

Article

Extended Chern–Simons Model for a Vector Multiplet

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Abstract: We consider a gauge theory of vector fields in 3D Minkowski space. At the free level, the dynamical variables are subjected to the extended Chern–Simons (ECS) equations with higher derivatives. If the color index takes n values, the third-order model admits a $2n$ -parameter series of second-rank conserved tensors, which includes the canonical energy–momentum. Even though the canonical energy is unbounded, the other representatives in the series have a bounded from below the 00-component. The theory admits consistent self-interactions with the Yang–Mills gauge symmetry. The Lagrangian couplings preserve the energy–momentum tensor that is unbounded from below, and they do not lead to a stable non-linear theory. The non-Lagrangian couplings are consistent with the existence of a conserved tensor with a 00-component bounded from below. These models are stable at the non-linear level. The dynamics of interacting theory admit a constraint Hamiltonian form. The Hamiltonian density is given by the 00-component of the conserved tensor. In the case of stable interactions, the Poisson bracket and Hamiltonian do not follow from the canonical Ostrogradski construction. Particular attention is paid to the “triply massless” ECS theory, which demonstrates instability even at the free level. It is shown that the introduction of extra scalar field, serving as Higgs, can stabilize the dynamics in the vicinity of the local minimum of energy. The equations of motion of the stable model are non-Lagrangian, but they admit the Hamiltonian form of dynamics with a Hamiltonian that is bounded from below.

Keywords: higher-derivative theories; extended Chern–Simons; Hamiltonian formalism; stability



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1. Introduction

The higher-derivative theories are well known for their better convergency properties at classical and quantum levels and wider symmetry. In most instances, these advantages come at the price of dynamic instability, which is the typical problem for the models in this class. At the classical level, the solutions to the equations of motion demonstrate runaway behavior (“collapse”). At the quantum level, ghost poles appear in the propagator, and so the unitarity of dynamics is an issue. These peculiarities follow from a single fact: the canonical energy is unbounded in every non-singular higher derivative theory. For a review of the problem, we cite recent articles [1–3] and references therein. The canonical energy of singular models can be bounded. The examples include $f(R)$ -theories of gravity [4–7]. These models do not demonstrate instability. The stability problem for constrained higher-derivative theories is discussed in [8]. In the majority of interesting (constrained or unconstrained) models, the instability has a special form: the classical dynamics are regular (no “collapse”), but the canonical energy is unbounded from below. The Pais–Uhlenbeck oscillator [9], Podolsky [10] and Lee–Wick [11,12] electrodynamics, ECS theory [13] and conformal gravity [14] are examples. With regard to the current studies, we refer readers to [15–18] and references therein. In all these models, the quantum theory is expected to be well defined [2,19], but the application of the canonical quantization procedure based on the Ostrogradski procedure (The canonical Hamiltonian formalism for the non-singular higher-derivative theories was first proposed in [20]. Its generalization

for singular theories was developed in [21]: see also [22]. For recent studies, see [23,24].) leads to a model with a spectrum of energy that is unbounded from below; that is why the stability and unitarity of quantum dynamics is the most important issue.

Recently, it has been recognized that the higher-derivative dynamics can be stable at the classical and quantum levels even if the canonical energy of the model is unbounded. The articles [25,26] explain the stability of Pais–Uhlenbeck theory from the perspective of the existence of the Hamiltonian form of dynamics with a bounded (non-canonical) Hamiltonian. The quantization of the model with an alternative Hamiltonian leads to a quantum theory with a well-defined vacuum state with the lowest energy. In [27,28], the authors use a special PT-symmetry for the construction of stable quantum mechanics in the higher-derivative oscillator model. In [1–3,29], the authors state that the non-linear higher derivative models may exhibit well-defined classical dynamics without “collapsing” trajectories with runaway behavior. In the latter articles, the existence of stable classical dynamics serves as a necessary prerequisite for the construction of a well-defined quantum theory. With regard to the studies of field theories, we refer readers to [30,31]. All the models mentioned in this paragraph have one common feature: they admit alternative (in contrast to the canonical energy) conserved quantities that are bounded from below. Quantum stability is achieved if the model admits the Hamiltonian form of dynamics, with the bounded conserved quantity being the Hamiltonian. This means that the stable higher-derivative theory is characterized by two ingredients: the bounded conserved quantity and the Poisson bracket on its phase space that brings the equations of motion to the Hamiltonian form with a Hamiltonian bounded from below.

In [32], the stability is studied in the class of models of the derived type. At the free level, the wave operator of the theory is given by a polynomial (characteristic polynomial) in another formally self-adjoint operator of lower order. It has been shown that each derived theory admits a series of conserved tensors, which includes the canonical energy–momentum [33]. The number of entries in the series grows with the order of the characteristic polynomial. Even though the canonical energy is unbounded, the other conserved tensors in the series can be bounded from below [33]. The bounded conserved quantity stabilizes the classical dynamics of the theory. The quantum stability is explained by the existence of the Lagrange anchor, which relates a bounded conserved quantity with the invariance of the model with respect to the time translations. (The concept of the Lagrange anchor was proposed in [34] in the context of the quantization of non-Lagrangian field theories. Later, it was recognized that the Lagrange anchor connects symmetries and conserved quantities [35].) In the first-order formalism, the Lagrange anchor defines the Poisson bracket [36]. The Hamiltonian is given by the conserved quantity in the series and is expressed in terms of phase-space variables. The linear higher-derivative theory is stable if a bounded conserved quantity, related to time-translation symmetry, is admitted by the model [32]. In [37,38], the authors consider the problem of the consistent deformations of symmetries and conserved quantities that preserve the stability of dynamics in the class of derived-type models. In all cases, interaction vertices do not come from the least-action principle with higher derivatives, but the equations of motion admit the Hamiltonian form of dynamics. The main difficulty of the procedures cited above is that they do not automatically preserve gauge invariance. This restricts their applications in gauge systems.

The ECS model [13] is the simplest gauge theory of the derived type. In current studies, it often serves as a prototype of the class of gauge theories with higher derivatives; see, e.g., [39–41]. The stability of the ECS model was first studied in [33]. It has been observed that the theory of order p admits a $p - 1$ -parameter series of second-rank conserved tensors, which includes the canonical energy–momentum. The canonical energy of the model is always unbounded from below. The other tensors in the series may have a 00-component that is bounded from below. The stability conditions for the theory are determined by the structure of the roots of the characteristic polynomial [33]. The theory is stable if all the nonzero roots of the characteristic polynomial are real and different and the zero root has a multiplicity of one or two. The stability of the ECS theory is confirmed in [42].

The ECS model was shown to be a multi-Hamiltonian at the free level in [43]. The series of Hamiltonians includes the canonical (Ostrogradski) Hamiltonian, which is unbounded, and the other representatives, which are bounded or unbounded depending on the model parameters. If the Hamiltonian is bounded, this ensures the stability of the model at both classical and quantum levels. The model admits the inclusion of stable non-Lagrangian interactions with scalar, fermionic and gravitational fields that preserve a selected representative in the series of conserved quantities of free model [38,43,44]. However, the gauge symmetry is abelian in the sector of the vector field in all these examples.

The concept of the consistency of interaction between the gauge fields was developed in [45]. The consistent couplings between gauge vectors were studied in [46,47]. It was shown that the Yang–Mills vertex is unique in the covariant setting. The Lagrangian self-interactions of the gauge vector multiplet subjected to the ECS equations at the free level were reconsidered in [42]. The most general consistent non-linear theory has been proven to have the Yang–Mills gauge symmetry, and the Lagrangian is given by the covariantization of the free ECS Lagrangian. The interacting model demonstrates Ostrogradski instability at the non-linear level in all instances, even though the free theory is stable. This result implies a no-go theorem for stable Lagrangian interactions in the ECS model. The conclusion is not surprising because the Lagrangian couplings preserve the conservation law of canonical energy, being unbounded already at the free level. The inclusion of non-Lagrangian interactions can solve the issue of the dynamic stability at the interacting level, because such couplings preserve bounded conserved quantities. However, the problem of the construction of stable and consistent non-Lagrangian (self-)couplings (stable or unstable) has not been studied in the ECS model in the literature before.

In the current work, we present a class of stable self-interactions in the theory of vector multiplets. The free fields are subjected to the ECS equations of the third order. The interaction is (in general) non-Lagrangian. The non-linear equations of motion are consistent with Yang–Mills gauge symmetry. A selected second-rank conserved tensor of the free model is preserved by the coupling. Depending on the values of coupling constants, this can be bounded or unbounded. The Lagrangian interaction vertex in [42] is unstable. The non-Lagrangian coupling can be consistent with the existence of the bounded conserved tensor. The equations of motion admit the Hamiltonian form of dynamics. On the shell, the Hamiltonian density is given by the 00-component of the conserved tensor. For Lagrangian interactions, the canonical formalism with the unbounded Ostrogradski Hamiltonian is reproduced. The bounded Hamiltonians do not follow from the Ostrogradski procedure, and thus we find non-canonical Hamiltonian formalism in this case. In all the instances, the Poisson bracket is a non-degenerate tensor, so the model admits a Hamiltonian action principle. The quantization of the first-order action with the bounded Hamiltonian leads to a stable quantum theory with a well-defined vacuum state.

In the case of resonance (multiple roots of a charlatanistic polynomial), the dynamics of the theory are unstable even at the free level. The inclusion of self-interaction does not improve the stability properties of dynamics of this model because the conserved quantities have the same structure in the free and non-linear models. In the current article, a model with the third-order zero resonance root is of interest. The free field is subjected to the “triply massless” Chern–Simons (CS) equations. The wave operator of free equations is given by a cube of the CS operator. To stabilize the dynamics at the non-linear level, we apply the “Higgs-like” mechanism described in [48]. Introducing an extra scalar, we generate a coupling such that the energy of the interacting theory obtains a local minimum for a solution with nonzero values of dynamical variables. The motions of small fluctuations in the vicinity of this solution are stable because the non-linear theory has no resonance. The dynamics of fluctuations admit the Hamiltonian form, with the Hamiltonian being given by the positive definite quadratic form of the fields. This means that the dynamics of the theory with a resonance can be stabilized by the inclusion of an appropriate interaction.

The article is organized as follows. In Section 2, we consider the third-order ECS model for a vector multiplet. Particular attention is paid to the structure of symmetries,

conserved quantities of the theory and the stability of the dynamics. In Section 3, we propose stable self-interactions between the multiplet of vector fields with the Yang–Mills gauge symmetry. The general model in the class model preserves a selected conserved tensor of the free theory, which can be bounded or unbounded from below depending on the values of coupling constants. The stable interactions are inevitably non-Lagrangian. In Section 4, we construct the constrained Hamiltonian formalism for the non-linear model. The density of the Hamiltonian is given by the 00-component of the conserved tensor, being expressed in terms of the phase-space variables. In Section 5, we consider the issue of the stability of “triply massless” ECS theory. We propose the class of consistent couplings, with the scalar field stabilizing the dynamics in the vicinity of equilibrium position. The equations of motion are non-Lagrangian, but they admit the constrained Hamiltonian form with a bounded Hamiltonian. The concluding section discusses the potential applications of the model that are considered in the article.

