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A Hopf Algebra on Permutations Arising from Super-Shuffle Product

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Abstract: In this paper, we first prove that any atom of a permutation obtained by the super-shuffle product of two permutations can only consist of some complete atoms of the original two permutations. Then, we prove that the super-shuffle product and the cut-box coproduct on permutations are compatible, which makes it a bialgebra. As this algebra is graded and connected, it is a Hopf algebra.

Keywords: permutation; global ascent; super-shuffle product; cut-box coproduct; Hopf algebra

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1. Introduction

The concept of Hopf algebra was derived from the study of algebraic topology and algebraic groups in the 1940s and 1950s [1–3]. The structure of Hopf algebras is relatively stable and has many properties worth studying, such as commutativity, cocommutativity, and free and cofree properties [4–6]. With the further study of these properties, many new Hopf algebra structures have been discovered, especially in algebraic combinatorics, such as Hopf algebra on permutations [6–8], Hopf algebra on graphs [9], Hopf algebra on trees [6,10], and Hopf algebra on the symmetric functions [11,12]. In recent years, they have been gradually applied to many subjects, such as free Lie algebras [13,14], Lie superalgebras [15,16], Clifford algebras [17–19], Drinfeld-Jimbo quantum groups [20], the Jacobian Conjecture for free metabelian algebras [21,22] and even nonlinear control theory [23].

In 1995, Malvenuto and Reutenauer [24] studied a pair of Hopf algebra on permutations, where one of the products $\star'$ is essentially the shuffle product ш in [25]. The shuffle product's ш is classic and has been studied by many teams recently. However, the product $\star'$ is not commutative on permutations. In 2020, Zhao and Li [26] introduced a new shuffle product ш_G on permutations based on the global descents, which makes it commutative. Then, they defined a deconcatenation coproduct Δ and proved that $(KS, \text{ш}_G, \mu, \Delta, \nu)$ is a graded connected bialgebra and, hence, a Hopf algebra [27]. Moreover, they figured out its dual Hopf algebra $(KS, \text{ш}_G^*, \mu, \Delta^*, \nu)$ and a closed-formula of its antipode [28].

The symmetric group on a set X, denoted as $Sym(X)$, is a group of all bijections from X to itself, which is widely applied to many areas, such as algebraic number theory [29] and substochastic matrices [30–33]. In combinatorics, a permutation of degree n is an arrangement of n elements. The symmetric group of degree n, denoted by S_n, contains all permutations of $[n] = \{1, 2, \ldots, n\}$. Let KS_n be the vector space spanned by S_n over field K, where S_n is the symmetric group of degree n. Define $KS := \bigoplus_{n\geq 0} KS_n$, where $S_0 = \{\epsilon\}$ and ϵ is the empty permutation. Then, KS is graded, and its nth component is KS_n.

In 2014, Vargas [34] defined the super-shuffle product $\underline{\text{ш}}$ on permutations, interpolating between the products $\star$ and $\star'$ given in [24], and a coproduct $\Delta_\diamond$, called the *cut-box coproduct* in our paper. He also put forward that the super-shuffle product $\underline{\text{ш}}$ is commutative and associative on permutations and claimed that $(KS, \underline{\text{ш}}, \mu, \Delta_\diamond, \nu)$ is a graded connected bialgebra and, hence, a Hopf algebra. However, he did not provide a proof.

In 2018, Bergeron, Ceballos, and Pilaud [35] gave a new concept of gaps on permutations in S_n. In 2020, Aval, Bergeron, and Machacek [36] introduced global ascents of permutations and the standard factorization of a permutation given by global ascents. Then, the cut-box coproduct $\Delta_\diamond$ of a permutation can be reinterpreted as using the tensor notation at one of its global ascents. In this paper, we prove that any atom of a permutation obtained by the super-shuffle product of two permutations can only consist of some complete atoms of the original two permutations, see Lemma 1. The main work of this paper is to prove that $(KS, \underline{\sqcup\!\sqcup}, \mu, \Delta_\diamond, \nu)$ is a bialgebra.

In Section 2, we recall some important definitions and notations on Hopf algebras and permutations. In Section 3, we introduce the definition of a super-shuffle product $\underline{\sqcup\!\sqcup}$ and reinterpret the cut-box coproduct $\Delta_\diamond$ on permutations. We prove that $\underline{\sqcup\!\sqcup}$ and $\Delta_\diamond$ satisfy the compatibility condition in Section 4. Therefore, $(KS, \underline{\sqcup\!\sqcup}, \mu, \Delta_\diamond, \nu)$ is a Hopf algebra since it is graded and connected. Finally, a summarization can be found in Section 5.

2. Preliminaries

2.1. Basic Definitions

Here, we recall some basic definitions related to Hopf algebra, see [3] for more details. Let R be a commutative ring and B be an R-module.

