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On Symmetric Brackets Induced by Linear Connections

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Abstract: In this note, we discuss symmetric brackets on skew-symmetric algebroids associated with metric or symplectic structures. Given a pseudo-Riemannian metric structure, we describe the symmetric brackets induced by connections with totally skew-symmetric torsion in the language of Lie derivatives and differentials of functions. We formulate a generalization of the fundamental theorem of Riemannian geometry. In particular, we obtain an explicit formula of the Levi-Civita connection. We also present some symmetric brackets on almost Hermitian manifolds and discuss the first canonical Hermitian connection. Given a symplectic structure, we describe symplectic connections using symmetric brackets. We define a symmetric bracket of smooth functions on skew-symmetric algebroids with the metric structure and show that it has properties analogous to the Lie bracket of Hamiltonian vector fields on symplectic manifolds.

Keywords: skew-symmetric algebroid; almost Lie algebroid; anchored vector bundle; connection; symmetric product; symmetrized covariant derivative; symmetric Lie derivative; symplectic connection; Hamiltonian vector fields; Bismut connection

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1. Introduction

Let M be a differential manifold and S^kT^*M denote the k-th symmetric power of the cotangent bundle of M. On $\mathsf{S}(TM) = \bigoplus_{k\geq 0} \mathsf{S}^kT^*M$, there exists the mapping $d^s : \mathsf{S}(TM) \to \mathsf{S}(TM)$ being the symmetrized covariant derivative of a connection ∇ on M, i.e., $d^s\eta = (k+1)\cdot(\mathrm{Sym}\circ\nabla)\eta$ for $\eta \in \Gamma(\mathsf{S}^kT^*M)$. This mapping can be written, for $\eta \in \Gamma(\mathsf{S}^kT^*M)$, $X_1,\ldots,X_{k+1} \in \Gamma(TM)$, as follows:

$$(d^s\eta)(X_1,\ldots,X_{k+1}) = \sum_{j=1}^{k+1} X_j(\eta(X_1,\ldots\widehat{X}_j\ldots,X_{k+1})) - \sum_{i<j}\eta(\langle X_i : X_j\rangle^{\nabla}, X_1,\ldots\widehat{X}_i\ldots\widehat{X}_j\ldots,X_{k+1}), \tag{1}$$

where

$$\langle X : Y\rangle^{\nabla} = \nabla_X Y + \nabla_Y X \tag{2}$$

for $X, Y \in \Gamma(TM)$. Thus, d^s can be written in the Koszul-type form (1) shown above. This form is a symmetric equivalent of the exterior derivative operator where the role of the Lie bracket of vector fields is taken over by the symmetric bracket (2). We add that the Koszul-type shape of d^s for tangent bundles was first obtained by Heydari, Boroojerdian, and Peyghan in [1] and next under the study of generalized gradients on Lie algebroids in the sense of Stein–Weiss in [2]. However, d^s in the case of tangent bundles was introduced by Sampson in [3]. This mapping on tangent bundles was discussed by several authors when studying the Lichnerowicz-type Laplacian on symmetric tensors, cf. [4,5].

Observe that the symmetric product $\langle X : Y\rangle^{\nabla} = \nabla_X Y + \nabla_Y X$ satisfies the Leibniz rule $\langle X : fY\rangle^{\nabla} = f\langle X : Y\rangle^{\nabla} + X(f)Y$ and

$$\nabla_X Y = \tfrac{1}{2}([X,Y] + \langle X : Y\rangle^{\nabla}) + \tfrac{1}{2}T^{\nabla}(X,Y)$$

for $X, Y \in \Gamma(TM)$, $f \in C^{\infty}(M)$, and where T^{∇} denote the torsion of ∇. Thus, the symmetric product induced by ∇ is a summand of the connection. Our goal is to examine symmetric brackets for connections. In particular, the area of our interest is the discovery of explicit forms of symmetric brackets. Torsion-free connections, such as the Levi-Civita connection or symplectic connections, can be described completely by suitable symmetric brackets. The Levi-Civita connection is the basis of many constructions of linear connections; therefore, the symmetric bracket of this connection will be one of the first objects of our interest. We will also give examples of symmetric products that define symplectic connections.

Linear connections are the subject of geometric problems not only in geometric structures on a tangent bundle to a differential manifold but also in others such as Lie algebroids, in particular Lie algebras, or more general structures such as anchored vector bundles with skew-symmetric brackets. Therefore, our general discussion of symmetric brackets related to linear connections is in the framework of skew-symmetric algebroids.

An *anchored vector bundle* (A, ϱ_A) over a manifold M is a vector bundle A over M equipped with a homomorphism of vector bundles $\varrho_A : A \to TM$ over the identity, which is called an *anchor*. If, additionally, in the space $\Gamma(A)$ of smooth sections of A we have $\mathbb{R}$-bilinear skew-symmetric mapping $[\cdot,\cdot] : \Gamma(A) \times \Gamma(A) \to \Gamma(A)$ associated with the anchor with the following derivation law

$$[X, f \cdot Y] = f \cdot [X,Y] + (\varrho_A \circ X)(f) \cdot Y \tag{3}$$

for $X, Y \in \Gamma(A)$, $f \in C^{\infty}(M)$, we say that $(A, \varrho_A, [\cdot,\cdot])$ is a *skew-symmetric algebroid* over M.

If the anchor preserves $[\cdot,\cdot]$ and the Lie bracket $[\cdot,\cdot]_{TM}$ of vector fields on M, i.e., $\varrho_A \circ [X,Y] = [\varrho_A \circ X, \varrho_A \circ Y]_{TM}$ for $X, Y \in \Gamma(A)$, a skew-symmetric algebroid is an *almost Lie algebroid*. Any skew-symmetric algebroid in which $[\cdot,\cdot]$ satisfies the Jacobi identity is a Lie algebroid in the sense of Pradines, who discovered them as infinitesimal parts of differentiable groupoids [6] (for the general theory of Lie algebroids, we refer to the Mackenzie monographs [7,8]). Thus, Lie algebroids are simultaneous generalizations of integrable distributions on the one hand and Lie algebras on the other. Anchored vector bundles, in particular almost Lie algebroids, are studied by Marcela Popescu and Paul Popescu, among others, in [9–12] and recently in [13], in which the Chern character for almost Lie algebroids is considered. However, the concept of skew-symmetric algebroids was introduced by Kosmann-Schwarzbach and Magri in [14] on the level of finitely generated projective modules over commutative and associative algebras with unit and under the name pre-Lie algebroids. Skew-symmetric algebroids (under the same name pre-Lie algebroids) were examined by Grabowski and Urbański in [15,16], where a concept of general algebroids, which have an important role in analytical mechanics, was also introduced. Using general algebroids instead of Lie algebroids, one can describe a larger family of systems, both in the Lagrangian and Hamiltonian formalisms [17]. In this paper, we use the terminology of skew-symmetric algebroid which comes from de León, Marrero, and de Diego in [18], in which linear almost Poisson structures (also discussed in [14–16]) are applied to nonholonomic mechanical systems.

Given an anchored vector bundle, we can associate a connection. Given a skew-symmetric algebroid, we can associate a connection with a torsion. An *A-connection* in a vector bundle $E \to M$ is an $\mathbb{R}$-bilinear map $\nabla : \Gamma(A) \times \Gamma(E) \to \Gamma(E)$ with the following properties:

$$\begin{aligned} \nabla_{f\cdot X}(u) &= f \cdot \nabla_X(u), \\ \nabla_X(f \cdot u) &= f \cdot \nabla_X(u) + (\varrho_A \circ X)(f) \cdot u \end{aligned}$$

for any $X \in \Gamma(A)$, $f \in C^{\infty}(M)$, $u \in \Gamma(E)$. The *torsion* of an A-connection ∇ in A is the tensor $T^{\nabla} \in \Gamma(\bigwedge^2 A^* \otimes A)$ defined by $T^{\nabla}(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$ for $X, Y \in \Gamma(A)$. We say that an A-connection is *torsion-free* if its torsion equals zero.

In the article [19] by Enrietti, Fino, and Vezzoni, the connections on Lie algebras, understood precisely as anchored bundles with skew-symmetric brackets, are examined. Such research motivates the indication of the properties of connections on more general structures involving both Lie algebras and differential manifolds.

An important algebraic structure that provides further motivations for the study of connections on skew-symmetric algebroids is the special algebroid structure determined by an almost complex manifold. Namely, an almost complex structure $J : TM \to TM$ on $2n$-dimensional manifold M defines a new skew-symmetric bracket $[\![X,Y]\!]^J = [JX,Y] + [X,JY] - J[X,Y]$ giving a skew-symmetric algebroid structure in TM with J as an anchor (cf. [14]). The fulfillment of Jacobi's identity by the bracket $[\![\cdot,\cdot]\!]^J$ is equivalent to the integrability of J. Thus, structures of skew-symmetric algebroids naturally appear in geometric problems.

We now describe the sections of this paper. In Section 2, we discuss the substitution operator, the Lie derivative operator, and the exterior derivative operator on the general structure of a skew-symmetric algebroid. We also consider the symmetrized covariant derivative d^s determined by a connection. Symmetrized covariant derivatives depend on symmetric products designated by the connection. In Section 3, we extend the concept of symmetric brackets to anchored bundles and the associated symmetric Lie derivative and d^s to the whole tensor bundle. We note that d^s satisfy the Cartan-type formulas analogous to those on exterior forms. The primary goal of Section 4 is to obtain the explicit formula for the symmetric bracket defined by the metric connection. As a result, we obtain an explicit formula for a symmetric bracket defined by a connection with totally skew-symmetric torsion. Section 5 deals with metric connections on skew-symmetric algebroids with an additional symmetric bracket. We show that the condition for connections with totally skew-symmetric torsion to be compatible with the metric is that the (alternating) Lie derivative of the metric should be equal to the minus of the symmetric Lie derivative of the metric. We also extend the fundamental theorem of Riemannian geometry to skew-symmetric algebroids equipped with a metric structure. In a particular case, this theorem implies the existence of the only torsion-free connection compatible with the metric, which is called the Levi-Civita connection associated with the metric. In consequence, we give an explicit formula for a metric connection with totally skew-symmetric torsion using the language of symmetric product. To describe this symmetric product, we use the Lie derivative and the exterior derivative operator induced by the structure of the skew-symmetric algebroid and their symmetric counterparts.

In Section 6, we consider an almost Hermitian structure and some symmetric brackets associated with connections that are compatible with the metric structure and the almost complex structure. We consider two structures of the skew-symmetric algebroid in the almost Hermitian manifold (M, g, J). The first structure is the tangent bundle with the identity as an anchor and with the Lie bracket of vector fields. The second skew-symmetric algebroid structure TM^J induced by the almost complex structure J, where J is the anchor and the bracket is associated with the Nijenhuis tensor, was introduced in [14]. We also discuss the first canonical Hermitian connection $\overline{\nabla}$ and obtain a formula for $\overline{\nabla}$ in the case of nearly Kähler manifolds using the properties of symmetric brackets. Moreover, we show the dependence of the Bismut connection in Hermitian manifolds on the structure of the skew-symmetric algebroid TM^J. The torsion of this connection depends on the exterior differential of the Kähler form in TM^J.

Section 7 deals with symplectic connections on skew-symmetric algebroids additionally equipped with a symplectic form. A symplectic connection as a torsion-free connection is determined completely by a skew-symmetric bracket in a given algebroid and some symmetric brackets. Finding symmetric brackets that define symmetric connections is our goal. We use the idea of constructing symplectic connections noticed by Tondeur in [20] and by Bieliavsky, Cahen, Gutt, Rawnsley, and Schwachhöfer in [21].

We show that this idea leads to connections that are determined by the affine sum of two symmetric brackets. The first of them is a symmetric bracket for a certain initial torsion-free connection ∇^0, while the second is a symmetric bracket of the connection, which is a dual to ∇^0 with respect to the symplectic form. In addition to general considerations, we consider symplectic connections on symplectic manifolds and on skew-symmetric algebras, which contain the family of symplectic Lie algebras. We give an example of the symplectic connection on a 4-dimensional symplectic Lie algebra $\mathfrak{r}_2\mathfrak{r}_2$ being the double direct product of 2-dimensional non-abelian Lie algebra $\mathfrak{r}_2 = \mathfrak{aff}(\mathbb{R})$ of the group of affine transformations of the real line (cf. [22,23]). We note that the symmetric bracket associated with this symplectic connection defines a structure of Jordan algebra in $\mathfrak{r}_2\mathfrak{r}_2$.

Motivations for the considerations in Section 8 come from Poisson geometry, in particular from symplectic geometry. In a given Poisson manifold, the Lie bracket of Hamiltonian vector fields is the Hamiltonian vector field defined by the Poisson bracket of smooth functions. We show that the analogical property holds on skew-symmetric algebroids over a manifold M with a metric g for the symmetric bracket $\langle\cdot,\cdot\rangle^{LC}$ defined by the Levi-Civita connection associated with g. We define a symmetric bracket $(\cdot,\cdot) : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ of smooth functions on M, which has the property that $\langle \operatorname{grad} f : \operatorname{grad} h\rangle^{LC} = \operatorname{grad}(f,h)$ for $f,h \in C^\infty(M)$. We consider in particular the case of a symplectic manifold where we introduce a symmetric bracket in the algebra of Hamiltonian vector fields and show that it has analogous properties to the Lie bracket of Hamiltonian vector fields.