2. Higher-Derivative Chern–Simons Model

We consider the most general third-order gauge theory of the vector multiplet $A^a = A^a_\mu(x)dx^\mu$, $\mu = 0, 1, 2$, $a = 1, 2, \dots, n$ in three-dimensional Minkowski space. The action functional of the model reads

$$S[A(x)] = \frac{1}{2} \int A^a_\mu (\alpha_1 F^{a\mu} + \alpha_2 G^{a\mu} + \alpha_3 K^{a\mu}) d^3x. \quad (1)$$

where the real numbers α_1 , α_2 and α_3 are model parameters. Without loss of generality, we assume that the coefficient at the higher derivative term is nonzero, $\alpha_3 \neq 0$. The vectors F^a_μ , G^a_μ and K^a_μ denote the (generalized) field strengths of the potential A^a_μ ,

$$F^a_\mu = \epsilon_{\mu\nu\rho} \partial^\nu A^{a\rho}, \quad G^a_\mu = \epsilon_{\mu\nu\rho} \partial^\nu F^{a\rho}, \quad K^a_\mu = \epsilon_{\mu\nu\rho} \partial^\nu G^{a\rho}, \quad (2)$$

where the Levi–Civita symbol $\epsilon_{\mu\nu\rho}$, $\epsilon_{012} = 1$ is antisymmetric with respect to indices. All the tensor indices are raised and lowered by the Minkowski metric. We use the mostly positive convention for the signature of metrics throughout the paper. The summation is implied over the isotopic indices $a = 1, \dots, n$ repeated at one level unless otherwise stated. For $\alpha_3 = \alpha_1 = 1$, $\alpha_2 = 0$, $n = 1$, the action functional Equation (1) was first proposed in the article [13]. We refer to the model Equation (1) as the ECS theory for a vector multiplet.

The least-action principle for the functional Equation (1) brings us to the following Lagrange equations for the field A_μ :

$$\frac{\delta S}{\delta A^{a\mu}} \equiv \alpha_1 F^a_\mu + \alpha_2 G^a_\mu + \alpha_3 K^a_\mu = 0, \quad a = 1, \dots, n. \quad (3)$$

These equations have the derived form [33] because the wave operator is a polynomial in the CS operator $*d$. The symbol $*$ denotes the Hodge star operator, and d is the de Rham differential. For the exposition of differential form theory concepts, we refer readers to the book [49]. The structure of the Poincare group representation, being described by the Equation (3), is determined by the roots of the characteristic polynomial

$$M(\alpha; z) = \alpha_1 z + \alpha_2 z^2 + \alpha_3 z^3. \quad (4)$$

The polynomial $M(\alpha; z)$ follows from Equation (3) after the formal replacement of the CS operator $*d$ by the complex-valued variable z . The following cases are distinguished in [33].

- (1a) $\alpha_1 \neq 0$, $(\alpha_2)^2 - 4\alpha_1\alpha_3 > 0$. The characteristic polynomial has a zero root and two different nonzero real roots. Equation (3) describes two massive spin-1 self-dual fields. (The theory of a massive spin-1 field being subjected to the self-duality condition was proposed in [50]. For the theory of representations of the 3D Poincare group, we refer readers to [51–53]). A zero root corresponds to the CS mode, which is a pure gauge;

- (1b) $\alpha_1 \neq 0, (\alpha_2)^2 - 4\alpha_1\alpha_3 < 0$. The characteristic polynomial has a zero root and two complex roots. The case is similar to (1a), but the masses of vector modes are complex. The Poincare group representation is non-unitary;
- (2a) $\alpha_2 \neq 0, \alpha_1 = 0$. The characteristic polynomial has multiple two zero roots and a simple nonzero root. The set of subrepresentations includes a massless spin-1 field and a massive spin-1 mode subjected to the self-duality condition;
- (2b) $\alpha_2 \neq 0, (\alpha_2)^2 - 4\alpha_1\alpha_3 = 0$. The characteristic polynomial has multiple two nonzero roots and a simple zero root. The subrepresentations describe a “doubly massive” massive mode and a gauge CS mode. The Poincare group representation is non-unitary;
- (3) $\alpha_1 = \alpha_2 = 0$. The characteristic polynomial has multiple three zero roots. Equation (3) describes the “triply” massless extended CS theory. The Poincare group representation is non-unitary.

As we see, the ECS model describes the unitary representation of the Poincare group if all the nonzero roots of the characteristic equation are different, and the zero root has a multiplicity of one or two. The dynamical degrees of freedom include the massive spin-1 vector subjected to the self-duality condition and/or spin-1 massless field, meeting the 3D Maxwell equations. In all instances, the model has two local physical degrees of freedom (four physical polarizations).

The action function Equation (1) is preserved by the $2n$ -parameter series of infinitesimal transformations:

$$\delta_{\xi;\beta} A^a{}_\mu = -\varepsilon_{\mu\nu\rho} \xi^\rho (\beta^a{}_1 F^{a\rho} + \beta^a{}_2 G^{a\rho}), \quad a = 1, \dots, n, \quad (5)$$

(no summation in a). The transformation parameters are the constant vector ξ^μ and real numbers $\beta^a{}_k$, $a = 0, \dots, n$, $k = 1, 2$. The series Equation (5) includes the space–time translations with the independent parameters for the individual vector $A^a{}_\mu$ in the multiplet and higher-order transformations, whose value is determined by $\beta^a{}_2$. The Noether theorem associates symmetries Equation (5) with the $2n$ -parameter series of second-rank conserved tensors, which has the form

$$\Theta^{\mu\nu}(\beta; \alpha) = \sum_{a=1}^n (\beta^a{}_1 \Theta^{a\mu\nu}{}_1(\alpha) + \beta^a{}_2 \Theta^{a\mu\nu}{}_2(\alpha)), \quad (6)$$

where

$$\Theta^{a\mu\nu}{}_1(\alpha) = \alpha_3 (G^{a\mu} F^{a\nu} + G^{a\nu} F^{a\mu} - g^{\mu\nu} G^a{}_\rho F^{a\rho}) + \alpha_2 (F^{a\mu} F^{a\nu} - \frac{1}{2} g^{\mu\nu} F^a{}_\rho F^{a\rho}); \quad (7)$$

$$\Theta^{a\mu\nu}{}_2(\alpha) = \alpha_3 (G^{a\mu} G^{a\nu} - \frac{1}{2} g^{\mu\nu} G^a{}_\rho G^{a\rho}) - \alpha_1 (F^{a\mu} F^{a\nu} - \frac{1}{2} g^{\mu\nu} F^a{}_\rho F^{a\rho}) \quad (8)$$

(no summation in a). The quantities $\Theta^{a\mu\nu}{}_1$ and $a = 1, \dots, n$ Equation (7) represent the canonical energy–momentum tensors of individual fields in the vector multiplet $A^a{}_\mu$. The tensors $\Theta^{a\mu\nu}{}_2$ and $a = 1, \dots, n$ Equation (8) are other conserved quantities. The total number of independent conserved tensors in the free theory is $2n$ because each field in the multiplet admits two different symmetries.

The 00-component of the conserved tensor Equation (6) reads

$$\begin{aligned} \Theta^{00}(\beta; \alpha) = \frac{1}{2} \sum_{a=1}^n \left\{ \beta_2 \alpha_3 (G^{a0} G^{a0} + G^{ai} G^{ai}) + 2\beta_1 \alpha_3 (G^{a0} F^{a0} + G^{ai} F^{ai}) + \right. \\ \left. + (\beta_1 \alpha_2 - \beta_2 \alpha_1) (F^{a0} F^{a0} + F^{ai} F^{ai}) \right\}. \end{aligned} \quad (9)$$

The summation over the index $i = 1, 2$ repeated at one level is implied. The quadratic form Equation (9) can be reduced to the principal axes as follows:

$$\Theta^{00}(\beta; \alpha) = \sum_{a=1}^n \left\{ \frac{\alpha_3 (X^{a0} X^{a0} + X^{ai} X^{ai})}{2\beta^a} + \frac{C(\beta^a; \alpha) (F^{a0} F^{a0} + F^{ai} F^{ai})}{2\beta^a} \right\}. \quad (10)$$

where $X_\mu^a = \beta^a_1 F_\mu^a + \beta^a_2 G_\mu$, and the following notation is used:

$$C(\beta^a; \alpha) = -(\beta^a_2)^2 \alpha_1 + \beta^a_2 \beta^a_1 \alpha_2 - (\beta^a_1)^2 \alpha_3. \quad (11)$$

In Section 4, we see that $X^{a0} X^{a0} + X^{ai} X^{ai}$ and $F^{a0} F^{a0} + F^{ai} F^{ai}$ depend on different initial data (See the detailed discussion in [43]). In this case, the 00-component Equation (9) is bounded from below if the coefficients at squares are positive:

$$\beta^a_2 \alpha_3 > 0, \quad C(\beta^a; \alpha) > 0, \quad a = 1, \dots, n. \quad (12)$$

We ignore the case of the semi-positive quadratic form because the degenerate conserved quantities do not ensure the stability of dynamics. Relations Equation (12) are consistent if and only if the model parameters $\alpha_1, \alpha_2, \alpha_3$ meet conditions (1a) and (2a) of classification on pages 4–5. In cases (1b), (2b) and (3) of this classification, the conditions in Equation (12) are inconsistent. This means that the free model Equation (1) is stable if the wave Equation (3) describes a unitary representation of the Poincare group. This result is quite natural because the theories that correspond to the unitary representations of the Poincare group should have stable dynamics.

3. Stable Interactions

In this section, we present an example of stable self-interactions in the model Equation (1). The interactions are non-Lagrangian. The dynamics of the non-linear theory are determined by the equations of motion. The interactions are associated with the deformations of free equations of motion that preserve the gauge symmetries and gauge identities of the free model. The interaction is consistent if the number of physical degrees of freedom is preserved by coupling. For details of the concept of the consistency of interaction in the class of unnecessary Lagrangian theories, we refer readers to [54].

We begin the construction of a non-linear theory by assuming that the dynamical fields take values in the Lie algebra of a semisimple Lie group with the generators $t^a, a = 1, \dots, n$,

$$\mathcal{A}_\mu = A^a_\mu(x) t^a dx^\mu, \quad [t^a, t^b] = f^{abc} t^c, \quad \text{tr}(t^a t^b) = \delta^{ab}. \quad (13)$$

The covariant analogs of the (generalized) field strength vectors Equation (2) are defined as follows:

$$\mathcal{F}_\mu = \epsilon_{\mu\nu\rho} (\partial^\nu \mathcal{A}^\rho + \frac{1}{2} [\mathcal{A}^\nu, \mathcal{A}^\rho]), \quad \mathcal{G}_\mu = \epsilon_{\mu\nu\rho} D^\nu \mathcal{F}^\rho, \quad \mathcal{K}_\mu = \epsilon_{\mu\nu\rho} D^\nu \mathcal{G}^\rho. \quad (14)$$

The vectors $\mathcal{F}_\mu, \mathcal{G}_\mu, \mathcal{K}_\mu$ lie in the Lie algebra of a semisimple Lie group, and D stands for the covariant derivative

$$D_\mu = \partial_\mu + [\mathcal{A}_\mu, \cdot]. \quad (15)$$

The vector $\mathcal{F}_\mu$ represents a dual of the standard Yang–Mills strength tensor. The generalized field strengths $\mathcal{G}_\mu$ and $\mathcal{K}_\mu$ are other covariant quantities that involve second and third derivatives of $\mathcal{A}_\mu$.