Define a *product* $\pi : B \otimes_R B \longrightarrow B$ and a *unit* $\mu : R \longrightarrow B$, respectively, satisfying the diagrams in Figure 1.

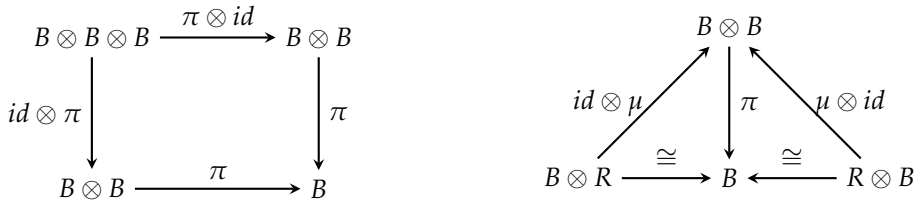


Figure 1. Associative law and unit.

Then, (B, π, μ) is an *R-algebra*.

The algebra B is *graded* if there is a direct sum decomposition $B = \bigoplus_{i \geq 0} B_i$ such that the product of homogeneous elements of degrees p and q is homogeneous of degree $p + q$, that is, $\pi(B_p \otimes B_q) \subseteq B_{p+q}$, and $\mu(R) \subseteq B_0$.

Define a *coproduct* $\Delta : B \longrightarrow B \otimes_R B$ and a *counit* $\nu : B \longrightarrow R$, respectively, satisfying the diagrams in Figure 2.

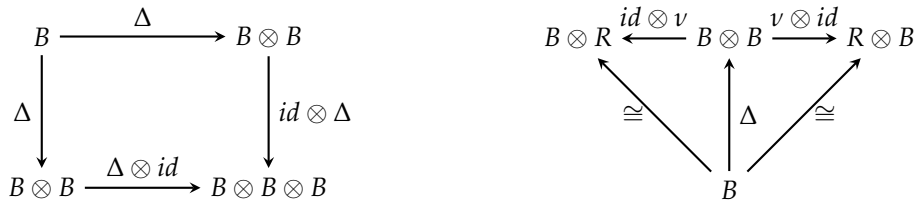


Figure 2. Coassociative Law and Counit.

Then, (B, Δ, ν) is an *R-coalgebra*.

The coalgebra B is *graded* if there is a direct sum decomposition $B = \bigoplus_{i \geq 0} B_i$ such that $\Delta(B_n) \subseteq \bigoplus(B_k \otimes B_{n-k})$ and $\nu(B_n) = 0$ if $n \geq 1$.

If B is both an R-algebra and an R-coalgebra, and satisfies one of the following equivalent conditions:

(1) Δ and ν are algebra homomorphisms;

(2) π and μ are coalgebra homomorphisms,

then we say the algebra and coalgebra structures on B are *compatible* and $(B, \pi, \mu, \Delta, \nu)$ is an *R-bialgebra*.

If $B = \bigoplus_{i\geq 0} B_i$ is both a graded algebra and a graded coalgebra, and satisfies the compatibility condition, then we say B is a *graded bialgebra*.

If there exists a linear map $\theta : B \longrightarrow B$ satisfying

$$\pi \circ (\theta \otimes id) \circ \Delta = \mu \circ \nu = \pi \circ (id \otimes \theta) \circ \Delta,$$

i.e., the diagram in Figure 3 commutes, then θ is an *antipode*. A bialgebra with an antipode is a *Hopf algebra*.

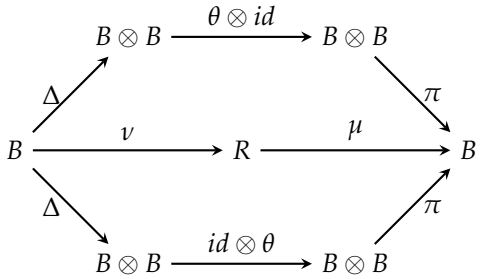


Figure 3. Antipode.

A bialgebra B over K is called *graded connected* if it is graded and satisfies $B_0 = K1_B$, where K is a field. In 2008, Manchon [27] proved that any graded connected bialgebra admits a unique antipode and is a Hopf algebra.

2.2. Basic Notations

Now, we recall some basic notations on permutations; see [34,36] for more details.

Let S_n be the symmetric group of degree n, which consists of all the permutations of degree n. A *permutation* ω in S_n is a rearrangement of $[n] := \{1, 2, \ldots, n\}$, denoted by $\omega = \omega_1 \cdots \omega_i \omega_{i+1} \cdots \omega_n$ in one-line notation, where $\omega_i = \omega(i) \in [n]$. Let $S := \biguplus_{n\geq 0} S_n$ be the disjoint union of S_n, where $S_0 = \{\epsilon\}$ and ϵ is the *empty permutation*.