2. The Exterior Derivative Operator and the Symmetrized Covariant Derivative

Let $(A, \varrho_A, [\cdot,\cdot])$ be a skew-symmetric algebroid over a manifold M. The *substitution* operator $i_X : \Gamma(\bigotimes^k A^*) \to \Gamma(\bigotimes^{k-1} A^*)$ for $X \in \Gamma(A)$ is defined by

$$(i_X\zeta)(X_1,\ldots,X_{k-1}) = \zeta(X,X_1,\ldots,X_{k-1})$$

for $\zeta \in \Gamma(\bigotimes^k A^*)$, $X, X_1,\ldots,X_{k-1} \in \Gamma(A)$.

The (*alternating*) *Lie derivative* $\mathcal{L}^a_X : \Gamma(\bigotimes^k A^*) \to \Gamma(\bigotimes^k A^*)$ for $X \in \Gamma(A)$ is defined by

$$(\mathcal{L}^a_X\Omega)(X_1,\ldots,X_k) = (\varrho_A \circ X)(\Omega(X_1,\ldots,X_k)) - \sum_{i=1}^{k} \Omega(X_1,\ldots,[X,X_i],\ldots,X_k)$$

for $\Omega \in \Gamma(\bigotimes^k A^*)$, $X_1,\ldots,X_k \in \Gamma(A)$. Notice that $\mathcal{L}^a_X(\eta) \in \Gamma(\bigwedge A^*)$ if $\eta \in \Gamma(\bigwedge A^*)$.

Moreover, let $\nabla : \Gamma(A) \times \Gamma(A) \to \Gamma(A)$ be an A-connection in A. We define the A-connection $\bar{\nabla}$ in the dual bundle in a classical way by the following formula

$$(\bar{\nabla}_X\omega)Y = (\varrho_A \circ X)(\omega(Y)) - \omega(\nabla_X Y)$$

for $\omega \in \Gamma(A^*)$, $X, Y \in \Gamma(A)$. Next, by the Leibniz rule, we extend this connection to the A-connection in the whole tensor bundle $\bigotimes A^*$, which will also be denoted by ∇. Then, for $\zeta \in \Gamma(\bigotimes^k A^*)$, $X, X_1,\ldots,X_k \in \Gamma(A)$,

$$(\nabla_X\zeta)(X_1,\ldots,X_k) = (\varrho_A \circ X)(\zeta(X_1,\ldots,X_k)) - \sum_{j=1}^{k} \zeta(X_1,\ldots,\nabla_X X_j,\ldots,X_k).$$

Now, we define the operator $\nabla : \Gamma(\bigotimes^k A^*) \to \Gamma(\bigotimes^{k+1} A^*)$ by

$$(\nabla\zeta)(X_1,X_2\ldots,X_{k+1}) = (\nabla_{X_1}\zeta)(X_2,\ldots,X_{k+1}).$$

We recall that the exterior derivative operator on the skew-symmetric algebroid $(A, \varrho_A, [\cdot,\cdot])$ is defined by

$$(d^a\eta)(X_1,\ldots,X_{k+1}) = \sum_{j=1}^{k+1}(-1)^{j+1}(\rho_A \circ X_j)\Big(\eta(X_1,\ldots\widehat{X}_j\ldots,X_{k+1})\Big) + \sum_{i<j}(-1)^{i+j}\eta\Big([X_i,X_j],X_1,\ldots\widehat{X}_i\ldots\widehat{X}_j\ldots,X_{k+1}\Big) \tag{4}$$

for $\eta \in \Gamma(\bigwedge^k A^*)$, $X_1,\ldots,X_{k+1} \in \Gamma(A)$. Associated with a skew-symmetric algebroid $(A, \varrho_A, [\cdot,\cdot])$ is the *Jacobiator* $\mathrm{Jac}_{[\cdot,\cdot]} : \Gamma(A)\times\Gamma(A)\times\Gamma(A) \to \Gamma(A)$ of the bracket $[\cdot,\cdot]$ given by

$$\mathrm{Jac}_{[\cdot,\cdot]}(X,Y,Z) = [[X,Y],Z] + [[Z,X],Y] + [[Y,Z],X]$$

for $X,Y,Z \in \Gamma(A)$. If the bracket $[\cdot,\cdot]$ satisfies the Jacobi identity, i.e., $\mathrm{Jac}_{[\cdot,\cdot]} = 0$, $d^a \circ d^a = 0$ (discussed in [24]). If ∇ is torsion-free A-connection in A, then d^a can be written as the alternation of the operator ∇ (cf. [2]), i.e., $d^a = (k+1)\cdot(\mathrm{Alt}\circ\nabla)$ on $\Gamma(\bigwedge^k A^*)$, where Alt is the *alternator* given by $(\mathrm{Alt}\,\zeta)(X_1,\ldots,X_k) = \frac{1}{k!}\sum_{\sigma\in S_k}\mathrm{sgn}\,\sigma\,\zeta(X_{\sigma(1)},\ldots,X_{\sigma(k)})$ for $\zeta \in \Gamma(\bigotimes^k A^*)$. Equivalently,

$$(d^a\eta)(X_1,\ldots,X_{k+1}) = \sum_{j=1}^{k+1}(-1)^{j+1}\Big(\nabla_{X_j}\eta\Big)\Big(X_1,\ldots\widehat{X}_j\ldots,X_{k+1}\Big)$$

for $\eta \in \Gamma(\bigwedge^k A^*)$, $X_1,\ldots,X_{k+1} \in \Gamma(A)$.

Here, we recall the classical Cartan's formulas:

Lemma 1. *For any $X,Y \in \Gamma(A)$,*

(a) $\mathcal{L}^a_X = i_X d^a + d^a i_X$ *and*
(b) $\mathcal{L}^a_X i_Y - i_Y\mathcal{L}^a_X = i_{[X,Y]}$.

The symmetrized covariant derivative is

$$d^s = (k+1)\cdot(\mathrm{Sym}\circ\nabla) : \Gamma(\mathsf{S}^k A^*) \to \Gamma(\mathsf{S}^{k+1}A^*)$$

which is the symmetrization of ∇ up to a constant on the symmetric power bundle, where Sym is the *symmetrizer* defined by $(\mathrm{Sym}\,\zeta)(X_1,\ldots,X_k) = \frac{1}{k!}\sum_{\sigma\in S_k}\zeta(X_{\sigma(1)},\ldots,X_{\sigma(k)})$ for $\zeta \in \Gamma(\bigotimes^k A^*)$. Equivalently,

$$(d^s\eta)(X_1,\ldots,X_{k+1}) = \sum_{j=1}^{k+1}\Big(\nabla_{X_j}\eta\Big)\Big(X_1,\ldots\widehat{X}_j\ldots,X_{k+1}\Big) \tag{5}$$

for $\eta \in \Gamma(\mathsf{S}^k A^*)$, $X_1,\ldots,X_{k+1} \in \Gamma(A)$. We recall that d^s in the case of tangent bundles was introduced by Sampson in [3], in which a symmetric version of Chern's theorem was proved. This mapping on tangent bundles was discussed in [1], in which a Frölicher–Nijenhuis bracket for vector-valued symmetric tensors was also discussed and in [25], in which the Dirac-type operator on symmetric tensors was considered. One can check that for $\eta \in \Gamma(\mathsf{S}^k A^*)$, $X_1,\ldots,X_{k+1} \in \Gamma(A)$, the following Koszul-type formula holds:

$$(d^s\eta)(X_1,\ldots,X_{k+1}) = \sum_{j=1}^{k+1}(\rho_A \circ X_j)\Big(\eta(X_1,\ldots\widehat{X}_j\ldots,X_{k+1})\Big) - \sum_{i<j}\eta\Big(\langle X_i : X_j\rangle^{\nabla},X_1,\ldots\widehat{X}_i\ldots\widehat{X}_j\ldots,X_{k+1}\Big),$$

where $\langle X : Y \rangle^{\nabla} = \nabla_X Y + \nabla_Y X$ for $X, Y \in \Gamma(A)$. This shape of d^s in the case $A = TM$ was discovered by Heydari, Boroojerdian, and Peyghan in [1]. The symmetric $\mathbb{R}$-bilinear form

$$\langle \cdot : \cdot \rangle^{\nabla} : \Gamma(A) \times \Gamma(A) \longrightarrow \Gamma(A), \quad \langle X : Y \rangle^{\nabla} = \nabla_X Y + \nabla_Y X$$

is called the **symmetric product** or the **symmetric bracket** induced by the A-connection ∇. The symmetric product in the case $A = TM$ was first introduced by Crouch in [26]. However, the symmetric product for Lie algebroids was first considered in the context of control systems by Cortés and Martínez in [27]. Observe that

$$\langle X : f \cdot Y \rangle^{\nabla} = f \cdot \langle X : Y \rangle^{\nabla} + (\varrho_A \circ X)(f) \cdot Y$$

for all $X, Y \in \Gamma(A)$ and $f \in C^{\infty}(M)$. Therefore, $\langle \cdot : \cdot \rangle$ satisfies the Leibniz-kind rule. We add that Lewis in [28] gives some interesting geometrical interpretation of the symmetric product associated with the geodesically invariant property of a distribution. We say that a smooth distribution D on a manifold M with an affine connection ∇^{TM} is *geodesically invariant* if for every geodesic $c : I \to M$ satisfying the property $c'(s) \in D_{c(s)}$ for some $s \in I$, we have $c'(s) \in D_{c(s)}$ for every $s \in I$. Lewis proved in [28] that a distribution D on a manifold M equipped with an affine connection ∇^{TM} is geodesically invariant if and only if the symmetric product induced by ∇^{TM} is closed under D.

3. Symmetric Bracket. Symmetric Lie Derivative

In this section, we introduce the concepts of a symmetric bracket and the related mapping d^s and the symmetric Lie derivative defined on the whole tensor bundle of a given skew-symmetric algebroid. Implemented operators satisfy the Cartan properties analogous to those fulfilled by the exterior derivative and the Lie derivative.

Let $(A, \varrho_A, [\cdot, \cdot])$ be a skew-symmetric algebroid over a manifold M. A *symmetric bracket* on the anchored vector bundle (A, ϱ_A) is an $\mathbb{R}$-bilinear symmetric mapping

$$\langle \cdot : \cdot \rangle : \Gamma(A) \times \Gamma(A) \to \Gamma(A)$$

satisfying the following Leibniz-kind rule:

$$\langle X : fY \rangle = f \langle X : Y \rangle + (\varrho_A \circ X)(f) Y$$

for $X, Y \in \Gamma(A)$, $f \in C^{\infty}(M)$. Let us assume that the skew-symmetric algebroid $(A, \varrho_A, [\cdot, \cdot])$ is equipped with a symmetric bracket $\langle \cdot : \cdot \rangle : \Gamma(A) \times \Gamma(A) \to \Gamma(A)$.

We define $d^s : \Gamma(\bigotimes^k A^*) \to \Gamma(\bigotimes^{k+1} A^*)$ on the whole tensor bundle by

$$\begin{aligned} (d^s\Omega)(X_1, \ldots, X_{k+1}) \quad &= \quad \sum_{j=1}^{k+1} (\rho_A \circ X_j)\Big(\Omega(X_1, \ldots \widehat{X}_j \ldots, X_{k+1})\Big) \\ &\quad - \sum_{i<j} \Omega\Big(X_1, \ldots \widehat{X}_i \ldots, \langle X_i : X_j \rangle, \ldots, X_{k+1}\Big) \end{aligned}$$

for $\Omega \in \Gamma(\bigotimes^k A^*)$, $X_1, \ldots, X_{k+1} \in \Gamma(A)$. We denote the restriction of d^s to the symmetric power bundle $\mathsf{S}(A)$ by the same symbol.

The *symmetric Lie derivative* $\mathcal{L}^s_X : \Gamma(\bigotimes^k A^*) \to \Gamma(\bigotimes^k A^*)$ for $X \in \Gamma(A)$ is defined by

$$(\mathcal{L}^s_X \Omega)(X_1, \ldots, X_k) = (\varrho_A \circ X)(\Omega(X_1, \ldots, X_k)) - \sum_{i=1}^{k} \Omega(X_1, \ldots, \langle X : X_i \rangle, \ldots, X_k)$$

for $\Omega \in \Gamma(\bigotimes^k A^*)$, $X_1, \ldots, X_k \in \Gamma(A)$. Notice that the image $\mathcal{L}^s_X(\varphi)$ of a symmetric tensor φ is also a symmetric tensor.

By using definitions, one can prove that the symmetric Lie derivative satisfies the following Cartan's identities analogous to these Cartan identities on exterior forms:

Lemma 2. *For any $X, Y \in \Gamma(A)$,*

(a) $\mathcal{L}^s_X = i_X d^s - d^s i_X$ *and*
(b) $\mathcal{L}^s_X i_Y - i_Y \mathcal{L}^s_X = i_{\langle X:Y \rangle}$.

Moreover, the symmetric Lie derivative has the following properties:

Lemma 3. *For $f \in C^\infty(M)$, $X \in \Gamma(A)$, $\omega \in \Gamma(A^*)$, we have*

(a) $\mathcal{L}^s_{f \cdot X}\omega = f \cdot \mathcal{L}^s_X \omega - (i_X\omega) \cdot d^s f$ *and*
(b) $\mathcal{L}^s_X(f \cdot \omega) = f \cdot \mathcal{L}^s_X \omega + (\varrho_A \circ X)(f) \cdot \omega$.

