{I}}$ such that $\mathcal{I}_x = \mathcal{J}_{f^{-1}(x)}$ for all $x \in \Omega_{\mathcal{I}}$. We then write $\mathcal{I} = f(\mathcal{J})$ and an equation analogous to (3) holds. For MMs $\mathcal{M}_1 = (H, K, \eta, \nu, F_1)$ and $\mathcal{M}_2 = (H, K, \eta, \nu, F_2)$, we say that $\mathcal{M}_1$ is *part of* $\mathcal{M}_2$ (and write $\mathcal{M}_1 \rightarrow \mathcal{M}_2$) if $F_1 \rightarrow F_2$. It follows that $F_1 = f(F_2)$ and we write $\mathcal{M}_1 = f(\mathcal{M}_2)$. We can also define “part of” for entities of different types. An observable $A \in \mathcal{O}(H)$ is *part of* $\mathcal{I} \in \text{In}(H)$ (written $A \rightarrow \mathcal{I}$) if $A \rightarrow \widehat{\mathcal{I}}$ and A is *part of* $\mathcal{M}$ (written $A \rightarrow \mathcal{M}$) if $A \rightarrow \widehat{\mathcal{M}}$ which is equivalent to $A \rightarrow \mathcal{M}^{\wedge\wedge}$. Finally, we say that $\mathcal{I}$ is *part of* $\mathcal{M}$ (written $\mathcal{I} \rightarrow \mathcal{M}$) if $\mathcal{I} \rightarrow \widehat{\mathcal{M}}$. Two entities α and β are *equivalent* (written $\alpha \cong \beta$) if $\alpha \rightarrow \beta$ and $\beta \rightarrow \alpha$. It is easy to check that $\cong$ is an equivalence relation and that $\alpha \cong \beta$ if and only if $\alpha = f(\beta)$ for f a bijection. Our first result summarizes properties possessed by “part of”. Some of these properties have been verified in [13], however, we give the full proof for completeness.

Theorem 1. (a) If α, β are of types 2 or 3 and $\alpha \rightarrow \beta$, then $\widehat{\alpha} \rightarrow \widehat{\beta}$; (b) $f(\widehat{\mathcal{I}}) = f(\mathcal{I})^{\wedge}$ and $f(\widehat{\mathcal{M}}) = f(\mathcal{M})^{\wedge}$; (c) If α, β, γ are of the same type and $\alpha = g(\beta)$, $\beta = f(\gamma)$, then $\alpha = (g \circ f)(\gamma)$; (d) The relation $\rightarrow$ is a partial order to within equivalence; (e) If α and β are of different types and $\alpha \rightarrow \beta$, then $\alpha = \widehat{\beta}_1$ where $\beta_1 \rightarrow \beta$.

Proof. (a) Let $\mathcal{I}, \mathcal{J} \in \text{In}(H)$ with $\mathcal{I} \rightarrow \mathcal{J}$. Then, there exists a surjection $f: \Omega_{\mathcal{J}} \rightarrow \Omega_{\mathcal{I}}$ such that $\mathcal{I} = f(\mathcal{J})$. We now show that $\widehat{\mathcal{I}} = f(\widehat{\mathcal{J}})$. Indeed, for any $\rho \in \mathcal{S}(H)$, $x \in \Omega_{\mathcal{I}}$ we have that:

$$\text{tr}(\rho \widehat{\mathcal{I}}_x) = \text{tr}[\mathcal{I}_x(\rho)] = \text{tr}[\mathcal{J}_{f^{-1}(x)}(\rho)] = \text{tr}[\rho \widehat{\mathcal{J}}_{f^{-1}(x)}] = \text{tr}[\rho f(\widehat{\mathcal{J}})_x]$$

Hence, $\widehat{\mathcal{I}} = f(\widehat{\mathcal{J}})$ so $\widehat{\mathcal{I}} \rightarrow \widehat{\mathcal{J}}$. Let $\mathcal{M}_1 = (H, K, \eta, \nu, F_1)$, $\mathcal{M}_2 = (H, K, \eta, \nu, F_2)$ be MMs where $F_1 = f(F_2)$. Then, for any $\rho \in \mathcal{S}(H)$, $x \in \Omega_{F_1}$ we have that:

$$\widehat{\mathcal{M}}_{1,x}(\rho) = \text{tr}_K[\nu(\rho \otimes \eta)(I \otimes F_{1,x})] = \text{tr}_K[\nu(\rho \otimes \eta)(I \otimes F_{2,f^{-1}(x)})]$$

$$= \widehat{\mathcal{M}}_{2, f^{-1}(x)}(\rho) = f(\widehat{\mathcal{M}}_2)$$

Hence, $\widehat{\mathcal{M}}_1 = f(\widehat{\mathcal{M}}_2)$ so $\widehat{\mathcal{M}}_1 \rightarrow \widehat{\mathcal{M}}_2$. If $\mathcal{I} \rightarrow \mathcal{M}$, then $\mathcal{I} \rightarrow \widehat{\mathcal{M}}$. As before, $\widehat{\mathcal{I}} \rightarrow \mathcal{M}^{\wedge\wedge}$ so $\widehat{\mathcal{I}} \rightarrow \widehat{\mathcal{M}}$.

(b) This was proved in (a). (c) We prove the result for observables A, B, C and the result for instruments and MMs is similar. We have that $A_x = B_{g^{-1}(x)}$ and $B_y = C_{f^{-1}(y)}$. Since $g: \Omega_B \rightarrow \Omega_A$ and $f: \Omega_C \rightarrow \Omega_B$, we have that $g \circ f: \Omega_C \rightarrow \Omega_A$. Hence:

$$A_x = B_{g^{-1}(x)} = C_{f^{-1}(g^{-1}(x))} = C_{(g \circ f)^{-1}(x)}$$

Hence, $A = (g \circ f)(C)$. (d) We only need to prove that if $\alpha \rightarrow \beta$ and $\beta \rightarrow \gamma$, then $\alpha \rightarrow \gamma$. If α, β, γ are of the same type, the $\alpha \rightarrow \gamma$ follows from (c). Suppose $A, B \in \mathcal{O}(H)$, $\mathcal{I} \in \text{In}(H)$ and $A \rightarrow B, B \rightarrow \mathcal{I}$. Then, $A \rightarrow B \rightarrow \widehat{\mathcal{I}}$ and these are the same type so $A \rightarrow \widehat{\mathcal{I}}$ and hence, $A \rightarrow \mathcal{I}$. Suppose $A \in \mathcal{O}(H)$, $\mathcal{I}, \mathcal{J} \in \text{In}(H)$ and $A \rightarrow \mathcal{I}, \mathcal{I} \rightarrow \mathcal{J}$. Then, $A \rightarrow \widehat{\mathcal{I}}$ and $\mathcal{I} \rightarrow \mathcal{J}$. By (a), we have $\widehat{\mathcal{I}} \rightarrow \widehat{\mathcal{J}}$. Since $A, \widehat{\mathcal{I}}, \widehat{\mathcal{J}}$ have the same type, $A \rightarrow \widehat{\mathcal{J}}$ and hence, $A \rightarrow \mathcal{J}$. Suppose that $A \rightarrow \mathcal{I}$ and $\mathcal{I} \rightarrow \mathcal{M}$. Then, $A \rightarrow \widehat{\mathcal{I}}$ and $\mathcal{I} \rightarrow \widehat{\mathcal{M}}$. By (a) $\widehat{\mathcal{I}} \rightarrow \mathcal{M}^{\wedge\wedge}$ so $A \rightarrow \widehat{\mathcal{I}}$ and $\widehat{\mathcal{I}} \rightarrow \mathcal{M}^{\wedge\wedge}$. Since these are the same type, we have that $A \rightarrow \mathcal{M}^{\wedge\wedge}$ so $A \rightarrow \mathcal{M}$. Similar reasoning holds for the cases $\mathcal{I} \rightarrow \mathcal{J} \rightarrow \mathcal{M}$ and $\mathcal{I} \rightarrow \mathcal{M}_1 \rightarrow \mathcal{M}_2$.

(e) If $A \in \mathcal{O}(H)$, $\mathcal{I} \in \text{In}(H)$ and $A \rightarrow \mathcal{I}$, then $A \rightarrow \widehat{\mathcal{I}}$ so $A = f(\widehat{\mathcal{I}})$ for some surjection $f: \Omega_{\widehat{\mathcal{I}}} \rightarrow \Omega_A$. By (b), we have that $f(\widehat{\mathcal{I}}) = f(\mathcal{I})^{\wedge}$ so letting $\mathcal{I}_1 = f(\mathcal{I})$ we have that $A = f(\mathcal{I})^{\wedge} = \widehat{\mathcal{I}}_1$. Hence, $\mathcal{I}_1 \rightarrow \mathcal{I}$. If $A \rightarrow \mathcal{M}$, then $A \rightarrow \mathcal{M}^{\wedge\wedge}$. By (b), $A = f(\mathcal{M}^{\wedge\wedge}) = [f(\widehat{\mathcal{M}})]^{\wedge}$. Letting $\mathcal{I} = f(\widehat{\mathcal{M}})$, we have that $A = \widehat{\mathcal{I}}, \mathcal{I} \rightarrow \widehat{\mathcal{M}} \rightarrow \mathcal{M}$. If $\mathcal{I} \rightarrow \mathcal{M}$, then $\mathcal{I} \rightarrow \widehat{\mathcal{M}}$. By (b) $\mathcal{I} = f(\mathcal{M}^{\wedge}) = f(\mathcal{M})^{\wedge}$. Letting $\mathcal{I}_1 = f(\mathcal{M})$, we have that $\mathcal{I} = \widehat{\mathcal{I}}_1$ and $\mathcal{I}_1 \rightarrow \mathcal{M}$. $\square$

For an entity α , we denote its set of parts by $\tilde{\alpha} = \{\beta: \beta \rightarrow \alpha\}$. We say that a set $\mathcal{A}$ of entities *coexist* if $\mathcal{A} \subseteq \tilde{\alpha}$ for some entity α . A coexistent set $\mathcal{A} \subseteq \tilde{\alpha}$ is thought of as being simultaneously measured by α . A related concept is that of joint measurability [14]. We say that observables $A^i \in \mathcal{O}(H)$ with outcome sets $\Omega_i, i = 1, 2, \dots, n$ are *jointly measurable* with *joint observable* $B \in \mathcal{O}(H)$ if $\Omega_B = \Omega_1 \times \dots \times \Omega_n$ and for all $x_i \in \Omega_i$ we have:

$$A_{x_i}^i = \sum \left\{ B_{(x_1, \dots, x_i, \dots, x_n)} : x_j \in \Omega_j, j \neq i \right\} \quad (4)$$

We interpret A^i as being the *i*th *marginal* of B as in classical probability theory [8,9,12]. Similar definitions can be made for the joint measurability of instruments and MMs.