For a permutation $\omega = \omega_1\omega_2 \cdots \omega_n$ in S_n in one-line notation, the position in front of the first element ω_1 is called *gap* 0 of ω, the position between ω_i and ω_{i+1} is called *gap* i of ω for all $0 < i < n$, and the position behind the last element ω_n is called *gap* n of ω. For example, the gaps of permutation $321465 \in S_6$ are the numbers in the circles: ${}_{⓪}3_{①}2_{②}1_{③}4_{④}6_{⑤}5_{⑥}$.

For a gap γ of ω, if $\omega([\gamma]) = [\gamma]$, then γ is called a *global ascent* of ω. The gaps 0 and n must be global ascents of any permutation of degree n. A nonempty permutation with no global ascents except 0 and n is called an *indecomposable permutation* or an *atom*. A *portion of an atom* is a nontrivial subsequence of k adjacent elements from an atom with m elements, where $0 < k < m$. For example, permutations $321, 312, 21$, and 1 are all atoms. The portions of 321 can only be $3, 2, 1, 32$, and 21.

A *sequence* of $[n]$ with length m is denoted by $a = a_1a_2 \cdots a_m$ $(1 \leq m \leq n)$, where a_i $(1 \leq i \leq m)$ are finitely distinct elements selected from $[n]$. When no element is selected, we have an *empty sequence* with length 0, also denoted by ϵ. Let $\mathrm{alph}(a)$ be the set consisting of all elements in a and A_m be the set of all the sequences of length m. The *standard form of sequence* a in A_m is a permutation in S_m, denoted by $\mathrm{st}(a)$, satisfying

$$\mathrm{st}(a_i) < \mathrm{st}(a_j) \Leftrightarrow a_i < a_j,$$

$$\mathrm{st}(a_i) > \mathrm{st}(a_j) \Leftrightarrow a_i > a_j.$$

For example, the standard form of sequence 432 is 321, i.e., $\mathrm{st}(432) = 321$. The standard form of the empty sequence is the empty permutation, i.e., $\mathrm{st}(\epsilon) = \epsilon$.

Consider a permutation $\alpha \in S_n$. By putting boxes $\diamond$ at all global ascents except 0 and n, we find $\alpha = \alpha_1 \diamond \cdots \diamond \alpha_l$, called the *factorization* of α, where α_i's are sequences of $[n]$ and the sum of the length of all α_i's is n. For example, let $\alpha = 2314657$, then its global ascents

are $0, 3, 4, 6, 7$ and its factorization is $\alpha = 231 \diamond 4 \diamond 65 \diamond 7$, where $\alpha_1 = 231, \alpha_2 = 4, \alpha_3 = 65$, $\alpha_4 = 7$.

By taking the standard form of each sequence α_i $(1 \leq i \leq l)$, we obtain α's *standard factorization* $\alpha = \mathrm{st}(\alpha_1) \diamond \cdots \diamond \mathrm{st}(\alpha_l)$. Each permutation has a unique standard factorization. Clearly, each $\mathrm{st}(\alpha_i)$ $(1 \leq i \leq l)$ in the standard factorization of α is an atom. The number of atoms in α is called the *length* of α, denoted as $|\alpha|_\diamond$, so $|\alpha|_\diamond = l$, which is always less than or equal to its degree n. From now on, the notation $\alpha = \alpha_1 \diamond \cdots \diamond \alpha_l$ always expresses α's standard factorization. For example, the standard factorization of $\alpha = 2314657$ is $\alpha = \mathrm{st}(231) \diamond \mathrm{st}(4) \diamond \mathrm{st}(65) \diamond \mathrm{st}(7) = 231 \diamond 1 \diamond 21 \diamond 1$, where $\alpha_1 = 231, \alpha_2 = 1, \alpha_3 = 21, \alpha_4 = 1$, and the length of α is $|\alpha|_\diamond = |231 \diamond 1 \diamond 21 \diamond 1|_\diamond = 4$.

From a standard factorization to the original permutation, we renumber the numbers in the atoms from left to right. For example,

$$21 \diamond 12 \diamond 1 \diamond 1 = 213456,$$

$$1 \diamond 12 \diamond 12 \diamond 1 = 123456.$$

3. Super-Shuffle Product and Cut-Box Coproduct

In 2014, Vargas defined two operations on permutations—the super-shuffle product and cut-box coproduct as follows [34].

Define the *cut-box coproduct* $\Delta_\diamond$ on KS by

$$\Delta_\diamond(\alpha) := \sum_{i=0}^{l} \alpha_1 \diamond \cdots \diamond \alpha_i \otimes \alpha_{i+1} \diamond \cdots \diamond \alpha_l,$$

for any permutation $\alpha = \alpha_1 \diamond \alpha_2 \diamond \cdots \diamond \alpha_l$.