We consider the interactions that are polynomial in the gauge invariants of Equation (14) and do not involve the highest derivative in the non-linear part. In this setting, the most general self-consistent non-linear theory is determined by the following equations of motion:

$$\mathbb{T}^\mu = \alpha_3 \mathcal{K}^\mu + \alpha_2 \mathcal{G}^\mu + \alpha_1 \mathcal{F}^\mu - \frac{\alpha_3^2}{2C(\beta; \alpha)} \epsilon^{\mu\nu\rho} [\beta_1 \mathcal{F}_\nu + \beta_2 \mathcal{G}_\nu, \beta_1 \mathcal{F}_\rho + \beta_2 \mathcal{G}_\rho] = 0. \quad (16)$$

We prove the uniqueness of this coupling in Appendix A. The model parameters are the real numbers $\alpha_k, k = 1, 2, 3, \beta_l, l = 1, 2$ and $f^{abc}, a, b, c = 1, \dots, n$. The constants α_1, α_2 determine the free limit of the Equation (16). The numbers β_1 and β_2 distinguish admissible couplings with the same gauge group. Throughout this section and below, we assume that $C(\beta; \alpha) \neq 0$. The equations of motion Equation (16) come from the least-action principle if $\beta_1 = 1, \beta_2 = 0$. In this case, the action functional reads

$$S[A(x)] = \frac{1}{2} \text{tr} \int \mathcal{F}_\mu (\alpha_1 A_\mu + \alpha_2 \mathcal{F}^\mu + \alpha_3 \mathcal{G}^\mu) d^3x. \quad (17)$$

This action functional was first derived in [42] (The higher-derivative Yang–Mills theory with a similar structure of the Lagrangian has been known for a long time [55]). The same paper tells us that Equation (17) is the most general form of consistent self-interaction in the gauge theory of vector fields. This means that the most general consistent Lagrangian coupling Equation (17) is included in the model Equation (16). If the parameter β_2 is nonzero, Equation (16) does not follow from the least-action principle for any functional with higher derivatives. The variational principle with auxiliary fields still exists, even if the higher-derivative model is non-Lagrangian. In the last case, the theory Equation (16) admits consequent quantization and the establishment of a relationship between symmetries and conserved quantities.

The concept of interaction consistency for theories that are not necessarily Lagrangian has been developed in [54]. This paper tells us that the non-Lagrangian interaction is consistent if the non-linear theory admits the same number of (i) gauge symmetries, (ii) gauge identities and (iii) physical degrees of freedom as a free model. All these facts are easily verified. At first, the equations of motion Equation (16) are preserved by the Yang–Mills gauge symmetry:

$$\delta_\zeta A_\mu = D_\mu \zeta, \quad \delta_\zeta \mathbb{T}_\mu = [\zeta, \mathbb{T}_\mu], \quad (18)$$

where $\zeta = (\zeta^a(x), a = 1, \dots, n)$ is the gauge transformation parameter. The free model Equation (17) is preserved by the standard gradient gauge symmetry, $\delta_\zeta A^a_\mu = \partial_\mu \zeta^a$. As required, the gauge symmetry Equation (18) is given by the deformation of the gradient gauge symmetry of free model. The important difference is that the gauge symmetry Equation (18) is non-abelian. Thus, the inclusion of interaction tends towards the model with non-abelian gauge symmetry. We have an obvious set of gauge identities between the Equation (16),

$$\mathbb{D}_\mu \mathbb{T}^\mu = 0, \quad \mathbb{D}_\mu = D_\mu + \frac{\alpha_3}{C(\beta; \alpha)} [\beta_1 \mathcal{F}_\mu + \beta_2 \mathcal{G}_\mu, \cdot]. \quad (19)$$

Again, the leading term of the gauge identity is given by the free contribution. This agrees with the concept of interaction consistency. At the final step of analysis, we verify that the physical degrees of freedom number is preserved by the coupling. Equation (8) of [54] expresses the number of physical degrees of freedom via the orders of gauge symmetries, gauge identities and equations of motion in the involutive of dynamics. The systems Equations (3) and (16) are involutive, and they have equal orders of gauge symmetries, gauge identities and equations of motion. Thus, they have to possess the same number of physical degrees of freedom. All the above implies that the non-Lagrangian interaction Equation (16) is consistent for the general values of the parameters β, α .

The theory Equation (16) admits a symmetric conserved tensor of second rank of the following form:

$$\Theta^{\mu\nu}(\beta; \alpha) = \text{tr} \left\{ \beta_2 \alpha_3 (\mathcal{G}^\mu \mathcal{G}^\nu - \frac{1}{2} g^{\mu\nu} \mathcal{G}_\rho \mathcal{G}^\rho) + \beta_1 \alpha_3 (\mathcal{G}^\mu \mathcal{F}^\nu + \mathcal{G}^\nu \mathcal{F}^\mu - \right. \\ \left. - g^{\mu\nu} \mathcal{G}_\rho \mathcal{F}^\rho) + (\beta_1 \alpha_2 - \beta_2 \alpha_1) (\mathcal{F}^\mu \mathcal{F}^\nu - \frac{1}{2} g^{\mu\nu} \mathcal{F}_\rho \mathcal{F}^\rho) \right\}, \quad (20)$$

where $\alpha_k, k = 1, 2, 3$ and $\beta_l, l = 1, 2$ are the model parameters. The divergence of the quantity $\Theta^{\mu\nu}(\beta; \alpha)$ reads

$$\partial_\nu \Theta^{\mu\nu} = -\text{tr}(\epsilon^{\mu\nu\rho}(\beta_1 \mathcal{F}_\nu + \beta_2 \mathcal{G}_\nu) \mathbb{T}_\rho). \quad (21)$$

Expression Equation (20) is the covariantization of a selected representative in the conserved tensor series: Equation (6),

$$\beta^a_1 = \beta_1, \quad \beta^a_2 = \beta_2, \quad a = 1, \dots, n. \quad (22)$$

As is seen, the model Equation (16) represents the class of deformations of free ECS Equation (3) that preserves a selected representative in the series Equation (6) of conserved quantities at the non-linear level. It is important to note that the other representatives in the series Equation (20) are no longer conserved in the non-linear theory Equation (16). This happens because the parameters $\beta_k, k = 1, 2$ in the conserved tensor Equation (20) are unambiguously fixed by the interaction.

As far as stability is concerned, the 00-component of the tensor Equation (20) is relevant. The latter reads

$$\begin{aligned} \Theta^{00}(\beta; \alpha) = & \text{tr} \left\{ \frac{1}{2} \beta_2 \alpha_3 (\mathcal{G}^0 \mathcal{G}^0 + \mathcal{G}^i \mathcal{G}^i) + \beta_1 \alpha_3 (\mathcal{G}^0 \mathcal{F}^0 + \mathcal{G}^i \mathcal{F}^i) + \right. \\ & \left. + \frac{1}{2} (\beta_1 \alpha_2 - \beta_2 \alpha_1) (\mathcal{F}^0 \mathcal{F}^0 + \mathcal{F}^i \mathcal{F}^i) \right\}. \end{aligned} \quad (23)$$

The conserved tensor is a bilinear form in (generalized) strengths $\mathcal{F}_\mu, \mathcal{G}_\mu$. The application of the quadratic form Equation (23) to the principal axes reveals that the model is stable if

$$\beta_2 \alpha_3 > 0, \quad C(\beta; \alpha) > 0. \quad (24)$$

These conditions can be consistent or inconsistent depending on the values of the model parameters $\alpha_k, k = 1, 2, 3$ and $\beta_l, l = 1, 2$. The Lagrangian interaction vertex Equation (17) does not meet stability requirements. This confirms the instability of the variational coupling proposed in [42]. The stable interactions in the class of models Equation (16) are inevitably non-Lagrangian. The similar form of stability conditions Equation (12), Equation (24) at the free and interacting cases implies that the linear and non-linear dynamics are stable or unstable simultaneously. In the class of theories of stable dynamics at the linear level, Equation (16) determines a class of non-linear models that preserve a selected bounded conserved quantity in the series Equation (6).

Now, we can return to the special case $C(\beta; \alpha) = 0$, which is excluded in Equation (16). The conserved quantity Equations (6) and (22) are a degenerate quadratic form of the initial data. The 00-component of the free conserved tensor can be bounded from below, but its degeneracy implies the existence of zero vectors. The motion of the system in the degenerate direction can be infinite, and the corresponding conserved quantity appears to be irrelevant to stability even at the free level. The formula Equation (16) prevents the construction of interacting theories that preserve the conserved tensors with a semi-definite 00-component.

4. Hamiltonian Formalism

In this section, we construct a constrained Hamiltonian formalism for the higher-derivative theory Equation (16).

Let us first explain what we understand by the constrained Hamiltonian formalism for the system of not necessarily Lagrangian field equations with higher derivatives. A general fact is that the higher-derivative system can be reduced to the first order by the introduction of extra fields that absorb the time derivatives of original dynamical variables. The set of original dynamical variables and extra fields is denoted by $\{\varphi^I(x, t), \lambda^A(x, t)\}$, where $t = x^0$

is the time, and $x = (x^1, x^2)$ stands for space coordinates. The multi-indices A, I label the phase-space variables. The first-order equations are said to be Hamiltonian if there exists a Hamiltonian function $\mathcal{H}(\varphi^I, \nabla \varphi^I, \nabla^2 \varphi^I, \dots)$ (∇ stands for derivatives by the space x) and a Poisson bracket $\{\varphi^I(x), \varphi^J(y)\}$ such that the equations constitute a constrained Hamiltonian system; i.e.,

$$\begin{aligned} \dot{\varphi}^I(x) &= \{\varphi^I(x), \int \mathcal{H} dy\}, \quad \theta_A(\varphi^I(x), \nabla \varphi^I(x), \dots) = 0; \\ \mathcal{H} &= \mathcal{H}_0(\varphi^I(x), \nabla \varphi^I(x), \dots) + \lambda^A(x) \theta_A(\varphi^I(x), \nabla \varphi^I(x), \dots). \end{aligned} \quad (25)$$

where the dot denotes the derivative by time

$$\dot{\varphi}^I = \frac{d\varphi^I}{dx^0}. \quad (26)$$

The system is multi-Hamiltonian if there exists a series of Hamiltonian $\mathcal{H}_\beta$ and Poisson brackets $\{\varphi^I(x), \varphi^J(y)\}_\beta$ parameterized by constraints $\beta_1, \dots, \beta_k$ such that they determine the same equations of motion for the dynamical fields. The existence of the constrained Hamiltonian formulation is not guaranteed for a system of general non-Lagrangian equations of motion. In particular, the Hamiltonian formulation for the ECS theory may not exist, at least for certain combinations of model parameters.