Theorem 2. A set of observables $A^i \in \mathcal{O}(H), i = 1, 2, \dots, n$ is jointly measurable if and only if the A^i coexist.

Proof. If $\{A^i: i = 1, 2, \dots, n\}$ are jointly measurable, there exists a joint observable $B \in \mathcal{O}(H)$ satisfying (4). Defining $f_i: \Omega_B \rightarrow \Omega_{A^i}$ by

$$f_i(x_1, \dots, x_i, \dots, x_n) = x_i$$

for $i = 1, 2, \dots, n$, then by (4), we have that $A_{x_i}^i = B_{f_i^{-1}(x_i)}$ for all $x_i \in \Omega_i$. Hence, $A^i = f_i(B)$, $i = 1, 2, \dots, n$, so $\{A^i\}$ coexist. Conversely, suppose that $\{A^i: i = 1, 2, \dots, n\}$ coexist so there exists an observable $C \in \mathcal{O}(H)$ such that $A^i \in \tilde{C}, i = 1, 2, \dots, n$. We then have surjections $f_i: \Omega_C \rightarrow \Omega_{A^i}$ such that $A^i = f_i(C), i = 1, 2, \dots, n$. Define $\Omega_B = \Omega_1 \times \dots \times \Omega_n$, a surjection $h: \Omega_C \rightarrow \Omega_B$ by $h(y) = (f_1(y), \dots, f_n(y))$ and let $B = h(C)$. For $i = 1, 2, \dots, n$, we obtain:

$$\begin{aligned} A_{x_i}^i &= C_{f_i^{-1}(x_i)} = \sum \{C_y: f_i(y) = x_i\} \\ &= \sum \{C_y: (f_1(y), \dots, f_n(y)) = \{x_1, \dots, x_i, \dots, x_n\}, x_j \in \Omega_j, j \neq i\} \end{aligned}$$

$$\begin{aligned}
&= \sum \{C_y: h(y) = (x_1, \dots, x_i, \dots, x_n): x_j \in \Omega_j, j \neq i\} \\
&= \sum \{C_{h^{-1}(x_1, \dots, x_i, \dots, x_n)}: x_j \in \Omega_j, j \neq i\} \\
&= \sum \{h(C)_{(x_1, \dots, x_i, \dots, x_n)}: x_j \in \Omega_j, j \neq i\} \\
&= \sum \{B_{(x_1, \dots, x_i, \dots, x_n)}: x_j \in \Omega_j, j \neq i\}
\end{aligned}$$

Thus, (4) holds so $\{A^i\}$ are jointly measurable. $\square$

Theorem 2 also holds for instruments and MMs. An important property of coexistent entities is that they have joint probability distributions Φ_ρ for all $\rho \in \mathcal{S}(H)$. For example, if $A, B \in \mathcal{O}(H)$ coexist, then $A = f(C)$, $B = g(C)$ for some $C \in \mathcal{O}(H)$. Then, for any $X \subseteq \Omega_A$, $Y \subseteq \Omega_B$, the joint probability becomes:

$$\Phi_\rho(A_X, B_Y) = \text{tr} \left[\rho \sum \{C_z: z \in f^{-1}(X) \cap g^{-1}(Y)\} \right] = \text{tr} \left[\rho C_{f^{-1}(X) \cap g^{-1}(Y)} \right]$$

As another example, if $A, B \rightarrow \mathcal{I}$, then $A, B \rightarrow \widehat{\mathcal{I}}$ so $A = f(\widehat{\mathcal{I}})$, $B = g(\widehat{\mathcal{I}})$ for surjections f, g . We then obtain:

$$\Phi_\rho(A_X, B_Y) = \text{tr} \left[\rho \widehat{\mathcal{I}}_{f^{-1}(X) \cap g^{-1}(Y)} \right] = \text{tr} \left[\mathcal{I}_{f^{-1}(X) \cap g^{-1}(Y)}(\rho) \right]$$

We can continue this for many coexistent entities. Moreover, the entities do not need to be of the same type. For instance, suppose $A, \mathcal{I} \rightarrow \mathcal{J}$ where $A = f(\widehat{\mathcal{J}})$ and $\mathcal{I} = g(\mathcal{J})$. Then, we have that:

$$\begin{aligned}
\Phi_\rho(A_X, \mathcal{I}_Y) &= \Phi_\rho \left[f(\widehat{\mathcal{J}})_X, g(\mathcal{J})_Y \right] = \Phi_\rho \left[\widehat{\mathcal{J}}_{f^{-1}(X)}, \mathcal{J}_{g^{-1}(Y)} \right] \\
&= \text{tr} \left[\mathcal{J}_{f^{-1}(X) \cap g^{-1}(Y)}(\rho) \right]
\end{aligned}$$

For $A \in \mathcal{O}(H)$, we define the probability distribution $\Phi_\rho^A(X) = \text{tr}(\rho A_X)$ for all $X \subseteq \Omega_A$, $\rho \in \mathcal{S}(H)$. In a similar way, if $\mathcal{I} \in \text{In}(H)$, we define $\Phi_\rho^\mathcal{I}(X) = \text{tr}[\mathcal{I}_X(\rho)]$ and if $\mathcal{M}$ is a MM, then $\Phi_\rho^\mathcal{M}(X) = \Phi_\rho^{\widehat{\mathcal{M}}}(X)$.

Lemma 1. If α is an entity and $f: \Omega_\alpha \rightarrow \Omega$ is a surjection, then $\Phi^{f(\alpha)} = \Phi^\alpha \circ f^{-1}$.

Proof. We give the proof for $A \in \mathcal{O}(H)$ and the proof for other entities is similar. For $x \in \Omega_A$, $\rho \in \mathcal{S}(H)$ we obtain:

$$\begin{aligned}
\Phi_\rho^{f(A)}(x) &= \text{tr}[\rho f(A)_x] = \text{tr}[\rho A_{f^{-1}(x)}] = \text{tr}[\rho \sum \{A_y: f(y) = x\}] \\
&= \sum \{\text{tr}(\rho A_y): f(y) = x\} = \sum \{\Phi_\rho^A(y): f(y) = x\} \\
&= \Phi_\rho^A[f^{-1}(x)] = \Phi_\rho^A \circ f^{-1}(x)
\end{aligned}$$

The result now follows. $\square$

We now consider sequential products of observables.

Theorem 3. If $A, B \in \mathcal{O}(H)$ and $h: \Omega_B \rightarrow \Omega$ is a surjection, then $A, (B \mid A)$ and $A \circ h(B)$ are parts of $A \circ B$.

Proof. Defining $f: \Omega_A \times \Omega_B \rightarrow \Omega_A$ by $f(x, y) = x$ we have that:

$$f(A \circ B)_x = (A \circ B)_{f^{-1}(x)} = \sum \left\{ (A \circ B)_{(y,z)}: f(y, z) = x \right\} = \sum_{z \in \Omega_B} (A \circ B)_{(x,z)}$$

$$= \sum_{z \in \Omega_B} A_x \circ B_z = A_x \circ 1 = A_x$$

Thus, $A = f(A \circ B)$ so $A \rightarrow A \circ B$. Defining $g: \Omega_A \times \Omega_B \rightarrow \Omega_B$ by $g(x, y) = y$ we obtain:

$$\begin{aligned} g(A \circ B)_y &= (A \circ B)_{g^{-1}(y)} = \sum \left\{ (A \circ B)_{(x,z)} : g(x, z) = y \right\} = \sum_{x \in \Omega_A} (A \circ B)_{(x,y)} \\ &= \sum_{x \in \Omega_A} A_x \circ B_y = (B \mid A)_y \end{aligned}$$

Hence, $(B \mid A) = g(A \circ B)$ so $(B \mid A) \rightarrow A \circ B$. Defining $u: \Omega_A \times \Omega_B \rightarrow \Omega_A \times \Omega$ by $u(x, y) = (x, h(y))$ we have that:

$$\begin{aligned} [u(A \circ B)]_{(x,y)} &= (A \circ B)_{u^{-1}(x,y)} = (A \circ B)_{(x,h^{-1}(y))} = A_x \circ B_{h^{-1}(y)} \\ &= A_x \circ h(B)_y = [A \circ h(B)]_{(x,y)} \end{aligned}$$

It follows that $A \circ h(B) = u(A \circ B)$. Hence, $A \circ h(B) \rightarrow A \circ B$. $\square$

Some results analogous to Theorem 3 hold for other entities.

Example 1. We consider the simplest nontrivial example of a sequential product $A \circ B$ of observables. Let $A = \{a_0, a_1\}$, $B = \{b_0, b_1\}$ be binary (diatomic) observables. Then, $\Omega_{A \circ B} = \{0, 1\} \times \{0, 1\}$ and:

$$A \circ B = \{a_0 \circ b_0, a_1 \circ b_0, a_0 \circ b_1, a_1 \circ b_1\}$$

Except in trivial cases, $A \circ B$ has precisely the following nine parts to within equivalence:

$$\begin{aligned} &A \circ B, \{a_0 \circ b_0, a_1 \circ a_0 \circ b_1\}, \{a_1 \circ b_0, a_0 \circ a_1 \circ b_1\}, \{a_0 \circ b_1, a_1 \circ a_0 \circ b_0\} \\ &\{a_1 \circ b_1, a_0 \circ a_1 \circ b_0\}, \{a_0 \circ b_0 + a_1 \circ b_0, a_0 \circ b_1 + a_1 \circ b_1\}, \{a_0, a_1\} \\ &\{a_0 \circ b_0 + a_1 \circ b_1, a_1 \circ b_0 + a_0 \circ b_1\}, \{1\} \end{aligned}$$

Notice that the sixth of the parts is $(B \mid A)$ and the seventh is A as required by Theorem 3. Each of the parts is a function of $A \circ B$. The parts listed correspond to the following functions $f_i: \{0, 1\} \times \{0, 1\} \rightarrow \{1, 2, 3, 4\}$, $i = 1, 2, \dots, 9$ (Table 1). $\square$

Table 1. Function values.