4. The Symmetric Brackets Induced by Connections Associated with a Metric Structure

Let $(A, \varrho_A, [\cdot,\cdot])$ be a skew-symmetric algebroid over a manifold M equipped with a pseudo-Riemannian metric $g \in \Gamma(S^2 A^*)$ in the vector bundle A and an A-connection ∇ in A. Let $\langle \cdot : \cdot \rangle^\nabla$ be the symmetric product induced by ∇ and d^s the symmetrized covariant derivative. A connection ∇ on is said to be *compatible with the metric g* if $\nabla g = 0$. The pseudo-Riemannian metric defines two homomorphisms of vector bundles

$$\flat : A \to A^*, \qquad \sharp : A^* \longrightarrow A$$

by

$$\flat(X) = i_X g, \quad g(\sharp(\omega), X) = \omega(X)$$

for $X \in \Gamma(A)$, $\omega \in \Gamma(A^*)$, respectively. For any $X \in \Gamma(A)$, the 1-form $i_X g = g(X, \cdot)$ will be denoted, briefly, by $X^\flat$.

We say that ∇ is a *connection with totally skew-symmetric torsion* with respect to a pseudo-Riemannian metric g if the tensor $T^g \in \Gamma(\bigotimes^3 A^*)$ given by

$$T^g(X, Y, Z) = g(T^\nabla(X, Y), Z)$$

for $X, Y, Z \in \Gamma(A)$, is a 3-form on A, i.e., $T^g \in \Gamma(\bigwedge^3 A^*)$.

Theorem 1. *Let $X, Z \in \Gamma(A)$. Then,*

$$\begin{aligned} g(\nabla_X X, Z) &= g(\sharp(\mathcal{L}^a_X X^\flat - \tfrac{1}{2} d^a(g(X,X)), Z) - g(T^\nabla(X,Z), X) \\ &\quad + (\nabla g)(Z, X, X) - \tfrac{1}{2}(d^s g)(X, X, Z). \end{aligned}$$

In particular, if ∇ is a connection with totally skew-symmetric torsion compatible with g, then

$$\nabla_X X = \sharp(\mathcal{L}^a_X X^\flat - \tfrac{1}{2} d^a(g(X,X)). \tag{6}$$

Proof. Let $X, Z \in \Gamma(A)$. First, observe that

$$(d^s g)(X, X, Z) = 2(\nabla g)(X, X, Z) + (\nabla g)(Z, X, X).$$

Therefore, we have

$$(\nabla g)(Z, X, X) - \tfrac{1}{2}(d^s g)(X, X, Z) = \tfrac{1}{2}(\nabla g)(Z, X, X) - (\nabla g)(X, X, Z).$$

Next, observe that

$$\begin{aligned} & \tfrac{1}{2}(\nabla g)(Z,X,X)-(\nabla g)(X,X,Z) \\ = & \tfrac{1}{2}(\nabla_Z g)(X,X)-(\nabla_X g)(X,Z) \\ = & \tfrac{1}{2}\varrho_A(Z)(g(X,X))-g(\nabla_Z X,X)-\varrho_A(X)(g(X,Z))+g(\nabla_X X,Z)+g(X,\nabla_X Z) \\ = & \tfrac{1}{2}\varrho_A(Z)(g(X,X))+g(\nabla_X Z-\nabla_Z X-[X,Z],X) \\ & -\varrho_A(X)(g(X,Z))+g([X,Z],X)+g(\nabla_X X,Z). \end{aligned}$$

Since

$$\varrho_A(Z)(g(X,X))=d^a(g(X,X))(Z)=g(\sharp(d^a(g(X,X))),Z)$$

and

$$\left(\mathcal{L}^a_X X^\flat\right)(Z)=\varrho_A(X)(g(X,Z))-g(X,[X,Z]),$$

we have

$$\begin{aligned} & \tfrac{1}{2}(\nabla g)(Z,X,X)-(\nabla g)(X,X,Z) \\ = & \tfrac{1}{2}d^a(g(X,X))(Z)+g(T^\nabla(X,Z),X)-\left(\mathcal{L}^a_X X^\flat\right)(Z)+g(\nabla_X X,Z). \end{aligned}$$

Moreover, if ∇ is a metric connection with totally skew-symmetric torsion, then $\nabla g=0$, $d^s g=0$, and

$$g(T^\nabla(X,Z),X)=-g(T^\nabla(X,X),Z)=0,$$

and, in consequence, we obtain (6). This completes the proof. □

Applying Theorem 1, we have

Theorem 2. *Let $X,Y,Z\in\Gamma(A)$ and let $\langle X:Y\rangle^\nabla$ be the symmetric bracket of sections induced by ∇. Then,*

$$\begin{aligned} g(\langle X:Y\rangle^\nabla,Z) = & g(\sharp(\mathcal{L}^a_X Y^\flat+\mathcal{L}^a_Y X^\flat-d^a(g(X,Y))),Z) \\ & -g(T^\nabla(X,Z),Y)-g(T^\nabla(Y,Z),X) \\ & +2(\nabla g)(Z,X,Y)-(d^s g)(X,Y,Z). \end{aligned} \tag{7}$$

Proof. Using the following polarization formula

$$\langle X:Y\rangle^\nabla=\nabla_{X+Y}(X+Y)-\nabla_X X-\nabla_Y Y$$

and Theorem 1, we obtain

$$\begin{aligned} g(\langle X:Y\rangle^\nabla,Z) = & g(\sharp(\mathcal{L}^a_{X+Y}(X+Y)^\flat-\tfrac{1}{2}d^a(g(X+Y,X+Y)),Z) \\ & -g(T^\nabla(X+Y,Z),X+Y)+(\nabla g)(Z,X+Y,X+Y) \\ & -\tfrac{1}{2}(d^s g)(X+Y,X+Y,Z)-g(\sharp(\mathcal{L}^a_X X^\flat-\tfrac{1}{2}d^a(g(X,X)),Z) \\ & +g(T^\nabla(X,Z),X)-(\nabla g)(Z,X,X)+\tfrac{1}{2}(d^s g)(X,X,Z) \\ & -g(\sharp(\mathcal{L}^a_Y Y^\flat-\tfrac{1}{2}d^a(g(Y,Y)) \\ & +g(T^\nabla(Y,Z),Y)-(\nabla g)(Z,Y,Y)+\tfrac{1}{2}(d^s g)(Y,Y,Z). \end{aligned}$$

First, observe that

$$\mathcal{L}^a_{X+Y}(X+Y)^\flat-\mathcal{L}^a_X X^\flat-\mathcal{L}^a_Y Y^\flat=\mathcal{L}^a_X Y^\flat+\mathcal{L}^a_Y X^\flat$$

and

$$-\tfrac{1}{2}d^a(g(X+Y,X+Y)+\tfrac{1}{2}d^a(g(X,X))+\tfrac{1}{2}d^a(g(Y,Y)=-d^a(g(X,Y)).$$

Since g is a symmetric tensor and T^{∇} is skew-symmetric, we conclude that

$$-g(T^{\nabla}(X+Y,Z),X+Y)+g(T^{\nabla}(X,Z),X)+g(T^{\nabla}(Y,Z),Y)$$

is equal to

$$-g(T^{\nabla}(X,Z),Y)-g(T^{\nabla}(Y,Z),X).$$

Moreover,

$$(\nabla g)(Z,X+Y,X+Y)-(\nabla g)(Z,X,X)-(\nabla g)(Z,Y,Y)=2(\nabla g)(Z,X,Y)$$

and

$$\begin{aligned}(d^s g)(X,Y,Z) &= \tfrac{1}{2}(d^s g)(X,Y,Z)+\tfrac{1}{2}(d^s g)(Y,X,Z)\\ &= \tfrac{1}{2}(d^s g)(X+Y,X+Y,Z)-\tfrac{1}{2}(d^s g)(X,X,Z)-\tfrac{1}{2}(d^s g)(Y,Y,Z).\end{aligned}$$

Hence, it is clear that some summands of $g(\langle X:Y\rangle^{\nabla},Z)$ cancel. This establishes (7). □

The formula in Theorem 2 gives an explicit one of symmetric bracket defined by any metric connection with totally skew-symmetric torsion.

Corollary 1. *Let ∇ be any metric A-connection in A with totally skew-symmetric torsion with respect to a pseudo-Riemannian metric g. Then,*

$$\langle X:Y\rangle^{\nabla}=\sharp(\mathcal{L}^a_X Y^{\flat}+\mathcal{L}^a_Y X^{\flat}-d^a(g(X,Y))$$

for $X,Y\in\ \Gamma(A)$.

5. A General Metric Compatibility Condition of Connections with Totally Skew-Symmetric Torsion. Fundamental Theorem of Pseudo-Riemannian Geometry and the Levi-Civita Connection

In this section, we consider skew-symmetric algebroids equipped with a metric structure and additionally with a symmetric bracket. The considerations in the last section show that the given skew-symmetric bracket and the metric define a symmetric bracket. We would like to note here that some properties hold for any given symmetric bracket. Thus, the discovery of the symmetric bracket leads to receiving new structure. Using the symmetric bracket setting by the metric, we will show a generalization of the fundamental theorem of the Riemannian geometry, which says that for a given metric and the 2-form Ω with values in a given algebroid, there is exactly one metric connection preserving the given metric and whose torsion is equal to Ω. In particular, we will obtain the form of a metric connection with totally skew-symmetric torsion and a formula for the Levi-Civita connection.

Let $(A,\varrho_A,[\cdot,\cdot])$ be a skew-symmetric algebroid over a manifold M equipped with a pseudo-Riemannian metric $g\in\Gamma(S^2A^*)$ in the vector bundle A and a symmetric bracket $\langle\cdot:\cdot\rangle:\Gamma(A)\times\Gamma(A)\to\Gamma(A)$. By definition, we recall that the symmetric bracket is an $\mathbb{R}$-bilinear symmetric mapping which satisfies the following Leibniz-kind rule:

$$\langle X:fY\rangle=f\langle X:Y\rangle+(\varrho_A\circ X)(f)\cdot Y$$

for $X,Y\in\ \Gamma(A)$, $f\in\ C^{\infty}(M)$.

Let $\mathcal{L}^s$ and d^s denote the symmetric Lie derivative and the symmetric covariant derivative, respectively, and both are induced by $\langle\cdot:\cdot\rangle$.

Theorem 3. *Let ∇ be an A-connection in A with totally skew-symmetric torsion with respect to a pseudo-Riemannian metric g on A given by*

$$\nabla_X Y=\tfrac{1}{2}([X,Y]+\langle X:Y\rangle)+\tfrac{1}{2}T(X,Y) \tag{8}$$

for $X, Y \in \Gamma(A)$, *and some* $T \in \Gamma(\bigwedge^2 A^* \otimes A)$. *Then,*

$$(i_X \circ \nabla)g = \frac{1}{2}(\mathcal{L}^a_X + \mathcal{L}^s_X)g \text{ for } X \in \Gamma(A).$$

Proof. Let $X, Y, Z \in \Gamma(A)$. Since $T \in \Gamma\left(\bigwedge^2 A^* \otimes A\right)$ is a 2-skew-symmetric tensor with the property that

$$g(Y, T(X, Z)) = g(T(X, Z), Y) = -g(T(X, Y), Z),$$

we have

$$\begin{aligned}(\nabla_X g)(Y, Z) &= \rho_A(X)(g(Y, Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) \\ &= \tfrac{1}{2}(\rho_A(X)(g(Y, Z)) - g([X, Y], Z) - g(Y, [X, Z])) \\ &\quad + \tfrac{1}{2}(\rho_A(X)(g(Y, Z)) - g(\langle X : Y\rangle, Z) - g(Y, \langle X : Z\rangle)) \\ &\quad - \tfrac{1}{2}g(T(X, Y), Z) - \tfrac{1}{2}g(Y, T(X, Z)) \\ &= \tfrac{1}{2}(\mathcal{L}^a_X g + \mathcal{L}^s_X g)(Y, Z).\end{aligned}$$

□

Hence, we can conclude the following condition on a connection with totally skew-symmetric torsion to be a metric connection:

Corollary 2. *If* ∇ *is an* A*-connection with totally skew-symmetric torsion with respect to* g *given by* (8)*, then* ∇ *is metric with respect to* g *if and only if*

$$\mathcal{L}^a_X g = -\mathcal{L}^s_X g \text{ for any } X \in \Gamma(A).$$

Now, we recall some properties of the (skew-symmetric) Lie derivative.

Lemma 4. *For* $f \in C^\infty(M)$, $X \in \Gamma(A)$, $\omega \in \Gamma(A^*)$, *we have*

(a) $\mathcal{L}^a_{f \cdot X}\omega = f \cdot \mathcal{L}^a_X \omega + (i_X \omega) \cdot d^a f$ *and*
(b) $\mathcal{L}^a_X(f \cdot \omega) = f \cdot \mathcal{L}^a_X \omega + (\varrho_A \circ X)(f) \cdot \omega$.

Theorem 4. *Given a skew-symmetric algebroid* $(A, \varrho_A, [\cdot, \cdot])$, *we define*

$$\langle X : Y\rangle^s : \Gamma(A) \times \Gamma(A) \to \Gamma(A)$$

by

$$\langle X : Y\rangle^s = \sharp(\mathcal{L}^a_X Y^\flat + \mathcal{L}^a_Y X^\flat - d^a(g(X, Y)) \tag{9}$$

for $X, Y \in \Gamma(A)$. *Then,* $\langle \cdot : \cdot\rangle^s$ *is a symmetric bracket that defines the symmetric Lie derivative* $\mathcal{L}^s$ *satisfying* $\mathcal{L}^s_X g = -\mathcal{L}^a_X g$.