We begin the construction of the Hamiltonian formalism with the reduction of order of Equation (16). The space components of the field strength $\mathcal{F}_i, i = 1, 2$, and generalized field strength of second order $\mathcal{G}_i, i = 1, 2$, are chosen as extra fields. By construction, they absorb the first and second time derivatives of space components of the vector field $\mathcal{A}_i, i = 1, 2$,

$$\begin{aligned} \mathcal{F}_i &= \epsilon_{ij}(\dot{\mathcal{A}}_j - \partial_i \mathcal{A}_0 + [\mathcal{A}_0, \mathcal{A}_j]), \\ \mathcal{G}_i &= -\ddot{\mathcal{A}}_i + \partial_i \dot{\mathcal{A}}_0 - [\dot{\mathcal{A}}_0, \mathcal{A}_i] - [\mathcal{A}_0, \dot{\mathcal{A}}_i] + [\mathcal{A}_0, \mathcal{A}_i - \partial_i \mathcal{A}_0 + [\mathcal{A}_0, \mathcal{A}_i]] - \\ &\quad - \epsilon_{ij} \epsilon_{kl} \partial_j (\partial_k \mathcal{F}_l + 1/2 [\mathcal{A}_k, \mathcal{A}_l]) + \epsilon_{ij} \epsilon_{kl} [\mathcal{A}_j, \partial_k \mathcal{F}_l + 1/2 [\mathcal{A}_k, \mathcal{A}_l]]. \end{aligned} \quad (27)$$

where $\epsilon_{ij}, \epsilon_{12} = 1$ is the 2D Levi-Civita symbol. The Latin indices i, j run over the values 1, 2. Summation over Latin indices repeated at the same level is implied. As can be seen from the equations, in Equation (27), the fields $\mathcal{F}_i$ absorb the first time derivatives of space components of original vector field $\mathcal{A}_i$. The vector $\mathcal{G}_i$ involves the second time derivatives of $\mathcal{A}_i$. The time components $\mathcal{F}_0, \mathcal{G}_0$ of vectors Equation (27) are functions of $\mathcal{A}_0, \mathcal{A}_i, \mathcal{F}_i$ and $\mathcal{G}_i$, with no new combinations of time derivatives being involved:

$$\mathcal{F}_0 \equiv \epsilon_{ij}(\partial_i \mathcal{A}_j + [\mathcal{A}_i, \mathcal{A}_j]), \quad \mathcal{G}_0 \equiv \epsilon_{ij}(\partial_i \mathcal{F}_j + [\mathcal{A}_i, \mathcal{G}_j]). \quad (28)$$

In the remaining part of the article, we associate the quantities $\mathcal{F}_0, \mathcal{G}_0$ with their expressions in terms of the phase-space variables $\mathcal{A}_0, \mathcal{A}_i$ and $\mathcal{F}_i$.

Substituting the extra variables in Equation (27) into Equation (16), we obtain the first-order equations for the fields $\mathcal{A}_i, \mathcal{F}_i, \mathcal{G}_i$:

$$\dot{\mathcal{A}}_i = \partial_i \mathcal{A}_0 - [\mathcal{A}_0, \mathcal{A}_i] - \epsilon_{ij} \mathcal{F}_j; \quad (29)$$

$$\dot{\mathcal{F}}_i = \partial_i \mathcal{F}_0 + [\mathcal{A}_i, \mathcal{F}_0] - [\mathcal{A}_0, \mathcal{F}_i] - \epsilon_{ij} \mathcal{G}_j; \quad (30)$$

$$\begin{aligned} \dot{\mathcal{G}}_i &= \partial_i \mathcal{G}_0 + \frac{\alpha_2}{\alpha_3} \epsilon_{ij} \mathcal{G}_j + \frac{\alpha_1}{\alpha_3} \epsilon_{ij} \mathcal{F}_j + [\mathcal{A}_i, \mathcal{G}_0] - [\mathcal{A}_0, \mathcal{G}_i] + \\ &\quad + \frac{\alpha_3^2}{C(\beta; \alpha)} [\beta_1 \mathcal{F}_0 + \beta_2 \mathcal{G}_0, \beta_1 \mathcal{F}_i + \beta_2 \mathcal{G}_i]. \end{aligned} \quad (31)$$

The evolutionary Equations (29)–(31) are supplemented by the constraint

$$\Theta \equiv \epsilon_{ij}(\alpha_1 \partial_i \mathcal{A}_j + \alpha_2 \partial_i \mathcal{F}_j + \alpha_3 \partial_i \mathcal{G}_j + [\mathcal{A}_i, \frac{1}{2} \alpha_1 \mathcal{A}_j + \alpha_2 \mathcal{F}_j + \alpha_3 \mathcal{G}_j]) - \frac{\alpha_3}{2C(\beta; \alpha)} [\beta_1 \mathcal{F}_i + \beta_2 \mathcal{G}_i, \beta_1 \mathcal{F}_j + \beta_2 \mathcal{G}_j] \approx 0. \quad (32)$$

The Equations (16), (29)–(32) are equivalent. Evolutionary Equations (29) and (30) express the auxiliary fields $\mathcal{F}_i$ and $\mathcal{G}_i$ in terms of time derivatives of the original vector potential $\mathcal{A}_\mu$. Solving them with respect to the unknown $\mathcal{F}_i$, $\mathcal{G}_i$, we obtain the relations in Equation (27). Equations (31) and (32) represent the space and time components of the original higher-derivative system in Equation (16), where all the higher derivatives of the vector potential are expressed in terms of extra variables $\mathcal{F}_i$, $\mathcal{G}_i$. The left-hand side of Equation (32) does not involve time derivatives, so we have the constraint $\Theta \approx 0$. The sign $\approx$ means the equality modulo constraint in Equation (32). Once the time evolution is preserved (see the gauge identity, Equation (19)), no secondary constraints are imposed on the fields.

We associate the on-shell Hamiltonian $\mathcal{H}_0$ with the 00-component of the conserved tensor in Equation (20). In terms of the phase-space variables $\mathcal{A}_i$, $\mathcal{F}_i$ and $\mathcal{G}_i$, it reads

$$\mathcal{H}_0 = \frac{1}{2} \text{tr} \left\{ \beta_2 \alpha_3 (\mathcal{G}^0 \mathcal{G}^0 + \mathcal{G}^i \mathcal{G}^i) + 2\beta_1 \alpha_3 (\mathcal{G}^0 \mathcal{F}^0 + \mathcal{G}^i \mathcal{F}^i) + (\beta_1 \alpha_2 - \beta_2 \alpha_1) (\mathcal{F}^0 \mathcal{F}^0 + \mathcal{F}^i \mathcal{F}^i) \right\}. \quad (33)$$

where the functions $\mathcal{F}^0$ and $\mathcal{G}^0$ are defined in Equation (28). The on-shell Hamiltonian $\mathcal{H}_0$ depends on the free model parameters $\alpha_k, k = 1, 2, 3$ and coupling constants $\beta_l, l = 1, 2$. Off-shell, the Hamiltonian is a sum of the on-shell part of Equation (33) and a linear combination of constraints. We chose the following ansatz for the total Hamiltonian:

$$\mathcal{H} = \mathcal{H}_0 + \text{tr} \left[\frac{C(\beta; \alpha)}{\beta_2 \alpha_2 - \beta_1 \alpha_3} \mathcal{A}_0 - \frac{\beta_2 \beta_1 \alpha_3}{\beta_2 \alpha_2 - \beta_1 \alpha_3} \mathcal{F}_0 - \frac{\beta_2^2 \alpha_3}{\beta_2 \alpha_2 - \beta_1 \alpha_3} \mathcal{G}_0 \right] \Theta, \quad (34)$$

where β, α are constant parameters. The on-shell vanishing terms included in the Hamiltonian do not contribute to the equations of motion of gauge-invariant quantities. The equations of motion do alter for non-gauge-invariant variables. As we are attempting to determine the Hamiltonian and Poisson bracket by literally reproducing the first-order Equation (29)–(32) of Equation (16), the on-shell vanishing contributions are kept under control in Equation (34).

The Hamiltonian is well-defined if the parameters β and α are subject to the following conditions:

$$C(\beta; \alpha) \neq 0, \quad \beta_1 \alpha_3 - \beta_2 \alpha_2 \neq 0. \quad (35)$$

These relations have a clear origin. The first condition in this set ensures that the on-shell Hamiltonian is a non-degenerate quadratic form of the phase-space variables $\mathcal{A}_i$, $\mathcal{F}_i$, $\mathcal{G}_i$. This requirement is reasonable because the degenerate Hamiltonian cannot generate the evolution of all physical degrees of freedom. The second relation of Equation (35) ensures that the numerical factor at the Lagrange multiplier $\mathcal{A}_0$ is non-singular. This is necessary to reproduce the correct gauge transformations for all the dynamical variables. We also note that the obstructions to the existence of the Hamiltonian remain valid in the free limit. This means that the inclusion of an interaction by the scheme presented in Section 3 does not restrict the class models that admit the Hamiltonian formulation. In other words, every theory in the class of Equation (16) admitting the Hamiltonian formulation with the Hamiltonian Equation (34) in the free limit is Hamiltonian at the interacting level. Hereafter, we assume that the relations in Equation (35) are satisfied. The Hamiltonian is on-shell bounded from below if the conditions in Equation (24) are satisfied. In this case, we expect

to construct the constrained Hamiltonian formulation with a bounded Hamiltonian. The corresponding quantum theory has a good chance to be stable.

Now, let us find the Poisson bracket between the fields $\mathcal{A}_i$, $\mathcal{F}_i$ and $\mathcal{G}_i$ that leads Equations (29)–(32) to adhere to the Hamiltonian form Equation (25) with the Hamiltonian Equation (34). Comparing the right-hand sides of Equations (25), (29)–(31), we obtain the following system of algebraic equations:

$$\left\{ \mathcal{A}_i, \int \mathcal{H}(\mathbf{y}) d\mathbf{y} \right\} \approx \partial_i \mathcal{A}_0 - [\mathcal{A}_0, \mathcal{A}_i] - \epsilon_{ij} \mathcal{F}_j; \quad (36)$$

$$\left\{ \mathcal{F}_i, \int \mathcal{H}(\mathbf{y}) d\mathbf{y} \right\} \approx \partial_i \mathcal{F}_0 + [\mathcal{A}_i, \mathcal{F}_0] - [\mathcal{A}_0, \mathcal{F}_i] - \epsilon_{ij} \mathcal{G}_j; \quad (37)$$

$$\begin{aligned} \left\{ \mathcal{G}_i, \int \mathcal{H}(\mathbf{y}) d\mathbf{y} \right\} \approx & \partial_i \mathcal{G}_0 + \frac{\alpha_2}{\alpha_3} \epsilon_{ij} \mathcal{G}_j + \frac{\alpha_1}{\alpha_3} \epsilon_{ij} \mathcal{F}_j + [\mathcal{A}_i, \mathcal{G}_0] - [\mathcal{A}_0, \mathcal{G}_i] + \\ & + \frac{\alpha_3^2}{C(\beta; \alpha)} [\beta_1 \mathcal{F}_0 + \beta_2 \mathcal{G}_0, \beta_1 \mathcal{F}_i + \beta_2 \mathcal{G}_i]. \end{aligned} \quad (38)$$

In Equations (36)–(38), the sign $\approx$ means the equality modulo constraint seen in Equation (32). The Poisson bracket defined by these equations depends on five independent arguments: the free model parameters α_3 , α_2 and α_1 and coupling constants β_2 and β_1 . The bracket has the following form:

$$\{\mathcal{G}_i^a(\mathbf{x}), \mathcal{G}_j^b(\mathbf{y})\} = \frac{\beta_1 \alpha_2^2 - \beta_1 \alpha_1 \alpha_3 - \beta_2 \alpha_2 \alpha_1}{\alpha_3^2 C(\beta; \alpha)} \epsilon_{ij} \delta^{ab} \delta^{(2)}(\mathbf{x} - \mathbf{y}); \quad (39)$$

$$\{\mathcal{F}_i^a(\mathbf{x}), \mathcal{G}_j^b(\mathbf{y})\} = \frac{\beta_2 \alpha_1 - \beta_1 \alpha_2}{\alpha_3 C(\beta; \alpha)} \epsilon_{ij} \delta^{ab} \delta^{(2)}(\mathbf{x} - \mathbf{y}); \quad (40)$$

$$\{\mathcal{F}_i^a(\mathbf{x}), \mathcal{F}_j^b(\mathbf{y})\} = \{\mathcal{A}_i^a(\mathbf{x}), \mathcal{G}_j^b(\mathbf{y})\} = \frac{\beta_1}{C(\beta; \alpha)} \epsilon_{ij} \delta^{ab} \delta^{(2)}(\mathbf{x} - \mathbf{y}); \quad (41)$$

$$\{\mathcal{A}_i^a(\mathbf{x}), \mathcal{F}_j^b(\mathbf{y})\} = \frac{-\beta_2}{C(\beta; \alpha)} \epsilon_{ij} \delta^{ab} \delta^{(2)}(\mathbf{x} - \mathbf{y}); \quad (42)$$

$$\{\mathcal{A}_i^a(\mathbf{x}), \mathcal{A}_j^b(\mathbf{y})\} = 0. \quad (43)$$

where $\delta^{(2)}(\mathbf{x} - \mathbf{y}) = \delta(x^1 - y^1) \delta(x^2 - y^2)$ is the 2D δ -function in the space coordinates. The Poisson bracket in Equations (39)–(43) is a covariant generalization of its free analog, which is derived in [43]. This result is not surprising. The free limit of expressions in Equations (39)–(43) is determined by the linear model, while the Poisson brackets with a polynomial dependence on fields contradict the structure of Equation (25). In the last case, the Poisson bracket with the Hamiltonian involves higher powers of fields than the right-hand side of first-order Equations (29)–(31).