Function	(0, 0)	(0, 1)	(1, 0)	(1, 1)
f_1	1	2	3	4
f_2	1	2	2	2
f_3	2	2	1	2
f_4	2	1	2	2
f_5	2	2	2	1
f_6	1	2	1	2
f_7	1	1	2	2
f_8	1	2	2	1
f_9	1	1	1	1

Example 2. Similarly to Example 1, for the two binary instruments $\mathcal{I} = \{\mathcal{I}_0, \mathcal{I}_1\}$, $\mathcal{J} = \{\mathcal{J}_0, \mathcal{J}_1\}$, we have the instrument $\mathcal{I} \circ \mathcal{J}$ with $\Omega_{\mathcal{I} \circ \mathcal{J}} = \{0, 1\} \times \{0, 1\}$ and:

$$\mathcal{I} \circ \mathcal{J} = \{\mathcal{I}_0 \circ \mathcal{J}_0, \mathcal{I}_1 \circ \mathcal{J}_0, \mathcal{I}_0 \circ \mathcal{J}_1, \mathcal{I}_1 \circ \mathcal{J}_1\}$$

The nine parts of $\mathcal{I} \circ \mathcal{J}$ to within equivalence are:

$$\begin{aligned} & \mathcal{I} \circ \mathcal{J}, \{\mathcal{I}_0 \circ \mathcal{J}_0, \mathcal{I}_0 \circ \mathcal{J}_1 + \mathcal{I}_1 \circ \mathcal{C}_{\mathcal{J}}\}, \{\mathcal{I}_1 \circ \mathcal{J}_0, \mathcal{I}_1 \circ \mathcal{J}_1 + \mathcal{I}_0 \circ \mathcal{C}_{\mathcal{J}}\} \\ & \{\mathcal{I}_0 \circ \mathcal{J}_1, \mathcal{I}_0 \circ \mathcal{J}_0 + \mathcal{I}_1 \circ \mathcal{C}_{\mathcal{J}}\}, \{\mathcal{I}_1 \circ \mathcal{J}_1, \mathcal{I}_1 \circ \mathcal{J}_0 + \mathcal{I}_0 \circ \mathcal{C}_{\mathcal{J}}\}, \{\mathcal{C}_{\mathcal{I}} \circ \mathcal{J}_0, \mathcal{C}_{\mathcal{I}} \circ \mathcal{J}_1\} \\ & \{\mathcal{I}_0 \circ \mathcal{C}_{\mathcal{J}}, \mathcal{I}_1 \circ \mathcal{C}_{\mathcal{J}}\}, \{\mathcal{I}_0 \circ \mathcal{J}_0 + \mathcal{I}_1 \circ \mathcal{J}_1, \mathcal{I}_1 \circ \mathcal{J}_0 + \mathcal{I}_0 \circ \mathcal{J}_1\}, \{\mathcal{C}_{\mathcal{I} \circ \mathcal{J}}\} \end{aligned}$$

As in Example 1, the sixth part is $(\mathcal{J} | \mathcal{I})$, however, unlike the observable case, the seventh part is not $\mathcal{I}$. In fact, unlike that case, $\mathcal{I}$ is not a part of $(\mathcal{I} \circ \mathcal{J})$. $\square$

If $A \in \mathcal{O}(H)$, the corresponding Lüders instrument $\mathcal{L}^A \in \text{In}(H)$ is defined by $\Omega_{\mathcal{L}^A} = \Omega_A$ and $\mathcal{L}_x^A(\rho) = A_x^{1/2} \rho A_x^{1/2}$ for all $\rho \in \mathcal{S}(H)$. It follows that [15]:

$$\mathcal{L}_X^A(\rho) = \sum_{x \in X} A_x^{1/2} \rho A_x^{1/2}$$

for all $\rho \in \mathcal{S}(H)$, $X \subseteq \Omega_A$. It is easy to check that $(\mathcal{L}^A)^\wedge = A$. Hence, for $B \in \mathcal{O}(H)$, we have that $B \rightarrow \mathcal{L}^A$ if and only if $B \rightarrow A$.

Theorem 4. (a) $\mathcal{L}^{A \circ B} = \mathcal{L}^A \circ \mathcal{L}^B$ if and only if $A_x B_y = B_y A_x$ for all $x \in \Omega_A$, $y \in \Omega_B$; (b) $(\mathcal{L}^{A \circ B})^\wedge = (\mathcal{L}^A \circ \mathcal{L}^B)^\wedge = A \circ B$; (c) An observable C satisfies $C \rightarrow \mathcal{L}^A \circ \mathcal{L}^B$ if and only if $C \rightarrow A \circ B$.

Proof. (a) For all $\rho \in \mathcal{S}(H)$, $(x, y) \in \Omega_A \times \Omega_B$ we have that:

$$(\mathcal{L}^A \circ \mathcal{L}^B)_{(x,y)}(\rho) = \mathcal{L}_y^B(\mathcal{L}_x^A(\rho)) = \mathcal{L}_y^B(A_x^{1/2} \rho A_x^{1/2}) = B_y^{1/2} A_x^{1/2} \rho A_x^{1/2} B_y^{1/2} \quad (5)$$

On the other hand:

$$\begin{aligned} (\mathcal{L}^{A \circ B})_{(x,y)}(\rho) &= (A \circ B)_{(x,y)}^{1/2} \rho (A \circ B)_{(x,y)}^{1/2} = (A_x \circ B_y)^{1/2} \rho (A_x \circ B_y)^{1/2} \\ &= (A_x^{1/2} B_y A_x^{1/2})^{1/2} \rho (A_x^{1/2} B_y A_x^{1/2})^{1/2} \end{aligned} \quad (6)$$

If $A_x B_y = B_y A_x$, we obtain:

$$\begin{aligned} (\mathcal{L}^{A \circ B})_{(x,y)}(\rho) &= (A_x B_y)^{1/2} \rho (A_x B_y)^{1/2} = B_y^{1/2} A_x^{1/2} \rho A_x^{1/2} B_y^{1/2} \\ &= (\mathcal{L}^A \circ \mathcal{L}^B)_{(x,y)}(\rho) \end{aligned}$$

so that $\mathcal{L}^{A \circ B} = \mathcal{L}^A \circ \mathcal{L}^B$. Conversely, if $\mathcal{L}^{A \circ B} = \mathcal{L}^A \circ \mathcal{L}^B$, letting $\rho = \frac{1}{n} 1$ where $n = \dim H$, we obtain from (5) and (6) that:

$$B_y \circ A_x = B_y^{1/2} A_x B_y^{1/2} = A_x^{1/2} B_y A_x^{1/2} = A_x \circ B_y$$

It follows that $B_y A_x = A_x B_y$ for all $x \in \Omega_A$, $y \in \Omega_B$ [7]. (b) We already pointed out that $(\mathcal{L}^{A \circ B})^\wedge = A \circ B$. To show that $(\mathcal{L}^A \circ \mathcal{L}^B)^\wedge = A \circ B$, applying (5) gives:

$$\begin{aligned} \text{tr} \left[\rho (\mathcal{L}^A \circ \mathcal{L}^B)_{(x,y)}^\wedge \right] &= \text{tr} \left[(\mathcal{L}^A \circ \mathcal{L}^B)_{(x,y)}(\rho) \right] = \text{tr} (\rho A_x^{1/2} B_y A_x^{1/2}) \\ &= \text{tr} (\rho A_x \circ B_y) = \text{tr} [\rho (A \circ B)_{(x,y)}] \end{aligned}$$

Hence, $(\mathcal{L}^A \circ \mathcal{L}^B)^\wedge = A \circ B$. (c) This follows from (b) and Theorem 1(a). $\square$

Example 3. We saw from Theorem 4(b) that $(\mathcal{L}^A \circ \mathcal{L}^B)^\wedge = (\mathcal{L}^A)^\wedge \circ (\mathcal{L}^B)^\wedge$. We now show that $(\mathcal{I} \circ \mathcal{J})^\wedge \neq \widehat{\mathcal{I}} \circ \widehat{\mathcal{J}}$ in general. Let $\delta, \gamma \in \mathcal{S}(H)$ and $A, B \in \mathcal{O}(H)$. The instruments $\mathcal{I}_x(\rho) = \text{tr}(\rho A_x) \delta$ and $\mathcal{J}_y(\rho) = \text{tr}(\rho B_y) \gamma$ are called trivial instruments with observables A, B and states δ, γ , respectively, [2]. We have that:

$$\mathrm{tr}(\rho \widehat{\mathcal{I}}_x) = \mathrm{tr}[\mathcal{I}_x(\rho)] = \mathrm{tr}[\mathrm{tr}(\rho A_x) \delta] = \mathrm{tr}(\rho A_x)$$

Hence, $\widehat{\mathcal{I}} = A$ and similarly $\widehat{\mathcal{J}} = B$. For all $\rho \in \mathcal{S}(H)$ we obtain:

$$\begin{aligned} \mathrm{tr}[\rho(\mathcal{I} \circ \mathcal{J})_{(x,y)}^\wedge] &= \mathrm{tr}[(\mathcal{I} \circ \mathcal{J})_{(x,y)}(\rho)] = \mathrm{tr}[\mathcal{J}_y(\mathcal{I}_x(\rho))] = \mathrm{tr}[\mathcal{J}_y(\mathrm{tr}(\rho A_x) \delta)] \\ &= \mathrm{tr}(\rho A_x) \mathrm{tr}[\mathcal{J}_y(\delta)] = \mathrm{tr}(\rho A_x) \mathrm{tr}[\mathrm{tr}(\delta B_y) \gamma] \\ &= \mathrm{tr}(\rho A_x) \mathrm{tr}(\delta B_y) \end{aligned} \quad (7)$$

On the other hand:

$$\mathrm{tr}[\rho \widehat{\mathcal{I}}_x \circ \widehat{\mathcal{J}}_y] = \mathrm{tr}(\rho A_x \circ B_y) \quad (8)$$

Since the right hand sides of (7) and (8) are different in general, we conclude that $(\mathcal{I} \circ \mathcal{J})^\wedge \neq \widehat{\mathcal{I}} \circ \widehat{\mathcal{J}}$. $\square$

We saw in Theorem 4(a) that $\mathcal{L}^{A \circ B} \neq \mathcal{L}^A \circ \mathcal{L}^B$, in general. The following lemma shows they can differ in a striking way.

Lemma 2. If $A_x = |\phi_x\rangle\langle\phi_x|$ and $B_y = |\psi_y\rangle\langle\psi_y|$ are atomic observables on H , then for all $\rho \in \mathcal{S}(H)$, there exist numbers $\lambda_{xy}(\rho) \in [0, 1]$ with $\sum_{x,y} \lambda_{xy}(\rho) = 1$ such that $\mathcal{L}_{(x,y)}^{A \circ B}(\rho) = \lambda_{xy}(\rho) A_x$ and $(\mathcal{L}^A \circ \mathcal{L}^B)_{(x,y)}(\rho) = \lambda_{xy}(\rho) B_y$ for all $(x, y) \in \Omega_A \times \Omega_B$.