Proof. It is evident that $\langle \cdot : \cdot\rangle^s$ is a symmetric and $\mathbb{R}$-bilinear mapping. Let $X, Y, Z \in \Gamma(A)$. Lemma 4 now gives

$$\mathcal{L}^a_X(fY)^\flat = f\mathcal{L}^a_X Y^\flat + (\varrho_A \circ X)(f)Y^\flat$$

and

$$\mathcal{L}^a_{fY} X^\flat = f\mathcal{L}^a_Y X^\flat + g(X, Y)d^a f.$$

Since

$$d^a(g(X, fY)) = fd^a(g(X, Y)) + g(X, Y)d^a f,$$

we conclude that $\langle\cdot:\cdot\rangle^s$ satisfies the Leibniz rule. In consequence, $\langle\cdot:\cdot\rangle^s$ is a symmetric bracket. Observe that

$$\begin{aligned} g(\langle X:Y\rangle^s, Z) &= \big(\langle X:Y\rangle^s\big)^\flat(Z) = (\mathcal{L}^a_X Y^\flat + \mathcal{L}^a_Y X^\flat - d^a(g(X,Y))(Z) \\ &= (\varrho_A\circ X)(g(Y,Z)) - g(Y,[X,Z]) \\ &\quad +(\varrho_A\circ Y)(g(X,Z)) - g(X,[Y,Z]) - (\varrho_A\circ Z)(g(X,Y)). \end{aligned}$$

Similarly,

$$\begin{aligned} g(Y,\langle X:Z\rangle^s) &= (\varrho_A\circ X)(g(Y,Z)) - g(Z,[X,Y]) \\ &\quad +(\varrho_A\circ Z)(g(X,Y)) - g(X,[Z,Y]) - (\varrho_A\circ Y)(g(X,Z)). \end{aligned}$$

Therefore,

$$\begin{aligned} (\mathcal{L}^s_X g)(Y,Z) &= (\varrho_A\circ X)(g(Y,Z)) - g(\langle X:Y\rangle^s, Z) - g(Y,\langle X:Z\rangle^s) \\ &= g(Y,[X,Z]) + g(X,[Y,Z]) \\ &\quad -(\varrho_A\circ X)(g(Y,Z)) + g(Z,[X,Y]) + g(X,[Z,Y]) \\ &= -(\varrho_A\circ X)(g(Y,Z)) + g([X,Y],Z) + g(Y,[X,Z]) \\ &\quad +g(X,[Y,Z]+[Z,Y]) \\ &= -(\mathcal{L}^a_X g)(Y,Z) + 0. \end{aligned}$$

□

Theorem 3 now yields:

Corollary 3. *The torsion-free connection* ∇ *given by*

$$\nabla_X Y = \tfrac{1}{2}\big([X,Y] + \langle X:Y\rangle^s\big),$$

where

$$\langle X:Y\rangle^s = \sharp(\mathcal{L}^a_X Y^\flat + \mathcal{L}^a_Y X^\flat - d^a(g(X,Y)) \tag{10}$$

for $X,Y\in\Gamma(A)$*, is compatible with* g*.*

Now, we show that for the skew-symmetric algebroid structure equipped with additional pseudometric g, the following generalization of the fundamental theorem of Riemannian geometry holds:

Theorem 5. *Let* g *be a pseudo-Riemannian metric in the vector bundle* A *and* $\Omega\in\Gamma(\bigwedge^2 A^*\otimes A)$ *be a* 2*-form on* A *with values in* A*. Then, there exists a unique connection* ∇ *on* A *compatible with* g *such that its torsion tensor equals* Ω*, i.e.,* $\nabla g = 0$ *and* $T^\nabla = \Omega$*, and is given by*

$$\nabla_X Y = \tfrac{1}{2}\big([X,Y] + \langle X:Y\rangle^s\big) + \tfrac{1}{2}\Omega(X,Y) + S(X,Y),$$

where

$$\langle X:Y\rangle^s = \sharp(\mathcal{L}^a_X Y^\flat + \mathcal{L}^a_Y X^\flat - d^a(g(X,Y)), \tag{11}$$

and $S\in\Gamma(\mathsf{S}^2 A^*\otimes A)$ *is the symmetric* 2*-tensor on* A *with values in* A *such that*

$$g(S(X,Y),Z) = g(\Omega(Z,X),Y) + g(\Omega(Z,Y),X), \quad \textit{for}\ \ X,Y,Z\in\Gamma(A).$$

Proof. Let $X,Y\in\Gamma(A)$. Consider the linear connection ∇^g given by

$$\nabla^g_X Y = \tfrac{1}{2}\big([X,Y] + \langle X:Y\rangle^s\big),$$

where

$$\langle X:Y\rangle^s = \sharp(\mathcal{L}^a_X Y^\flat + \mathcal{L}^a_Y X^\flat - d^a(g(X,Y)).$$

Let ∇ be a linear connection compatible with g and with torsion $T^\nabla = \Omega$. Observe that

$$\nabla_X Y = \nabla^g_X Y + \Phi(X,Y)$$

for some 2-tensor $\Phi \in \Gamma(\otimes^2 A^* \otimes A)$ and that

$$T^\nabla(X,Y) = \Phi(X,Y) - \Phi(Y,X).$$

Therefore,

$$\nabla_X Y = \nabla^g_X Y + \tfrac{1}{2}\Omega(X,Y) + S(X,Y),$$

where $S \in \Gamma(\mathrm{S}^2 A^* \otimes A)$ is some symmetric tensor. So,

$$\langle X:Y\rangle^\nabla = \langle X:Y\rangle^s + S(X,Y). \tag{12}$$

This shows at once that S is determined uniquely. Since $\nabla g = 0$, Theorem 2 and (12) now lead to

$$g(\langle X:Y\rangle^s + S(X,Y), Z) = g(\langle X:Y\rangle^s, Z) - g(T^\nabla(X,Z),Y) - g(T^\nabla(Y,Z),X).$$

From this and skew-symmetricity of the torsion $T^\nabla = \Omega$, it follows that

$$g(S(X,Y),Z) = g(\Omega(Z,X),Y) + g(\Omega(Z,Y),X).$$

□

One can immediately see that the result of Theorem 5 allows us to write formulas of some connections related to the given 2-skew-symmetric form on A with values in A. In the case, if ∇ is a metric A-connection in the bundle A with torsion $T \in \Gamma(\bigwedge^2 A^* \otimes A)$ which is totally skew-symmetric with respect to g, we can write the form of this connection as

$$\nabla_X Y = \tfrac{1}{2}\big([X,Y] + \langle X:Y\rangle^s\big) + \tfrac{1}{2}T(X,Y),$$

where $\langle X:Y\rangle^s$ is given in (9).

Given the bundle metric g on A, there is a unique A-connection in A which is torsion-free and metric-compatible (i.e., $T^\nabla = 0$ and $\nabla g = 0$). We call such an A-connection the *Levi-Civita connection* with respect to g. Of course, the explicit formula of the Levi-Civita connection compatible with g is written in Corollary 3.

6. Symmetric Brackets on Almost Hermitian Manifolds

In this section, we consider various symmetric brackets induced by the structures of almost Hermitian manifolds. We would like to show here that symmetric brackets are related to the symmetrized covariant derivatives and use the observed relationships to show some classical properties of the first canonical Hermitian connection, in particular in the case of nearly Kähler manifolds.

An almost complex structure (M,g,J) defines a skew-symmetric bracket on vector fields other than the usual Lie bracket of vector fields introducing a new skew-symmetric algebroid structure TM^J into the tangent bundle. We note the relationship of this structure with connections compatible with a given Riemannian structure or an almost complex structure with totally skew-symmetric torsion. We consider the Bismut connection on the Hermitian manifold noting that the torsion of this connection depends on the differential d^J of the Kähler form, where d^J is the exterior differential operator in the algebroid TM^J.

Let (M, g, J) be an almost Hermitian manifold, i.e., (M, g) is a $2n$-dimensional Riemannian manifold admitting an orthogonal almost complex structure $J : TM \to TM$. Associated with the structures g and J are the *Kähler form* $\Omega \in \Gamma(\bigwedge^2 T^*M)$ given by

$$\Omega(X, Y) = g(JX, Y) \tag{13}$$

for $X, Y \in \Gamma(TM)$ and the *Nijenhuis tensor* $N_J \in \Gamma(\bigwedge^2 T^*M \otimes TM)$ of J, which is defined by

$$N_J(X, Y) = J[JX, Y] + J[X, JY] + [X, Y] - [JX, JY]$$

for $X, Y \in \Gamma(TM)$.

Kosmann-Schwarzbach and Magri introduced in [14] (cf. also [15]) the bracket $[\![\cdot, \cdot]\!]^J$ on TM defined by

$$[\![X, Y]\!]^J = [JX, Y] + [X, JY] - J[X, Y]. \tag{14}$$

One can observe that for any $X, Y \in \Gamma(TM)$, we have

$$N_J(X, Y) = J[\![X, Y]\!]^J - [JX, JY].$$

Since

$$[\![X, fY]\!]^J = f[\![X, Y]\!]^J + (JX)(f)Y$$

for $X, Y \in \Gamma(TM)$ and $f \in C^\infty(M)$, the tangent bundle together with the almost complex structure J as an anchor and the mapping $[\![\cdot, \cdot]\!]^J$ given in (14) as a skew-symmetric bracket is a skew-symmetric algebroid, which we denote by TM^J. It is obvious that if $N_J = 0$, then $[\![[\![X, Y]\!]^J, Z]\!]^J = -J[[JX, JY], JZ]$ for $X, Y, Z \in \Gamma(TM)$ and so $\mathrm{Jac}_{[\![\cdot,\cdot]\!]^J}(X, Y, Z) = -J\,\mathrm{Jac}_{[\cdot,\cdot]}(JX, JY, JZ) = 0$ for $X, Y, Z \in \Gamma(TM)$. In consequence, if the almost complex structure J is integrable, then the skew-symmetric algebroid $\big(TM, J, [\![\cdot, \cdot]\!]^J\big)$ is a Lie algebroid over M.

Now, we define some symmetric brackets on almost Hermitian manifolds. First, we note the general properties for any skew-symmetric algebroid structures.

Let $\big(TM, \rho, [\cdot, \cdot]^\rho\big)$ be a structure of skew-symmetric algebroid, and let $\langle \cdot : \cdot \rangle^\rho$ be a symmetric bracket in this algebroid. By definition,

$$\langle X : fY\rangle^\rho = f\langle X : Y\rangle^\rho + (\rho \circ X)(f)Y$$

for $X, Y \in \Gamma(TM)$.

We define $\mathbb{R}$-bilinear symmetric operators $P^\rho, Q^\rho : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$,

$$P^\rho(X, Y) = -J\big([X, JY]^\rho + [Y, JX]^\rho\big)$$

and

$$Q^\rho(X, Y) = -J\big(\langle X : JY\rangle^\rho + \langle Y : JX\rangle^\rho\big)$$

for $X, Y \in \Gamma(TM)$.

Lemma 5. *For any $X, Y \in \Gamma(TM)$, $f \in C^\infty(M)$, we have*

(a) $P^\rho(X, f \cdot Y) = f \cdot P^\rho(X, Y) + (\rho \circ X)(f) \cdot Y + (\rho \circ JX)(f) \cdot JY$ *and*
(b) $Q^\rho(X, f \cdot Y) = f \cdot Q^\rho(X, Y) + (\rho \circ X)(f) \cdot Y - (\rho \circ JX)(f) \cdot JY$.

Proof. Compute directly,

$$\begin{aligned} P^\rho(X, f \cdot Y) &= -J\big([X, f \cdot JY]^\rho + [f \cdot Y, JX]^\rho\big) \\ &= -J\big(f \cdot [X, JY]^\rho + (\rho \circ X)(f) \cdot JY + f \cdot [Y, JX]^\rho - (\rho \circ JX)(f) \cdot Y\big) \\ &= f \cdot P^\rho(X, Y) + (\rho \circ X)(f) \cdot Y + (\rho \circ JX)(f) \cdot JY \end{aligned}$$

and

$$\begin{aligned} Q^{\rho}(X, f\cdot Y) &= -J\big(\langle X : f\cdot JY\rangle^{\rho} + \langle f\cdot Y : JX\rangle^{\rho}\big) \\ &= -J\big(f\cdot\langle X : JY\rangle^{\rho} + (\rho\circ X)(f)\cdot JY + f\cdot\langle Y : JX\rangle^{\rho} + (\rho\circ JX)(f)\cdot Y\big) \\ &= f\cdot Q^{\rho}(X,Y) + (\rho\circ X)(f)\cdot Y - (\rho\circ JX)(f)\cdot JY. \end{aligned}$$

□

In consequence of Lemma 5, we immediately get the following results.

Theorem 6. *The mapping*

$$\tfrac{1}{2}(P^{\rho} + Q^{\rho})$$

is a symmetric bracket in the skew-symmetric algebroid $\big(TM, \rho, [\cdot,\cdot]^{\rho}\big)$.

Corollary 4. *The mapping* $\langle\cdot : \cdot\rangle : \Gamma(TM)\times\Gamma(TM)\to\Gamma(TM)$ *given by*

$$\langle X : Y\rangle = -\tfrac{1}{2}J\Big([X,JY] + [Y,JX] + \sharp(\mathcal{L}^a_X(JY)^\flat + \mathcal{L}^a_Y(JX)^\flat + \mathcal{L}^a_{JX}Y^\flat + \mathcal{L}^a_{JY}X^\flat)\Big)$$

is a symmetric bracket in the Lie algebroid $(TM, \mathrm{Id}_{TM}, [\cdot,\cdot])$, *where* $[\cdot,\cdot]$ *is the Lie bracket of vector fields on* M *and* $\mathcal{L}^a$ *is the Lie derivative on* M.