Example 1. *For permutation* 214356*, its global ascents are* $0, 2, 4, 5, 6$*, and its standard factorization is* $21 \diamond 21 \diamond 1 \diamond 1$*. Then,*

$$\begin{aligned}
\Delta_\diamond(214356) =& \Delta_\diamond(21 \diamond 21 \diamond 1 \diamond 1) \\
=& \epsilon \otimes 21 \diamond 21 \diamond 1 \diamond 1 + 21 \otimes 21 \diamond 1 \diamond 1 + 21 \diamond 21 \otimes 1 \diamond 1 \\
&+ 21 \diamond 21 \diamond 1 \otimes 1 + 21 \diamond 21 \diamond 1 \diamond 1 \otimes \epsilon \\
=& \epsilon \otimes 214356 + 21 \otimes 2134 + 2143 \otimes 12 \\
&+ 21435 \otimes 1 + 214356 \otimes \epsilon.
\end{aligned}$$

From above, in the coproduct of a permutation the tensor notation $\otimes$ can only appear in one of its global ascents.

The *counit* map on KS is defined by $\nu : KS \longrightarrow K, \nu(\epsilon) = 1$ and $\nu(\alpha) = 0$ for any $\alpha \in S_n, n \geq 1$.

Let $I, J \subseteq [n]$, and $[n]$ be the disjoint union of I and J, denoted by $I \uplus J = [n]$.

For a permutation $\omega = \omega_1 \cdots \omega_n \in S_n$, let $\omega_I = \omega_{i_1} \cdots \omega_{i_k}$ with $I = \{i_1 < \cdots < i_k\} \subseteq [n]$.

Define *super-shuffle product* on KS by

$$\alpha \text{ш} \beta := \sum_{\substack{I \uplus J = [n+p] \\ st(\omega_I) = \alpha, st(\omega_J) = \beta \\ alpha(\omega_I) \uplus alpha(\omega_J) = [n+p]}} \omega$$

for any $\alpha \in S_n, \beta \in S_p$.

Example 2. *Let* $\alpha = 12 \in S_2, \beta = 1 \in S_1$.

$$\begin{aligned} 12 \underline{\sqcup\!\sqcup} 1 =& 123 + 132 + 312 \\ &+ 132 + 123 + 213 \\ &+ 231 + 213 + 123. \end{aligned}$$

Here we color the elements of α red and the elements of β blue, then the same numbers in α and β are different. Thus, we can claim that all permutations in the super-shuffle product are distinct even if they are same. From now on, when we compute super-shuffle product of two permutations we will color them in different colors.

For $\alpha = \alpha_1 \cdots \alpha_q \in S_q$, define $\hat{\alpha}_i$ $(1 \leq i \leq q)$ to be the renumbering of α_i satisfying that, if $r \neq t$ and $\alpha_r < \alpha_t$, then $\hat{\alpha}_r < \hat{\alpha}_t$, for all $r, t \in [q]$.

Remark 1. *For* $\alpha = \alpha_1 \cdots \alpha_q \in S_q, \beta = \beta_1 \cdots \beta_p \in S_p$. *If* $T := \{\hat{\alpha}_i | 1 \leq i \leq q\} \subseteq [q+p]$ *and* $\{\hat{\beta}_j | 1 \leq j \leq p\} = [q+p] \setminus T$, *then*

$$\pi := \hat{\alpha}_1 \cdots \hat{\alpha}_r \hat{\beta}_1 \cdots \hat{\alpha}_t \cdots \hat{\beta}_k \cdots \hat{\alpha}_q \hat{\beta}_s \cdots \hat{\beta}_p$$

is a permutation in the super-shuffle product $\alpha \underline{\sqcup\!\sqcup} \beta$, *where* $0 \leq r \leq q, r+1 \leq t \leq q, 1 \leq k \leq s-1, 1 \leq s \leq p+1$. *In particular, when* $r = 0$ *the first element of* π *is* $\hat{\beta}_1$, *and when* $s = p+1$, *the last element of* π *is* $\hat{\alpha}_q$.

Therefore, $\hat{1}\hat{2}\hat{1}$ is one of permutations in $\{123, 132, 231\}$, $\hat{1}\hat{1}\hat{2}$ is one of permutations in $\{132, 123, 213\}$, and $\hat{1}\hat{1}\hat{2}$ is one of permutations in $\{312, 213, 123\}$.

Define the *unit* map $\mu : K \longrightarrow KS$ on KS by $\mu(1) = \epsilon$.