Proof. For all $\rho \in \mathcal{S}(H)$ we have that:

$$\begin{aligned} (\mathcal{L}^A \circ \mathcal{L}^B)_{(x,y)}(\rho) &= \mathcal{L}_y^B(\mathcal{L}_x^A(\rho)) = B_y A_x \rho A_x B_y \\ &= |\psi_y\rangle\langle\psi_y| |\phi_x\rangle\langle\phi_x| \rho |\phi_x\rangle\langle\phi_x| |\psi_y\rangle\langle\psi_y| \\ &= |\langle\phi_x, \psi_y\rangle|^2 \langle\phi_x, \rho \phi_x\rangle B_y \end{aligned}$$

Since:

$$A_x B_y A_x = |\phi_x\rangle\langle\phi_x| |\psi_y\rangle\langle\psi_y| |\phi_x\rangle\langle\phi_x| = |\langle\phi_x, \psi_y\rangle|^2 A_x$$

we obtain:

$$(A_x B_y A_x)^{1/2} = |\langle\phi_x, \psi_y\rangle| A_x$$

Hence:

$$(\mathcal{L}^{A \circ B})_{(x,y)}(\rho) = (A_x B_y A_x)^{1/2} \rho (A_x B_y A_x)^{1/2} = |\langle\phi_x, \psi_y\rangle|^2 \langle\phi_x, \rho \phi_x\rangle A_x$$

Letting $\lambda_{xy}(\rho) = |\langle\phi_x, \psi_y\rangle|^2 \langle\phi_x, \rho \phi_x\rangle$, the result follows. $\square$

4. Composite Systems

Let H_1 and H_2 be Hilbert spaces with $\dim H_1 = n_1$ and $\dim H_2 = n_2$. If H_1, H_2 represent quantum systems, we call $H = H_1 \otimes H_2$ a *composite* quantum system. For $a \in \mathcal{E}(H)$, we define the *reduced effects* $a^1 \in \mathcal{E}(H_1)$, $a^2 \in \mathcal{E}(H_2)$ by $a^1 = \frac{1}{n_2} \mathrm{tr}_2(a)$, $a^2 = \frac{1}{n_1} \mathrm{tr}_1(a)$. We view a^i to be the effect a as measured in a system $i = 1, 2$. The map $a \mapsto a^1$ is a surjective effect algebra morphism from $\mathcal{E}(H)$ onto $\mathcal{E}(H_1)$ and similarly for $a \mapsto a^2$ [6,7]. Conversely, if $a \in \mathcal{E}(H_1)$, $b \in \mathcal{E}(H_2)$, then $a \otimes b \in \mathcal{E}(H)$ and:

$$(a \otimes b)^1 = \frac{1}{n_2} \mathrm{tr}_2(a \otimes b) = \frac{1}{n_2} \mathrm{tr}(b) a$$

Similarly, $(a \otimes b)^2 = \frac{1}{n_1} \mathrm{tr}_1(a \otimes b)$. It follows that:

$$(a^1 \otimes a^2)^1 = \frac{1}{n_2} \mathrm{tr}(a^2) a^1$$

and

$$(a^1 \otimes a^2)^2 = \frac{1}{n_1} \text{tr}(a^1) a^2$$

An effect $a \in \mathcal{E}(H)$ is *factorized* if $a = b \otimes c$ for $b \in \mathcal{E}(H_1)$, $c \in \mathcal{E}(H_2)$ [2].

Lemma 3. If $a \in \mathcal{E}(H)$ with $a \neq 0$, then a is factorized if and only if:

$$a = \frac{n_1 n_2}{\text{tr}(a)} a^1 \otimes a^2 \quad (9)$$

Proof. If (9) holds, then a is factorized. Conversely, suppose a is factorized with $a = b \otimes c$, $b \in \mathcal{E}(H_1)$, $c \in \mathcal{E}(H_2)$. Then, $a^1 = \frac{1}{n_2} \text{tr}(c) b$ and $a^2 = \frac{1}{n_1} \text{tr}(b) c$. Hence, $b = \frac{n_2}{\text{tr}(c)} a^1$ and $c = \frac{n_1}{\text{tr}(b)} a^2$. We conclude that:

$$a = \frac{n_1 n_2}{\text{tr}(b) \text{tr}(c)} a^1 \otimes a^2 = \frac{n_1 n_2}{\text{tr}(a)} a^1 \otimes a^2 \quad \square$$

Corollary 1. If $a \in \mathcal{E}(H)$, then $a = a^1 \otimes a^2$ if and only if $a = 0$ or $a = 1$.

Proof. If $a = 0$ or $a = 1$, then clearly $a = a^1 \otimes a^2$. Conversely, if $a = a^1 \otimes a^2$, then by Lemma 3, $a = 0$ or $\text{tr}(a) = n_1 n_2$. In the latter case, $a = 1$. $\square$

An effect is *indecomposable* if it has the form $a = \lambda b$ where $0 \leq \lambda \leq 1$ and b is an atom.

Theorem 5. Let $a \in \mathcal{E}(H)$ be an atom $a = P_\psi$ where $H = H_1 \otimes H_2$: (a) a is factorized if and only if a^1 and a^2 are indecomposable; (b) We can arrange the nonzero eigenvalues $\alpha_1, \alpha_2, \dots, \alpha_n$ of a^1 and the nonzero eigenvalues $\beta_1, \beta_2, \dots, \beta_n$ of a^2 so that $\alpha_i = \frac{n_1}{n_2} \beta_i$, $i = 1, 2, \dots, n$. Hence, if $n_1 = n_2$, then the eigenvalues of a^1 and a^2 are identical.

Proof. The unit vector $\psi \in H$ has a Schmidt decomposition $\psi = \sum_{i=1}^m \lambda_i \psi_i \otimes \phi_i$, $\lambda_i \geq 0$, $\sum \lambda_i^2 = 1$. We have that:

$$\begin{aligned} a = |\psi\rangle\langle\psi| &= \left| \sum \lambda_i \psi_i \otimes \phi_i \right\rangle \left\langle \sum \lambda_j \psi_j \otimes \phi_j \right| = \sum_{i,j} \lambda_i \lambda_j |\psi_i \otimes \phi_i\rangle \langle \psi_j \otimes \phi_j| \\ &= \sum_{i,j} \lambda_i \lambda_j |\psi_i\rangle \langle \psi_j| \otimes |\phi_i\rangle \langle \phi_j| \end{aligned}$$

Hence:

$$\begin{aligned} a^1 &= \frac{1}{n_2} \text{tr}_2(a) = \frac{1}{n_2} \sum_{i,j} \lambda_i \lambda_j \text{tr}_2(|\psi_i\rangle \langle \psi_j| \otimes |\phi_i\rangle \langle \phi_j|) \\ &= \frac{1}{n_2} \sum_{i,j} \lambda_i \lambda_j \delta_{ij} |\psi_i\rangle \langle \psi_j| = \frac{1}{n_2} \sum \lambda_i^2 P_{\psi_i} \end{aligned} \quad (10)$$

and similarly:

$$a^2 = \frac{1}{n_1} \sum \lambda_i^2 P_{\phi_i} \quad (11)$$

Now a is factorized if and only if ψ is factorized which is equivalent to $m = 1$ and $\psi = \psi_1 \otimes \phi_1$. Applying (10) and (11), we conclude that a is factorized if and only if $a^1 = \frac{1}{n_2} \lambda_1^2 P_{\psi_1}$ and $a^2 = \frac{1}{n_1} \lambda_1^2 P_{\phi_1}$, in which case a^1 and a^2 are indecomposable. This completes the proof of (a). To prove (b), we see from (10), (11) that the eigenvalues of a^1, a^2 are $\alpha_i = \frac{1}{n_2} \lambda_i^2$ and $\beta_i = \frac{1}{n_1} \lambda_i^2$. It follows that $\alpha_i = \frac{n_1}{n_2} \beta_i$. $\square$

If $A \in \mathcal{O}(H_1 \otimes H_2)$ we define the *reduced observables* $A^1 \in \mathcal{O}(H_1)$, $A^2 \in \mathcal{O}(H_2)$ by $A^1 = \{A_x^1: x \in \Omega_A\}$ and $A^2 = \{A_x^2: x \in \Omega_A\}$. Note that $A^1(A^2)$ is indeed an observable because:

$$\sum_{x \in \Omega_A} A_x^1 = \sum_{x \in \Omega_A} \frac{1}{n_2} \text{tr}_2(A_x) = \frac{1}{n_2} \text{tr}_2 \left(\sum_{x \in \Omega_A} A_x \right) = \frac{1}{n_2} \text{tr}_2(1_1 \otimes 1_2) = 1_1$$

Lemma 4. If $A \in \mathcal{O}(H_1 \otimes H_2)$ and $\rho_1 \in \mathcal{S}(H_1)$, then:

$$\Phi_{\rho_1}^{A^1} = \Phi_{\rho_1 \otimes 1_2/n_2}^A$$

Proof. For $X \subseteq \Omega_A$ we have that:

$$\begin{aligned} \Phi_{\rho_1}^{A^1}(X) &= \text{tr}(\rho_1 A_x^1) = \text{tr} \left[\rho_1 \frac{1}{n_2} \text{tr}_2(A_x) \right] = \frac{1}{n_2} \text{tr}[\rho_1 \text{tr}_2(A_x)] \\ &= \frac{1}{n_2} \text{tr}[A_x(\rho_1 \otimes 1_2)] = \text{tr} \left[\left(\rho_1 \otimes \frac{1}{n_2} 1_2 \right) A_x \right] = \Phi_{\rho_1 \otimes 1_2/n_2}(X) \end{aligned}$$

The result now follows. $\square$

In a similar way:

$$\Phi_{\rho_2}^{A^2} = \Phi_{1_1/n_1 \otimes \rho_2}^A$$

For $A \in \mathcal{O}(H_1)$, we define the *A-random measure* on Ω_A by

$$\mu^A(X) = \frac{1}{n_1} \text{tr}(A_X) = \text{tr} \left(\frac{1_1}{n_1} A_X \right) = \Phi_{1_1/n_1}^A(X)$$

for all $X \subseteq \Omega_A$. Thus, μ^A is the distribution of A in the random state $1_1/n_1$. If $A_1 \in \mathcal{O}(H_1)$, $A_2 \in \mathcal{O}(H_2)$, we define the *composite observable*:

$$B_{(x,y)} = A_{1,x} \otimes A_{2,y} \in \mathcal{O}(H_1 \otimes H_2)$$

In this case, $\Omega_B = \Omega_{A_1} \times \Omega_{A_2}$ and for $Z \subseteq \Omega_B$, we have that:

$$B_Z = \sum_{(x,y) \in Z} B_{(x,y)}$$

Hence, $B_{X \times Y} = A_{1,X} \otimes A_{2,Y}$.

