Article

Preserving the Shape of Functions by Applying Multidimensional Schoenberg-Type Operators

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Abstract: The paper presents a multidimensional generalization of the Schoenberg operators of higher order. The new operators are powerful tools that can be used for approximation processes in many fields of applied sciences. The construction of these operators uses a symmetry regarding the domain of definition. The degree of approximation by sequences of such operators is given in terms of the first and the second order moduli of continuity. Extending certain results obtained by Marsden in the one-dimensional case, the property of preservation of monotonicity and convexity is proved.

Keywords: multidimensional splines; multidimensional Schoenberg-type operators; order of approximation; monotonicity; convexity

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1. Introduction

The theory of splines approximation was founded by Schoenberg and became one of the main chapters of approximation theory. Now there is a vast literature dedicated to spline approximation. We refer the reader to the monograph of Schumaker [1] for historical notes. The success of this type of approximation is due both to the nice mathematical theory and to the great efficiency in practical applications. In practice, the spline approximation is more efficient than the polynomial approximation.

In [2], Schoenberg considered also a particular method of approximation of functions by splines, with the aid of certain positive linear operators, which are named the Schoenberg operators. Important contributions in the study of these operators are due to Marsden [3].

In more recent times, the topic of one-dimensional Schoenberg spline operators are presented in papers written for instance in Gonska [4–7], Tachev [4–8], Beutel [5–7,9], Kacsó [4–7] and our papers [10–12].

The subject of multidimensional spline is developed in many papers. We can specify here the paper [13] where the approximation of functions using multivariate splines is presented and the monograph [14], which is dedicated to the theory of multivariate splines. We mention also the paper [15] where the multivariate polynomial interpolation is approached, the paper [16] where a computationally effective way to construct stable bases on general non-degenerate lattices is presented, Reference [17] where the subject of Hermite-vector splines and multi-wavelets is developed and the paper [18] in which a generalization of bases, namely B-spline frames, is approached. Estimates of approximation by linear operators in the multidimensional case are established in [19].

As exemplification of the application of the Schoenberg operators in practice we mention the recent paper [20] where one-dimensional Schoenberg spline operators were used, obtaining a substantial improvement of the clear sky models which estimate the direct solar irradiance.

The present paper is a continuation of paper [12], where two-dimensional Schoenberg operators were considered. Now we extend this definition in multidimensional case and we establish certain properties of them.

Several important connections with symmetry exist in this study. Because these operators present a symmetry in their construction, the computation of their moments is made by symmetry. The symmetry is also used in establishing the estimates with second order moduli, which are defined with the aid of finite symmetric differences. On the other hand, we study the property of preservation of convexity and this property can be described using the Hessian of functions, which is a symmetric quadratic form.

2. Multidimensional Schoenberg-Type Operators on Arbitrary Nodes

We consider the integers $j, m, 1 \leq j \leq m; n_j > 0; k_j > 0$; the vector $(x_1, \dots, x_m) \in [0, 1]^m$ and the knots sequences Δ_{n_j, k_j}

$$0 = v_{-k_j} = v_{-k_j+1} = \dots = v_{0_j} < v_{1_j} < v_{2_j} < \dots < v_{n_j} = v_{n_j+1} = \dots = v_{n_j+k_j} = 1. \quad (1)$$

The Greville abscissas associated with division $\Delta_{n_j, k_j}, 1 \leq j \leq m$ have the next form

$$\xi_{i_j} := \frac{v_{i_j+1} + v_{i_j+2} + \dots + v_{i_j+k_j}}{k_j}, \quad -k_j \leq i_j \leq n_j - 1. \quad (2)$$

The B-splines $N_{i_j, k_j}(x_j), 1 \leq j \leq m$ depending on Δ_{n_j, k_j} are

$$N_{i_j, k_j}(x_j) = (v_{i_j+k_j+1} - v_{i_j})[v_{i_j}, v_{i_j+1}, \dots, v_{i_j+k_j+1}](\cdot - x_j)_+^{k_j}, \quad -k_j \leq i_j \leq n_j - 1. \quad (3)$$

When $x_j \in [v_{q_j}, v_{q_j+1}]$ with $0 \leq q_j \leq n_j - 1, 1 \leq j \leq m$ we have:

$$N_{i_j, k_j}(x_j) = 0, \text{ for } i_j < q_j - k_j \text{ or } i_j \geq q_j + 1, \text{ and } N_{i_j, k_j}(x_j) \geq 0, \text{ for } q_j - k_j \leq i_j \leq q_j.$$

The next relations take place

$$\sum_{i_j=-k_j}^{n_j-1} N_{i_j, k_j}(x_j) = 1, \text{ for } x_j \in [0, 1] \quad (4)$$

and

$$\sum_{i_j=-k_j}^{n_j-1} \xi_{i_j} N_{i_j, k_j}(x_j) = x_j \text{ for } x_j \in [0, 1], \quad (5)$$

where $1 \leq j \leq m$.

We consider $\Delta^* = \Delta_{n_1, k_1} \times \Delta_{n_2, k_2} \times \dots \times \Delta_{n_m, k_m}$

$$\Delta^* := \{(v_{i_1}, \dots, v_{i_m}), -k_j \leq i_j \leq n_j + k_j, 1 \leq j \leq m\}. \quad (6)$$

Definition 1. Multidimensional Schoenberg-type operator associated with Δ^* has the form

$$\begin{aligned} & (S_{\Delta^*} f)(x_1, \dots, x_m) \\ &= \sum_{i_1=-k_1}^{n_1-1} \dots \sum_{i_m=-k_m}^{n_m-1} (N_{i_1, k_1} \otimes \dots \otimes N_{i_m, k_m})(x_1, \dots, x_m) f(\xi_{i_1}, \dots, \xi_{i_m}), \end{aligned} \quad (7)$$

where $f : [0, 1]^m \rightarrow \mathbb{R}$, and $x = (x_1, \dots, x_m) \in [0, 1]^m$.

Remark 1.

- (i) Symmetrizing the knots v_{i_j} on each components by function $\sigma(x) = 1 - x, x \in [0, 1]$ one obtains also a Schoenberg-type operators of the same degree. If the knots are equidistant, one obtains the same Schoenberg-type operators.

(ii) For $x_j \in [v_{q_j}, v_{q_j+1}]$, with $0 \leq q_j \leq n_j - 1$ and $1 \leq j \leq m$, then

$$(S_{\Delta^*} f)(x_1, \dots, x_m) = \sum_{i_1=q_1-k_1}^{q_1} \dots \sum_{i_m=q_m-k_m}^{q_m} N_{i_1,k_1}(x_1) \dots N_{i_m,k_m}(x_m) f(\xi_{i_1}, \dots, \xi_{i_m}). \quad (8)$$

(iii) Multidimensional Schoenberg-type operators are linear and positive.

(iv) S_{Δ^*} is a polynomial of degree at most k_j in each variable x_j , $1 \leq j \leq m$, on each domain $[v_{i_1-1}, v_{i_1}] \times \dots \times [v_{i_m-1}, v_{i_m}]$, with $0 \leq i_j \leq n_j - 1$ and $1 \leq j \leq m$.

(v) S_{Δ^*} is a B-spline in each variable.