Proof. Let d^a be the exterior derivative on manifold M. Taking $\rho = \mathrm{Id}_{TM}$ in Theorem 6 and using Theorem 4, we deduce that the formula

$$\begin{aligned} \langle X : Y\rangle &= -\tfrac{1}{2}J([X,JY] + [Y,JX]) - \tfrac{1}{2}(J\circ\sharp)(\mathcal{L}^a_X(JY)^\flat + \mathcal{L}^a_{JY}X^\flat - d^a(g(X,JY)) \\ &\quad -\tfrac{1}{2}(J\circ\sharp)(\mathcal{L}^a_{JX}Y^\flat + \mathcal{L}^a_Y(JX)^\flat - d^a(g(JX,Y)) \end{aligned}$$

defines a symmetric bracket in the tangent bundle with Id_{TM} as an anchor and with the classical Lie bracket. Since Ω is a skew-symmetric 2-form on M, it follows that

$$g(X,JY) + g(JX,Y) = \Omega(Y,X) + \Omega(Y,X) = 0.$$

Therefore,

$$\begin{aligned} \langle X : Y\rangle &= -\tfrac{1}{2}J([X,JY] + [Y,JX]) \\ &\quad -\tfrac{1}{2}(J\circ\sharp)(\mathcal{L}^a_X(JY)^\flat + \mathcal{L}^a_Y(JX)^\flat + \mathcal{L}^a_{JX}Y^\flat + \mathcal{L}^a_{JY}X^\flat). \end{aligned}$$

□

Let ∇ be the Levi-Civita connection in $(TM, \mathrm{Id}_{TM}, [\cdot,\cdot])$ with respect to g determining the symmetric bracket $\langle\cdot : \cdot\rangle^{\nabla}$, i.e.,

$$\nabla_X Y = \tfrac{1}{2}\Big([X,Y] + \langle X : Y\rangle^{\nabla}\Big).$$

It is obvious that the bracket in Corollary 4 is a totally symmetric part of the connection $\nabla^J : \Gamma(TM)\times\Gamma(TM)\to\Gamma(TM)$ defined by

$$\nabla^J_X Y = -\tfrac{1}{2}J\Big([X,JY] + \langle X : JY\rangle^{\nabla}\Big).$$

Hence,

$$\nabla^J_X Y = -J\nabla_X(JY).$$

One can observe that the affine sum

$$\overline{\nabla} = \tfrac{1}{2}\Big(\nabla + \nabla^J\Big)$$

of connections ∇ and ∇^J is Lichnerowicz's first canonical Hermitian connection (cf. [29]), which is compatible with both the metric structure and the almost complex structure. In fact, since $\nabla^J J = -\nabla J$ and $(\nabla^J g)(X,Y,Z) = (\nabla g)(X, JY, JZ)$ for $X,Y,Z \in \Gamma(TM)$, we conclude that $\overline{\nabla} J = 0$ and $\nabla^J g = 0$, and consequently $\overline{\nabla} g = \frac{1}{2}(\nabla + \nabla^J)g = 0$.

We will now consider some further properties of ∇^J and $\overline{\nabla}$. For an A-connection ∇ on A, we define the operators

$$d^a_\nabla, d^s_\nabla : \Gamma(\bigotimes^k T^*M \otimes TM) \to \Gamma(\bigotimes^{k+1} T^*M \otimes TM)$$

as the alternation and the symmetrization of ∇, respectively, i.e., for $\zeta \in \Gamma(\bigotimes^k T^*M)$, $X_1, \ldots, X_{k+1} \in \Gamma(TM)$, we have

$$(d^a_\nabla \zeta)(X_1, \ldots, X_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i+1} (\nabla_{X_i} \zeta)(X_1, \ldots \widehat{X}_i \ldots, X_{k+1})$$

and

$$(d^s_\nabla \zeta)(X_1, \ldots, X_{k+1}) = \sum_{i=1}^{k+1} (\nabla_{X_i} \zeta)(X_1, \ldots \widehat{X}_i \ldots, X_{k+1}).$$

We say that an almost Hermitian manifold (M, g, J) is *nearly Kähler* if $(\nabla_X J)Y = -(\nabla_Y J)Y$ for $X, Y \in \Gamma(TM)$ (cf. [30]). Thus, we have the following lemma.

Lemma 6. *An almost Hermitian manifold (M, g, J) is nearly Kähler if and only if $d^s_\nabla J = 0$.*

Moreover, if (M, g, J) is nearly Kähler, $\overline{\nabla}$ is a Hermitian connection with totally skew-symmetric torsion (cf. [31]).

Now, we compare the symmetric brackets induced by ∇ and $\overline{\nabla}$. We will denote by $\langle \cdot : \cdot \rangle^{\overline{\nabla}}$ the symmetric product of $\overline{\nabla}$.

Theorem 7. *For $X, Y \in \Gamma(TM)$, we have $J((d^s_\nabla J)(X,Y)) = \langle X : Y \rangle^\nabla - \langle X : Y \rangle^{\nabla^J}$.*

Proof. We first observe that

$$\begin{aligned} (d^s_\nabla J)(X,Y) &= (\nabla_X J)Y + (\nabla_Y J)X \\ &= \nabla_X(JY) + \nabla_Y(JX) - J(\nabla_X Y + \nabla_Y X) \\ &= \nabla_X(JY) + \nabla_Y(JX) - J\langle X : Y \rangle^\nabla. \end{aligned}$$

From this equality, we obtain

$$J((d^s_\nabla J)(X,Y)) = J\nabla_X(JY) + J\nabla_Y(JX) + \langle X : Y \rangle^\nabla = -\langle X : Y \rangle^{\nabla^J} + \langle X : Y \rangle^\nabla.$$

□

Theorem 8. *For $X, Y \in \Gamma(TM)$, we have $\langle X : Y \rangle^{\overline{\nabla}} = \langle X : Y \rangle^\nabla - \frac{1}{2} J((d^s_\nabla J)(X,Y))$.*

Proof. Since $\overline{\nabla} = \frac{1}{2}(\nabla + \nabla^J)$ is an affine sum of connections ∇ and ∇^J,

$$\langle X : Y \rangle^{\overline{\nabla}} = \frac{1}{2}\langle X : Y \rangle^\nabla + \frac{1}{2}\langle X : Y \rangle^{\nabla^J}.$$

From this result and Theorem 7, we see that

$$\begin{aligned} \langle X : Y \rangle^{\overline{\nabla}} &= \frac{1}{2}\langle X : Y \rangle^\nabla + \frac{1}{2}\Big(\langle X : Y \rangle^\nabla - J((d^s_\nabla J)(X,Y))\Big) \\ &= \langle X : Y \rangle^\nabla - \frac{1}{2} J((d^s_\nabla J)(X,Y)). \end{aligned}$$

□

Since $\overline{\nabla} = \frac{1}{2}(\nabla + \nabla^J)$ and ∇ is torsion-free, we have

$$T^{\overline{\nabla}} = T^{\frac{1}{2}\nabla + \frac{1}{2}\nabla^J} = \frac{1}{2}T^{\nabla} + \frac{1}{2}T^{\nabla^J} = \frac{1}{2}T^{\nabla^J}. \tag{15}$$

Theorem 9. $T^{\nabla^J} = -J \circ (d^a_\nabla J)$.

Proof. Let $X, Y \in \Gamma(TM)$. Then,

$$(d^a_\nabla J)(X,Y) = (\nabla_X J)Y - (\nabla_Y J)X = \nabla_X(JY) - \nabla_Y(JX) - J[X,Y].$$

Hence,

$$\begin{aligned} -J((d^a_\nabla J)(X,Y)) &= -J\nabla_X(JY) - (-J\nabla_Y(JX)) + J^2[X,Y] \\ &= \nabla^J_X Y - \nabla^J_Y X - [X,Y] = T^{\nabla^J}(X,Y). \end{aligned}$$

□

Theorem 10. *For $X, Y \in \Gamma(TM)$, we have*

$$2T^{\nabla^J}(X,Y) = -N_J(X,Y) + (d^s_\nabla J)(X,JY) - (d^s_\nabla J)(JX,Y).$$

In particular, if (M,g,J) is nearly Kähler, then $T^{\nabla^J} = -\frac{1}{2}N_J$.

Proof. Let $X, Y \in \Gamma(TM)$. Then (e.g., [31] shows the first equality),

$$\begin{aligned} -N_J(X,Y) &= (\nabla_X J)JY - (\nabla_Y J)JX + (\nabla_{JX} J)Y - (\nabla_{JY} J)X \\ &= (\nabla_X J)JY - (\nabla_Y J)JX - (\nabla_Y J)JX + (d^s_\nabla J)(JX,Y) \\ &\quad +(\nabla_X J)JY - (d^s_\nabla J)(X,JY) \\ &= 2((\nabla_X J)JY - (\nabla_Y J)JX) + (d^s_\nabla J)(JX,Y) - (d^s_\nabla J)(X,JY). \end{aligned}$$

Moreover,

$$\begin{aligned} (\nabla_X J)JY - (\nabla_Y J)JX &= -\nabla_X Y - J(\nabla_X(JY)) + \nabla_Y X + J(\nabla_Y(JX)) \\ &= -J(\nabla_X(JY)) - (-J(\nabla_Y(JX))) - \nabla_X Y + \nabla_Y X \\ &= \nabla^J_X Y - \nabla^J_Y X - [X,Y] = T^{\nabla^J}(X,Y). \end{aligned}$$

It follows that

$$-N_J(X,Y) = 2T^{\nabla^J}(X,Y) + (d^s_\nabla J)(JX,Y) - (d^s_\nabla J)(X,JY).$$

□

Since $\overline{\nabla}$ is a totally skew-symmetric connection, Theorems 8 and (15) now lead to

$$\begin{aligned} \overline{\nabla}_X Y &= \frac{1}{2}\Big([X,Y] + \langle X : Y\rangle^{\overline{\nabla}}\Big) + \frac{1}{2}T^{\overline{\nabla}}(X,Y) \\ &= \frac{1}{2}\Big([X,Y] + \langle X : Y\rangle^{\nabla} - J((d^s_\nabla J)(X,Y))\Big) + \frac{1}{2}T^{\overline{\nabla}}(X,Y) \\ &= \nabla_X Y - \frac{1}{2}J((d^s_\nabla J)(X,Y)) + \frac{1}{4}T^{\nabla^J}(X,Y). \end{aligned} \tag{16}$$

Combining (16) with Lemma 6 and Theorems 9 and 10, we get the following result:

Corollary 5. *If (M,g,J) is nearly Kähler, then $d^s_\nabla J = 0$, and in consequence,*

$$\overline{\nabla} = \nabla - \frac{1}{4}J \circ (d^a_\nabla J) = \nabla - \frac{1}{8}N_J.$$

Now, we would like to show the relationship of the Bismut connection [32] with the structure of the algebroid TM^J. The Bismut connection is the unique connection ∇^B on a complex Hermitian manifold (M, g, J) (J is integrable, i.e., the Nijenhuis tensor vanishes) with totally skew-symmetric torsion such that $\nabla^B g = 0$ and $\nabla^B J = 0$.

It is proved in (Theorem 10.1 [33]) by Friedrich and Ivanov that a Hermitian connection ∇ on an almost complex manifold (M, g, J) with totally skew-symmetric torsion exists if and only if the Nijenhuis tensor is totally skew-symmetric, and if the Nijenhuis tensor is totally skew-symmetric, the unique Hermitian connection with torsion T is given by $\nabla = \nabla^{LC} + \frac{1}{2}T$, where ∇^{LC} is the Levi-Civita connection associated with g and

$$g(T(X,Y),Z) = d\Omega(JX, JY, JZ) + g(N(X,Y),Z)$$

for $X, Y, Z \in \Gamma(TM)$. The idea comes from [34] by Gauduchon (cf. also [31]).

We will now show the relation linking the torsion of such a connection with the structure of the algebroid TM^J.

Lemma 7. *Let (M, g, J) be a $2n$-dimensional almost Hermitian manifold with the Kähler form Ω given by (13), $X, Y, Z \in \Gamma(TM)$. Then,*

$$(d\Omega)(JX, JY, JZ) = (d^J\Omega)(X,Y,Z) + \sum_{cycl} g(N_J(X,Y),Z),$$

where d^J is the exterior derivative operator on the skew-symmetric algebroid $\big(TM, J, [\![\cdot,\cdot]\!]^J\big)$.

Proof. Let $X, Y, Z \in \Gamma(TM)$. Since $[JX, JY] = J[\![X,Y]\!]^J - N_J(X,Y)$, we have

$$\begin{aligned}
(d\Omega)(JX, JY, JZ) &= (JX)(\Omega(JX,JY)) - (JY)(\Omega(JY,JZ)) + (JZ)(\Omega(JX,JY)) \\
&\quad -\Omega([JX,JY],JZ) + \Omega([JX,JZ],JY) - \Omega([JY,JZ],JX) \\
&= (JX)(\Omega(X,Y)) - (JY)(\Omega(Y,Z)) + (JZ)(\Omega(X,Y)) \\
&\quad -\Omega(J[\![X,Y]\!]^J, JZ) + \Omega(J[\![X,Z]\!]^J, JY) - \Omega(J[\![Y,Z]\!]^J, JX) \\
&\quad +\Omega(N_J(X,Y),JZ) - \Omega(N_J(X,Z),JY) + \Omega(N_J(Y,Z),JY) \\
&= (JX)(\Omega(X,Y)) - (JY)(\Omega(Y,Z)) + (JZ)(\Omega(X,Y)) \\
&\quad -\Omega([\![X,Y]\!]^J, Z) + \Omega([\![X,Z]\!]^J, Y) - \Omega([\![Y,Z]\!]^J, X) \\
&\quad +g(N_J(X,Y),Z) - g(N_J(X,Z),Y) + g(N_J(Y,Z),X) \\
&= (d^J\Omega)(X,Y,Z) + \sum_{cycl} g(N_J(X,Y),Z).
\end{aligned}$$

□

As a conclusion, we obtain that the Bismut connection is determined by the Levi-Civita connection ∇^{LC} on M with respect to g and is related to $d^J\Omega$ as follows:

$$g(\nabla^B_X Y, Z) = g(\nabla^{LC}_X Y, Z) + \tfrac{1}{2}(d^J\Omega)(X,Y,Z).$$

In view of Theorem 5, we can say that the Bismut connection is the only one connection with totally skew-symmetric torsion such that its torsion T^B satisfies $g(T^B(X,Y),Z) = (d^J\Omega)(X,Y,Z)$ for $X, Y \in \Gamma(TM)$.