Define the *relative shuffle product* $\sqcup\!\sqcup_\diamond$ on KS recursively by

$$\alpha \sqcup\!\sqcup_\diamond \beta = \alpha_1 \diamond (\alpha_2 \diamond \cdots \diamond \alpha_l \sqcup\!\sqcup_\diamond \beta) + \beta_1 \diamond (\alpha \sqcup\!\sqcup_\diamond \beta_2 \diamond \cdots \beta_k),$$

for $\alpha = \alpha_1 \diamond \cdots \diamond \alpha_l$, $\beta = \beta_1 \diamond \cdots \diamond \beta_k$ and $\alpha \sqcup\!\sqcup_\diamond \epsilon = \epsilon \sqcup\!\sqcup_\diamond \alpha = \alpha$.

For example,

$$(21 \diamond 1) \sqcup\!\sqcup_\diamond 1 = 21 \diamond 1 \diamond 1 + 21 \diamond 1 \diamond 1 + 1 \diamond 21 \diamond 1 = 2134 + 2134 + 1324.$$

For $\alpha \in S_n$ and $\beta \in S_p$, the permutations obtained by $\alpha \underline{\sqcup\!\sqcup} \beta$ of degree $n+p$, which are the permutations obtained by $\alpha \sqcup\!\sqcup_\diamond \beta$ and some others with length less than $|\alpha|_\diamond + |\beta|_\diamond$.

For example, let $\alpha = 12 \in S_2, \beta = 1 \in S_1$, and then $|\alpha|_\diamond = |1 \diamond 1|_\diamond = 2, |\beta|_\diamond = 1, |\alpha|_\diamond + |\beta|_\diamond = 3$. Then,

$$12 \sqcup\!\sqcup_\diamond 1 = (1 \diamond 1) \sqcup\!\sqcup_\diamond 1 = 1 \diamond 1 \diamond 1 + 1 \diamond 1 \diamond 1 + 1 \diamond 1 \diamond 1 = 123 + 123 + 123,$$

$$12 \underline{\sqcup\!\sqcup} 1 = \underbrace{123 + 123 + 123}_{12 \sqcup\!\sqcup_\diamond 1} + \underbrace{132 + 132 + 213 + 213}_{|\pi|_\diamond = 2} + \underbrace{231 + 312}_{|\pi|_\diamond = 1}.$$

4. The Hopf Algebra Arising from the Super-Shuffle

By the definition, we have that the super-shuffle product of any two permutations of degrees n and p is a sum of permutations of degree $n+p$. Therefore, $(KS, \underline{\sqcup\!\sqcup}, \mu)$ is a graded algebra. Since $\Delta_\diamond(KS_n) \subseteq \bigoplus_{k=0}^{n}(KS_k \otimes KS_{n-k}), \nu(K) = KS_0$, $(KS, \Delta_\diamond, \nu)$ is a graded coalgebra.

Vargas [34] put forward that $(KS, \underline{\sqcup\!\sqcup}, \mu, \Delta_\diamond, \nu)$ is a graded bialgebra, and thus it is a graded Hopf algebra. However, there was no proof. Here, we give a proof.

Lemma 1. *Let* $\sigma = \sigma_1 \cdots \sigma_s, \rho = \rho_1 \cdots \rho_t$ *be any two nonempty permutations. If* π *is a permutation obtained by* $\sigma \underline{\sqcup\!\sqcup} \rho$, *then any atom of* π *can only consist of some complete atoms of* σ *and* ρ.

Proof of Lemma 1. Equivalently, we will prove that any atom of π does not contain any portion of atoms of σ or ρ.

Let $\pi = \pi_1\pi_2\cdots\pi_{s+t}$ in one-line notation be a permutation obtained by $\sigma \underline{\sqcup\!\sqcup} \rho$ and $\mathrm{st}(\pi_u\cdots\pi_v)$ is an atom of π, where $1 \le u \le v \le s+t$.

Suppose $\pi_u\cdots\pi_v$ contains a portion of an atom from the original permutation σ. Let $\mathrm{st}(\sigma_1\cdots\sigma_p)$ be the portion, where $\sigma_p\cdots\sigma_q$ is corresponding to the portion in the one-line notation of σ. Then neither $\hat{\sigma}_{p-1}$ nor $\hat{\sigma}_{q+1}$ is an element in $\pi_u\cdots\pi_v$.