(vi) Multidimensional Schoenberg-type operators admit partial continuous derivatives on $[0, 1]^m$, since

$$\begin{aligned} & \left(\frac{\partial^{i_1+\dots+i_m}}{\partial x_1^{i_1} \dots \partial x_m^{i_m}} (S_{\Delta^*} f) \right) (x_1, \dots, x_m) \\ &= \sum_{i_1=-k_1}^{n_1-1} \frac{\partial^{i_1}}{\partial x_1^{i_1}} N_{i_1,k_1}(x_1) \dots \sum_{i_m=-k_m}^{n_m-1} \frac{\partial^{i_m}}{\partial x_m^{i_m}} N_{i_m,k_m}(x_m) f(\xi_{i_1}, \dots, \xi_{i_m}), \end{aligned}$$

where $0 \leq i_j \leq k_j$, $1 \leq j \leq m$.

We consider the next functions: $e_0 \in C([0, 1]^m)$, $e_0(x_1, \dots, x_m) = 1$ and $\pi_j \in C([0, 1]^m)$, $\pi_j(x_1, \dots, x_m) = x_j$, for $(x_1, \dots, x_m) \in [0, 1]^m$, $1 \leq j \leq m$.

Proposition 1. For $(x_1, \dots, x_m) \in [0, 1]^m$ we have

- (i) $(S_{\Delta^*} e_0)(x_1, \dots, x_m) = 1$;
- (ii) $(S_{\Delta^*} \pi_j)(x_1, \dots, x_m) = x_j$, $1 \leq j \leq m$;
- (iii)

$$S_{\Delta^*} \left(\prod_{j=1}^m \pi_j \right) (x_1, \dots, x_m) = \prod_{j=1}^m x_j.$$

We use the next notations: $e_1(t) = t$, $t \in [0, 1]$; e_0 for the constant function equal to 1, on $[0, 1]^m$, $1 \leq j \leq m$ and Δ_j , $1 \leq j \leq m$ denotes the knot sequence use to one-dimensional Schoenberg operators.

Proposition 2. For $(x_1, \dots, x_m) \in [0, 1]^m$ we have

- (i) $(S_{\Delta^*} (\pi_j - x_j e_0))(x_1, \dots, x_m) = 0$,
- (ii)

$$\left(S_{\Delta^*} \left(\prod_{j=1}^m (\pi_j - x_j e_0) \right) \right) (x_1, \dots, x_m) = 0,$$

- (iii) $(S_{\Delta^*} (\pi_j - x_j e_0)^m)(x_1, \dots, x_m) = (S_{\Delta_j} (e_1 - x_j e_0)^m)(x_j)$, $1 \leq j \leq m$.

Theorem 1. For multidimensional Schoenberg-type operators

$$(S_{\Delta^*} f)(x_1, \dots, x_m) = \sum_{i_1=-k_1}^{n_1-1} \dots \sum_{i_m=-k_m}^{n_m-1} N_{i_1,k_1}(x_1) \dots N_{i_m,k_m}(x_m) f(\xi_{i_1}, \dots, \xi_{i_m}), \quad (9)$$

to converge uniformly on $[0, 1]^m$ to continuous function f , it is sufficient that for any $\eta > 0$

$$\sum_{\substack{-k_j \leq i_j \leq n_j-1, 1 \leq j \leq m \\ \|(\xi_{i_1}, \dots, \xi_{i_m}) - (x_1, \dots, x_m)\| \geq \eta}} N_{i_1, k_1}(x_1) \dots N_{i_m, k_m}(x_m) \rightarrow 0, \quad (10)$$

uniformly for $0 \leq x_j \leq 1$, when $n_j \rightarrow \infty, 1 \leq j \leq m$.

Proof. We consider (10) is fulfilled.

From f continue function on $[0, 1]^m$, we have $\forall \varepsilon > 0, \exists \eta_\varepsilon > 0$ such that for any $(x_{i_1}, \dots, x_{i_m}) \in [0, 1]^m$ and $(x'_{i_1}, \dots, x'_{i_m}) \in [0, 1]^m$ with $\|(x'_{i_1}, \dots, x'_{i_m}) - (x_{i_1}, \dots, x_{i_m})\| < \eta_\varepsilon$ it results $\|f(x'_{i_1}, \dots, x'_{i_m}) - f(x_{i_1}, \dots, x_{i_m})\| < \frac{\varepsilon}{2}$.

Also $\exists M > 0$ such that $\|f(x_1, \dots, x_m)\| \leq M, (x_1, \dots, x_m) \in [0, 1]^m$.

Let $n_{\varepsilon_j} \in \mathbb{N}$, such that:

$$\sum_{\substack{-k_j \leq i_j \leq n_j-1, 1 \leq j \leq m \\ \|(\xi_{i_1}, \dots, \xi_{i_m}) - (x_1, \dots, x_m)\| \geq \eta_\varepsilon}} N_{i_1, k_1}(x_1) \dots N_{i_m, k_m}(x_m) < \frac{\varepsilon}{4M},$$

for $n \geq n_{\varepsilon_j}$. For such n we obtain

$$\begin{aligned} & |(S_{\Delta} f)(x_1, \dots, x_m) - f(x_1, \dots, x_m)| \\ & \leq \sum_{\substack{-k_j \leq i_j \leq n_j-1, 1 \leq j \leq m \\ \|(\xi_{i_1}, \dots, \xi_{i_m}) - (x_1, \dots, x_m)\| \geq \eta_\varepsilon}} N_{i_1, k_1}(x_1) \dots N_{i_m, k_m}(x_m) |f(\xi_{i_1}, \dots, \xi_{i_m}) - f(x_1, \dots, x_m)| \\ & \quad + \sum_{\substack{-k_j \leq i_j \leq n_j-1, 1 \leq j \leq m \\ \|(\xi_{i_1}, \dots, \xi_{i_m}) - (x_1, \dots, x_m)\| < \eta_\varepsilon}} N_{i_1, k_1}(x_1) \dots N_{i_m, k_m}(x_m) |f(\xi_{i_1}, \dots, \xi_{i_m}) - f(x_1, \dots, x_m)| \\ & \leq 2M \sum_{\substack{-k_j \leq i_j \leq n_j-1, 1 \leq j \leq m \\ \|(\xi_{i_1}, \dots, \xi_{i_m}) - (x_1, \dots, x_m)\| \geq \eta_\varepsilon}} N_{i_1, k_1}(x_1) \dots N_{i_m, k_m}(x_m) \\ & \quad + \frac{\varepsilon}{2} \sum_{\substack{-k_j \leq i_j \leq n_j-1, 1 \leq j \leq m \\ \|(\xi_{i_1}, \dots, \xi_{i_m}) - (x_1, \dots, x_m)\| < \eta_\varepsilon}} N_{i_1, k_1}(x_1) \dots N_{i_m, k_m}(x_m) \\ & \leq 2M \cdot \frac{\varepsilon}{4M} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

□

The norm of the division Δ^* is

$$\|\Delta^*\| := \|\Delta_1\| + \|\Delta_2\| + \dots + \|\Delta_m\|, \quad (11)$$

where $\|\Delta_j\| = \max_{i_j} (x_{i_{j+1}} - x_{i_j})$.