7. Examples of Symmetric Product Associated with a Symplectic Connection

In this section, we show the existence of a symplectic connection for a skew-symmetric algebroid equipped with a symplectic form. We base our considerations on this general framework because we want to apply them to specific cases. Such structures include symplectic manifolds on the one hand, and symplectic Lie algebras on the other hand, understood as Lie algebroids with zero anchors. We will show that a symplectic form determines a symplectic connection understood as a linear torsion-free connection preserving the symplectic form. We add that the existence of such a connection is not unique. Each connection with zero torsion is determined by some symmetric brackets. We designate such brackets and see that the considered examples are related to a certain linear connection and its dual with respect to the symplectic form. More precisely, the resulting symmetric bracket is a certain affine combination of two symmetric brackets corresponding to the selected connection and to its dual connection.

7.1. Some Symplectic Connection on a Skew-symmetric Algebroid with Symplectic Form

Let $(A, \varrho_A, [\cdot,\cdot])$ be a skew-symmetric algebroid over a manifold M equipped with a symplectic form $\omega \in \Gamma(\bigotimes^2 A^*)$, i.e., a nondegenerated and 2-skew-symmetric form ω on A which is closed with respect to the exterior differential operator d^a given in (4). We define $\sharp_\omega : A^* \to A, \omega(\sharp_\omega(\alpha), Y) = \alpha(Y)$, which is an isomorphism with the inverse map $\flat_\omega : A \to A^*$ defined by the contraction $\flat_\omega(X) = i_X\omega$. We will use the symbol X^ω to denote $i_X\omega$ for $X \in \Gamma(A)$.

A *symplectic connection* on $(A, \varrho_A, [\cdot,\cdot], \omega)$ is a torsion-free connection ∇ which is compatible with ω, i.e., $\nabla\omega = 0$.

We find the construction of a symplectic connection on symplectic manifolds primarily in [20] by Tondeur and its application in [21]. When looking for a symplectic connection, a good starting point is to take some torsion-free connection ∇^0. When ∇^0 is torsion-free and $\langle\cdot:\cdot\rangle^0$ is the symmetric bracket determined by ∇^0, the construction is to find a symmetric 2-tensor $S \in \Gamma(\mathrm{S}^2 A^* \otimes A)$ such that $\langle\cdot:\cdot\rangle^0 + S$ is a symmetric product associated with the connection ∇ we are looking for, which means that

$$\nabla_X Y = \tfrac{1}{2}\Big([X,Y] + \langle X:Y\rangle^0 + S(X,Y)\Big) \tag{17}$$

$(X, Y \in \Gamma(A))$ defines a torsion-free connection compatible with ω. This approach uses the equality $\mathrm{Alt}(\widetilde{\nabla})(\eta) = d^a\eta + d^T\eta$, where $(d^T\eta)(X,Y,Z) = -\sum_{cycl} \eta(T^{\widetilde{\nabla}}(X,Y),Z)$ for any A-connection $\widetilde{\nabla}$ in A and $\eta \in \Gamma(\bigwedge^2 A^*)$.

Let ∇^0 be an A-connection in A with zero torsion, $T^{\nabla^0} = 0$. Our search for a symplectic connection and its corresponding symmetric bracket will be based on the mentioned construction (cf. [20,21]) with ∇^0 as the initial connection. We take the symmetric tensor $S \in \Gamma(\mathrm{S}^2 A^* \otimes A)$ uniquely determined by

$$\omega(S(X,Y),Z) = \tfrac{2}{3}\Big(\Big(\nabla^0_X\omega\Big)(Y,Z) + \Big(\nabla^0_Y\omega\Big)(X,Z)\Big)$$

for $X, Y, Z \in \Gamma(A)$. Then, $(\nabla_X\omega)(Y,Z) = \frac{1}{3}\mathrm{Alt}(\nabla)(\omega) = \frac{1}{3}d^a\omega = 0$ because ∇ is torsion-free and ω is closed. Thus, in fact, the formula in (17) describes a symplectic connection. We will write S as a linear combination of two symmetric brackets. The first is the symmetric bracket corresponding to ∇^0, while the second is the symmetric bracket of its *dual connection* $(\nabla^0)^*$ with respect to ω defined by

$$\omega((\nabla^0)^*_X Y, Z) = (\varrho_A \circ X)(\omega(Y,Z)) - \omega(Y, \nabla^0_X Z)$$

for $X, Y \in \Gamma(A)$.

Let us denote by $\langle\cdot:\cdot\rangle^*$ the symmetric bracket induced by $(\nabla^0)^*$. The key to determining a symmetric bracket is the result written in the lemma below.

Lemma 8. *Let $\langle\cdot:\cdot\rangle^*$ be the symmetric bracket determined by the connection $(\nabla^0)^*$. Then,*

$$S(X,Y) = \tfrac{2}{3}\cdot\Big(-\langle X:Y\rangle^0 + \langle X:Y\rangle^*\Big), \quad X, Y \in \Gamma(A). \tag{18}$$

Proof. Let us take an arbitrary $X, Y, Z \in \Gamma(A)$. The computation goes as follows:

$$\begin{aligned} & (\nabla^0_X \omega)(Y, Z) + (\nabla^0_Y \omega)(X, Z) \\ = \; & (\varrho_A \circ X)(\omega(Y, Z)) - \omega(\nabla^0_X Y, Z) - \omega(Y, \nabla^0_X Z) \\ & + (\varrho_A \circ Y)(\omega(X, Z)) - \omega(\nabla^0_Y X, Z) - \omega(X, \nabla^0_Y Z) \\ = \; & -\omega(\langle X : Y \rangle^0, Z) + (\varrho_A \circ X)(\omega(Y, Z)) - \omega(Y, \nabla^0_X Z) \\ & + (\varrho_A \circ Y)(\omega(X, Z)) - \omega(X, \nabla^0_Y Z) \\ = \; & \omega(-\langle X : Y \rangle^0, Z) + \omega((\nabla^0)^*_X Y, Z) + \omega((\nabla^0)^*_Y X, Z) \\ = \; & \omega(-\langle X : Y \rangle^0, Z) + \omega(\langle X : Y \rangle^*, Z). \end{aligned}$$

□

Now, we can write the formula on the symplectic connection ∇ in the language of symmetric brackets.

Theorem 11. *For any $X, Y \in \Gamma(A)$, one has*

$$\nabla_X Y = \frac{1}{2}\left([X, Y] + \frac{1}{3} \cdot \langle X : Y \rangle^0 + \frac{2}{3} \cdot \langle X : Y \rangle^* \right), \tag{19}$$

where $\langle \cdot : \cdot \rangle^0$ and $\langle \cdot : \cdot \rangle^*$ are symmetric brackets determined by ∇^0 and $\left(\nabla^0\right)^*$, respectively.

Proof. Combining (17) with (18), we get (19). □

Corollary 6. *The symmetric bracket determined by ∇ is some affine sum of symmetric brackets associated with ∇^0 and $\left(\nabla^0\right)^*$, namely*

$$\langle X : Y \rangle^{\nabla} = \frac{1}{3} \cdot \langle X : Y \rangle^0 + \frac{2}{3} \cdot \langle X : Y \rangle^* \;\; \textit{for} \;\; X, Y \in \Gamma(A).$$

Remark 1. The construction of a symplectic connection used here consists in looking for $a \in \mathbb{R}$ such that the torsion-free connection given by

$$\nabla_X Y = \frac{1}{2}\left([X, Y] + a \cdot \langle X : Y \rangle^0 + (1 - a) \cdot \langle X : Y \rangle^* \right) \tag{20}$$

is compatible with ω. Therefore, we will see below why for $a = \frac{1}{3}$ the connection ∇ is compatible with the symplectic form. For this purpose, let us note some properties of ∇^0, ∇ and the skew-symmetric and corresponding symmetric Lie derivatives written in the lemma below.

Lemma 9. *Let $a \in \mathbb{R}$ and $\mathcal{L}^{s,0}$, $\mathcal{L}^{s,*}$, and $\mathcal{L}^{s}$ denote the symmetric Lie derivatives determined by symmetric brackets $\langle \cdot : \cdot \rangle^0$, $\langle \cdot : \cdot \rangle^*$, and $\langle \cdot : \cdot \rangle^s = a\langle \cdot : \cdot \rangle^0 + (1 - a)\langle \cdot : \cdot \rangle^*$, respectively. Let $\mathcal{L}^{alt}$ be the Lie derivative associated with the skew-symmetric bracket $[\cdot, \cdot]$, $X \in \Gamma(A)$, and ∇ be given in (20). Then,*

(a) $\nabla_X \omega = \frac{1}{2}(\mathcal{L}^{alt}_X + \mathcal{L}^{s}_X)\omega$,
(b) $\mathcal{L}^{s}_X = a\mathcal{L}^{s,0}_X + (1 - a)\mathcal{L}^{s,*}_X$,
(c) $(\mathcal{L}^{alt}_X + \mathcal{L}^{s,*}_X)\omega = -\nabla^0_X \omega$, *and*
(d) $(\mathcal{L}^{s,0}_X - \mathcal{L}^{s,*}_X)\omega = 3\nabla^0_X \omega$.

Proof. Since ∇^0 is torsion-free, $\nabla_X Y = \frac{1}{2}([X, Y] + \langle X : Y \rangle^s)$ for $X, Y \in \Gamma(A)$. Hence, (a) and (b) immediately follow. From Lemma 8, it follows that

$$\omega(-\langle X : Y \rangle^0 + \langle X : Y \rangle^*, Z) = \left(\nabla^0_X \omega\right)(Y, Z) + \left(\nabla^0_Y \omega\right)(X, Z).$$

From this equality and the fact that $d^a\omega = 0$, one can obtain (c) and (d). □

Properties from Lemma 9 are helpful in determining the relationship between $\nabla\omega$ and $\nabla^0\omega$, which allows us to notice how important it is for the connection given in (20) to be compatible with the symplectic form; this is the influence of constant $a = \frac{1}{3}$. We present this relationship in the corollary below.

Corollary 7. $\nabla\omega = \frac{3a-1}{2} \cdot \nabla^0\omega$.

Proof. Let $X \in \Gamma(A)$. Using successively the properties (a)–(d) from Lemma 9, we get

$$\begin{aligned} \nabla_X\omega &= \tfrac{1}{2}(\mathcal{L}^{alt}_X + \mathcal{L}^s_X)\omega \\ &= \tfrac{1}{2}\Big(\mathcal{L}^{alt}_X + a\mathcal{L}^{s,0}_X + (1-a)\mathcal{L}^{s,*}_X\Big)\omega \\ &= \tfrac{1}{2}\Big(\mathcal{L}^{alt}_X + \mathcal{L}^{s,*}_X\Big)\omega + \tfrac{a}{2}(\mathcal{L}^{s,0}_X - \mathcal{L}^{s,*}_X)\omega \\ &= -\tfrac{1}{2} \cdot \nabla^0_X\omega + \tfrac{3a}{2} \cdot \nabla^0_X\omega \\ &= \tfrac{3a-1}{2} \cdot \nabla^0_X\omega. \end{aligned}$$

□

In Sections 7.2 and 7.3, we give examples of symplectic connections and the corresponding symmetric brackets in two cases: a symplectic manifold and a symplectic algebra (in particular, a symplectic Lie algebra).

7.2. The Case of Symplectic Manifold

Let (M, ω) be a symplectic manifold, i.e., the manifold M is equipped with a nondegenerated and closed exterior 2-form ω on M. The form ω is then called a *symplectic form* on M. We define $\sharp_\omega : T^*M \to TM, \omega(\sharp_\omega(\alpha), Y) = \alpha(Y)$, which is an isomorphism with the inverse map $\flat_\omega : TM \to T^*M$ defined by the contraction $\flat_\omega(X) = i_X\omega$. We will use the symbol X^ω to denote $i_X\omega$ for $X \in \Gamma(TM)$.

A *symplectic connection* on (M, ω) is a torsion-free connection ∇ which is compatible with ω, i.e., $\nabla\omega = 0$.

We use the construction discussed in Section 7.1. We take the Levi-Civita connection associated with g as a starting connection. Let $J : TM \to TM$ be an almost complex structure compatible with ω and g be an associated pseudo-Riemannian metric, i.e., $g(X, Y) = \omega(X, JY)$ for $X, Y \in \Gamma(TM)$. Let ∇^{LC} be the Levi-Civita connection induced by g and $\langle\cdot,\cdot\rangle^{LC}$ be the symmetric product defined by ∇^{LC}. Let $\sharp_g : T^*M \to TM$ denote the sharp operator for g, i.e., $g(\sharp_g(\alpha), Y) = \alpha(Y)$ for $\alpha \in \Gamma(T^*M)$, $Y \in \Gamma(TM)$.