The Poisson bracket in Equations (39)–(43) is a non-degenerate tensor, so it has an inverse, being a symplectic two-form. The latter defines a Hamiltonian action functional

$$\begin{aligned} S_h = \int \left\{ \text{tr} \left(\frac{C(\beta; \alpha)}{\beta_2 \alpha_2 - \beta_1 \alpha_3} \epsilon_{ij} (\alpha_1 \mathcal{A}_i + 2\alpha_2 \mathcal{F}_i + 2\alpha_3 \mathcal{G}_i) \dot{\mathcal{A}}_j + \frac{\beta_1^2 \alpha_3}{\beta_2 \alpha_2 - \beta_1 \alpha_3} \times \right. \right. \\ \left. \left. \times \epsilon_{ij} (\beta_1 \mathcal{F}_i + \beta_2 \mathcal{G}_i) (\beta_1 \mathcal{F}_j + \beta_2 \mathcal{G}_j) \right) - \mathcal{H} \right\} dx dt, \end{aligned} \quad (44)$$

where $\mathcal{H}$ denotes the Hamiltonian Equation (34). In the case $\beta_2 = 0, \beta_1 = 1$, Equation (44) reproduces the Ostrogradski action for the variational model Equation (17),

$$S_c = \int \left\{ \text{tr}(\epsilon_{ij}(\alpha_1 \mathcal{A}_i + 2\alpha_2 \mathcal{F}_i + 2\alpha_3 \mathcal{G}_i) \dot{\mathcal{A}}_j - \epsilon_{ij} \mathcal{F}_i \dot{\mathcal{F}}_j) - \mathcal{H}_c \right\} dx dt, \quad (45)$$

where $\mathcal{H}_c$ is the canonical Hamiltonian,

$$\mathcal{H}_c = \frac{1}{2} \text{tr} \left(\alpha_3 (\mathcal{G}^0 \mathcal{F}^0 + \mathcal{G}^i \mathcal{F}^i) + \alpha_2 (\mathcal{F}^0 \mathcal{F}^0 + \mathcal{F}^i \mathcal{F}^i) + \mathcal{A}_0 \Theta \right). \quad (46)$$

In the free case, the Hamiltonian formulation Equation (44) was first proposed in [43]. Later, it was re-derived in [42] by the direct application of the Ostrogradski procedure to the action functional Equation (1). The comparison of Equations (12) and (35) implies that the Hamiltonian Equation (34) is on-shell unbounded for all the Lagrangian interactions. For non-Lagrangian interactions, the Hamiltonian Equation (34) can be on-shell bounded or unbounded depending on the value of model parameters. If the Hamiltonian is on-shell bounded, the interacting theory is stable at the quantum level. To our knowledge, the ECS theory Equation (16) is the first higher-derivative model with a non-abelian gauge symmetry admitting an alternative Hamiltonian formulation with a bounded Hamiltonian. This means that the concept of the stabilization of dynamics by means of an alternative Hamiltonian formalism applies beyond the linear level.

In the free limit, the Hamiltonian Equation (34) and Poisson bracket in Equations (39)–(43) depend on the parameters β_1 and β_2 , while the equations of motion in Equations (29)–(32) do not. This means that the free ECS theory Equation (1) is a multi-Hamiltonian theory. The general representative of the series of Hamiltonian formulations does not follow from the Ostrogradski procedure. The reason is that the bounded and unbounded Hamiltonian cannot be connected by the change of coordinates in the phase space. The number of entries in the free Hamiltonian series equals $2n$, where n is the number of color indices. The on-shell Hamiltonian is given by the 00-component Equation (9) of the conserved tensor Equation (6). The Poisson bracket between the phase-space variables A^a_i, F^a_i, G^a_i is determined by the formulas Equations (39)–(43), but the parameters β_1, β_2 are replaced by β^a_1, β^a_2 for each field in the multiplet. At the non-linear level, a selected Poisson bracket, as shown in Equations (39)–(43), is preserved. If the respective Hamiltonian is bounded, the Poisson bracket cannot follow from the Ostrogradski construction; thus, the Hamiltonian formalism for the stable non-linear theories does not follow from the canonical formulation.

In the conclusion of this section, we present the Poisson brackets between the constraints Equations (39)–(43):

$$\{\Theta^a(x), \Theta^b(y)\} = \frac{\beta_2 \alpha_2 - \beta_1 \alpha_3}{C(\beta; \alpha)} f^{abc} \Theta^c(x) \delta(x - y). \quad (47)$$

The Hamiltonian is gauge-invariant,

$$\{\Theta^a(x), \mathcal{H}_0(y)\} = 0, \quad a = 1, \dots, n. \quad (48)$$

As can be seen, the first-order model, Equation (44), is a gauge theory of a special form in the whole range, Equation (35), of admissible values of the parameters α, β . Equation (44) determines the least-action principle for the model. The quantization of this theory can be performed by means of the well-known procedures [56]. In all instances with a Hamiltonian that is bounded from below, we expect the stability of the model. This means that Equation (16) determines a higher-derivative theory, which can be stable at the classical and quantum level. Because of the existence of first-order formalism, the model is as good as the Lagrangian theories. In particular, this admits consequent quantization and correspondence between symmetries and conserved quantities. The Hamiltonian serves as a quantity that is associated to the invariance of the model in Equation (44) with respect to the time translations. The Hamiltonian Equation (34) is on-shell bounded if the conditions in Equation (24) are satisfied.

5. Resonance Case

In case of resonance (positions (2b), (3) of classification on pages 4–5), the non-linear theory Equation (3) admits the Hamiltonian form of dynamics, but the Hamiltonian is unbounded in all instances (conditions Equation (24) are inconsistent). Thus, the model has to be considered to be unstable. In this section, we demonstrate that the dynamics of the theory with resonance can be stabilized by means of the inclusion of an interaction with an extra dynamical field. We apply the “Higgs-like” mechanism, which was first proposed in the context of study of the “doubly massless” generalized Podolsky electrodynamics in the paper [48]. Here, we use it in the theory with non-abelian gauge symmetry for the first time. We mostly consider the model (Equation (16)) with the third-order resonance. We chose the following values for the parameters of free theory, Equation (1): $\alpha_1 = \alpha_2 = 0$, $\alpha_3 = -1$. This choice does not restrict the generality of our work, because the constant α_3 accounts for the possibility of multiplication of equations of motion by an overall factor. The wave operator of the free model (Equation (3)) appears to be cube of the CS operator, so we have a sort of “triply massless” extended theory.

We begin the construction of the interaction by extending the set of dynamical variables by a real scalar field $\phi(x)$. The non-linear theory of the vector multiplet $\mathcal{A}_\mu(x)$ and scalar field $\phi(x)$ is determined by the equations of motion, which have the following form:

$$\begin{aligned} \mathbb{T}_\mu &= \epsilon_{\mu\nu\rho} \left\{ D^\nu [(\tilde{\gamma}^2 \phi^2 - 1) \mathcal{G} + \gamma^2 \phi^2 \mathcal{F}]^\rho + \frac{(\tilde{\gamma}^2 \phi^2 - 1)^2}{2\beta_1^2} [\beta_1 \mathcal{F}^\nu + \beta_2 \mathcal{G}^\nu, \beta_1 \mathcal{F}^\rho + \beta_2 \mathcal{G}^\rho] \right\} = 0, \\ \mathbb{T} &= \left\{ \partial_\mu \partial^\mu - (m^2 - \tilde{\gamma}^2 \frac{(\beta_1 \mathcal{F}_\rho + \beta_2 \mathcal{G}_\rho)(\beta_1 \mathcal{F}^\rho + \beta_2 \mathcal{G}^\rho)}{\beta_1^2} + \phi^2 \right\} \phi = 0. \end{aligned} \quad (49)$$

where the vectors $\mathcal{F}_\mu$ and $\mathcal{G}_\mu$ are defined in Equation (2), and the abbreviation $\tilde{\gamma}^2 = \gamma^2 \beta_2 / \beta_1$ is used. The constants β_2 , β_1 and γ , m are model parameters and are real numbers. Throughout this section, we assume that $m, \gamma > 0$ and $\beta_1 \neq 0$. The option $\beta_1 = 0$ is not admissible because the self-coupling Equation (16) between the vector fields becomes inconsistent in this case. The value β_2 can be an arbitrary real number (positive, negative or zero). Only positive values of β_2 , β_1 lead to stable couplings. This justifies the notation $\tilde{\gamma}^2$ for the quantity $\gamma^2 \beta_2 / \beta_1$. Without loss of generality, we put a unit coefficient at ϕ^3 -term. An overall factor at the ϕ^3 -vertex can be absorbed by the scaling of the scalar field, $\phi \mapsto \lambda \phi$, with the appropriate $\lambda \neq 0$.

Equation (49) has a clear meaning. The first line of the system in Equation (49) describes the motion of the vector multiplet. The linear term in the fields in the equations corresponds to the “triply massless” ECS theory. The coupling includes the self-interaction term of Equation (16) and an extra contribution involving the scalar field. It is convenient to think that Equation (49) follows from Equation (16) after the formal redefinition of the model parameters:

$$\alpha_1 = 0, \quad \alpha_2 = \gamma^2 \phi^2, \quad \alpha_3 = \tilde{\gamma}^2 \phi^2 - 1. \quad (50)$$

The characteristic polynomial in (Equation (4)) for the model reads

$$M(\alpha; z) = \gamma^2 \phi^2 z^2 + (\tilde{\gamma}^2 \phi^2 - 1) z^3. \quad (51)$$

If the scalar field is set to a nonzero constant from the outset, we obtain Equation (16) with the second-order resonance for the zero root (case (2a) of classification on pages 4–5). As explained above, the corresponding model describes a massless spin-1 field and a massive spin-1 vector subjected to the self-duality condition. It is stable at the classical and quantum levels. The second equation in the system of Equation (49) describes the motion of the scalar field. This equation includes ϕ^3 -coupling, which ensures the existence of a nonzero stationary solution for $\mathcal{A}_\mu = 0$. This means that ϕ serves as the Higgs field in the model of Equation (49).