We use the first order modulus of continuity:

$$\omega_1(f, \rho) := \sup\{|f(x) - f(y)|, x, y \in [0, 1]^m, \|x - y\| \leq \rho\}, \quad (12)$$

where $f \in C([0, 1]^m), \rho > 0$.

Theorem 2. For any $f \in C([0, 1]^m)$, operators S_{Δ^*} given in (7) satisfy inequality

$$\|(S_{\Delta^*} f) - f\| \leq \omega_1(f, \vartheta \|\Delta^*\|), \quad (13)$$

where $\vartheta = \frac{1}{2} \max_{1 \leq j \leq m} \{k_j + 1\}$.

Proof. Let the continuous function f and $(x_1, \dots, x_m) \in [0, 1]^m$. For any $1 \leq j \leq m$ there is $q_j \in \{0, 1, \dots, n_j - 1\}$, such that $x_j \in [v_{q_j}, v_{q_j+1}]$. Then $N_{i_j, k_j}(x_j) = 0$, for $-k_j \leq i_j < q_j - k_j$ and $q_j < i_j \leq n_j - 1$.

Let $q_j - k_j \leq i_j \leq q_j$, $1 \leq j \leq m$. Then

$$\begin{aligned} x_j - \xi_{i_j} &\leq v_{q_j+1} - \frac{v_{i_j+1} + \dots + v_{i_j+k_j}}{k_j} \leq v_{q_j+1} - \frac{v_{q_j-k_j+1} + \dots + v_{q_j}}{k_j} \\ &\leq \frac{1}{k_j}(k_j + (k_j - 1) + \dots + 1)\|\Delta_j\| = \frac{k_j + 1}{2}\|\Delta_j\| \end{aligned}$$

and

$$\begin{aligned} x_j - \xi_{i_j} &\geq v_{q_1} - \frac{v_{i_j+1} + \dots + v_{i_j+k_j}}{k_j} \geq v_{q_j} - \frac{v_{q_j+1} + \dots + v_{q_j+k_j}}{k_j} \\ &\geq -\frac{1}{k_j}(1 + 2 + \dots + k_j)\|\Delta_j\| = -\frac{k_j + 1}{2}\|\Delta_j\|. \end{aligned}$$

Therefore,

$$|x_j - \xi_{i_j}| \leq \frac{k_j + 1}{2}\|\Delta_j\|, \text{ for } q_j - k_j \leq i_j \leq q_j, 1 \leq j \leq m.$$

Then, for $q_j - k_j \leq i_j \leq q_j$, $1 \leq j \leq m$ we have

$$\|(x_1, \dots, x_m) - (\xi_1, \dots, \xi_m)\| \leq \sum_{j=1}^m |x_j - \xi_{i_j}| \leq \vartheta \|\Delta^*\|.$$

It results

$$\begin{aligned} &|(S_{\Delta^*}f)(x_1, \dots, x_m) - f(x_1, \dots, x_m)| \\ &= \left| \sum_{i_1=-k_1}^{n_1-1} \dots \sum_{i_m=-k_m}^{n_m-1} N_{i_1, k_1}(x_1) \dots N_{i_m, k_m}(x_m) f(\xi_{i_1}, \dots, \xi_{i_m}) - f(x_1, \dots, x_m) \right| \\ &\leq \sum_{i_1=-k_1}^{n_1-1} \dots \sum_{i_m=-k_m}^{n_m-1} N_{i_1, k_1}(x_1) \dots N_{i_m, k_m}(x_m) |f(\xi_{i_1}, \dots, \xi_{i_m}) - f(x_1, \dots, x_m)| \\ &= \sum_{i_1=q_1-k_1}^{q_1} \dots \sum_{i_m=q_m-k_m}^{q_m} N_{i_1, k_1}(x_1) \dots N_{i_m, k_m}(x_m) |f(\xi_{i_1}, \dots, \xi_{i_m}) - f(x_1, \dots, x_m)| \\ &\leq \sum_{i_1=q_1-k_1}^{q_1} \dots \sum_{i_m=q_m-k_m}^{q_m} N_{i_1, k_1}(x_1) \dots N_{i_m, k_m}(x_m) \omega_1(f, \|(\xi_{i_1}, \dots, \xi_{i_m}) - (x_1, \dots, x_m)\|) \\ &\leq \sum_{i_1=q_1-k_1}^{q_1} \dots \sum_{i_m=q_m-k_m}^{q_m} N_{i_1, k_1}(x_1) \dots N_{i_m, k_m}(x_m) \omega_1(f, \vartheta \|\Delta^*\|) \\ &= \omega_1(f, \vartheta \|\Delta^*\|). \end{aligned}$$

□

Corollary 1. Multidimensional Schoenberg-type operators

$$(S_{\Delta^*}f)(x_1, \dots, x_m) = \sum_{i_1=-k_1}^{n_1-1} \dots \sum_{i_m=-k_m}^{n_m-1} N_{i_1, k_1}(x_1) \dots N_{i_m, k_m}(x_m) f(\xi_{i_1}, \dots, \xi_{i_m})$$

converge uniformly on $[0, 1]^m$ to f , for any continuous function f if $\|\Delta^*\| \rightarrow 0$.

3. Preservation of Monotonicity and Convexity by Multidimensional Schoenberg-Type Operators with Equidistant Knots

In this section, we will extend some results obtained by Marsden in the case of one-dimensional Schoenberg operators.

Let $k \in \mathbb{N}$. We denote by $(S_\Delta \varphi)(x)$ the one-dimensional Schoenberg operators of degree k associated with the knot sequence $\Delta = \{\nu_i\}_{-k}^{n+k}$, where $\nu_{-k} = \dots = \nu_0 = 0 < \nu_1 < \dots < \nu_{n-1} < 1 = \nu_n = \dots = \nu_{n+k}$, and the Greville abscissas $\xi_j = \frac{\nu_{j+1} + \dots + \nu_{j+k}}{k}$, $(\varphi : [0, 1] \rightarrow \mathbb{R}, x \in [0, 1])$. The B-spline of degree k associated to Δ is denoted by $N_j(x)$, $-k \leq j \leq n-1$. Next, denote $\Delta^- = \{\nu_i\}_{-k+1}^{n+k-1}$. The corresponding B-splines of degree $k-1$ associated with the knot sequence Δ^- by $N_j^-(x)$, $1-k \leq j \leq n-1$ and the corresponding Greville abscissas is denoted by ξ_j^- . In addition, for $k \geq 2$, denote $\Delta^= = \{\nu_i\}_{-k+2}^{n+k-2}$ and the corresponding B-splines of degree $k-2$ associated with the knot sequence $\Delta^=$ by $N_j^=(x)$, $2-k \leq j \leq n-1$. Using these notations, the following relations are given in [3]:

$$D(S_\Delta \varphi)(x) = \sum_{j=1-k}^{n-1} \frac{f(\xi_j) - f(\xi_{j-1})}{\xi_j - \xi_{j-1}} N_j^-(x), \quad (14)$$

$$D^2(S_\Delta \varphi)(x) = \sum_{j=2-k}^{n-1} D^2 \varphi(\eta_j) \frac{\xi_j - \xi_{j-2}}{2(\xi_j^- - \xi_{j-1}^-)} N_j^=(x), \text{ where } \xi_{j-2} < \eta_j < \xi_j. \quad (15)$$

In the following theorems it is considered that n and k are variable.