Since $\pi_u\cdots\pi_v$ is an atom, we have $\pi([u-1]) = [u-1]$ and $\pi([v]) = [v]$. So, $\{\hat{\sigma}_1,\ldots,\hat{\sigma}_{p-1}\} \subseteq [u-1]$,$\{\hat{\sigma}_p,\ldots,\hat{\sigma}_s\} \subseteq [s+t]\setminus[u-1]$,$\{\hat{\sigma}_1,\ldots,\hat{\sigma}_q\} \subseteq [v]$ and $\{\hat{\sigma}_{q+1},\ldots,\hat{\sigma}_s\} \subseteq [s+t]\setminus[v]$. By the definition of renumbering, $\max\{\sigma_1,\ldots,\sigma_{p-1}\} < \min\{\sigma_p,\ldots,\sigma_s\}$, $\max\{\sigma_1,\ldots,\sigma_q\} < \min\{\sigma_{q+1},\ldots,\sigma_s\}$ and $\{\sigma_1,\ldots,\sigma_s\} = [n]$. Thus, $\sigma([p-1]) = [p-1]$ and $\sigma([q]) = [q]$. Clearly, $\sigma([k]) \neq [k]$ for all $p < k < q$. Therefore, $p-1$ and q are global ascents of σ, and there is no global ascent between p and $q-1$, i.e., $\mathrm{st}(\sigma_p\cdots\sigma_q)$ is an atom in σ, contradiction to that $\mathrm{st}(\sigma_p\cdots\sigma_q)$ is a portion.

Similarly, $\pi_u\cdots\pi_v$ cannot contain a portion of any atom from the original permutation ρ. □

Example 3. *For any permutation* $\alpha = 21$ *and* $\beta = 1\diamond 1$,

$$\begin{aligned}
21\underline{\sqcup\!\sqcup}1\diamond 1 =& 4312 + 4132 + 4123 + 1\diamond 432 + 1\diamond 423 + 1\diamond 2\diamond 43\\
&+ 4213 + 4123 + 4132 + 1\diamond 423 + 1\diamond 432 + 1\diamond 342\\
&+ 4123 + 4213 + 4231 + 2413 + 2431 + 2341\\
&+ 321\diamond 4 + 312\diamond 4 + 3142 + 1\diamond 32\diamond 4 + 1\diamond 342 + 1\diamond 432\\
&+ 312\diamond 4 + 321\diamond 4 + 3241 + 231\diamond 4 + 2341 + 2431\\
&+ 21\diamond 3\diamond 4 + 231\diamond 4 + 2341 + 321\diamond 4 + 3241 + 3421.
\end{aligned}$$

In our example, the atoms of each permutation obtained by $21\underline{\sqcup\!\sqcup}1\diamond 1$ all consist of some complete atoms of α and β.

Theorem 1. $(KS, \underline{\sqcup\!\sqcup}, \mu, \Delta_\diamond, \nu)$ *is a bialgebra.*

Proof of Theorem 1. We know that $(KS, \underline{\sqcup\!\sqcup}, \mu, \Delta_\diamond, \nu)$ is a graded algebra and a graded coalgebrae. It is easy to check that ν is an algebra homomorphism. Thus, to prove $(KS, \underline{\sqcup\!\sqcup}, \mu, \Delta_\diamond, \nu)$ is a graded bialgebra we only need to prove it satisfies the compatibility, i.e.,

$$\Delta_\diamond(\alpha \underline{\sqcup\!\sqcup} \beta) = \Delta_\diamond(\alpha)\underline{\sqcup\!\sqcup}\Delta_\diamond(\beta),$$

for any $\alpha = \alpha_1\diamond\cdots\diamond\alpha_l$ and $\beta = \beta_1\diamond\cdots\diamond\beta_t$.

From Lemma 1, we have that any atom of a permutation π obtained by $\alpha\underline{\sqcup\!\sqcup}\beta$ can only consist of some complete atoms from the original two permutations α and β.

Denote $\Delta_\diamond(\alpha\underline{\sqcup\!\sqcup}\beta)|_{i=x}$ as the sum of all items in $\Delta_\diamond(\alpha\underline{\sqcup\!\sqcup}\beta)$ with x atoms from α on the left side of tensor notation $\otimes$, $0 \le x \le l$. Hence, we have

$$\Delta_\diamond(\alpha\underline{\sqcup\!\sqcup}\beta) = \Delta_\diamond(\alpha\underline{\sqcup\!\sqcup}\beta)|_{i=0} + \Delta_\diamond(\alpha\underline{\sqcup\!\sqcup}\beta)|_{i=1} + \Delta_\diamond(\alpha\underline{\sqcup\!\sqcup}\beta)|_{i=2} + \cdots + \Delta_\diamond(\alpha\underline{\sqcup\!\sqcup}\beta)|_{i=l},$$