The interaction defined by Equation (49) is consistent. The equations are preserved by the standard Yang–Mills gauge symmetry in Equation (18). The scalar field is preserved by the gauge transformation. The gauge identity has a slightly different form:

$$\mathbb{D}_\mu \mathbb{T}^\mu = 0, \quad \mathbb{D}_\mu = D_\mu + \frac{\tilde{\gamma}^2 \phi^2 - 1}{\beta_1^2} [\beta_1 \mathcal{F}_\mu + \beta_2 \mathcal{G}_\mu, \cdot]. \quad (52)$$

We note that the gauge generator involves the scalar field explicitly. The model has the same number of physical degrees of freedom as the free theory because the orders of equations of motion in Equation (49), gauge identities in Equation (52) and gauge symmetries in Equation (18) are preserved by coupling. Equations (49) do not follow from the least-action principle for a function with higher derivatives unless $\beta_2 = 0$. In the case of Lagrangian coupling, the action principle reads

$$S[\phi(x), \mathcal{A}(x)] = \frac{1}{2} \int \left\{ \mathcal{F}_\mu (\mathcal{G}^\mu + \gamma^2 \phi^2 \mathcal{F}^\mu) + \partial_\mu \phi \partial^\mu \phi + m^2 \phi^2 - \frac{1}{2} \phi^4 \right\} d^3x. \quad (53)$$

The dynamics of the Lagrangian theory are unstable. In the case of non-Lagrangian couplings, the dynamics of the model can be stable. In the remaining part of this section, we address an issue of the construction of the constrained Hamiltonian formalism with a bounded Hamiltonian for the higher-derivative Equation (49). The existence of such a formalism implies the classical and quantum stability of the model. To avoid repetition, we outline the most crucial steps of the construction, while the details are provided above.

We introduce a special notation for the linear combination of generalized strengths $\mathcal{F}_\mu, \mathcal{G}_\mu$ entering the free part of Equation (49),

$$\mathcal{W}_\mu = (\tilde{\gamma}^2 \phi^2 - 1) \mathcal{G}_\mu + \gamma^2 \phi^2 \mathcal{F}_\mu. \quad (54)$$

We choose the space components of the vectors $\mathcal{F}_i$ and $\mathcal{W}_i$, $i = 1, 2$ Equation (27) as variables absorbing the first and second derivatives of the original dynamical field $\mathcal{A}_\mu$. In the sector of the scalar field, we introduce the canonical momentum $\pi = \dot{\phi}$. We consider $\mathcal{W}_0$ as a special notation for the combination Equation (54) of the derivatives of phase-space variables $\mathcal{A}_\mu, \mathcal{F}_i$. In terms of the variables $\phi, \pi, \mathcal{A}_\mu, \mathcal{F}_i, \mathcal{G}_i$, the first-order equations eventually read

$$\dot{\mathcal{A}}_i = \partial_i \mathcal{A}_0 - [\mathcal{A}_0, \mathcal{A}_i] - \epsilon_{ij} \mathcal{F}_j; \quad (55)$$

$$\dot{\mathcal{F}}_i = \partial_i \mathcal{F}_0 + [\mathcal{A}_i, \mathcal{F}_0] - [\mathcal{A}_0, \mathcal{F}_i] - \epsilon_{ij} \frac{\mathcal{W}_j - \gamma^2 \phi^2 \mathcal{F}_j}{\tilde{\gamma}^2 \phi^2 - 1}; \quad (56)$$

$$\dot{\mathcal{W}}_i = \partial_i \mathcal{W}_0 + [\mathcal{A}_i, \mathcal{W}_0] - [\mathcal{A}_0, \mathcal{W}_i] + \epsilon_{ij} [\beta_2 \mathcal{W}_0 - \beta_1 \mathcal{F}_0, \beta_2 \mathcal{W}_j - \beta_1 \mathcal{F}_j]; \quad (57)$$

$$\dot{\pi} = \left\{ \partial_i \partial_i - m^2 + \tilde{\gamma}^2 \frac{(\beta_2 \mathcal{W}_\mu - \beta_1 \mathcal{F}_\mu)(\beta_2 \mathcal{W}^\mu - \beta_1 \mathcal{F}^\mu)}{(\tilde{\gamma}^2 \phi^2 - 1)^2} + \phi^2 \right\} \phi; \quad (58)$$

$$\dot{\phi} = \pi. \quad (59)$$

The evolutionary Equations (55)–(59) are supplemented by the constraint

$$\Theta \equiv \epsilon_{ij} \left(\partial_i \mathcal{W}_j + [\mathcal{A}_i, \mathcal{W}_j] + \frac{1}{2\beta_2^2} [\beta_2 \mathcal{W}_i - \beta_1 \mathcal{F}_i, \beta_2 \mathcal{W}_j - \beta_1 \mathcal{F}_j] \right) = 0. \quad (60)$$

The constraint conserves the on-shell property if the identity Equation (52) is taken into account. The higher-derivative system Equation (49) follows from

Equations (55)–(57) and (60) if all the extra variables are expressed in terms of the derivatives of original dynamical fields $\phi, \mathcal{A}$.

The model in Equation (49) admits a second-rank symmetric conserved tensor:

$$\Theta^{\mu\nu}(\beta; \alpha) = \text{tr} \left\{ \frac{\tilde{\gamma}^2 \phi^2 - 1}{\beta_2} \left((\beta_1 \mathcal{F}^\mu + \beta_2 \mathcal{G}^\mu)(\beta_1 \mathcal{F}^\nu + \beta_2 \mathcal{G}^\nu) - \frac{1}{2} g^{\mu\nu} (\mathcal{F}_\rho + \mathcal{G}_\rho)(\mathcal{F}^\rho + \mathcal{G}^\rho) \right) + \right. \\ \left. + \frac{\beta_2^2}{\beta_1} \left(\mathcal{F}^\mu \mathcal{F}^\nu - \frac{1}{2} g^{\mu\nu} \mathcal{F}_\rho \mathcal{F}^\rho \right) \right\} + \partial^\mu \phi \partial^\nu \phi + g^{\mu\nu} \left(-\frac{1}{2} \partial_\rho \phi \partial^\rho \phi - \frac{1}{2} m^2 \phi^2 + \frac{1}{4} \phi^4 \right); \quad (61)$$

$$\partial_\nu \Theta^{\mu\nu}(\beta; \alpha) = -\text{tr}(\epsilon^{\mu\nu\rho}(\beta_1 \mathcal{F}_\nu + \beta_2 \mathcal{G}_\nu) \mathbb{T}_\rho) + \partial^\mu \phi \cdot \mathbb{T}. \quad (62)$$

The Hamiltonian is given by a sum of the 00-component of the conserved tensor and is expressed in terms of variables $\phi, \pi, \mathcal{A}_\mu, \mathcal{F}_i, \mathcal{W}_i$ Equations (27) and (54) and a constraint term:

$$\mathcal{H} = \text{tr} \left\{ \frac{(\beta_2 \mathcal{W}^0 - \beta_1 \mathcal{F}^0)(\beta_2 \mathcal{W}^0 - \beta_1 \mathcal{F}^0) + (\beta_2 \mathcal{W}^i - \beta_1 \mathcal{F}^i)(\beta_2 \mathcal{W}^i - \beta_1 \mathcal{F}^i)}{2(\tilde{\gamma}^2 \phi^2 - 1)\beta_2} + \right. \\ \left. + \frac{\beta_1^2}{2\beta_2} (\mathcal{F}^0 \mathcal{F}^0 + \mathcal{F}^i \mathcal{F}^i) - (\mathcal{A}_0 - \beta_1 \mathcal{F}_0 + \beta_2 \mathcal{W}_0) \Theta \right\} + \\ + \frac{1}{2} (\pi \pi + \partial^i \phi \partial_i \phi) - \frac{1}{2} m^2 \phi^2 + \frac{1}{4} \phi^4. \quad (63)$$

where the quantities $\mathcal{F}_0, \mathcal{W}_0$ denote abbreviations of Equations (28) and (54), and the numbers m, γ are model parameters. The Hamiltonian is on-shell bounded if

$$\tilde{\gamma}^2 \phi^2 - 1 > 0, \quad \beta_2 > 0. \quad (64)$$

These conditions involve the scalar field ϕ . Once the initial value of ϕ is a Cauchy datum for Equation (49), the Hamiltonian cannot be globally bounded. However, the Hamiltonian is given by a positive definite quadratic form in the variables $\mathcal{F}, \mathcal{G}$ in the range $|\tilde{\gamma}\phi| > 1$ of values of the scalar field. This corresponds to the case of a stability island.

The Poisson bracket between the fields $\phi, \pi, \mathcal{A}_\mu, \mathcal{F}_i$ and $\mathcal{G}_i$ is determined by the following system of equations:

$$\left\{ \mathcal{A}_i, \int \mathcal{H}(\mathbf{y}) d\mathbf{y} \right\} \approx \partial_i \mathcal{A}_0 - [\mathcal{A}_0, \mathcal{A}_i] - \epsilon_{ij} \mathcal{F}_j; \quad (65)$$

$$\left\{ \mathcal{F}_i, \int \mathcal{H}(\mathbf{y}) d\mathbf{y} \right\} \approx \partial_i \mathcal{F}_0 + [\mathcal{A}_i, \mathcal{F}_0] - [\mathcal{A}_0, \mathcal{F}_i] - \epsilon_{ij} \frac{\mathcal{W}_j - \gamma^2 \phi^2 \mathcal{F}_j}{\tilde{\gamma}^2 \phi^2 - 1}; \quad (66)$$

$$\left\{ \mathcal{G}_i, \int \mathcal{H}(\mathbf{y}) d\mathbf{y} \right\} \approx \partial_i \mathcal{W}_0 + [\mathcal{A}_i, \mathcal{W}_0] - [\mathcal{A}_0, \mathcal{G}_i] + \frac{\epsilon_{ij}}{\beta_1^2} [\beta_2 \mathcal{W}_0 - \beta_1 \mathcal{F}_0, \beta_2 \mathcal{W}_j - \beta_1 \mathcal{F}_j]. \quad (67)$$

$$\left\{ \pi, \int \mathcal{H}(\mathbf{y}) d\mathbf{y} \right\} \approx \partial_i \partial_i \phi + \left(m^2 - \tilde{\gamma}^2 \frac{(\beta_2 \mathcal{W}_\mu - \beta_1 \mathcal{F}_\mu)(\beta_2 \mathcal{W}^\mu - \beta_1 \mathcal{F}^\mu)}{(\tilde{\gamma}^2 \phi^2 - 1)^2} - \phi^2 \right) \phi; \quad (68)$$

$$\left\{ \phi, \int \mathcal{H}(\mathbf{y}) d\mathbf{y} \right\} \approx \pi. \quad (69)$$

All the equalities are considered modulo terms that remove the modulo constraint of Equation (60). The solution for Equations (65)–(69) reads

$$\{\mathcal{W}^a_i(\mathbf{x}), \mathcal{W}^b_j(\mathbf{y})\} = \{\mathcal{F}^a_i(\mathbf{x}), \mathcal{W}^b_j(\mathbf{y})\} = \{\mathcal{A}^a_i(\mathbf{x}), \mathcal{A}^b_j(\mathbf{y})\} = 0; \quad (70)$$

$$\{\mathcal{A}_i^a(\mathbf{x}), \mathcal{W}_j^b(\mathbf{y})\} = -\{\mathcal{F}_i^a(\mathbf{x}), \mathcal{F}_j^b(\mathbf{y})\} = \frac{1}{\beta_1} \epsilon_{ij} \delta^{ab} \delta^{(2)}(\mathbf{x} - \mathbf{y}); \quad (71)$$

$$\{\mathcal{A}_i^a(\mathbf{x}), \mathcal{F}_j^b(\mathbf{y})\} = \frac{\beta_2}{\beta_1^2} \epsilon_{ij} \delta^{ab} \delta^{(2)}(\mathbf{x} - \mathbf{y}); \quad (72)$$

$$\{\phi(\mathbf{x}), \pi(\mathbf{y})\} = \delta^{(2)}(\mathbf{x} - \mathbf{y}). \quad (73)$$

All the Poisson brackets between ϕ , π and $\mathcal{A}_i$, $\mathcal{F}_i$ and $\mathcal{G}_i$ vanish. The relations of Equations (70)–(72) can be obtained from Equations (39)–(42) after the substitution of Equation (50). We note that the Poisson bracket between the fields $\mathcal{A}_i$, $\mathcal{F}_i$, $\mathcal{W}_i$ is constant, even though the theory involves extra fields π , ϕ .