Theorem 3 ([3]). Let $\varphi \in C^1[0, 1]$ and $\frac{\|\Delta\|}{k} \rightarrow 0$, $\liminf k > 1$. Then:

- (i) $(S_\Delta \varphi)(x) \rightarrow \varphi(x)$ uniformly on $[0, 1]$;
- (ii) $D(S_\Delta \varphi)(x) \rightarrow D\varphi(x)$ uniformly on $[0, 1]$.

Theorem 4 ([3]). Let $\varphi \in C^2[0, 1]$ and $x_i = \frac{i}{n}$, $0 < i < n$ the interior knots of Δ . Let $n+k \rightarrow \infty$, $\liminf n > 1$, $\liminf k > 1$. Then $\lim D^2(S_\Delta \varphi)(x) = D^2 \varphi(x)$, $0 < x < 1$. The convergence is uniform on compact subsets of $(0, 1)$.

Theorem 5 ([3]). Let $\varphi \in C^2[0, 1]$ and $k > 2$. Then:

- (i) If $D\varphi(x) \geq 0$ on $[0, 1]$ then $D(S_\Delta \varphi)(x) \geq 0$ on $[0, 1]$;
- (ii) If $D^2 \varphi(x) \geq 0$ on $[0, 1]$ then $D^2(S_\Delta \varphi)(x) \geq 0$ on $[0, 1]$.

We are interested in generalizing these above results in the case of multidimensional Schoenberg-type operators.

Let an integer $m \geq 1$. Denote $D = [0, 1]^m$.

We consider now multidimensional Schoenberg-type operators with equidistant knots on D of the form

$$(S_{n,k}^m f)(x) = \sum_{i_1, \dots, i_m = -k}^{n-1} N_{i_1}(x_1) \dots N_{i_m}(x_m) f(\xi_{i_1}, \dots, \xi_{i_m}), \quad (16)$$

where $f : D \rightarrow \mathbb{R}$, $x = (x_1, \dots, x_m) \in D$, $\xi_i^j = \frac{\nu_{i+1} + \dots + \nu_{i+k}}{k}$ and

$$0 = \nu_{-k} = \dots = \nu_0 < \nu_1 = \frac{1}{n} < \nu_2 = \frac{2}{n} < \dots < \nu_{n-1} = \frac{n-1}{n} < \nu_n = \dots = \nu_{n+k} = 1.$$

On D consider the following partial order. If $a = (a_1, \dots, a_m) \in D$, $b = (b_1, \dots, b_m) \in D$, we write $a \leq b$, iff $a_i \leq b_i$, for $1 \leq i \leq m$. A function $f : D \rightarrow \mathbb{R}$ is said to be *increasing* if for any $a, b \in D$, such that $a \leq b$, we have $f(a) \leq f(b)$.

Theorem 6. For any integers $m \geq 1$, and $k \geq 1$, if $f : D \rightarrow \mathbb{R}$ is increasing then $S_{n,k}^m f$ is increasing on D , for any $n \geq 1$.

Proof. Show that

$$(S_{n,k}^m f)(a) \leq (S_{n,k}^m f)(b), \quad a, b \in D, \quad a \leq b. \quad (17)$$

Let $a, b \in D, a \leq b, a \neq b$. Let $v = b - a$. Write $v = (v_1, \dots, v_m)$. Then $v_i \geq 0, i = 1, m$. In order to show (17) it suffices to show that function $g(t) = (S_{n,k}^m f)(a + tv), t \in [0, 1]$, is increasing, and for this it suffices to have $\frac{d}{dt}(S_{n,k}^m f)(a + tv) \geq 0$, for $t \in [0, 1]$.

We have

$$\frac{d}{dt}(S_{n,k}^m f)(a + tv) = \sum_{j=1}^m \frac{\partial}{\partial x_j}(S_{n,k}^m f)(a + tv)v_j.$$

Because $v_i \geq 0, 1 \leq i \leq m$ it suffices to show:

$$\frac{\partial}{\partial x_j}(S_{n,k}^m f)(a + tv) \geq 0, \quad 1 \leq j \leq m.$$

Denoting $x = a + tv, x = (x_1, \dots, x_m)$ one obtains

$$\begin{aligned} \frac{\partial}{\partial x_j}(S_{n,k}^m f)(x) &= \frac{\partial}{\partial x_j} \sum_{i_1, \dots, i_m = -k}^{n-1} N_{i_1}(x_1) \cdots N_{i_m}(x_m) f(\xi_{i_1}, \dots, \xi_{i_m}) \\ &= \sum_{-k \leq i_1, \dots, i_{j-1}, i_{j+1}, \dots, i_m \leq n-1} \prod_{l=1, m, l \neq j} N_{i_l}(x_l) \frac{d}{dx_j} \sum_{i_j = -k}^{n-1} N_{i_j}(x_j) f(\xi_{i_1}, \dots, \xi_{i_m}). \end{aligned}$$

Fix the indices $i_1, \dots, i_{j-1}, i_{j+1}, \dots, i_m \in \{-k, \dots, n-1\}$ and define function $h : [0, 1] \rightarrow \mathbb{R}$, given by

$$h(t) = f(\xi_{i_1}, \dots, \xi_{i_{j-1}}, t, \xi_{i_{j+1}}, \dots, \xi_{i_m}), \quad t \in [0, 1]. \quad (18)$$

Using formula (14) we obtain

$$\frac{d}{dx_j} \sum_{i_j = -k}^{n-1} N_{i_j}(x_j) f(\xi_{i_1}, \dots, \xi_{i_m}) = \frac{d}{dx_j} \sum_{i_j = -k}^{n-1} N_{i_j}(x_j) h(\xi_{i_j}) = \sum_{i_j = -k}^{n-1} \frac{h(\xi_{i_j}) - h(\xi_{i_j-1})}{\xi_{i_j} - \xi_{i_j-1}} N_{i_j}^-(x_j).$$

However, $h(\xi_{i_j}) - h(\xi_{i_j-1}) \geq 0$, since f is increasing. In addition, taking into account that $N_{i_j}^-(x_j) \geq 0, 1 - k \leq i_j \leq n - 1$ and $N_{i_l}(x_l) \geq 0, -k \leq i_l \leq n - 1, 1 \leq l \leq m, l \neq j$, relation (17) is true. $\square$

In the next two theorems we give generalizations of Theorem 4. We mention that $\overset{\circ}{D}$ means the interior of the set D .