We will calculate the dual connection and the symmetric bracket designated by it, also analogously to the Levi-Civita connection using the Lie derivative and the differential operator. Let us recall that in the geometry of Hermitian manifolds, a special role is played by the connection ∇^J associated with ∇^{LC} and the almost complex structure by

$$\nabla^J_X Y = -J\Big(\nabla^{LC}_X(JY)\Big) \quad \text{for } X, Y \in \Gamma(TM).$$

The dual connection $\left(\nabla^{LC}\right)^*$ to the Levi-Civita connection is just ∇^J. Indeed, note that since ∇^{LC} preserves the metric, we have

$$\begin{aligned} \omega\Big(\Big(\nabla^{LC}\Big)^*_X Y, Z\Big) &= X(\omega(Y,Z)) - \omega(Y, \nabla^{LC}_X Z) = X(g(JY,Z)) - g(JY, \nabla^{LC}_X Z) \\ &= g(\nabla^{LC}_X(JY), Z) = \omega(\nabla^{LC}_X(JY), JZ) = -\omega(J(\nabla^{LC}_X(JY)), Z) \\ &= \omega(\nabla^J_X Y, Z) \end{aligned}$$

for $X, Y, Z \in \Gamma(TM)$.

Let $\langle \cdot : \cdot \rangle^{LC}$ and $\langle \cdot : \cdot \rangle^{J}$ denote the symmetric brackets determined by ∇^{LC} and ∇^{J}, respectively. Theorem 11 shows immediately that an example of a symplectic connection is a torsion-free connection ∇ that defines a symmetric bracket, which is the following affine combination $\frac{1}{3}\langle \cdot : \cdot \rangle^{LC} + \frac{2}{3}\langle \cdot : \cdot \rangle$, i.e.,

$$\nabla_X Y = \frac{1}{2}\Big([X,Y] + \frac{1}{3}\cdot \langle X : Y\rangle^{LC} + \frac{2}{3}\cdot \langle X : Y\rangle^{J}\Big)$$

for $X, Y \in \Gamma(TM)$. We will now designate the symmetric bracket $\langle \cdot : \cdot \rangle^{J}$ setting by ∇^{J}. Note that using the formula (11) of $\langle \cdot : \cdot \rangle^{LC}$ we can write explicitly the symmetric bracket of the connection ∇^{J}:

$$\begin{aligned}
\langle X : Y\rangle^{J} &= -J\Big(\nabla^{LC}_X(JY) + \nabla^{LC}_Y(JX)\Big) \\
&= -\frac{1}{2}J\Big([X,JY] + \sharp_g(\mathcal{L}^a_X(JY)^\flat + \mathcal{L}^a_{JY}X^\flat - d^a(g(X,JY)\Big) \\
&\quad -\frac{1}{2}J\Big([Y,JX] + \sharp_g(\mathcal{L}^a_Y(JX)^\flat + \mathcal{L}^a_{JX}Y^\flat - d^a(g(Y,JX)\Big) \\
&= -\frac{1}{2}(J[X,JY] + J[Y,JX]) + \frac{1}{2}\sharp_\omega(\mathcal{L}^a_X Y^\omega + \mathcal{L}^a_Y X^\omega) \\
&\quad -\frac{1}{2}\sharp_\omega\Big(\mathcal{L}^a_{JX}(JY)^\omega + \mathcal{L}^a_{JY}(JX)^\omega\Big)
\end{aligned}$$

because $g(X,JY) + (g(Y,JX) = \omega(Y,X) + \omega(X,Y) = 0$, $\sharp_\omega = -J \circ \sharp_g$, $(JX)^\flat = X^\omega$, and $X^\flat = -(JX)^\omega$.

Remark 2. When looking for symmetric brackets, we notice that properties written in Lemmas 4 and 3 imply that one of the brackets is

$$(X,Y)^s = \frac{1}{2}\sharp_\omega(\mathcal{L}^a_X Y^\omega + \mathcal{L}^s_X Y^\omega + \mathcal{L}^a_Y X^\omega + \mathcal{L}^s_Y X^\omega),$$

where the symmetric Lie derivative is defined for $\langle \cdot, \cdot \rangle^{LC}$, i.e., for the symmetric product associated with ∇^{LC}. Observe that $(\cdot,\cdot)^s$ is the symmetric product defined by the connection ∇' given by

$$\nabla'_X Y := \frac{1}{2}\sharp_\omega(\mathcal{L}^a_X Y^\omega + \mathcal{L}^s_X Y^\omega)$$

for $X, Y \in \Gamma(TM)$. One can check that ∇' is actually the Levi-Civita connection associated with g.

7.3. The Case of Symplectic Skew-symmetric Algebra

We say that $(\mathfrak{g}, [\cdot,\cdot])$ is a *skew-symmetric algebra* if $\mathfrak{g}$ is a real vector space and $[\cdot,\cdot]$ is a bilinear skew-symmetric mapping. A skew-symmetric algebra can be regarded as a skew-symmetric algebroid with a zero anchor. In particular, any finite-dimensional Lie algebra is such an algebra. We call each skew-symmetric algebra $(\mathfrak{g}, [\cdot,\cdot])$ together with a bilinear skew-symmetric nondegenerated closed form $\omega \in \bigwedge^2 \mathfrak{g}^*$ a *symplectic skew-symmetric algebra*. Then, ω is said to be a *symplectic form* on $\mathfrak{g}$. A Lie algebra $(\mathfrak{g}, [\cdot,\cdot])$ endowed with a symplectic form ω is called a *symplectic Lie algebra*.

Let $(\mathfrak{g}, [\cdot,\cdot])$ be a skew-symmetric algebra together with a bilinear skew-symmetric nondegenerated form $\omega \in \bigwedge^2 \mathfrak{g}^*$. Let $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ be the adjoint representation of $\mathfrak{g}$, i.e., $\mathrm{ad}_x(y) = [x,y]$ for $x, y \in \mathfrak{g}$. Our goal is to admit a certain symplectic connection using the result of Theorem 11. Thus, the starting point is a certain torsion-free connection. As a starting connection, let us take $\nabla^0 = \frac{1}{2}\,\mathrm{ad}$. Next, we take the connection $\nabla^* = \frac{1}{2}\,\mathrm{ad}^*$, where ad^* is dual to ad with respect to the symplectic form ω. To be more precise, ∇^* is determined as follows:

$$\omega(\nabla^*_x y, z) = \omega\Big(\frac{1}{2}\,\mathrm{ad}^*_x(y), z\Big) = -\frac{1}{2}\omega(y, [x,z])$$

for $x, y, z \in \mathfrak{g}$. Hence, the symmetric bracket $\langle \cdot : \cdot \rangle^*$ of $\frac{1}{2}\,\mathrm{ad}^*$ is given by

$$\omega\big(\langle x : y\rangle^*, z\big) = -\tfrac{1}{2}\Big(\omega\big([z,x],y\big) + \omega\big([z,y],x\big)\Big). \tag{21}$$

Since $\langle \cdot : \cdot \rangle^{\nabla^0} = 0$ holds, Theorem 11, indicates that

$$\nabla_x y = \tfrac{1}{2}[x,y] + \tfrac{1}{3}\langle x : y\rangle^*, \quad x, y \in \mathfrak{g}, \tag{22}$$

defines a symplectic connection with respect to the symplectic form ω. From (21), we conclude that

$$\omega\big(\nabla_x y, z\big) = \tfrac{1}{2}\omega([x,y],z) - \tfrac{1}{6}\omega([z,x],y) - \tfrac{1}{6}\omega([z,y],x) \tag{23}$$

for $x, y, z \in \mathfrak{g}$. Using the condition $d\omega = 0$, we can rewrite (23) as

$$\omega(\nabla_x y, z) = -\tfrac{2}{3}\omega([z,x],y) + \tfrac{1}{3}\omega([z,y],x), \quad x, y, z \in \mathfrak{g}.$$

Example 1. We will designate a symplectic connection of the symplectic algebra $\mathfrak{g} = \mathfrak{r}_2\mathfrak{r}_2$ from [23], where the classification of four dimensional symplectic Lie algebras is given. We consider a 4-dimensional vector space $\mathfrak{g}$ with a basis $\{e_1, e_2, e_3, e_3\}$. Let $\{e^1, e^2, e^3, e^4\}$ be its dual basis of $\mathfrak{g}^*$. We define the skew-symmetric bracket $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying the following rules $[e_1, e_2] = e_2$, $[e_3, e_4] = e_4$. The Lie algebra defined in this way is isomorphic to the direct product $\mathfrak{r}_2 \times \mathfrak{r}_2$, where $\mathfrak{r}_2 = \mathfrak{aff}(\mathbb{R})$ is 2-dimensional non-abelian Lie algebra of the group of affine transformations of the real line (cf. [22,23]).
Let $a, b, c \in \mathbb{R}$. Take the skew-symmetric bilinear form

$$\omega = ae^1 \wedge e^2 + be^1 \wedge e^3 + ce^3 \wedge e^4 \in \textstyle\bigwedge^2 \mathfrak{g}^*.$$

Note that the determinant of the matrix $[\omega(e_i, e_j)]$ of ω in the frame $B = (e_1, e_2, e_3, e_3)$ is equal to a^2c^2. It follows that if $a \neq 0$ and $c \neq 0$, then ω is nondegenerate, and in consequence, ω is a symplectic form on $\mathfrak{g}$. We further assume that a and c are nonzero reals. The symmetric bracket $\langle \cdot : \cdot \rangle^*$ of $\frac{1}{2}\,\mathrm{ad}^*$ is given by (21). Therefore, one can check that $\langle \cdot : \cdot \rangle^*$ is fully designated by the following values: $\big(\langle e_1 : e_1\rangle^*\big)^\omega = -a\,\mathrm{pr}_{2,B}$, $\big(\langle e_2 : e_1\rangle^*\big)^\omega = \frac{a}{2}\,\mathrm{pr}_{1,B}$, $\big(\langle e_3 : e_3\rangle^*\big)^\omega = -c\,\mathrm{pr}_{4,B}$, and $\big(\langle e_4 : e_3\rangle^*\big)^\omega = \frac{c}{2}\,\mathrm{pr}_{3,B}$, where $\mathrm{pr}_{i,B} : \mathfrak{g} \to \mathbb{R}$ denotes the frame $B = (e_1, e_2, e_3, e_3)$ dependent projection given by $\mathrm{pr}_{i,B}(z) = z^i$ if $z = \sum_{k=1}^4 z^k e_k$, and zero in other cases $\langle e_i : e_j\rangle$. Hence,

$$\begin{array}{llll}
\langle e_1 : e_1\rangle^* = -e_1 - \frac{b}{c}e_4, & & & \\
\langle e_2 : e_1\rangle^* = -\frac{1}{2}e_2, & \langle e_2 : e_2\rangle^* = 0, & & \\
\langle e_3 : e_1\rangle^* = 0, & \langle e_3 : e_2\rangle^* = 0, & \langle e_3 : e_3\rangle^* = \frac{b}{a}e_2 - e_3, & \\
\langle e_4 : e_1\rangle^* = 0, & \langle e_4 : e_2\rangle^* = 0, & \langle e_4 : e_3\rangle^* = -\frac{1}{2}e_4, & \langle e_4 : e_4\rangle^* = 0.
\end{array}$$

Therefore, a symplectic connection ∇ given in (22) and (23) can be described on vectors from the frame B as follows:

$$\begin{array}{llll}
\nabla_{e_1}e_1 = -\frac{1}{3}(e_1 + \frac{b}{c}e_4), & \nabla_{e_1}e_2 = \frac{1}{3}e_2, & \nabla_{e_1}e_3 = 0, & \nabla_{e_1}e_4 = 0, \\
\nabla_{e_2}e_1 = -\frac{2}{3}e_2, & \nabla_{e_2}e_2 = 0, & \nabla_{e_2}e_3 = 0, & \nabla_{e_2}e_4 = 0, \\
\nabla_{e_3}e_1 = 0, & \nabla_{e_3}e_2 = 0, & \nabla_{e_3}e_3 = \frac{1}{3}(\frac{b}{a}e_2 - e_3), & \nabla_{e_3}e_4 = \frac{1}{3}e_4, \\
\nabla_{e_4}e_1 = 0, & \nabla_{e_4}e_2 = 0, & \nabla_{e_4}e_3 = -\frac{2}{3}e_4, & \nabla_{e_4}e_4 = 0.
\end{array}$$

One can check that $\left\langle \langle x : y\rangle^* : \langle x : x\rangle^* \right\rangle^* = \left\langle x : \left\langle y : \langle x : x\rangle^* \right\rangle^* \right\rangle^*$ for $x, y \in \mathfrak{g}$. This means that the symmetric bracket $\langle \cdot : \cdot \rangle^*$ satisfies the Jordan identity and thus introduces the structure of a Jordan algebra into $\mathfrak{g} = \mathfrak{r}_2\mathfrak{r}_2$.