Lemma 5. $B_{X \times Y}^1 = \mu^{A^2}(Y) A_{1,X}$ and $B_{X \times Y}^2 = \mu^{A^1}(X) A_{2,Y}$.

Proof. For $x \in \Omega_{A_1}$, $y \in \Omega_{A_2}$ we obtain:

$$B_{(x,y)}^1 = \frac{1}{n_2} \text{tr}_2[B_{(x,y)}] = \frac{1}{n_2} \text{tr}_2(A_{1,x} \otimes A_{2,y}) = \frac{1}{n_2} \text{tr}(A_{2,y}) A_{1,x}$$

Hence:

$$B_{X \times Y}^1 = \frac{1}{n_2} \text{tr}(A_{2,Y}) A_{1,X} = \mu^{A^2}(Y) A_{1,X}$$

The second equation is similar. $\square$

A *transition probability* from Ω_1 to Ω_2 is a map $\nu: \Omega_1 \times \Omega_2 \rightarrow [0, 1]$ satisfying $\sum_{y \in \Omega_2} \nu_{xy} = 1$ for all $x \in \Omega_1$. (The matrix $[\nu_{xy}]$ is called a *stochastic matrix*.) Let $A \in \mathcal{O}(H_1)$ with outcome-space Ω_1 and let ν be a transition probability from Ω_1 to Ω_2 . Then, $(\nu \cdot A)_y = \sum_{x \in \Omega_1} \nu_{xy} A_x$ is an observable on H_1 with outcome-space Ω_2 called a *post-processing* of A from Ω_1 to Ω_2 [12]. If we also have $B \in \mathcal{O}(H_2)$ with outcome-space Ω_3 and μ a transition probability from Ω_3 to Ω_4 , we can form the post-processing $\mu \cdot B$.

Theorem 6. (a) $(\nu \cdot A) \otimes (\mu \cdot B) \in \mathcal{O}(H_1 \otimes H_2)$ with outcome-space $\Omega_2 \times \Omega_4$ and is a post-processing $\alpha \cdot (A \otimes B)$ from $\Omega_1 \times \Omega_3$ to $\Omega_2 \times \Omega_4$ where $\alpha((x, r), (y, s)) = \nu_{xy}\mu_{rs}$; (b) If $A \in \mathcal{O}(H_1 \otimes H_2)$, then $(\nu \cdot A)^1 = \nu \cdot A^1$ and $(\nu \cdot A)^2 = \nu \cdot A^2$.

Proof. (a) The map $\alpha: \Omega_1 \times \Omega_3 \rightarrow \Omega_2 \times \Omega_4$ is a transition probability because $\alpha((x, r), (y, s)) \geq 0$ and:

$$\sum_{(y, s) \in \Omega_2 \times \Omega_4} \alpha((x, r), (y, s)) = \sum_{y, s} \nu_{xy}\mu_{rs} = 1$$

Moreover, $(\nu \cdot A) \otimes (\mu \cdot B) \in \mathcal{O}(H_1 \otimes H_2)$ with outcome-space $\Omega_2 \times \Omega_4$ and we have that:

$$\begin{aligned} [(\nu \cdot A) \otimes (\mu \cdot B)]_{(y, s)} &= (\nu \cdot A)_y \otimes (\mu \cdot B)_s \\ &= \left(\sum_{x \in \Omega_1} \nu_{xy} A_x \right) \otimes \left(\sum_{r \in \Omega_3} \mu_{rs} B_r \right) \\ &= \sum_{x \in \Omega_1} \sum_{r \in \Omega_3} \nu_{xy} \mu_{rs} A_x \otimes B_r \\ &= \sum_{x, r} \alpha((x, r), (y, s)) A_x \otimes B_r = [\alpha \cdot (A \otimes B)]_{(y, s)} \end{aligned}$$

Hence, $(\nu \cdot A) \otimes (\mu \cdot B) = \alpha \cdot (A \otimes B)$. (b) This follows from:

$$\begin{aligned} (\nu \cdot A)_y^1 &= \left(\sum_x \nu_{xy} A_x \right)^1 = \frac{1}{n_2} \text{tr}_2 \left(\sum_x \nu_{xy} A_x \right) = \frac{1}{n_2} \sum_x \nu_{xy} \text{tr}_2(A_x) \\ &= \sum_x \nu_{xy} A_x^1 = (\nu \cdot A^1)_y \end{aligned}$$

That $(\nu \cdot A)^2 = \nu \cdot A^2$ is similar. $\square$

We have seen in Theorem 2 that coexistence is equivalent to joint measurability. This is used in the next theorem [13].

Theorem 7. (a) If $A_1, B_1 \in \mathcal{O}(H_1)$ coexist with joint observable C_1 and A_2, B_2 coexist with joint observable C_2 , then $A_1 \otimes A_2, B_1 \otimes B_2$ coexist with joint observable $C = C_1 \otimes C_2$; (b) If $A, B \in \mathcal{O}(H_1 \otimes H_2)$ coexist with joint observable C , then A^1, B^1 coexist with joint observable C^1 and A^2, B^2 coexist with joint observable C^2 .

Proof. (a) We write $C_{1,(x,y)}$ for $(x, y) \in \Omega_{A_1} \times \Omega_{B_1}$ and $C_{2,(x',y')}$ for $(x', y') \in \Omega_{A_2} \times \Omega_{B_2}$. Then:

$$C_{(x,y,x',y')} = C_{1,(x,y)} \otimes C_{2,(x',y')}$$

and we have that:

$$\sum_{(y,y')} C_{(x,y,x',y')} = \sum_y C_{1,(x,y)} \otimes \sum_{y'} C_{2,(x',y')} = A_{1,x} \otimes A_{2,x'} = (A_1 \otimes A_2)_{(x,x')}$$

Moreover:

$$\sum_{(x,x')} C_{(x,y,x',y')} = \sum_x C_{1,(x,y)} \otimes \sum_{x'} C_{2,(x',y')} = B_{1,y} \otimes B_{2,y'} = (B_1 \otimes B_2)_{(y,y')}$$

and the result follows. (b) For all $(x, y) \in \Omega_A \times \Omega_B$, we obtain:

$$A_x^1 = \left[\sum_y C_{(x,y)} \right]^1 = \frac{1}{n_2} \text{tr}_2 \left[\sum_y C_{(x,y)} \right] = \sum_y \left[\frac{1}{n_2} \text{tr}(C_{(x,y)}) \right]$$

$$= \sum_y C_{(x,y)}^1$$

Similarly, $B_y^1 = \sum_x C_{(x,y)}^1$ so A^1, B^1 coexist with joint observable C^1 . The result for A^2, B^2 is similar. $\square$

For an instrument $\mathcal{I} \in \text{In}(H_1 \otimes H_2)$ on the composite system, the *reduced instrument* on system 1 is defined by [9,10]

$$\mathcal{I}_x^1(\rho_1) = \frac{1}{n_2} \text{tr}_2[\mathcal{I}_x(\rho_1 \otimes 1_2)]$$

for all $\rho_1 \in \mathcal{S}(H_1), x \in \Omega_{\mathcal{I}}$. Similarly:

$$\mathcal{I}_x^2(\rho_1) = \frac{1}{n_1} \text{tr}_1[\mathcal{I}_x(1_1 \otimes \rho_2)]$$

for all $\rho_2 \in \mathcal{S}(H_2), x \in \Omega_{\mathcal{I}}$.

Theorem 8. $(\mathcal{I}^1)^\wedge = (\hat{\mathcal{I}})^1$ and $(\mathcal{I}^2)^\wedge = (\hat{\mathcal{I}})^2$.

Proof. For all $\rho_1 \in \mathcal{S}(H_1)$, we have that:

$$\begin{aligned} \text{tr}[\rho_1(\hat{\mathcal{I}})_x^1] &= \frac{1}{n_2} \text{tr}[\rho_1 \text{tr}_2(\hat{\mathcal{I}})_x] = \frac{1}{n_2} \text{tr}[(\rho_1 \otimes 1_2)\hat{\mathcal{I}}_x] \\ &= \frac{1}{n_2} \text{tr}[\mathcal{I}_x(\rho_1 \otimes 1_2)] = \text{tr}[\mathcal{I}_x^1(\rho_1)] = \text{tr}[\rho_1(\mathcal{I}^1)_x^\wedge] \end{aligned}$$

We conclude that $(\mathcal{I}^1)^\wedge = (\hat{\mathcal{I}})^1$ and similarly, $(\mathcal{I}^2)^\wedge = (\hat{\mathcal{I}})^2$. $\square$

For $\mathcal{I} \in \text{In}(H_1)$ we define the $\mathcal{I}$ -random measure on $\Omega_{\mathcal{I}}$ by

$$\mu^{\mathcal{I}}(X) = \frac{1}{n_1} \text{tr}[\mathcal{I}_X(1_1)]$$

For $\mathcal{I}_1 \in \text{In}(H_1), \mathcal{I}_2 \in \text{In}(H_2)$ we define $\mathcal{J} = \mathcal{I}_1 \otimes \mathcal{I}_2 \in \text{In}(H_1 \otimes H_2)$ with outcome-space $\Omega_{\mathcal{I}_1} \times \Omega_{\mathcal{I}_2}$ by $\mathcal{J}_{(x,y)} = \mathcal{I}_{1,x} \otimes \mathcal{I}_{2,y}$. It is easy to check that $\mathcal{J}$ is indeed an instrument.

Theorem 9. Let $\mathcal{J} = \mathcal{I}_1 \otimes \mathcal{I}_2 \in \text{In}(H_1 \otimes H_2)$. (a) $\mathcal{J}_{(x,y)}^1(\rho_1) = \mu^{\mathcal{I}_2}(y)\mathcal{I}_{1,x}(\rho_1)$ for all $\rho_1 \in \mathcal{S}(H_1)$ and $\mathcal{J}_{(x,y)}^2(\rho_2) = \mu^{\mathcal{I}_1}(x)\mathcal{I}_{2,y}(\rho_2)$ for all $\rho_2 \in \mathcal{S}(H_2)$. (b) $(\mathcal{I}_1 \otimes \mathcal{I}_2)^\wedge = \hat{\mathcal{I}}_1 \otimes \hat{\mathcal{I}}_2$.

