



Article Generalized Variational Principle for the Fractal (2 + 1)-Dimensional Zakharov–Kuznetsov Equation in Quantum Magneto-Plasmas

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Abstract: In this paper, we propose the fractal (2 + 1)-dimensional Zakharov–Kuznetsov equation based on He's fractal derivative for the first time. The fractal generalized variational formulation is established by using the semi-inverse method and two-scale fractal theory. The obtained fractal variational principle is important since it not only reveals the structure of the traveling wave solutions but also helps us study the symmetric theory. The finding of this paper will contribute to the study of symmetry in the fractal space.

Keywords: fractal variational principle; semi-inverse method; two-scale fractal theory; symmetry

1. Introduction

Nonlinear differential equations are widely used to describe various complex phenomena arising in physics, biology, chemistry, and other fields [1-4]. The study of their solutions has always been the focus of the researchers. In this work, we mainly study the well-known (2 + 1)-dimensional Zakharov–Kuznetsov (Z-K) equation, which is first derived by Zakharov and Kuznetsov and can well describe the propagation of nonlinear ionic-sonic waves in a magnetized plasma composed of cold ions and hot isothermal electrons. The (2 + 1)-dimensional Z-K equation is illustrated in detail in [5-7] as:

$$u_t + 2muu_x + nu_{xxx} + ku_{xyy} = 0, (1)$$

where *m*, *n*, and *k* are non-zero constants. In recent years, the exact solutions of Equation (1) have been studied by many researchers, and many effective solutions have been obtained, such as the generalized exponential rational function method [8], Lie group analysis [9], group analysis approach [10], Exp-function method [11], Coupled Burgers' equations method [12], extended tanh method [13], and so on [14–17]. These results are important and can help us study the (2 + 1)-dimensional Z-K equation. But the research on its variational principle is relatively less. The study of the variational principle is also of great significance because it can help us study the symmetries, reveals the possible energy conservation of the whole solution domain, and plays a key role in the numerical and analytical analysis of practical problems involved in chemistry [18], mechanics [19,20], nano system [21], economics [22,23], life science [24], mathematics [25,26] and so on [27–29].

Recently, fractal calculus is a hot topic and has been used widely to model many complex phenomena involved in fractal filtering [30–32], physics [33,34], circuit [35], biomedical science [36,37], and so on. Motivated by the recent research on fractal calculus, this paper, for the first time, proposes the fractal (2 + 1)-dimensional Z-K equation that can describe the propagation of nonlinear ionic-sonic waves with non-smooth boundary (such as the fractal boundary in Figure 1) via replacing the smooth space (x, y, t) in Equation (1) by a



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Copyright: © 2021 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https:// creativecommons.org/licenses/by/ 4.0/). fractal space $(x^{\beta}, y^{\gamma}, t^{\rho})$, where β , γ , and ρ are, respectively, fractal dimensions in space and time. So, in the fractal space, Equation (1) can be modified as:

$$\frac{\partial}{\partial t^{\rho}}u + 2mu\frac{\partial}{\partial x^{\beta}}u + n\frac{\partial^{3}}{\partial x^{3\beta}}u + k\frac{\partial}{\partial x^{\beta}}\frac{\partial^{2}}{\partial y^{2\gamma}}u = 0,$$
(2)

where $\frac{\partial}{dt^{\beta}}$, $\frac{\partial}{dx^{\beta}}$, and $\frac{\partial}{dy^{\gamma}}$ are the He's fractal derivatives with respect to *t*, *x*, and *y* that are defined as [38,39]:

$$\frac{\partial}{dt^{\rho}}u(x,y,t_{0}) = \Gamma(1+\rho) \lim_{\substack{t-t_{0} = \Delta t \\ \Delta t \neq 0}} \frac{u(x,y,t) - u(x,y,t_{0})}{(t-t_{0})^{\rho}}$$
(3)