where

$$\begin{aligned}
\Delta_\diamond(\alpha\underline{\sqcup\!\sqcup}\beta)|_{i=x} =& \alpha_1\diamond\cdots\diamond\alpha_x \otimes (\alpha_{x+1}\diamond\cdots\diamond\alpha_l\underline{\sqcup\!\sqcup}\beta)\\
&+ (\alpha_1\diamond\cdots\diamond\alpha_x\underline{\sqcup\!\sqcup}\beta_1) \otimes (\alpha_{x+1}\diamond\cdots\diamond\alpha_l\underline{\sqcup\!\sqcup}\beta_2\diamond\cdots\diamond\beta_t)\\
&+ (\alpha_1\diamond\cdots\diamond\alpha_x\underline{\sqcup\!\sqcup}\beta_1\diamond\beta_2) \otimes (\alpha_{x+1}\diamond\cdots\diamond\alpha_l\underline{\sqcup\!\sqcup}\beta_3\diamond\cdots\diamond\beta_t)\\
&+ \cdots\\
&+ (\alpha_1\diamond\cdots\diamond\alpha_x\underline{\sqcup\!\sqcup}\beta) \otimes (\alpha_{x+1}\diamond\cdots\diamond\alpha_l\underline{\sqcup\!\sqcup}\epsilon)
\end{aligned}$$

$$=\sum_{j=0}^{t}(\alpha_1\diamond\cdots\diamond\alpha_x\underline{\sqcup\!\sqcup}\beta_1\diamond\cdots\diamond\beta_j)\otimes(\alpha_{x+1}\diamond\cdots\diamond\alpha_l\underline{\sqcup\!\sqcup}\beta_{j+1}\diamond\cdots\diamond\beta_t)$$

$$=(\alpha_1\diamond\cdots\diamond\alpha_x\otimes\alpha_{x+1}\diamond\cdots\diamond\alpha_l)\underline{\sqcup\!\sqcup}\sum_{j=0}^{t}(\beta_1\diamond\cdots\diamond\beta_j\otimes\beta_{j+1}\diamond\cdots\diamond\beta_t).$$

Thus, we have

$$\begin{aligned}\Delta_\diamond(\alpha\underline{\sqcup\!\sqcup}\beta)=&\Delta_\diamond(\alpha\underline{\sqcup\!\sqcup}\beta)|_{i=0}+\Delta_\diamond(\alpha\underline{\sqcup\!\sqcup}\beta)|_{i=1}+\Delta_\diamond(\alpha\underline{\sqcup\!\sqcup}\beta)|_{i=2}+\cdots+\Delta_\diamond(\alpha\underline{\sqcup\!\sqcup}\beta)|_{i=l}\\
=&(\epsilon\otimes\alpha_1\diamond\cdots\diamond\alpha_l)\underline{\sqcup\!\sqcup}\sum_{j=0}^{t}(\beta_1\diamond\cdots\diamond\beta_j\otimes\beta_{j+1}\diamond\cdots\diamond\beta_t)\\
&+(\alpha_1\otimes\alpha_2\diamond\cdots\diamond\alpha_l)\underline{\sqcup\!\sqcup}\sum_{j=0}^{t}(\beta_1\diamond\cdots\diamond\beta_j\otimes\beta_{j+1}\diamond\cdots\diamond\beta_t)\\
&+(\alpha_1\diamond\alpha_2\otimes\alpha_3\diamond\cdots\diamond\alpha_l)\underline{\sqcup\!\sqcup}\sum_{j=0}^{t}(\beta_1\diamond\cdots\diamond\beta_j\otimes\beta_{j+1}\diamond\cdots\diamond\beta_t)\\
&+\cdots\\
&+(\alpha_1\diamond\cdots\diamond\alpha_l\otimes\epsilon)\underline{\sqcup\!\sqcup}\sum_{j=0}^{t}(\beta_1\diamond\cdots\diamond\beta_j\otimes\beta_{j+1}\diamond\cdots\diamond\beta_t)\\
=&\begin{bmatrix}\epsilon\otimes\alpha_1\diamond\cdots\diamond\alpha_l+\alpha_1\otimes\alpha_2\diamond\cdots\diamond\alpha_l\\+\cdots+\alpha_1\diamond\cdots\diamond\alpha_l\otimes\epsilon\end{bmatrix}\underline{\sqcup\!\sqcup}\sum_{j=0}^{t}(\beta_1\diamond\cdots\diamond\beta_j\otimes\beta_{j+1}\diamond\cdots\diamond\beta_t)\\
=&\sum_{i=0}^{l}\alpha_1\diamond\cdots\diamond\alpha_i\otimes\alpha_{i+1}\diamond\cdots\diamond\alpha_l\underline{\sqcup\!\sqcup}\sum_{j=0}^{t}(\beta_1\diamond\cdots\diamond\beta_j\otimes\beta_{j+1}\diamond\cdots\diamond\beta_t)\\
=&\Delta_\diamond(\alpha)\underline{\sqcup\!\sqcup}\Delta_\diamond(\beta).\end{aligned}$$

Hence, the cut-box coproduct $\Delta_\diamond$ is an algebra homomorphism, i.e., the super-shuffle product and the cut-box coproduct are compatible. Therefore, $(KS,\underline{\sqcup\!\sqcup},\mu,\Delta_\diamond,\nu)$ is a bialgebra. □