Now, we can demonstrate the phenomenon of dynamic stabilization by means of the “Higgs-like” mechanism proposed in [48]. Equation (49) admits a nonzero stationary solution:

$$\mathcal{A}(\mathbf{x}) = \mathcal{F}(\mathbf{x}) = \mathcal{G}(\mathbf{x}) = 0, \quad \pi(\mathbf{x}) = 0, \quad \phi(\mathbf{x}) = \pm m. \quad (74)$$

Introducing the notation $\phi^*(\mathbf{x}) = \phi(\mathbf{x}) \mp m$ and expanding Equation (49) in the vicinity of this vacua, we obtain the linearized equations for the fields

$$\mathbb{T}_\mu = (m^2 \tilde{\gamma}^2 - 1) \mathcal{K}_\mu + m^2 \gamma^2 \mathcal{G}_\mu + \dots = 0, \quad (75)$$

$$\mathbb{T} = (\partial_\mu \partial^\mu + 5m^2) \phi^* + \dots = 0.$$

The dots denote the terms that are at least quadratic in the fields. As we see, the dynamics of the field $\mathcal{A}$ are described by the higher-derivative of Equation (3) with $\alpha_3 = m^2 \tilde{\gamma}^2 - 1$, $\alpha_2 = m^2 \gamma^2$, $\alpha_1 = 0$. In this case, the system has a second-order resonance for the zero root, which does not affect the stability of the dynamics, at least at the free level. The on-shell Hamiltonian reads

$$\begin{aligned} \mathcal{H} = \text{tr} \Big\{ & \frac{1}{2(m^2 \tilde{\gamma}^2 - 1) \beta_2} ((\beta_1 \mathcal{W}^0 - \beta_1 \mathcal{F}^0)(\beta_1 \mathcal{W}^0 - \beta_1 \mathcal{F}^0) + (\beta_2 \mathcal{W}^i - \beta_1 \mathcal{F}^i)(\beta_2 \mathcal{W}^i - \beta_1 \mathcal{F}^i)) + \\ & + \frac{\beta_1^2}{2\beta_2} (\mathcal{F}^0 \mathcal{F}^0 + \mathcal{F}^i \mathcal{F}^i) - (\mathcal{A}_0 - \beta_1 \mathcal{F}_0 + \beta_2 \mathcal{W}_0) \Theta \Big\} + \\ & + \frac{1}{2} (\pi \pi + \partial^i \phi^* \partial_i \phi^* + 5m^2 (\phi^*)^2) + \dots \end{aligned} \quad (76)$$

The dots denote the terms that are at least cubic in the fields. The quadratic part of the Hamiltonian is on-shell bounded if $m \tilde{\gamma} > 1$. This means that the dynamics of small fluctuations in the vicinity of vacua in Equation (74) are stable. Once local stability is sufficient for the construction of stable quantum theory, the “Higgs-like” mechanism can stabilize the dynamics of the non-linear theory of Equation (16) with the third-order resonance.

6. Conclusions

The results of the article demonstrate that the inclusion of non-Lagrangian couplings can solve the problem of the inclusion of stable interactions in higher-derivative models. Equation (16) introduces a two-parameter series of couplings in the theory of vector fields with Yang–Mills gauge symmetry. The dynamics of the model are stable if the coupling parameters meet the conditions in Equation (24). To our knowledge, Equation (16) represents a first example of a stable non-linear higher derivative model with non-abelian gauge symmetry whose canonical energy is unbounded from below in the free

limit (Here, we mean the class of models that demonstrate the Ostrogradski instability at the free level. In the $f(R)$ -theories of gravity [4,5], the canonical energy is bounded from below in the free approximation). The stable couplings do not follow from the least-action principle, but they admit the Hamiltonian form of dynamics with a Hamiltonian that is bounded from below. The presence of the Hamiltonian action principle allows the consecutive construction of quantum theory, which is expected to be free of ghosts. The model Equation (16) can be considered as the stable 3D generalization of the conventional higher derivative Yang–Mills symmetry [55]. The theory admits the inclusion of stable interactions with the scalar, spinor and gravitational fields in a similar manner to [38,43,44]. The constructed models can serve as the 3D higher-derivative generalizations of scalar electrodynamics, quantum chromodynamics and gravity with higher-derivative matter. These theories can mimic the actual 4D models that describe the real world. Finally, the procedure of the inclusion of non-Lagrangian couplings can be generalized to the case of arbitrary space–time dimensions. This will lead to the construction of new models of fundamental interactions with the non-Lagrangian equations of motion for the fields. In future studies, these models can be applied to unsolved problems such as the physics beyond the standard model [57] or dark matter [58].

The study of the “triply massless” ECS provides a new insight into the well-known problem of the stability of Pais–Uhlenbeck-type theories with equal frequencies [9,28]. Even though the theories with resonance are usually considered as unstable even at the free level, our results propose a constructive procedure of the stabilization of dynamics by the inclusion of coupling with an extra field. Equations of motion of the model are given in Equation (49). The dynamics are metastable in the vicinity of the energy minimum, while global stability in the model with resonance is impossible. A similar phenomenon is observed in higher-derivative gravities [31]. The ECS model provides a simple setting for the study of the metastability phenomenon of dynamics in the class of gauge theories. The conclusions of the study can be applied to gravity models of interest. We also mention that the inclusion of an interaction in the model with resonance does not change the gauge symmetry of vector fields, which remains of the Yang–Mills form. The gauge symmetry-preserving deformations of equations of motion are known as “gaugings” [59,60]. The results of this article tell us that the “gaugings” are essential in the context of the stabilization of higher-derivative dynamics with resonance. This application of this special type of interactions has never been considered before. Further studies will allow the construction of a new class of stable vector-scalar models with higher derivatives.

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Abbreviations

The following abbreviations are used in this manuscript:

CS Chern–Simons
ECS Extended Chern–Simons

Appendix A. Uniqueness of Consistent Interaction Vertex

In this Appendix, we demonstrate the fact of the uniqueness of the interaction vertex in Equation (16) in the class of Poincare-covariant couplings that are polynomial in the invariants $\mathcal{F}_\mu, \mathcal{G}_\mu$ without higher derivatives. We apply the method of the inclusion of not necessarily Lagrangian consistent interactions in [54].

Let us first explain the concept of interaction consistency in the class of non-Lagrangian field theories. The dynamics of the theory are determined by a system of partial differential equations (equations of motion) imposed onto the dynamical fields $\varphi^I(x)$:

$$\mathbb{T}_a(\varphi^I(x), \partial\varphi^I(x), \partial^2\varphi^I(x), \dots) = 0. \quad (\text{A1})$$

The equations of motion do not necessarily follow from the least-action principle for any functional $S[\varphi(x)]$, so I and a may run over different sets. For the free model, the left-hand side of Equation (A1) is supposed to be linear in the fields. The interactions are associated with the deformations of system Equation (A1) by non-linear terms. The equations of motion for the theory with coupling are polynomial in the fields

$$\mathbb{T}_a = \mathbb{T}_a^{(0)} + \mathbb{T}_a^{(1)} + \mathbb{T}_a^{(2)} + \dots = 0. \quad (\text{A2})$$

where $\mathbb{T}_a^{(0)}$, $\mathbb{T}_a^{(1)}$ and $\mathbb{T}_a^{(2)}$ are linear, quadratic and cubic in the dynamical variables. Throughout the section, the system Equation (A1) (or, equivalently, Equation (A2)) is supposed to be involutive. The concept of involution implies that Equation (A1) has no differential consequences of lower order (hidden integrability conditions). The ECS Equation (16) is involutive.

The defining relations for gauge symmetries and gauge identities in the system of partial derivative Equation (A1) read

$$\delta_\epsilon \varphi^I = R^I_\alpha \epsilon^\alpha, \quad \delta_\epsilon \mathbb{T}_a|_{\mathbb{T}=0} = 0; \quad (\text{A3})$$

$$L^a_A \mathbb{T}_a \equiv 0. \quad (\text{A4})$$

where R^I_α, L^a_A are certain differential operators. For non-Lagrangian equations, the gauge symmetries and gauge identities are not related to each other, so the multi-indices A, α are different. In the perturbative setting, the gauge symmetry and gauge identity generators are supposed to be polynomial in fields

$$R^I_\alpha = R^{(0)I}_\alpha + R^{(1)I}_\alpha + R^{(2)I}_\alpha + \dots, \quad L^a_A = L^{(0)a}_A + L^{(1)a}_A + L^{(2)a}_A + \dots \quad (\text{A5})$$

where $R^{(0)}, L^{(0)}, R^{(1)}, L^{(1)}$ and $R^{(2)}, L^{(2)}$ are field-independent, linear and quadratic in the dynamical variables. The interaction is consistent if all the gauge symmetries and gauge identities of the free model are preserved by the deformation of equations of motion. Equation (8) of paper [54] provides a simple formula for the computation of the number of physical degrees of freedom based on the orders of derivatives involved in the equations

of models, gauge symmetries and gauge identities. The free and non-linear theory must have the same number of physical degrees of freedom.