$$\frac{\partial u}{\partial x^{\beta}}u(x_0, y, t) = \Gamma(1+\beta) \lim_{\substack{x - x_0 = \Delta x \\ \Delta x \neq 0}} \frac{u(x, y, t) - u(x_0, y, t)}{(x - x_0)^{\beta}}$$
(4)

$$\frac{\partial u}{\partial y^{\gamma}}u(x,y_0,t) = \Gamma(1+\gamma) \lim_{\substack{y-y_0 = \Delta y \\ \Delta y \neq 0}} \frac{u(x,y,t) - u(x_0,y,t)}{(y-y_0)^{\gamma}}$$
(5)



Figure 1. The non-smooth boundary (fractal boundary).

There are the following chain rules:

$$\frac{\partial^2}{\partial x^{2\beta}} = \frac{\partial}{\partial x^{\beta}} \frac{\partial}{\partial x^{\beta}}$$
(6)

$$\frac{\partial^3}{\partial x^{3\beta}} = \frac{\partial}{\partial x^\beta} \frac{\partial}{\partial x^\beta} \frac{\partial}{\partial x^\beta} \tag{7}$$

Noting that Equation (2) becomes the classic (2 + 1)-dimensional Z-K equation when $\gamma = \beta = \rho = 1$.

2. Two-Scale Fractal Theory

The two-scale fractal theory is a powerful tool to solve fractal equations. Now we introduce the two-scale transforms in the fractal time and spatial respectively as [40,41]:

$$T = t^{\rho} \tag{8}$$

$$X = x^{\beta} \tag{9}$$

$$Y = y^{\gamma} \tag{10}$$

where *x*, *y*, *t* are for the small scale and *X*, *Y*, *T* for large scale, ρ , β , γ are the two-scale dimensions.

Applying the above transforms to Equation (2), Equation (2) can be converted into the following form:

$$u_T + 2muu_X + nu_{XXX} + ku_{XYY} = 0 \tag{11}$$

In the following content, we mainly use the semi-inverse method to develop the variational principle of Equation (2).

3. The Fractal Variational Principle

In order to use the semi-inverse method [42–50], we first re-write Equation (11) into the conserved form as:

$$u_T + \left(mu^2 + nu_{XX} + ku_{YY}\right)_X = 0 \tag{12}$$

Now we introduce a new function ω , which satisfies:

$$\frac{\partial\omega}{\partial X} = u \tag{13}$$

$$\frac{\partial\omega}{\partial T} = -\left(mu^2 + nu_{XX} + ku_{YY}\right) \tag{14}$$

Then we aim to structure a variational formulation for Equation (12) as:

$$J(\omega, u) = \iiint L(u, u_T, u_X, u_Y, u_{TT}, u_{XX}, u_{YY}, \omega, \omega_X, \omega_Y) dT dX dY$$
(15)

where *L* is the trial-Lagrange function.

According to the semi-inverse method, we suppose the trial-Lagrange function with the following form:

$$L = u\omega_T + \left(mu^2 + nu_{XX} + ku_{YY}\right)\omega_X + \varepsilon$$
(16)

here ε is an unknown function of u, and/or ω , and/or their derivatives.

Taking a variation on Equation (16) with respect to ω , yields:

$$-u_T - \left(mu^2 + nu_{XX} + ku_{YY}\right)_X + \frac{\delta\varepsilon}{\delta\omega} = 0$$
(17)

where $\delta \varepsilon / \delta \omega$ is called the variational derivative, which takes the following form in this paper:

$$\frac{\delta\varepsilon}{\delta\omega} = \frac{\partial\varepsilon}{\partial\omega} + \frac{\partial^2}{\partial X^2} \left(\frac{\partial\varepsilon}{\partial\omega_{XX}}\right) + \frac{\partial^2}{\partial y^2} \left(\frac{\partial\varepsilon}{\partial\omega_{YY}}\right)$$
(18)