Proof. (a) For all $\rho_1 \in \mathcal{S}(H_1)$ we have that:

$$\begin{aligned} \mathcal{J}_{(x,y)}^1(\rho_1) &= \frac{1}{n_2} \text{tr}_2[\mathcal{J}_{(x,y)}(\rho_1 \otimes 1_2)] = \frac{1}{n_2} \text{tr}_2[\mathcal{I}_{1,x} \otimes \mathcal{I}_{2,y}(\rho_1 \otimes 1_2)] \\ &= \frac{1}{n_2} \text{tr}_2[\mathcal{I}_{1,x}(\rho_1) \otimes \mathcal{I}_{2,y}(1_2)] = \frac{1}{n_2} \text{tr}[\mathcal{I}_{2,y}(1_2)]\mathcal{I}_{1,x}(\rho_1) \\ &= \mu^{\mathcal{I}_2}(y)\mathcal{I}_{1,x}(\rho_1) \end{aligned}$$

Similarly, $\mathcal{J}_{(x,y)}^2(\rho_2) = \mu^{\mathcal{I}_1}(x)\mathcal{I}_{2,y}(\rho_2)$ for all $\rho_2 \in \mathcal{S}(H_2)$. (b) For all $\rho_1 \in \mathcal{S}(H_1), \rho_2 \in \mathcal{S}(H_2)$ we have that:

$$\begin{aligned} \text{tr}[\rho_1 \otimes \rho_2(\mathcal{I}_1 \otimes \mathcal{I}_2)_{(x,y)}^\wedge] &= \text{tr}[\mathcal{I}_{1,x} \otimes \mathcal{I}_{2,y}(\rho_1 \otimes \rho_2)] = \text{tr}[\mathcal{I}_{1,x}(\rho_1) \otimes \mathcal{I}_{2,y}(\rho_2)] \\ &= \text{tr}[\mathcal{I}_{1,x}(\rho_1)]\text{tr}[\mathcal{I}_{2,y}(\rho_2)] = \text{tr}[\rho_1 \hat{\mathcal{I}}_{1,x}]\text{tr}[\rho_2 \hat{\mathcal{I}}_{2,y}] \\ &= \text{tr}[\rho_1 \otimes \rho_2(\hat{\mathcal{I}}_{1,x} \otimes \hat{\mathcal{I}}_{2,y})] \end{aligned}$$

and the result follows. $\square$

A Kraus instrument is an instrument of the form $\mathcal{I}_x(\rho) = S_x \rho S_x^*$ where $\sum_x S_x^* S_x = 1$, $x \in \Omega_{\mathcal{I}}$. The operators S_x are called Kraus operators for $\mathcal{I}$ [5].

Lemma 6. Let $\mathcal{I}_1 \in \text{In}(H_1)$, $\mathcal{I}_2 \in \text{In}(H_2)$ be Kraus instruments with operators $S_{1,x}$, $S_{2,y}$, respectively. (a) $\mathcal{J} = \mathcal{I}_1 \otimes \mathcal{I}_2$ is a Kraus instrument with operators $S_{1,x} \otimes S_{2,y}$; (b) $\mathcal{J}^1, \mathcal{J}^2$ are Kraus instruments with operators:

$$\begin{aligned} T_{(x,y)} &= \left[\frac{1}{n_2} \text{tr}(S_{2,y} S_{2,y}^*) \right]^{1/2} S_{1,x} \\ R_{(x,y)} &= \left[\frac{1}{n_1} \text{tr}(S_{1,x} S_{1,x}^*) \right]^{1/2} S_{2,y} \end{aligned}$$

Proof. (a) For all $\rho_1 \in \mathcal{S}(H_1)$, $\rho_2 \in \mathcal{S}(H_2)$ we have that:

$$\begin{aligned} \mathcal{J}_{(x,y)}(\rho_1 \otimes \rho_2) &= (\mathcal{I}_{1,x} \times \mathcal{I}_{2,y})(\rho_1 \otimes \rho_2) = \mathcal{I}_{1,x}(\rho_1) \otimes \mathcal{I}_{2,y}(\rho_2) \\ &= S_{1,x} \rho_1 S_{1,x}^* \otimes S_{2,y} \rho_2 S_{2,y}^* \\ &= S_{1,x} \otimes S_{2,y} (\rho_1 \otimes \rho_2) S_{1,x}^* \otimes S_{2,y}^* \end{aligned}$$

and the result follows. (b) For $\rho \in \mathcal{S}(H_1)$ we obtain:

$$\mathcal{J}_{(x,y)}^1(\rho_1) = \frac{1}{n_2} \text{tr}[\mathcal{I}_{2,y}(1_2)] \mathcal{I}_{1,x}(\rho_1) = \frac{1}{n_2} \text{tr}(S_{2,y} S_{2,y}^*) S_{1,x} \rho_1 S_{1,x}^*$$

This can be considered to be a Kraus instrument with operators $T_{(x,y)}$ given above. The result for $\mathcal{J}^2$ is similar. $\square$

Notice that a Lüders instrument defined by $\mathcal{L}_x^A(\rho_1) = A_x^{1/2} \rho_1 A_x^{1/2}$ for all $\rho_1 \in \mathcal{S}(H_1)$ is a particular case of a Kraus instrument with operators $A_x^{1/2}$ [15].

Corollary 2. Let $A \in \mathcal{O}(H_1)$, $B \in \mathcal{O}(H_2)$: (a) $\mathcal{L}_x^A \otimes \mathcal{L}_y^B = \mathcal{L}_{(x,y)}^{A \otimes B}$; (b) $(\mathcal{L}_x^A \otimes \mathcal{L}_y^B)^1 = \mathcal{L}_{(x,y)}^C$ where $C = \frac{1}{n_2} \text{tr}(B_y) A_x$ and $(\mathcal{L}_x^A \otimes \mathcal{L}_y^B)^2 = \mathcal{L}_{(x,y)}^D$ where $D = \frac{1}{n_2} \text{tr}(A_x) B_y$.

We say that a Kraus instrument $\mathcal{I} \in \text{In}(H_1 \otimes H_2)$ with operators R_x is factorized if $R_x = S_x \otimes T_x$ for all $x \in \Omega_{\mathcal{I}}$. We conjecture that if $\mathcal{I} \in \text{In}(H_1 \otimes H_2)$ is Kraus, then $\mathcal{I}^1$ and $\mathcal{I}^2$ need not be Kraus. However, we do have the following result.

Lemma 7. If $\mathcal{I} \in \text{In}(H_1 \otimes H_2)$ is Kraus and factorized, then $\mathcal{I}^1$ and $\mathcal{I}^2$ are Kraus.

Proof. If the operators R_x for $\mathcal{I}$ satisfy $R_x = S_x \otimes T_x$, then for all $\rho_1 \in \mathcal{S}(H_1)$, we have that:

$$\begin{aligned} \mathcal{I}_x^1(\rho_1) &= \frac{1}{n_2} \text{tr}_2[\mathcal{I}_x(\rho_1 \otimes 1_2)] = \frac{1}{n_2} \text{tr}_2[R_x(\rho_1 \otimes 1_2) R_x^*] \\ &= \frac{1}{n_2} \text{tr}_2[S_x \otimes T_x(\rho_1 \otimes 1_2) S_x^* \otimes T_x^*] \\ &= \frac{1}{n_2} \text{tr}_2(S_x \rho_1 S_x^* \otimes T_x T_x^*) = \frac{1}{n_2} \text{tr}(T_x T_x^*) S_x \rho_1 S_x^* \end{aligned}$$

Hence, $\mathcal{I}^1$ is Kraus with operators $\left[\frac{1}{n_2} \text{tr}(T_x T_x^*) \right]^{1/2} S_x$. Similarly, $\mathcal{I}^2$ is Kraus with operators $\left[\frac{1}{n_1} \text{tr}(S_x S_x^*) \right]^{1/2} T_x$. $\square$

We do not know if the converse of Lemma 7 holds. We now consider trivial instruments (see Example 3).

Lemma 8. Let $\mathcal{I}_1 \in \text{In}(H_1)$, $\mathcal{I}_2 \in \text{In}(H_2)$ be trivial instruments with:

$$\mathcal{I}_{1,x}(\rho_1) = \text{tr}(\rho_1 A_x) \alpha, \quad \mathcal{I}_{2,y}(\rho_2) = \text{tr}(\rho_2 B_y) \beta$$

(a) $\mathcal{I}_1 \otimes \mathcal{I}_2 \in \text{In}(H_1 \otimes H_2)$ is trivial with observable $A \otimes B$ and state $\alpha \otimes \beta$ (b) $(\mathcal{I}_1 \otimes \mathcal{I}_2)^1$, $(\mathcal{I}_1 \otimes \mathcal{I}_2)^2$ are trivial with observables $\mu^B(y)A_x$, $\mu^A(x)B_y$ and states α , β , respectively.

Proof. (a) For all $(x, y) \in \Omega_{\mathcal{I}_1} \times \Omega_{\mathcal{I}_2}$, $\rho_1 \in \mathcal{S}(H_1)$, $\rho_2 \in \mathcal{S}(H_2)$, we have that:

$$\begin{aligned} (\mathcal{I}_1 \otimes \mathcal{I}_2)_{(x,y)}(\rho_1 \otimes \rho_2) &= \mathcal{I}_{1,x}(\rho_1) \otimes \mathcal{I}_{2,y}(\rho_2) = \text{tr}(\rho_1 A_x) \alpha \otimes \text{tr}(\rho_2 B_y) \beta \\ &= \text{tr}(\rho_1 A_x) \text{tr}(\rho_2 B_y) \alpha \otimes \beta = \text{tr}(\rho_1 A_x \otimes \rho_2 B_y) \alpha \otimes \beta \\ &= \text{tr}(\rho_1 \otimes \rho_2 A \otimes B_{(x,y)}) \alpha \otimes \beta \end{aligned}$$

The result now follows. (b) This follows from:

$$\begin{aligned} (\mathcal{I}_1 \otimes \mathcal{I}_2)_{(x,y)}^1(\rho_1) &= \frac{1}{n_2} \text{tr}_2 [\mathcal{I}_{1,x} \otimes \mathcal{I}_{2,y}(\rho_1 \otimes 1_2)] = \frac{1}{n_2} \text{tr}_2 [\mathcal{I}_{1,x}(\rho_1) \otimes \mathcal{I}_{2,y}(1_2)] \\ &= \frac{1}{n_2} \text{tr} [\mathcal{I}_{2,y}(1_2)] \mathcal{I}_{1,x}(\rho_1) = \frac{1}{n_2} \text{tr} [\text{tr}(1_2 B_y) \beta] \text{tr}(\rho_1 A_x) \alpha \\ &= \frac{1}{n_2} \text{tr}(B_y) \text{tr}(\rho_1 A_x) \alpha = \text{tr} [\rho_1 \mu^B(y) A_x] \alpha \end{aligned}$$

and similarly:

$$(\mathcal{I}_1 \otimes \mathcal{I}_2)_{(x,y)}^2(\rho_2) = \text{tr} [\rho_2 \mu^A(x) B_y] \beta \quad \square$$

Lemma 9. Let $\mathcal{I} \in \text{In}(H_1 \otimes H_2)$ be trivial with $\mathcal{I}_x(\rho) = \text{tr}(\rho A_x) \alpha$. (a) $\mathcal{I}^1, \mathcal{I}^2$ are trivial with observables A_x^1, A_x^2 and states $\text{tr}_2(\alpha), \text{tr}_1(\alpha)$, respectively; (b) Letting $\mathcal{J} = \mathcal{I}^1 \otimes \mathcal{I}^2$ we have that $\mathcal{J}$ is trivial with observable $A^1 \otimes A^2$ and state $\text{tr}_2(\alpha) \otimes \text{tr}_1(\alpha)$. Moreover, $\mathcal{J}_{(x,y)}^1 = \mathcal{I}_x^1$ and $\mathcal{J}_{(x,y)}^2 = \mathcal{I}_y^2$ for all $(x, y) \in \Omega_{\mathcal{J}}$.