Relations in Equations (A3) and (A4) imply the following consistency conditions for $\mathbb{T}^{(k)}$, $R^{(k)}$, $L^{(k)}$ and $k = 0, 1, 2, \dots$:

$$R^{(0)I}{}_{\alpha} \partial_I \mathbb{T}^{(0)}{}_a = 0; \quad (\text{A6})$$

$$R^{(0)I}{}_{\alpha} \partial_I \mathbb{T}^{(1)}{}_a + R^{(1)I}{}_{\alpha} \partial_I \mathbb{T}^{(0)}{}_a = 0; \quad (\text{A7})$$

$$R^{(0)I}{}_{\alpha} \partial_I \mathbb{T}^{(2)}{}_a + R^{(1)I}{}_{\alpha} \partial_I \mathbb{T}^{(1)}{}_a + R^{(2)I}{}_{\alpha} \partial_I \mathbb{T}^{(0)}{}_a = 0 \quad (\text{A8})$$

(where the symbol ∂_I denotes a variational derivative with respect to the field φ^I);

$$L^{(0)a}{}_A \mathbb{T}^{(0)}{}_a = 0; \quad (\text{A9})$$

$$L^{(0)a}{}_A \mathbb{T}^{(1)}{}_a + L^{(1)a}{}_A \mathbb{T}^{(0)}{}_a = 0; \quad (\text{A10})$$

$$L^{(0)a}{}_A \mathbb{T}^{(2)}{}_a + L^{(1)a}{}_A \mathbb{T}^{(1)}{}_a + L^{(2)a}{}_A \mathbb{T}^{(0)}{}_a = 0. \quad (\text{A11})$$

Equations (A6) and (A9) determine the gauge symmetry and gauge identity generators $R^{(0)}$, $L^{(0)}$ of the free theory. These quantities are usually given from the outset. Relations in Equations (A7) and (A10) determine the first-order corrections to the equations of motion $T^{(1)}$, gauge symmetry generators $R^{(1)}$ and gauge identity generators $L^{(1)}$. Relations of Equations (A7) and (A10) determine the second-order corrections to the equations of motion $T^{(2)}$, gauge symmetry generators $R^{(2)}$ and gauge identity generators $L^{(2)}$. The procedure of interaction construction can be extended to the third and higher orders. Once the most general covariant ansatz is applied for $T^{(k)}$, $k = 1, 2, \dots$ the procedure of Equations (A6)–(A11), ... allows a complete classification of consistent interactions in a given field theory. An important subtlety of this procedure is that some lower-order couplings can be inconsistent at the higher orders of perturbation theory. The first critical step is the extension of the first-order (quadratic) interaction vertex to the second order of perturbation theory.

Equation (3) determines the left-hand side of the free ECS equations $\mathbb{T}^{(0)}$:

$$\mathbb{T}^{(0)a}{}_{\mu} = \alpha_1 F^a{}_{\mu} + \alpha_2 G^a{}_{\mu} + \alpha_3 K^a{}_{\mu} = 0. \quad (\text{A12})$$

The gauge symmetries and gauge identities are defined by the gradient and divergence operators:

$$R^{(0)}{}_{\mu} = \partial_{\mu}, \quad L^{(0)\mu} = \partial^{\mu}. \quad (\text{A13})$$

The free gauge identity of Equation (A6) and free gauge transformation read

$$\partial_{\mu} \mathbb{T}^{(0)\mu} \equiv 0, \quad \delta_{\epsilon} A^a{}_{\mu} = \partial_{\mu} \zeta^a, \quad (\text{A14})$$

where ζ values are gauge transformation parameters. We consider the Poincare-covariant interactions that are expressed in terms of gauge covariants $\mathcal{F}_{\mu}$, $\mathcal{G}_{\mu}$, $\mathcal{K}_{\mu}$ Equation (14) with no higher-derivative terms being included in the coupling. In this case, the equations of motion are automatically preserved by the Yang–Mills gauge symmetry (Equations (A6)–(A8) are satisfied). Consistent interaction vertices of first and second orders are selected by the conditions in Equations (A10)–(A11). We elaborate on this problem below.

We assume that the equations of motion are polynomial in gauge covariants $\mathcal{F}_\mu, \mathcal{G}_\mu, \mathcal{K}_\mu$. The linear term is given by the covariantization of the free Equation (3). The most general covariant first-order interaction vertex without higher-derivatives reads

$$\begin{aligned} & \mathbb{T}^{(0)}_\mu + \mathbb{T}^{(1)}_\mu = \\ & = \alpha_1 \mathcal{F}_\mu + \alpha_2 \mathcal{G}_\mu + \alpha_3 \mathcal{K}_\mu + \epsilon_{\mu\nu\rho} \left(\frac{1}{2} k_1 [\mathcal{F}^\nu, \mathcal{F}^\rho] + k_2 [\mathcal{F}^\nu, \mathcal{G}^\rho] + \frac{1}{2} k_3 [\mathcal{G}^\nu, \mathcal{G}^\rho] \right), \end{aligned} \quad (\text{A15})$$

where $k_l, l = 1, 2, 3$ are constants. The covariant divergence of equations of motion reads

$$\begin{aligned} D_\mu \mathbb{T}^\mu &= -\frac{1}{\alpha_3} [k_2 \mathcal{F}_\mu + k_3 \mathcal{G}_\mu, \mathbb{T}^\mu] + \\ &+ \frac{1}{\alpha_3} (\alpha_3^2 - \alpha_3 k_1 + \alpha_2 k_2 - \alpha_1 k_3) [\mathcal{F}_\mu, \mathcal{G}^\mu] + (k_2^2 - k_3 k_1) \epsilon_{\mu\nu\rho} [\mathcal{F}^\mu, [\mathcal{F}^\nu, \mathcal{G}^\rho]]. \end{aligned} \quad (\text{A16})$$

The gauge identity is satisfied in the first-order approximation in Equation (A10) if the coefficients $k_l, l = 1, 2, 3$ satisfy the relation

$$\alpha_3^2 - \alpha_3 k_1 + \alpha_2 k_2 - \alpha_1 k_3 = 0. \quad (\text{A17})$$

The general solution to this equations reads

$$k_1 = -\frac{\beta_1^2 \alpha_3^2}{C(\beta; \alpha)} + \alpha_1 \beta_3, \quad k_2 = -\frac{\beta_2^2 \beta_1 \alpha_3^2}{C(\beta; \alpha)}, \quad k_3 = -\frac{\beta_2^2 \alpha_3^2}{C(\beta; \alpha)} + \alpha_3 \beta_1, \quad (\text{A18})$$

where $\beta_1, \beta_2, \beta_3$ are coupling parameters. The parameters β_1, β_2 determine the coupling vertex of Equation (16) (two constants determine a single coupling because the ratio β_1/β_2 is relevant). The constant β_3 is responsible for another interaction vertex, which is consistent at the first order of perturbation theory. The interaction vertex of Equation (16) is self-consistent, with no higher-order corrections required for the equations of motion. The other coupling needs cubic corrections to the equations of motion in the fields. To prove the uniqueness of the interaction of Equation (16), we should demonstrate that the ansatz of Equations (A15) and (A18) is inconsistent at the second order of perturbation theory for $\beta_3 \neq 0$.

The most general second-order covariant interaction vertex reads

$$\begin{aligned} \mathbb{T}^{(2)}_\mu &= l_1 [\mathcal{F}_\nu, [\mathcal{F}_\mu, \mathcal{F}^\nu]] + l_2 [\mathcal{F}_\nu, [\mathcal{G}_\mu, \mathcal{F}^\nu]] + l_3 [\mathcal{F}_\nu, [\mathcal{F}_\mu, \mathcal{G}^\nu]] + \\ &+ l_4 [\mathcal{G}_\nu, [\mathcal{F}_\mu, \mathcal{G}^\nu]] + l_5 [\mathcal{G}_\nu, [\mathcal{G}_\mu, \mathcal{F}^\nu]] + l_6 [\mathcal{G}_\nu, [\mathcal{G}_\mu, \mathcal{G}^\nu]], \end{aligned} \quad (\text{A19})$$

where $l_p, p = \overline{1, 6}$ are constants. The covariant divergence of equations of motion reads (only cubic terms are written out)

$$\begin{aligned} D_\mu \mathbb{T}^\mu &= \frac{1}{\alpha_3} \epsilon_{\mu\nu\rho} [\mathcal{S}^{\mu\nu}, \mathbb{T}^\rho] + \frac{1}{2} C_1(k; l) \epsilon_{\mu\nu\rho} [\mathcal{F}^\mu, [\mathcal{F}^\nu, \mathcal{G}^\rho]] + \frac{1}{2} C_2(k; l) \epsilon_{\mu\nu\rho} [\mathcal{G}^\mu, [\mathcal{G}^\nu, \mathcal{F}^\rho]] + \\ &+ \frac{l_1}{2} [\mathcal{F}_\mu, [\mathcal{F}_\nu, D^\mu \mathcal{F}^\nu + D^\nu \mathcal{F}^\mu]] + \frac{l_3 - l_2}{2} [\mathcal{G}_\mu, [\mathcal{F}_\nu, D^\mu \mathcal{F}^\nu + D^\nu \mathcal{F}^\mu]] + \frac{2l_2 - l_3}{2} [\mathcal{F}_\mu, [\mathcal{G}_\nu, D^\mu \mathcal{F}^\nu + D^\nu \mathcal{F}^\mu]] + \\ &+ \frac{l_5}{2} [\mathcal{G}_\mu, [\mathcal{G}_\nu, D^\mu \mathcal{F}^\nu + D^\nu \mathcal{F}^\mu]] + \frac{l_3}{2} [\mathcal{F}_\mu, [\mathcal{F}_\nu, D^\mu \mathcal{G}^\nu + D^\nu \mathcal{G}^\mu]] + \frac{l_5 - l_4}{2} [\mathcal{F}_\mu, [\mathcal{G}_\nu, D^\mu \mathcal{G}^\nu + \\ &+ D^\nu \mathcal{G}^\mu]] + \frac{2l_4 - l_5}{2} [\mathcal{G}_\mu, [\mathcal{F}_\nu, D^\mu \mathcal{G}^\nu + D^\nu \mathcal{G}^\mu]] + \frac{l_6}{2} [\mathcal{G}_\mu, [\mathcal{G}_\nu, D^\mu \mathcal{G}^\nu + D^\nu \mathcal{G}^\mu]]. \end{aligned} \quad (\text{A20})$$

Here, the following notation is used:

$$C_1(k; l) = k_2^2 - k_1 k_3 - 3l_1 - \frac{\alpha_1}{\alpha_3} l_4 - \frac{\alpha_1}{\alpha_3} l_5 - \frac{\alpha_1}{\alpha_3} l_6, \quad C_2(k; l) = 3l_2 + l_3 - \frac{\alpha_2}{\alpha_3} l_5 - \frac{\alpha_1}{\alpha_3} l_6; \quad (\text{A21})$$

$$S^{\mu\nu} = l_3[\mathcal{F}^\mu, [\mathcal{F}^\nu, \cdot]] + l_4[\mathcal{G}^\mu, [\mathcal{F}^\nu, \cdot]] + l_6[\mathcal{G}^\mu, [\mathcal{G}^\nu, \cdot]] - l_4[[\mathcal{F}^\mu, \mathcal{F}^\nu], \cdot] - l_5[[\mathcal{F}^\mu, \mathcal{G}^\nu], \cdot] - l_6[[\mathcal{G}^\mu, \mathcal{G}^\nu], \cdot]. \quad (\text{A22})$$

The interaction is consistent at the second order of perturbation theory if the right-hand side of this expression causes the modulo free Equation (3) to vanish. The critical observation is that the expressions of the form $[\mathcal{X}_\mu, [\mathcal{Y}_\nu, D^\mu \mathcal{Z}^\nu + D^\nu \mathcal{Z}^\mu]]$, where $\mathcal{Y}_\mu, \mathcal{Z}_\mu, \mathcal{Y}_\mu = \mathcal{F}_\mu$ or $\mathcal{G}_\mu$ represent on-shell independent combinations of fields and their derivatives. Once they are made to vanish, we conclude

$$k_2^2 - k_3 k_1 = 0, \quad l_p = 0, \quad p = \overline{1, 6}. \quad (\text{A23})$$

The general solution to these equations has the form (A18) with $\beta_3 = 0$. Taking account of this fact, the interaction (16) is unique in the class of covariant couplings without higher derivatives.

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