By carefully comparing Equations (17) and (12), it is easy to find that when $\frac{\delta \varepsilon}{\delta \omega} = 0$, Equation (17) becomes Equation (12). So, we set:

$$\frac{\delta\varepsilon}{\delta\omega} = 0 \tag{19}$$

Making a variation on Equation (16) with respect to *u*, we have:

$$\omega_T + 2mu\omega_X + n\omega_{XXX} + k\omega_{XYY} + \frac{\delta\varepsilon}{\delta u} = 0$$
⁽²⁰⁾

where $\delta \varepsilon / \delta u$ is the variational derivative. In this paper, it can be written as:

$$\frac{\delta\varepsilon}{\delta u} = \frac{\partial\varepsilon}{\partial u} + \frac{\partial^2}{\partial X^2} \left(\frac{\partial\varepsilon}{\partial u_{XX}}\right) + \frac{\partial^2}{\partial Y^2} \left(\frac{\partial\varepsilon}{\partial u_{YY}}\right)$$
(21)

In the view of Equations (13) and (14), there are:

$$= -\omega_T - 2mu\omega_X - n\omega_{XXX} - k\omega_{XYY}$$

= $(mu^2 + nu_{XX} + ku_{YY}) - 2mu^2 - nu_{XX} - ku_{YY}$
= $-mu^2$ (22)

So, ε can be identified as:

$$\varepsilon = -\frac{1}{3}mu^3 \tag{23}$$

Finally, the following Lagrange function can be obtained:

$$L = u\omega_X + \left(mu^2 + nu_{XX} + ku_{YY}\right)\omega_X - \frac{1}{3}mu^3$$
(24)

Then we get the variational formulation for Equation (13) as:

$$J(\omega, u) = \iiint \left\{ u\omega_T + \left(mu^2 + nu_{XX} + ku_{YY} \right) \omega_X - \frac{1}{3}mu^3 \right\} dT dX dY$$
(25)

which is subject to Equation (13).

Proof. The Euler–Lagrange equations of Equation (25) are:

$$u_T + \left(mu^2 + nu_{XX} + ku_{YY}\right)_X = 0,$$
 (26)

$$\omega_T + 2mu\omega_X + n\omega_{XXX} + k\omega_{XYY} - mu^2 = 0 \tag{27}$$

Under the constraint condition given by Equation (13), it can be proven that Equations (26) and (27) are equivalent to Equations (12) and (14), respectively. In the view of the two-scale transforms of Equations (8)–(10), we can get the fractal variational formulation of Equation (2) as:

$$J(\omega, u) = \iiint \left\{ u \frac{\partial \omega}{\partial t^{\rho}} + \left(mu^2 + n \frac{\partial u}{\partial x^{2\beta}} + k \frac{\partial u}{\partial y^{2\gamma}} \right) \frac{\partial \omega}{\partial x^{\beta}} - \frac{1}{3} mu^3 \right\} dt^{\rho} dx^{\beta} dy^{\gamma}$$
(28)

The obtained fractal variational principle in Equation (28) can help us study the symmetries and the structure of the traveling wave solutions for the fractal (2 + 1)-dimensional Z-K equation in plasma physics.

4. Conclusions and Future Recommendation

Based on He's fractal derivative, we propose the fractal (2 + 1)-dimensional Z-K equation with a fractal boundary for the first time. By using the two-scale fractal theory and semi-inverse method, we successfully establish its fractal generalized variational formulation. And the obtained variational formulation is proven correct by minimizing the trial-Lagrange function with the calculus of variations. The whole derivation process is given in detail.

Furthermore, the variational formulation is the theoretical basis of the variational method [51–54] to seek the traveling wave solutions since it can reveal the solution structures. How to use the variational method to obtain the traveling wave solutions of Equation (2) is the focus of our future research. The obtained results in this work are expected to be helpful for the study of the symmetry and traveling wave theory in the fractal space.

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