Proof. (a) For all $\rho_1 \in \mathcal{S}(H_1)$ and $x \in \Omega_{\mathcal{I}}$, we have that:

$$\begin{aligned} \mathcal{I}_x^1(\rho_1) &= \frac{1}{n_2} \text{tr}_2 [\mathcal{I}_x(\rho_1 \otimes 1_2)] = \frac{1}{n_2} \text{tr}_2 \{ \text{tr}[(\rho_1 \otimes 1_2) A_x] \alpha \} \\ &= \frac{1}{n_2} \text{tr}[(\rho_1 \otimes 1_2) A_x] \text{tr}_1(\alpha) = \frac{1}{n_2} \text{tr}[\text{tr}_2(A_x) \rho_1] \text{tr}_2(\alpha) \\ &= \text{tr} \left[\rho_1 \frac{1}{n_2} \text{tr}_2(A_x) \right] \text{tr}_2(\alpha) = \text{tr}(\rho_1 A_x^1) \text{tr}_2(\alpha) \end{aligned}$$

Similarly, $\mathcal{I}_x^2(\rho_2) = \text{tr}(\rho_2 A_x^2) \text{tr}_1(\alpha)$ so the result follows. (b) This result follows from Lemma 8(b). $\square$

We now consider MMs for composite systems. A single probe MM on $H = H_1 \otimes H_2$ has the form $\mathcal{M} = (H, K, \eta, \nu, F)$ as defined before. As discussed earlier, $\widehat{\mathcal{M}} \in \text{In}(H)$ is the instrument measured by $\mathcal{M}$. Then, $\widehat{\mathcal{M}}^1 \in \text{In}(H_1)$ and for $\rho_1 \in \mathcal{S}(H_1)$, we obtain:

$$\begin{aligned} \widehat{\mathcal{M}}_x^1(\rho_1) &= \frac{1}{n_2} \text{tr}_2 [\widehat{\mathcal{M}}_x(\rho_1 \otimes 1_2)] \\ &= \frac{1}{n_2} \text{tr}_2 \{ \text{tr}_K [\nu(\rho_1 \otimes 1_2 \otimes \eta)(1_1 \otimes 1_2 \otimes F_x)] \} \end{aligned} \quad (12)$$

We have a similar expression for $\widehat{\mathcal{M}}^2 \in \text{In}(H_2)$.

Corresponding to $\mathcal{M}$, we define the reduced MM $\mathcal{M}_1 = (H_1, K, \eta, \nu_1, F)$ where $\nu_1 \in \mathcal{S}(H_1 \otimes K)$ is given by

$$\nu_1(\rho_1 \otimes \eta) = \frac{1}{n_2} \text{tr}_2 [\nu(\rho_1 \otimes 1_2 \otimes \eta)]$$

We then have for $\rho_1 \in \mathcal{S}(H_1)$ that:

$$\begin{aligned} \widehat{\mathcal{M}}_{1,x}(\rho_1) &= \text{tr}_K [\nu_1(\rho_1 \otimes \eta)(1_1 \otimes F_x)] \\ &= \frac{1}{n_2} \text{tr}_K \{ \text{tr}_2 [\nu(\rho_1 \otimes 1_2 \otimes \eta)] (1_1 \otimes F_x) \} \end{aligned} \quad (13)$$

Similarly, we define $\mathcal{M}_2 = (H_2, K, \eta, \nu_2, F)$ and an analogous formula for $\widehat{\mathcal{M}}_2$. Notice that (12) and (13) are quite similar and they are essentially an interchange of the two partial traces. We now show that they coincide.

Theorem 10. (a) Let H_1, H_2, H_3 be finite-dimensional Hilbert spaces and let $A \in \mathcal{L}(H_1 \otimes H_2 \otimes H_3)$, $B \in \mathcal{L}(H_3)$. Then:

$$\text{tr}_2[\text{tr}_3(A(1_1 \otimes 1_2 \otimes B))] = \text{tr}_3[(\text{tr}_2(A))(1_1 \otimes B)] \quad (14)$$

$$(b) \widehat{\mathcal{M}}^1 = \widehat{\mathcal{M}}_1 \text{ and } \widehat{\mathcal{M}}^2 = \widehat{\mathcal{M}}_2.$$

Proof. (a) First suppose that $A = A_1 \otimes A_2 \otimes A_3$ is factorized. We then obtain:

$$\begin{aligned} \text{tr}_2[\text{tr}_3(A(1_1 \otimes 1_2 \otimes B))] &= \text{tr}_2[\text{tr}_3(A_1 \otimes A_2 \otimes A_3(1_1 \otimes 1_2 \otimes B))] \\ &= \text{tr}_2[\text{tr}_3(A_1 \otimes A_2 \otimes A_3 B)] \\ &= \text{tr}_2[A_1 \otimes A_2 \text{tr}(A_3 B)] = \text{tr}(A_3 B) \text{tr}_2(A_1 \otimes A_2) \\ &= \text{tr}(A_3 B) \text{tr}(A_2) A_1 = \text{tr}(A_2) \text{tr}_3(A_1 \otimes A_3 B) \\ &= \text{tr}_3[\text{tr}(A_2)(A_1 \otimes A_3)(1_1 \otimes B)] \\ &= \text{tr}_3[(\text{tr}_2(A_1 \otimes A_2 \otimes A_3))(1_1 \otimes B)] \\ &= \text{tr}_3[(\text{tr}_2(A))(1_1 \otimes B)] \end{aligned}$$

Hence, (14) holds when A is factorized. Since any $A \in \mathcal{L}(H_1 \otimes H_2 \otimes H_3)$ is a linear combination of factorized operators, (14) holds in general. (b) Letting $A = \nu(\rho_1 \otimes 1_2 \otimes \eta)$, $B = F_x$ and $K = H_3$ in (14), we conclude that (12) and (13) coincide. Hence, $\widehat{\mathcal{M}}^1 = \widehat{\mathcal{M}}_1$ and similarly, $\widehat{\mathcal{M}}^2 = \widehat{\mathcal{M}}_2$. $\square$

We considered single probe composite MMs. We now briefly discuss general composite MMs. Let $\mathcal{M}_i = (H_i, K_i, \eta_i, \nu_i, F_i)$, $i = 1, 2$ be two MMs. Define the unitary swap operator [2]:

$$U: H_1 \otimes H_2 \otimes K_1 \otimes K_2 \rightarrow H_1 \otimes K_1 \otimes H_2 \otimes K_2$$

by

$$U(\phi_1 \otimes \phi_2 \otimes \psi_1 \otimes \psi_2) = \phi_1 \otimes \psi_1 \otimes \phi_2 \otimes \psi_2$$

We now define the channel $\nu_1 \otimes \nu_2 \in \mathcal{C}(H_1 \otimes H_2 \otimes K_1 \otimes K_2)$ by

$$\nu_1 \otimes \nu_2(\rho_1 \otimes \rho_2 \otimes \eta_1 \otimes \eta_2) = U^*[\nu_1(\rho_1 \otimes \eta_2) \otimes \nu_2(\rho_2 \otimes \eta_1)]U \quad (15)$$

The composite of $\mathcal{M}_1$ and $\mathcal{M}_2$ is declared to be:

$$\mathcal{M} = \mathcal{M}_1 \otimes \mathcal{M}_2 = (H_1 \otimes H_2, K_1 \otimes K_2, \eta_1 \otimes \eta_2, \nu_1 \otimes \nu_2, F_1 \otimes F_2)$$

For $\rho \in \mathcal{S}(H_1 \otimes H_2)$, we have that:

$$\widehat{\mathcal{M}}_{(x,y)}(\rho) = \text{tr}_{K_1 \otimes K_2}[\nu_1 \otimes \nu_2(\rho \otimes \eta_1 \otimes \eta_2)(1_1 \otimes 1_2 \otimes F_{1,x} \otimes F_{2,y})] \quad (16)$$

The next result shows that $\mathcal{M}$ has desirable properties.

Theorem 11. (a) For $\rho = \rho_1 \otimes \rho_2 \in \mathcal{S}(H_1 \otimes H_2)$ we have:

$$\widehat{\mathcal{M}}_{(x,y)}(\rho) = \widehat{\mathcal{M}}_{1,x}(\rho_1) \otimes \widehat{\mathcal{M}}_{2,y}(\rho_2)$$

(b) defining $\widehat{\mathcal{M}}^1$ and $\widehat{\mathcal{M}}^2$ in the usual way, we obtain:

$$\widehat{\mathcal{M}}_{(x,y)}^1(\rho_1) = \frac{1}{n_2} \operatorname{tr} [\widehat{\mathcal{M}}_{2,y}(1_2)] \widehat{\mathcal{M}}_{1,x}(\rho_1)$$

and:

$$\widehat{\mathcal{M}}_{(x,y)}^2(\rho_2) = \frac{1}{n_1} \operatorname{tr} [\widehat{\mathcal{M}}_{1,x}(1_1)] \widehat{\mathcal{M}}_{2,y}(\rho_2)$$

for all $\rho_1 \in \mathcal{S}(H_1)$, $\rho_2 \in \mathcal{S}(H_2)$.