Theorem 7. Let $k \geq 2$. Let $f : D \rightarrow \mathbb{R}$. For $1 \leq j \leq m$, if f admits the continuous derivatives $\frac{\partial^2}{\partial x_j^2}$ on D , then for any compact set $K \subset \overset{\circ}{D}$ we have

$$\lim_{n \rightarrow \infty} \frac{\partial^2}{\partial x_j^2}(S_{n,k}^m f)(x) = \frac{\partial^2 f}{\partial x_j^2}(x), \quad \text{uniformly for } x \in K \quad (19)$$

Proof. It suffices to consider only compacts of the form $K = [a, b]^m$, where $[a, b] \subset (0, 1)$, because for any compact $K \subset \overset{\circ}{D}$ there exists an interval $[a, b] \subset (0, 1)$, such that $K \subset [a, b]^m$. Using the function h given in (18), we obtain (denoting $i = i_j$):

$$\begin{aligned} \frac{\partial^2}{\partial x_j^2} (S_{n,k}^m f)(x) &= \sum_{i_1, \dots, i_{j-1}, i_{j+1}, \dots, i_m = -k}^{n-1} \prod_{l=1, m, l \neq j} N_l(x_l) \frac{d^2}{dx_j^2} \sum_{i_j = -k}^{n-1} N_{i_j}(x_j) f(\xi_{i_1}, \dots, \xi_{i_m}) \\ &= \sum_{-k \leq i_1, \dots, i_{j-1}, i_{j+1}, \dots, i_m \leq n-1} \prod_{l=1, m, l \neq j} N_l(x_l) \frac{d^2}{dx_j^2} \sum_{i = -k}^{n-1} N_i(x_j) h(\xi_i). \end{aligned} \quad (20)$$

Using formula (15) it follows

$$\frac{d^2}{dx_j^2} \sum_{i=-k}^{n-1} N_i(x_j) h(\xi_i) = \sum_{i_j=2-k}^{n-1} \frac{\xi_i - \xi_{i-2}}{2(\xi_i^- - \xi_{i-1}^-)} N_i^-(x_j) D^2 h(\eta_i)$$

where $\xi_{i-2} < \eta_i < \xi_i$.

For $0 \leq i \leq n-k$, we have $\xi_i = \xi_i^- = \frac{\nu_{i+1} + \dots + \nu_{i+k}}{k} = \nu_i + \frac{1 + \dots + k}{kn} = \nu_i + \frac{k+1}{2n}$. The support of a function N_i is the interval $[\nu_i, \dots, \nu_{i+k+1}]$. Then, if $\frac{k+1}{n} \leq a$ and $b + \frac{k+1}{n} \leq 1$, and $x_j \in [a, b]$, then $N_i^-(x_j) = 0$, for $-k \leq i \leq -1$, and $n-k+1 \leq i \leq n-1$. In addition, for $2 \leq i \leq n-k$ we obtain $\frac{\xi_i - \xi_{i-2}}{2(\xi_i^- - \xi_{i-1}^-)} = 1$, so that we can write more simply

$$\frac{d^2}{dx_j^2} \sum_{i=-k}^{n-1} N_i(x_j) h(\xi_i) = \sum_{i=2}^{n-k} N_i^-(x_j) D^2 h(\eta_i), \quad x_j \in [a, b]. \quad (21)$$

Consider the moduli of continuity of functions $D^2 h$ and $\frac{\partial^2 f}{\partial x_j^2}$:

$$\begin{aligned} \omega_1(D^2 h, \rho) &= \sup\{|D^2 h(s_1) - D^2 h(s_2)|, s_1, s_2 \in [0, 1], |s_1 - s_2| \leq \rho\} \\ \omega_1\left(\frac{\partial^2 f}{\partial x_j^2}, \rho\right) &= \sup\left\{\left|\frac{\partial^2 f}{\partial x_j^2}(y_1) - \frac{\partial^2 f}{\partial x_j^2}(y_2)\right|, y_1, y_2 \in D, \|y_1 - y_2\| \leq \rho\right\}. \end{aligned}$$

where $\rho > 0$. Because $D^2 h$ is a restriction of function $\frac{\partial^2 f}{\partial x_j^2}$ we obtain

$$\omega_1(D^2 h, \rho) \leq \omega_1\left(\frac{\partial^2 f}{\partial x_j^2}, \rho\right), \quad \forall \rho > 0.$$

In addition, since f has continuous partial second derivatives on D , it results that function $\frac{\partial^2 f}{\partial x_j^2}$ is uniformly continuous and consequently

$$\lim_{\rho \rightarrow 0} \omega_1\left(\frac{\partial^2 f}{\partial x_j^2}, \rho\right) = 0.$$

We have

$$\begin{aligned} &\left| \sum_{i=2}^{n-k} N_i^-(x_j) D^2 h(\eta_i) - \sum_{i=2}^{n-k} N_i^-(x_j) D^2 h(\xi_i) \right| \leq \sum_{i=2}^{n-k} N_i^-(x_j) \omega_1(D^2 h, |\eta_i - \xi_i|) \\ &\leq \omega_1\left(\frac{\partial^2 f}{\partial x_j^2}, \frac{2}{n}\right), \end{aligned}$$

since $\sum_{i=2}^{n-k} N_i^-(x_j) = 1$, for $x_j \in [a, b]$. It follows

$$\lim_{n \rightarrow \infty} \left(\sum_{i=2}^{n-k} N_i^-(x_j) D^2 h(\eta_i) - \sum_{i=2}^{n-k} N_i^-(x_j) D^2 h(\xi_i) \right) = 0, \quad (22)$$

uniformly with regard to indices $i_1, \dots, i_{j-1}, i_{j+1}, \dots, i_m$ and $x \in [a, b]^m$. On the other hand, consider the sequence of one-dimensional Schoenberg operators $L_n : C[0, 1] \rightarrow C[0, 1]$

$$(L_n \varphi)(x) = \sum_{i=2}^{n-k} N_i^-(x_j) \varphi(\xi_i), \quad \varphi \in C[0, 1], \quad x_j \in [0, 1].$$

This sequence of positive linear operators approximates uniformly on $[a, b]$ any function $\varphi \in C[0, 1]$. Moreover, $L_n e_0 = e_0$, $L e_1 = e_1$. Using the well known estimate of Shisha and Mond, we obtain

$$\|L_n(D^2 h) - D h^2\|_{[a, b]} \leq 2\omega_1 \left(D^2 h, \sqrt{\|L_n e_2 - e_2\|} \right) \leq 2\omega_1 \left(\frac{\partial^2 f}{\partial x_j^2}, \sqrt{\|L_n e_2 - e_2\|} \right).$$

Since $\lim_{n \rightarrow \infty} \|L_n e_2 - e_2\| = 0$ we get

$$\lim_{n \rightarrow \infty} \sum_{i=2}^{n-k} N_i^-(x_j) D^2 h(\eta_i) = D^2 h(x_j) \quad (23)$$

uniformly with regard to $x_j \in [a, b]$ and indices $i_1, \dots, i_{j-1}, i_{j+1}, \dots, i_m$. From (22) and (23) we deduce

$$\lim_{n \rightarrow \infty} \sum_{i=2}^{n-k} N_i^-(x_j) D^2 h(\eta_i) = \frac{\partial^2 f}{\partial x_j^2}(\xi_{i_1}, \dots, \xi_{i_{j-1}}, x_j, \xi_{i_{j+1}}, \dots, \xi_{i_m}),$$

uniformly with regard to $x \in [a, b]^m$ and indices $i_1, \dots, i_{j-1}, i_{j+1}, \dots, i_m$. Taking into account relations (20) and (21) we obtain the uniform limit with regard to $x \in [a, b]^m$:

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{\partial^2}{\partial x_j^2} (S_{n,k}^m f)(x) \\ &= \lim_{n \rightarrow \infty} \sum_{-k \leq i_1, \dots, i_{j-1}, i_{j+1}, \dots, i_m \leq n-1} \prod_{l=1, m, l \neq j} N_l(x_l) \frac{\partial^2 f}{\partial x_j^2}(\xi_{i_1}, \dots, \xi_{i_{j-1}}, x_j, \xi_{i_{j+1}}, \dots, \xi_{i_m}). \end{aligned}$$

Now consider the $m-1$ -dimensional Schoenberg operator $U_n : C([0, 1]^{m-1}) \rightarrow C([a, b]^{m-1})$, given by

$$\begin{aligned} & (U_n \varphi)(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_m) \\ &= \sum_{-k \leq i_1, \dots, i_{j-1}, i_{j+1}, \dots, i_m \leq n-1} \prod_{l=1, m, l \neq j} N_l(x_l) \varphi(\xi_{i_1}, \dots, \xi_{i_{j-1}}, \xi_{i_{j+1}}, \dots, \xi_{i_m}), \end{aligned}$$

where $\varphi \in C([0, 1]^{m-1})$ and $(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_m) \in [a, b]^{m-1}$. With the choice $\varphi(u_1, \dots, u_{j-1}, u_{j+1}, \dots, u_m) = \frac{\partial^2 f}{\partial x_j^2}(u_1, \dots, u_{j-1}, x_j, u_{j+1}, \dots, u_m)$, for fixed $x_j \in [0, 1]$ and $(u_1, \dots, u_{j-1}, x_j, u_{j+1}, \dots, u_m) \in [a, b]^{m-1}$ and using Theorem 2 we obtain, for fixed x_j :

$$\left| (U_n \varphi)(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_m) - \frac{\partial^2 f}{\partial x_j^2}(x_1, \dots, x_m) \right| \leq \omega_1 \left(\varphi, \frac{km}{2n} \right).$$

Since $\omega_1\left(\varphi, \frac{km}{2n}\right) \leq \omega_1\left(\frac{\partial^2 f}{\partial x_j^2}, \frac{km}{2n}\right)$ one obtains the uniform majorization with regard to $(x_1, \dots, x_m) \in [a, b]^m$:

$$\begin{aligned} & \left| \sum_{-k \leq i_1, \dots, i_{j-1}, i_{j+1}, \dots, i_m \leq n-1} \prod_{l=1, m, l \neq j} N_l(x_l) \frac{\partial^2 f}{\partial x_j^2}(\xi_{i_1}, \dots, \xi_{i_{j-1}}, x_j, \xi_{i_{j+1}}, \dots, \xi_{i_m}) \right. \\ & \quad \left. - \frac{\partial^2 f}{\partial x_j^2}(x_1, \dots, x_m) \right| \\ & \leq \omega_1\left(\frac{\partial^2 f}{\partial x_j^2}, \frac{km}{2n}\right). \end{aligned}$$

Finally, it results that (19) is true. $\square$

Theorem 8. Let $k \geq 2$ and $f : D \rightarrow \mathbb{R}$. For indices $1 \leq j_1 < j_2 \leq m$ if f admits the continuous second derivative $\frac{\partial^2 f}{\partial x_{j_1} \partial x_{j_2}}$ on D , then we have

$$\lim_{n \rightarrow \infty} \frac{\partial^2}{\partial x_{j_1} \partial x_{j_2}} (S_{n,k}^m f)(x) = \frac{\partial^2 f}{\partial x_{j_1} \partial x_{j_2}}(x), \text{ uniformly for } x \in D. \quad (24)$$

Proof. For $x = (x_1, \dots, x_m) \in D$ we find

$$\begin{aligned} \frac{\partial^2}{\partial x_{j_1} \partial x_{j_2}} (S_{n,k}^m f)(x) &= \sum_{-k \leq i_l \leq n-1, 1 \leq l \leq m, l \neq j_1, j_2} \prod_{l=1, m, l \neq j_1, j_2} N_l(x_l) \\ &\quad \times \frac{d}{dx_{j_1}} \sum_{r=-k}^{n-1} N_r(x_{j_1}) \frac{d}{dx_{j_2}} \sum_{s=-k}^{n-1} N_s(x_{j_2}) g(\xi_r, \xi_s), \end{aligned}$$

where we denoted by g , the function

$$g(u, v) = f(\xi_1, \dots, \xi_{j_1-1}, u, \xi_{j_1+1}, \dots, \xi_{j_2-1}, v, \xi_{j_2+1}, \dots, \xi_m), \quad (u, v) \in [0, 1]^2$$

and $r = i_{j_1}$ and $s = i_{j_2}$. Using formula (14) two times, it follows that we can write

$$\begin{aligned} \frac{d}{dx_{j_1}} \sum_{r=-k}^{n-1} N_r(x_{j_1}) \frac{d}{dx_{j_2}} \sum_{s=-k}^{n-1} N_s(x_{j_2}) g(\xi_r, \xi_s) &= \sum_{r=-k}^{n-1} \sum_{s=-k}^{n-1} N_r^-(x_{j_1}) N_s^-(x_{j_2}) \\ &\quad \times \frac{g(\xi_r, \xi_s) - g(\xi_r, \xi_{s-1}) - g(\xi_{r-1}, \xi_s) + g(\xi_{r-1}, \xi_{s-1})}{(\xi_r - \xi_{r-1})(\xi_s - \xi_{s-1})}. \end{aligned}$$

We have the limit

$$\lim_{n \rightarrow \infty} \left(\frac{g(\xi_r, \xi_s) - g(\xi_r, \xi_{s-1}) - g(\xi_{r-1}, \xi_s) + g(\xi_{r-1}, \xi_{s-1})}{(\xi_r - \xi_{r-1})(\xi_s - \xi_{s-1})} - \frac{\partial^2}{\partial u \partial v} g(\xi_r, \xi_s) \right) = 0,$$

uniformly with regard to index ξ_l , $1 \leq l \leq m$ and $(x_1, \dots, x_m) \in D$. It follows that the limit

$$\begin{aligned} & \lim_{n \rightarrow \infty} \sum_{r=-k}^{n-1} \sum_{s=-k}^{n-1} N_r^-(x_{j_1}) N_s^-(x_{j_2}) \frac{g(\xi_r, \xi_s) - g(\xi_r, \xi_{s-1}) - g(\xi_{r-1}, \xi_s) + g(\xi_{r-1}, \xi_{s-1})}{(\xi_r - \xi_{r-1})(\xi_s - \xi_{s-1})} \\ &= \lim_{n \rightarrow \infty} \sum_{r=-k}^{n-1} \sum_{s=-k}^{n-1} N_r^-(x_{j_1}) N_s^-(x_{j_2}) \frac{\partial^2}{\partial u \partial v} g(\xi_r, \xi_s), \end{aligned}$$

is uniform with regard to ξ_l , $1 \leq l \leq m$ and $(x_1, \dots, x_m) \in D$.