Example 4. *Let* $\alpha=12=1\diamond1$, $\beta=1$. *Then,*

$$\begin{aligned}\Delta_\diamond(12\underline{\sqcup\!\sqcup}1)=&\Delta_\diamond(231+213+123+132+123+213+123+132+312)\\
=&\Delta_\diamond\begin{bmatrix}231+21\diamond1+1\diamond1\diamond1+1\diamond21+1\diamond1\diamond1\\+21\diamond1+1\diamond1\diamond1+1\diamond21+312\end{bmatrix}\\
=&\epsilon\otimes\begin{bmatrix}231+21\diamond1+1\diamond1\diamond1+1\diamond21+1\diamond1\diamond1\\+21\diamond1+1\diamond1\diamond1+1\diamond21+312\end{bmatrix}+1\otimes1\diamond1\\
&+1\otimes(21+1\diamond1+21)+(21+1\diamond1+21)\otimes1\\
&+1\diamond1\otimes1+\begin{bmatrix}231+21\diamond1+1\diamond1\diamond1+1\diamond21+1\diamond1\diamond1\\+21\diamond1+1\diamond1\diamond1+1\diamond21+312\end{bmatrix}\otimes\epsilon\\
=&\epsilon\otimes(1\diamond1\underline{\sqcup\!\sqcup}1)+1\otimes(1\diamond1)\\
&+1\otimes(1\underline{\sqcup\!\sqcup}1)+(1\underline{\sqcup\!\sqcup}1)\otimes1\\
&+1\diamond1\otimes1+(1\diamond1\underline{\sqcup\!\sqcup}1)\otimes\epsilon,\end{aligned}$$

and

$$\begin{aligned}\Delta_\diamond(12)\underline{\sqcup\!\sqcup}\Delta_\diamond(1)=&(\epsilon\otimes1\diamond1)\underline{\sqcup\!\sqcup}(\epsilon\otimes1)+(1\otimes1)\underline{\sqcup\!\sqcup}(\epsilon\otimes1)+(1\diamond1\otimes\epsilon)\underline{\sqcup\!\sqcup}(\epsilon\otimes1)\\
&+(\epsilon\otimes1\diamond1)\underline{\sqcup\!\sqcup}(1\otimes\epsilon)+(1\otimes1)\underline{\sqcup\!\sqcup}(1\otimes\epsilon)+(1\diamond1\otimes\epsilon)\underline{\sqcup\!\sqcup}(1\otimes\epsilon)\end{aligned}$$

$$\begin{aligned}=&(\epsilon \underline{\sqcup\!\sqcup} \epsilon) \otimes (1 \diamond 1 \underline{\sqcup\!\sqcup} 1) + (1 \underline{\sqcup\!\sqcup} \epsilon) \otimes (1 \underline{\sqcup\!\sqcup} 1) + (1 \diamond 1 \underline{\sqcup\!\sqcup} \epsilon) \otimes (\epsilon \underline{\sqcup\!\sqcup} 1)\\ &+ (\epsilon \underline{\sqcup\!\sqcup} 1) \otimes (1 \diamond 1 \underline{\sqcup\!\sqcup} \epsilon) + (1 \underline{\sqcup\!\sqcup} 1) \otimes (1 \underline{\sqcup\!\sqcup} \epsilon) + (1 \diamond 1 \underline{\sqcup\!\sqcup} 1) \otimes (\epsilon \underline{\sqcup\!\sqcup} \epsilon).\end{aligned}$$

Thus, we have $\Delta_\diamond(12 \underline{\sqcup\!\sqcup} 1) = \Delta_\diamond(12) \underline{\sqcup\!\sqcup} \Delta_\diamond(1)$.

Corollary 1. $(KS, \underline{\sqcup\!\sqcup}, \mu, \Delta_\diamond, \nu)$ *is a Hopf algebra.*

Proof of Corollary 1. From Theorem 1, we know $(KS, \underline{\sqcup\!\sqcup}, \mu, \Delta_\diamond, \nu)$ is a graded bialgebra. Since $(KS, \underline{\sqcup\!\sqcup}, \mu, \Delta_\diamond, \nu)$ is graded and connected, $(KS, \underline{\sqcup\!\sqcup}, \mu, \Delta_\diamond, \nu)$ is a Hopf algebra. □

5. Conclusions

First, we proved that any atom of a permutation π obtained by $\alpha \underline{\sqcup\!\sqcup} \beta$ can only consist of some complete atoms of α and β. Secondly, we proved that the super-shuffle product $\underline{\sqcup\!\sqcup}$ and the cut-box coproduct $\Delta_\diamond$ on KS are compatible, and hence $(KS, \underline{\sqcup\!\sqcup}, \mu, \Delta_\diamond, \nu)$ is a bialgebra; Finally, since this bialgebra is graded and connected, it is also a Hopf algebra.

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