Proof. (a) Applying (15) and (16) we obtain:

$$\begin{aligned} \widehat{\mathcal{M}}_{(x,y)}(\rho) &= \operatorname{tr}_{K_1 \otimes K_2} [U^* v_1(\rho_1 \otimes \eta_1) \otimes v_2(\rho_2 \otimes \eta_2) U(1_1 \otimes 1_2 \otimes F_{1,x} \otimes F_{2,y})] \\ &= \operatorname{tr}_{K_1 \otimes K_2} [v_1(\rho_1 \otimes \eta_1) \otimes v_2(\rho_2 \otimes \eta_2) U(1_1 \otimes 1_2 \otimes F_{1,x} \otimes F_{2,y}) U^*] \\ &= \operatorname{tr}_{K_1} \operatorname{tr}_{K_2} [v_1(\rho_1 \otimes \eta_1) \otimes v_2(\rho_2 \otimes \eta_2) (1_1 \otimes F_{1,x} \otimes 1_2 \otimes F_{2,y})] \\ &= \operatorname{tr}_{K_1} \operatorname{tr}_{K_2} [v_1(\rho_1 \otimes \eta_1)(1_1 \otimes F_{1,x}) \otimes v_2(\rho_2 \otimes \eta_2)(1_2 \otimes F_{2,y})] \\ &= \widehat{\mathcal{M}}_{1,x}(\rho_1) \otimes \widehat{\mathcal{M}}_{2,y}(\rho_2) \end{aligned}$$

(b) For $\rho_1 \in \mathcal{S}(H_1)$ we have that:

$$\begin{aligned} \mathcal{M}_{(x,y)}^1(\rho_1) &= \frac{1}{n_2} \operatorname{tr}_2 [\widehat{\mathcal{M}}_{(x,y)}(\rho_1 \otimes 1_2)] = \frac{1}{n_2} \operatorname{tr}_2 [\widehat{\mathcal{M}}_{1,x}(\rho_1) \otimes \widehat{\mathcal{M}}_{2,y}(1_2)] \\ &= \frac{1}{n_2} \operatorname{tr} [\widehat{\mathcal{M}}_{2,y}(1_2)] \widehat{\mathcal{M}}_{1,x}(\rho_1) \end{aligned}$$

The expression for $\widehat{\mathcal{M}}^2$ is similar. $\square$

5. Concluding Remarks

In this article, we only considered finite-dimensional Hilbert spaces. One reason for this was to avoid various measure theoretic details and thus greatly simplify the exposition. A second reason was that the direction of quantum investigations has largely changed over the last twenty years. This modern research is mainly concerned with more practical matters involving quantum computation and information theory [2–4,13]. Although it is restrictive to only consider finite-dimensional Hilbert spaces, the resulting structures are general enough to include these modern theories. Nevertheless, with more work, many of our results extend to the infinite-dimensional case. For example, in this situation, an *observable* is defined as an effect-valued measure $A: \mathcal{B}(\mathbb{R}) \rightarrow \mathcal{E}(H)$, where $\mathcal{B}(\mathbb{R})$ is the collection of Borel subsets of $\mathbb{R}$ [1–3]. If $B: \mathcal{B}(\mathbb{R}) \rightarrow \mathcal{E}(H)$ is another observable, we can define their *sequential product* $A \circ B: \mathcal{B}(\mathbb{R}^2) \rightarrow \mathcal{E}(H)$ by

$$A \circ B(X \times Y) = A(X) \circ B(Y) \quad (17)$$

for all $X, Y \in \mathcal{B}(\mathbb{R})$. Some measure theoretic details are required to show that $A \circ B$ as defined in (17) for product sets $X \times Y$ extends to an effect-valued measure on $\mathcal{B}(\mathbb{R}^2)$. In a similar way, we define an *instrument* as an operation-valued measure on $\mathcal{B}(\mathbb{R})$. Just as we did before, an instrument $\mathcal{I}$ measures a unique observable $\widehat{\mathcal{I}}$. Moreover, it is straightforward to define measurement models for infinite-dimensional Hilbert spaces. One can define parts of observables in a natural way and employing measure theory, Theorem 1 extends to infinite-dimensions. We leave it to the reader to check which other theorems extend.

We now summarize the main results of this article. The basic concept of our work is an effect $a \in \mathcal{E}(H)$. An effect represents the simplest type of experiment in which the result of a measurement is either yes or no (also called true or false). An example would be to

flip a switch: a light either goes on or does not go on. An observable A corresponds to a more complicated experiment that has a finite number of possible outcomes $x \in \Omega_A$. If a measurement of A results in outcome x , then the effect A_x is true (has answer yes) and otherwise A_x is false (has answer no). If our underlying physical system is in a state $\rho \in \mathcal{S}(H)$, then the probability that a measurement of A results in outcome x is given by Born's rule $\text{tr}(\rho A_x)$. Various physical apparatuses may be employed to measure an observable A . The most common of these is $\mathcal{I} = \{\mathcal{I}_x : x \in \Omega_A\}$ where $\mathcal{I}_x : \mathcal{S}(H) \rightarrow \mathcal{S}(H)$ is a completely positive linear map. We say that $\mathcal{I}$ measures the observable A if $\text{tr}[\mathcal{I}_x(\rho)] = \text{tr}(\rho A_x)$. Thus, $\mathcal{I}$ and A have the same probability distribution. However, $\mathcal{I}$ gives more information than its unique measured observable $\hat{\mathcal{I}} = A$ because $\mathcal{I}$ determines the updated state $\mathcal{I}_x(\rho)/\text{tr}(\rho A_x)$ when A has outcome x .

Finally, we have the concept of a finite measurement model (MM) given by $\mathcal{M} = (H, K, \eta, \nu, F)$. In this case, H is the Hilbert space describing the system being observed and K is the Hilbert space describing the measuring apparatus which is in the initial state $\eta \in \mathcal{S}(K)$. In order to perform the measurement, the two systems interact, which is described by the tensor product $H \otimes K$ and a channel ν on $H \otimes K$. The initial state $\rho \in \mathcal{S}(H)$ of the observed system combines with η to form the state $\rho \otimes \eta \in H \otimes K$. This state is sent through the channel ν and the probe observable $F \in \mathcal{O}(K)$ is applied to give a measurement outcome. The model $\mathcal{M}$ measures a unique instrument $\hat{\mathcal{M}} \in \text{In}(H)$ given by

$$\hat{\mathcal{M}}_x(\rho) = \text{tr}_K[\nu(\rho \otimes \eta)(I \otimes F_x)]$$

and a unique observable $\mathcal{M}_x^{\wedge} \in \mathcal{O}(H)$.

We defined a function $A = f(B)$ of an observable B and call A a *part* of B . We can then think of A as being obtained from B by piecing together parts of B . This concept is extended to parts of instruments and MM's. Theorem 1 shows that parts are preserved under measurements. For example, $f(\hat{\mathcal{I}}) = f(\mathcal{I})^{\wedge}$ for all instruments $\mathcal{I}$ and $f(\hat{\mathcal{M}}) = f(\mathcal{M})^{\wedge}$ for all MM's $\mathcal{M}$. We then consider the coexistence of observables and Theorem 2 shows that observables coexist if and only if they are jointly measurable. We next introduce the concepts of sequential products $A \circ B$ and conditioning $(B | A)$ of observables $A, B \in \mathcal{O}(H)$. We interpret $A \circ B$ as the observable obtained by first measuring A and then measuring B and $(B | A)$ as the observable B conditioned on first measuring the observable A . Theorem 3 shows that $A, (B | A)$ and $A \circ f(B)$ are parts of $A \circ B$. Sequential products of instruments are also discussed. For an observable $A \in \mathcal{O}(H)$, we consider the corresponding Lüders instrument $\mathcal{L}^A \in \text{In}(H)$. Theorem 4 shows that $\mathcal{L}^{A \circ B} = \mathcal{L}^A \circ \mathcal{L}^B$ if and only if A and B commute. Moreover, $(\mathcal{L}^{A \circ B})^{\wedge} = A \circ B$ and C is a part of $\mathcal{L}^{A \circ B}$ if and only if C is a part of $A \circ B$.

Section 4 considers composite systems $H = H_1 \otimes H_2$. If $a \in \mathcal{E}(H)$, we define the *reduced effects* $a^1 \in \mathcal{E}(H_1), a^2 \in \mathcal{E}(H_2)$, where a^i is thought of as the effect as measured in the system $H_i, i = 1, 2$. This concept is extended to the reduced observables $A^i \in \mathcal{O}(H_i), i = 1, 2$, for $A \in \mathcal{O}(H_1 \otimes H_2)$. Theorem 6 proves results concerning post-processing for composite systems and Theorem 7 shows that coexistence is preserved on taking tensor products and under reducing observables. Reduced instruments $\mathcal{I}^i \in \text{In}(H_i), i = 1, 2$, for $\mathcal{I} \in \text{In}(H_1 \otimes H_2)$ are also defined and Theorems 8 and 9 show (among other things) that $(\mathcal{I}^1)^{\wedge} = (\hat{\mathcal{I}})^1, (\mathcal{I}^2)^{\wedge} = (\hat{\mathcal{I}})^2$ and $(\mathcal{I}_1 \otimes \mathcal{I}_2)^{\wedge} = \hat{\mathcal{I}}_1 \otimes \hat{\mathcal{I}}_2$. Finally, we consider MM's for composite systems. A *single probe* MM on $H = H_1 \otimes H_2$ has the usual form $\mathcal{M} = (H, K, \eta, \nu, F)$. Then, for $\rho_1 \in \mathcal{S}(H_1)$, we obtain $\hat{\mathcal{M}}^1 \in \text{In}(H_1)$ given by

$$\hat{\mathcal{M}}_x^1(\rho_1) = \frac{1}{n_2} \text{tr}_2\{\text{tr}_K[\eta(\rho_1 \otimes 1_2 \otimes \eta)(1_1 \otimes 1_2 \otimes F_x)]\} \quad (18)$$

where $n_2 = \dim H_2$. There is a similar expression for $\hat{\mathcal{M}}^2 \in \text{In}(H_2)$. Corresponding to $\mathcal{M}$, we define the *reduced* MM given by $\mathcal{M}_1 = (H_1, K, \eta, \nu_1, F)$ where $\nu_1 \in \mathcal{S}(H_1 \otimes K)$ is given by

$$\nu_1(\rho_1 \otimes \eta) = \frac{1}{n_2} \text{tr}_2[\nu(\rho_1 \otimes 1_2 \otimes \eta)]$$

We think of $\mathcal{M}_1$ as the MM $\mathcal{M}$ as measured by the H_1 system. We then obtain:

$$\widehat{\mathcal{M}}_{1,x}(\rho_1) = \frac{1}{n_2} \text{tr}_K \{ \text{tr}_2 [v(\rho_1 \otimes 1_2 \otimes \eta)] (1_1 \otimes F_x) \} \quad (19)$$

Similarly, we define $\mathcal{M}_2 = (H_2, K, \eta, v_2, F)$ and obtain an analogous formula for $\widehat{\mathcal{M}}_2$. Now, $\widehat{\mathcal{M}}_x^1$ and $\widehat{\mathcal{M}}_{1,x}$ essentially represent the same object and (18), (19) are quite similar. In fact, Theorem 10 shows that they coincide. This paper closes with a definition of a general composite MM and Theorem 11 shows that this definition has desirable properties.

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