Using the property of uniform approximation of two-dimensional Schoenberg operators

$$\sum_{r=-k}^{n-1} \sum_{s=-k}^{n-1} N_r^-(x_{j_1}) N_s^-(x_{j_2}) \varphi(\xi_r), \quad \varphi \in C([0, 1]^2),$$

the Shisha and Mond estimate and the inequality $\omega_1(g, \rho) \leq \omega_1\left(\frac{\partial^2 f}{\partial x_{j_1} \partial x_{j_2}}, \rho\right)$, for $\rho > 0$, we get, like in Theorem 7

$$\lim_{n \rightarrow \infty} \sum_{r=-k}^{n-1} \sum_{s=-k}^{n-1} N_r^-(x_{j_1}) N_s^-(x_{j_2}) \frac{\partial^2}{\partial u \partial v} g(\xi_r, \xi_s) = g(x_{j_1}, x_{j_2})$$

and this limit is uniform with regard to $x \in D$ and indices i_l , $1 \leq l \leq m$, $l \neq i_{j_1}, i_{j_2}$. It follows that

$$\lim_{n \rightarrow \infty} \frac{d}{dx_{j_1}} \sum_{r=-k}^{n-1} N_r(x_{j_1}) \frac{d}{dx_{j_2}} \sum_{s=-k}^{n-1} N_s(x_{j_2}) g(\xi_r, \xi_s) = \frac{\partial^2}{\partial u \partial v} g(x_{j_1}, x_{j_2})$$

and the limit is uniform with regard to $x \in D$ and indices i_l , $1 \leq l \leq m$, $l \neq i_{j_1}, i_{j_2}$. Then we have the limit

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{\partial^2}{\partial x_{j_1} \partial x_{j_2}} (S_{n,k}^m f)(x) \\ &= \lim_{n \rightarrow \infty} \sum_{-k \leq i_l \leq n-1, 1 \leq l \leq m, l \neq j_1, j_2} \prod_{l=1, m, l \neq j_1, j_2} N_l(x_l) \frac{\partial^2}{\partial u \partial v} g(x_{j_1}, x_{j_2}) \\ &= \lim_{n \rightarrow \infty} \sum_{-k \leq i_l \leq n-1, 1 \leq l \leq m, l \neq j_1, j_2} \prod_{l=1, m, l \neq j_1, j_2} N_l(x_l) \\ & \quad \times \frac{\partial^2}{\partial x_{j_1} \partial x_{j_2}} f(\xi_1, \dots, \xi_{j_1-1}, x_{j_1}, \xi_{j_1+1}, \dots, \xi_{j_2-1}, x_{j_2}, \xi_{j_2+1}, \dots, \xi_m), \end{aligned}$$

uniform with regard to $x \in D$. Finally, we apply the property of uniform approximation of the Schoenberg operators of degree $m - 2$:

$$\sum_{-k \leq i_l \leq n-1, 1 \leq l \leq m, l \neq j_1, j_2} \prod_{l=1, m, l \neq j_1, j_2} N_l(x_l) \psi(\xi_1, \dots, \xi_{j_1-1}, \xi_{j_1+1}, \dots, \xi_{j_2-1}, \xi_{j_2+1}, \dots, \xi_m),$$

to the function

$$\begin{aligned} & \psi(u_1, \dots, u_{j_1-1}, u_{j_1+1}, \dots, u_{j_2-1}, u_{j_2+1}, \dots, u_m) \\ &= \frac{\partial^2}{\partial x_{j_1} \partial x_{j_2}} f(u_1, \dots, u_{j_1-1}, x_{j_1}, u_{j_1+1}, \dots, u_{j_2-1}, x_{j_2}, u_{j_2+1}, \dots, u_m), \end{aligned}$$

with x_{j_1} and x_{j_2} fixed. One obtains relation (24). $\square$

Theorem 9. If $f : D \rightarrow \mathbb{R}$ is strictly convex and has continuous partial second derivatives on D , then for any compact convex set $K \subset \overset{\circ}{D}$ there exists an indice n_0 , depending on f and K , such that $S_{n,k}^m(f)$ is convex for each $n \geq n_0$ on K .

Proof. From the hypothesis we obtain the following symmetric positive definite quadratic form:

$$F(x, v) := \sum_{i,j=1}^n \frac{\partial^2 f(x)}{\partial x_i \partial x_j} v_i v_j > 0, \quad \forall x \in D, \quad \forall v = (v_1, \dots, v_m) \neq 0. \quad (25)$$

Denote $B = \{x \in \mathbb{R}^m, \|x\| = 1\}$. Because D is compact and B is compact we obtain that $D \times B$ is compact. Because the function $F : D \times B \rightarrow \mathbb{R}$ is continuous and strictly

positive on the domain of definition, from the Weierstrass theorem one obtains that there exists $\mu > 0$ such that

$$\sum_{i,j=1}^n \frac{\partial^2 f(x)}{\partial x_i \partial x_j} v_i v_j \geq \mu, \quad \forall x \in D, \quad \forall v \in B. \quad (26)$$

Using Theorems 7 and 8 we obtain

$$\lim_{n \rightarrow \infty} \frac{\partial^2}{\partial x_i \partial x_j} (S_{n,k}^m f)(x) = \frac{\partial^2 f(x)}{\partial x_i \partial x_j}, \quad \text{uniformly for } x \in K,$$

for any indices $1 \leq i, j \leq m$.

Then it results

$$\lim_{n \rightarrow \infty} \sum_{i,j=1}^n \frac{\partial^2}{\partial x_i \partial x_j} (S_{n,k}^m f)(x) v_i v_j = \sum_{i,j=1}^n \frac{\partial^2 f(x)}{\partial x_i \partial x_j} v_i v_j,$$

uniformly for $x \in K$ and for $v \in B$. Therefore, there exists $n_0 \in \mathbb{N}$, such that

$$\sum_{i,j=1}^n \frac{\partial^2}{\partial x_i \partial x_j} (S_{n,k}^m f)(x) v_i v_j \geq \frac{\mu}{2}, \quad x \in K, \quad v \in B. \quad (27)$$

Inequality (27) says that $S_{n,k}^m(f)$ is convex on K . $\square$

4. Multidimensional Schoenberg-Type Operators of Degree Three on Equidistant Knots

Let one consider the case with $k_j = 3$; $n_j = n$; the equidistant knots $v_{i_j} = \frac{i_j}{n}$, $0 \leq i_j \leq n$, where $1 \leq j \leq m$; the extra-knots $v_{i_{-3}} = v_{i_{-2}} = v_{i_{-1}} = 0$ and $v_{i_{n+1}} = v_{i_{n+2}} = v_{i_{n+3}} = 1$.

The Greville abscissas are

$$\xi_{i_j} := \frac{v_{i_j+1} + v_{i_j+2} + v_{i_j+3}}{3} = \begin{cases} v_{i_j+2}, & i_j \in \{-3, \dots, n-1\} \setminus \{-2, n-2\} \\ \frac{1}{3n}, & i_j = -2, \\ \frac{3n-1}{3n}, & i_j = n-2 \end{cases} \quad (28)$$

with $1 \leq j \leq m$.

The B-splines are

$$N_{i_j,3}(x_j) = (v_{i_j+4} - v_{i_j})[v_{i_j}, v_{i_j+1}, v_{i_j+2}, v_{i_j+3}, v_{i_j+4}](\cdot - x_j)_+^3, \quad (29)$$

with $1 \leq j \leq m$.

Multidimensional Schoenberg-type operators with equidistant knots, denoted in the sequel by $S_{n,3}^m$, for $k_j = 3$, $n_j = n$, $1 \leq j \leq m$, are:

$$(S_{n,3}^m f)(x_1, \dots, x_m) = \sum_{i_1=-3}^{n_1-1} \dots \sum_{i_m=-3}^{n_m-1} N_{i_1,3}(x_1) \dots N_{i_m,3}(x_m) f(\xi_{i_1}, \dots, \xi_{i_m}). \quad (30)$$

In this section we present certain special results for the cubic splines, which can be proved analogously as in [11].

Lemma 1. *The second moment of the multidimensional Schoenberg-type operators $S_{n,3}^m$, with $n \geq 5$ and $(x_1, \dots, x_m) \in [0, 1]^m$, verifies the relations*

$$\left(S_{n,3}^m \left(\sum_{j=0}^m (\pi_j - x_j e_0)^2 \right) \right) (x_1, \dots, x_m) \leq \frac{m}{3n^2}. \quad (31)$$

Moreover,

$$(S_{n,3}^m(\sum_{j=0}^m(\pi_j - x_j e_0)^2))(x_1, \dots, x_m) = \frac{m}{3n^2}, \quad (32)$$

for $x_j \in [\frac{2}{n}, \frac{n-2}{n}]$ and $1 \leq j \leq m$.

Using Lemma and the inequality given in [21]:

$$(S_{n,k}(e_1 - x_j e_0)^4)(x_j) \leq (\frac{k+1}{2n})^2 (S_{n,k}(e_1 - x_j e_0)^2)(x_j),$$

where $S_{n,k}$ denotes the Schoenberg one-dimensional operator of order k with equidistant knots, one obtains:

Lemma 2. For $n \geq 5$ we have

$$(S_{n,3}^m(\sum_{j=1}^m(\pi_j - x_j e_0)^2))(x_1, \dots, x_m) \leq \frac{4m}{3n^4}. \quad (33)$$

From Lemma 1 and Lemma 2 and the fact that Schoenberg preserves linear functions, one can deduce the following Voronovskaja-type result, in a similar mode as in [11].

Theorem 10. The following limit is true:

$$\lim_{n \rightarrow \infty} n^2 ((S_{n,3}^m f)(x_1, \dots, x_m) - f(x_1, \dots, x_m)) = \frac{1}{6} \sum_{j=1}^m \frac{\partial^2 f}{\partial x_j^2}(x_1, \dots, x_m), \quad (34)$$

for any $f \in C^2([0, 1]^m)$, $(x_1, \dots, x_m) \in (0, 1)^m$.

Because Schoenberg preserves linear functions there exists the possibility of expressing the degree of approximation in a more refined mode, using second order moduli of continuity. The following estimates can be obtained similarly to [11] by applying certain general estimates with moduli of continuity proved in [19].

Firstly, consider the usual second order modulus

$$\tilde{\omega}_2(f, h) := \sup \left\{ \left| f(u) - 2f\left(\frac{u+v}{2}\right) + f(v) \right|, u, v \in D, \|u - v\| \leq 2h \right\} \quad (35)$$

where $f \in C(D)$, $h > 0$

One obtains:

Theorem 11.

$$|(S_{n,3}^m f)(x_1, \dots, x_m) - f(x_1, \dots, x_m)| \leq m \left(1 + \frac{1}{6h^2 n^2}\right) \tilde{\omega}_2(f, h), \quad (36)$$

where $f \in C([0, 1]^m)$, $h > 0$, $(x_1, \dots, x_m) \in [0, 1]^m$, $n \in \mathbb{N}$, $n \geq 5$.

Consequently:

$$\|(S_{n,3}^m f) - f\| \leq \frac{7m}{6} \tilde{\omega}_2\left(f, \frac{1}{n}\right), f \in C([0, 1]^m), n \in \mathbb{N}, n \geq 5. \quad (37)$$

A global second modulus of continuity can be defined by:

$$\begin{aligned}\tilde{\omega}_2^*(f, \rho) = \sup \left\{ \left| \sum_{i=1}^n \lambda_i f(\alpha_i) - f(\alpha) \right|, \alpha \in \mathcal{A}, \alpha_i \in D, \|\alpha_i - \alpha\| \leq h, \right. \\ \left. \lambda_i \in (0, 1), 0 \leq i \leq n, \lambda_1 + \dots + \lambda_n = 1 \right\}.\end{aligned}\quad (38)$$

Using this modulus one can obtain an estimate which is independent on the dimension m :

Theorem 12. *Let the function f continue on $[0, 1]^m$ and $h > 0$. We have*

$$|(S_{n,3}^m f)(x_1, \dots, x_m) - f(x_1, \dots, x_m)| \leq \left(1 + \frac{m}{3n^2 h^2}\right) \tilde{\omega}_2^*(f, h). \quad (39)$$

Consequently,

$$\|(S_{n,3}^m f) - f\| \leq \left(1 + \frac{m}{3}\right) \tilde{\omega}_2^*\left(f, \frac{1}{n}\right), f \in C([0, 1]^m), n \in \mathbb{N}. \quad (40)$$

Remark 2. The usual polynomial operators used in approximation have an approximation order only of the type $O\left(\omega_2\left(f, \frac{1}{\sqrt{n}}\right)\right)$, $f \in C[0, 1]$.

5. Conclusions

The Schoenberg operators are practical tools to approximate functions, knowing the values of them in a finite number of points. Schoenberg operators attach to a function a particular type of spline of a freely chosen degree. It is not necessary to use high-degree splines in order to obtain a desired approximation order. It is usually sufficient to use 3rd order splines. This makes the calculation volume substantially lower than in the case of polynomial approximation.

In approximation of functions, the degree of approximation is not the unique objective. The preservation of certain shape properties of functions is also worth studying. Among these supplementary preservation proprieties, two special types are usually studied: the possibility of simultaneous approximation of functions and of their derivatives of different orders and the preservation of convexity of different orders, including the monotonicity and the usual convexity. These types of properties are known to be true for the one-dimensional case of Schoenberg operators. We put in evidence that they are true in great measure for the multidimensional case. It is well known also that maybe the more important polynomial approximation operators, namely the Bernstein operators, have very good properties for preserving different behaviors of functions. In fact, it is natural that the Schoenberg operators, which can be regarded as generalizations of Bernstein operators, maintain at least in part these good properties. On the other hand, by taking into account that Schoenberg operators offer a great improvement of the order of approximation for the same order of computation, they turn out to be a very powerful tool in the theory of function approximation. In this direction, other properties of preserving certain classes of functions or the simultaneous approximation can be taken into account for further studies.

The results obtained in this paper are connected to the notion of symmetry in several aspects, namely, in the construction of operators, in using the symmetric tools in estimates and in property of convexity, which is given using symmetrical expressions. We believe that this paper can offer a useful tool to specialists with concerns in many areas of practical approximation.

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