Example 2. Let $(\mathfrak{g}, [\cdot,\cdot])$ be a finite dimensional Lie algebra. In the direct sum $\mathfrak{h} = \mathfrak{g} \oplus \mathfrak{g}^*$, we consider a structure of algebra with skew-symmetric bracket defined by

$$[\![x \oplus \alpha, y \oplus \beta]\!] = [x,y] \oplus \tfrac{1}{2}\left(\alpha \circ \mathrm{ad}_y - \beta \circ \mathrm{ad}_x\right)$$

for $x, y \in \mathfrak{g}$, $\alpha, \beta \in \mathfrak{g}^*$, and where $\mathrm{ad}_y(x) = [x,y]$, i.e., ad is the adjoint representation of the Lie algebra $\mathfrak{g}$. We recall that any skew-symmetric algebra is a skew-symmetric algebroid with the zero anchor. So, we have the exterior differential operator $d : \bigwedge^k \mathfrak{h}^* \to \bigwedge^{k+1} \mathfrak{h}^*$ given by $(d\eta)(a_1, \ldots, a_{k+1}) = \sum_{i<j}(-1)^{i+j}\eta([\![a_i, a_j]\!], a_1, \ldots \widehat{a}_i \ldots \widehat{a}_j \ldots, a_{k+1})$ for $\eta \in \bigwedge^k \mathfrak{h}^*, a_1, \ldots, a_{k+1} \in \mathfrak{h}$.
We equip $\mathfrak{h}$ with a nondegenerate skew-symmetric bilinear form $\omega \in \bigwedge^2 \mathfrak{h}^*$ defined by

$$\omega(x \oplus \alpha, y \oplus \beta) = \alpha(y) - \beta(x).$$

One can verify that $d\omega = 0$. Thus, ω is a symplectic form in $\mathfrak{h} = \mathfrak{g} \oplus \mathfrak{g}^*$. We would have to find a symplectic connection in $\mathfrak{h}$ compatible with ω starting from the torsion-free connection $\nabla^0 = \frac{1}{2}\mathrm{ad}_{\mathfrak{h}}$. Since the connection $\left(\nabla^0\right)^* = \left(\frac{1}{2}\mathrm{ad}_{\mathfrak{h}}\right)^*$ which is dual with respect to ω is described by

$$\omega\left(\left(\nabla^0\right)^*_{x\oplus\alpha}(y \oplus \beta), z \oplus \gamma\right) = -\tfrac{1}{4}\alpha[y,z] - \tfrac{1}{2}\beta[x,z] - \tfrac{1}{4}\gamma[x,y],$$

it follows that $\left(\nabla^0\right)^*$ and the symmetric bracket $\langle \cdot : \cdot \rangle^*$ of $\left(\nabla^0\right)^*$ are given explicitly by

$$\left(\nabla^0\right)^*_{x\oplus\alpha}(y \oplus \beta) = \left(\tfrac{1}{4}[x,y]\right) \oplus \left(-\tfrac{1}{4}\alpha \circ \mathrm{ad}_y - \tfrac{1}{2}\beta \circ \mathrm{ad}_x\right),$$

$$\langle x \oplus \alpha : y \oplus \beta\rangle^* = 0 \oplus \left(-\tfrac{3}{4}\right)\left(\alpha \circ \mathrm{ad}_y + \beta \circ \mathrm{ad}_x\right)$$

for $x, y \in \mathfrak{g}$, $\alpha, \beta \in \mathfrak{g}^*$.
The formula given in (22) defines now a symplectic connection ∇ in $\mathfrak{h}$, which we can write as follows:

$$\begin{aligned}
\nabla_{x\oplus\alpha}(y \oplus \beta) &= \tfrac{1}{2}[\![x \oplus \alpha, y \oplus \beta]\!] + \tfrac{1}{3}\langle x \oplus \alpha : y \oplus \beta\rangle^* \\
&= \tfrac{1}{2}[x,y] \oplus \tfrac{1}{4}\left(\alpha \circ \mathrm{ad}_y - \beta \circ \mathrm{ad}_x\right) - 0 \oplus \tfrac{1}{4}\left(\alpha \circ \mathrm{ad}_y + \beta \circ \mathrm{ad}_x\right) \\
&= \tfrac{1}{2}([x,y] \oplus (-\beta \circ \mathrm{ad}_x))
\end{aligned}$$

for $x \oplus \alpha, y \oplus \beta \in \mathfrak{h}$.
We remark that one can also take ∇^1 given by $\nabla^1_{x\oplus\alpha}(y \oplus \beta) = \frac{1}{2}\left([x,y] \oplus \left(\alpha \circ \mathrm{ad}_y\right)\right)$ as a torsion-free initiating connection. One can check that

$$\begin{aligned}
\langle x \oplus \alpha : y \oplus \beta\rangle^{\nabla^1} &= 0 \oplus \tfrac{1}{2}\left(\alpha \circ \mathrm{ad}_y + \beta \circ \mathrm{ad}_x\right), \\
\left(\nabla^1\right)^*_{x\oplus\alpha}(y \oplus \beta) &= 0 \oplus \frac{1}{2}\left(-\alpha \circ \mathrm{ad}_y - \beta \circ \mathrm{ad}_x\right), \\
\langle x \oplus \alpha : y \oplus \beta\rangle^{\left(\nabla^1\right)^*} &= 0 \oplus \left(-\alpha \circ \mathrm{ad}_y - \beta \circ \mathrm{ad}_x\right).
\end{aligned}$$

According to Theorem 11, $\nabla^{sp}_A B = \frac{1}{2}[\![A, B]\!] + \frac{1}{3}\langle A : B\rangle^{\nabla^1} + \frac{2}{3}\langle A : B\rangle^{\left(\nabla^1\right)^*}$, $A, B \in \mathfrak{h}$, is a symplectic connection in $\mathfrak{h}$. However $\nabla^{sp} = \nabla$, which means that the procedure described in Section 7.1 gives the same symplectic connection for both of the initial torsion-free connections. In $\mathfrak{h}$ we have a natural metric $g \in S^2\mathfrak{h}^*$ given by

$$g(x \oplus \alpha, y \oplus \beta) = \alpha(y) + \beta(x)$$

for $x \oplus \alpha, y \oplus \beta \in \mathfrak{h}$. To discover a symplectic connection with respect to ω, one can take the Levi-Civita connection with respect to g as an initial connection. It can be checked that ∇ is actually the Levi-Civita ∇^{LC} connection with respect to g. Thus, the Levi-Civita connection related to g is also a symplectic connection in $\mathfrak{h}$ with respect to the symplectic form ω.

8. The Bochner Bracket of Smooth Functions

In this section, given the skew-symmetric algebroid endowed with a metric, we consider a symmetric product $(\cdot, \cdot)$ of smooth functions determined naturally by the metric. We consider a linear connection ∇ on a given algebroid and examine whether for a symmetric product $\langle \cdot : \cdot \rangle^\nabla$ determined by this connection there is the property $\langle \operatorname{grad} f : \operatorname{grad} h \rangle^\nabla = \operatorname{grad}(f, h)$ for any smooth functions f, h. Thus, we investigate whether there is an analogous property that holds for Hamiltonian vector fields on a Poisson manifold. We will see that this property holds for the connection with totally skew-symmetric torsion. The symmetric bracket defined by the Levi-Civita connection is essential.

Let $(A, \varrho_A, [\cdot, \cdot])$ be a skew-symmetric algebroid over a manifold M equipped with a pseudo-Riemannian metric $g \in \Gamma(S^2 A^*)$ in the vector bundle A. Moreover, let $\flat : A \to A^*$ and $\sharp : A^* \to A$ be the musician morphisms induced by g (see the beginning of Section 4). We extend $\sharp$ to the morphism

$$\sharp : S(A^*) \to \bigoplus_{k \geq 0} \mathsf{S}^k A$$

on the whole bundle $S(A^*)$ by

$$\sharp(\Omega)(\omega_1, \ldots, \omega_k) = (-1)^k \Omega(\sharp\omega_1, \ldots, \sharp\omega_k)$$

for $\Omega \in \Gamma(S^k A^*), \omega_1, \ldots, \omega_k \in \Gamma(A^*)$.

Define the 2-symmetric tensor

$$G = \sharp(g) \in \Gamma(S^2 A),$$

i.e.,

$$G(\omega, \eta) = g(\sharp\omega, \sharp\eta), \quad \omega, \eta \in \Gamma(A^*).$$

By the *gradient of a smooth function* $f \in C^\infty(M)$ with respect to g we mean the section

$$\operatorname{grad} f = \sharp(d^a f) \in \Gamma(A),$$

where d^a is the exterior derivative operator in the a skew-symmetric algebroid $(A, \varrho_A, [\cdot, \cdot])$, i.e., $(d^a f)(X) = (\varrho_A \circ X)(f)$ for $X \in \Gamma(A)$.

Now, define the symmetric bracket on smooth functions

$$(\cdot, \cdot) : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$$

by

$$(f, h) = G(df, dh) = g(\operatorname{grad} f, \operatorname{grad} h). \tag{24}$$

This bracket will be called the *Bochner bracket* of smooth functions, following Crouch [26]. Observe that $(\cdot, \cdot)$ is a symmetric $\mathbb{R}$-bilinear mapping with the property

$$(f_1, f_2 \cdot f_3) = f_2 \cdot (f_1, f_3) + (f_1, f_2) \cdot f_3$$

for all $f_1, f_2, f_3 \in C^\infty(M)$. So, $(f_1, \cdot)$ is a derivation for any $f_1 \in C^\infty(M)$.

We would now like to examine under what conditions the linear connection ∇ defines a symmetric bracket $\langle \cdot : \cdot \rangle^{\nabla}$, for which the gradient of the Bochner bracket of smooth functions defined in (24) is the bracket of the gradients. We notice in the lemma below the relationship of the Lie derivative with the symmetric bracket of functions. These properties are related to the symmetric bracket defined by the metric connection and given in (11).

Lemma 10. *If $f, h \in C^{\infty}(M)$ and $X = \operatorname{grad} f$, $Y = \operatorname{grad} h$, then*

$$\sharp(\mathcal{L}_X Y^{\flat}) = \sharp(d(g(X,Y))) = \sharp(\mathcal{L}_Y X^{\flat}) = \operatorname{grad}(f,h). \tag{25}$$

Proof. First observe that

$$(\mathcal{L}_X(dh))(Z) = X(dh(Z)) - dh([X,Z]) = Z(X(h)) = d(X(h))(Z)$$

for $Z \in \Gamma(TM)$. So,

$$\mathcal{L}_X Y^{\flat} = d(X(h)) = d(g(X,Y))$$

since $g(X,Y) = g(X, \sharp dh) = (dh)(X)$. Moreover, by definition, $(f,g) = G(df, gh) = g(\operatorname{grad} f, \operatorname{grad} h)$. Therefore, we have (25). □

As a consequence of the last result and Theorem 5, we obtain the following:

Theorem 12. *If ∇ is a metric A-connection in A with totally skew-symmetric torsion with respect to a pseudo-Riemannian metric g and $f, h \in C^{\infty}(M)$, then*

$$\langle \operatorname{grad} f : \operatorname{grad} h \rangle^{\nabla} = \operatorname{grad}(f,h).$$

Proof. Let $f, h \in C^{\infty}(M)$. Define $X = \operatorname{grad} f$ and $Y = \operatorname{grad} h$. By Theorem 5 and Lemma 10, we have at once that

$$\begin{aligned} \langle \operatorname{grad} f : \operatorname{grad} h \rangle^{\nabla} &= \langle X : Y \rangle^{\nabla} \\ &= \sharp(\mathcal{L}^a_X Y^{\flat} + \mathcal{L}^a_Y X^{\flat} - d^a(g(X,Y)) \\ &= \operatorname{grad}(f,h) + \operatorname{grad}(f,h) - \operatorname{grad}(f,h) = \operatorname{grad}(f,h). \end{aligned}$$

□

Corollary 8. *The space of all gradients* $\operatorname{Grad}(A,g)$ *together with the restricted symmetric bracket* $\langle \cdot : \cdot \rangle^{LC} : \operatorname{Grad}(A,g) \times \operatorname{Grad}(A,g) \to \operatorname{Grad}(A,g)$ *of the Levi-Civita connection associated with g forms a symmetric algebra. The multiplication $\langle \cdot : \cdot \rangle^{LC}$ satisfies*

$$\langle X : fY \rangle^{LC} = f \langle X : Y \rangle^{LC} + g(\operatorname{grad} f, X) Y$$

for $X, Y \in \operatorname{Grad}(A,g)$, $f \in C^{\infty}(M)$.

The Case of Symplectic Manifolds.

Let M continue to be a symplectic manifold with a symplectic form ω. Fix an almost complex structure $J : TM \to TM$ compatible with ω, i.e., J is a bundle morphism with $J^2 = -\mathrm{id}_{TM}$ and $g \in \Gamma(S^2 A^*)$ defined by $g(X,Y) = \omega(X, JY)$ is a Riemannian metric. These conditions imply that $J : TM \to TM$ is an isometry both with respect to ω (we then say that J is a symplectomorphism) and with respect to g, i.e., $\omega(JX, JY) = \omega(X,Y)$ and $g(JX, JY) = g(X,Y)$ for $X, Y \in \Gamma(TM)$, respectively. Let $\sharp_{\omega} : T^*M \to TM$ and $\flat_{\omega} : TM \to T^*M$ be the mappings defined at the beginning of Section 7.2 and determined by the symplectic form ω. On the other hand, the maps $\sharp_g : T^*M \to TM$, $g(\sharp_g(\alpha), Y) = \alpha(Y)$ and $\flat_{\omega} : TM \to T^*M$, $\flat_{\omega}(X) = i_X \omega$ are determined by the metric g. For any $X \in \Gamma(A)$, the forms $i_X g$ and $i_X \omega$ will be denoted, briefly, by $X^{\flat}$ and X^{ω}, respectively.

For every $f \in C^\infty(M)$, there is a corresponding unique vector field $H_f \in \Gamma(TM)$ such that

$$i_{H_f}\omega = df.$$

The vector field H_f is called the *Hamiltonian vector field* with *Hamiltonian function* f. The space of all Hamiltonian vector fields on (M,ω) is denoted by $\mathfrak{X}^{\mathrm{Ham}}(M)$. Since $[X,Y] = H_{\omega(Y,X)}$ for any Hamiltonian vector fields X, Y, $\mathfrak{X}^{\mathrm{Ham}}(M)$ is a Lie algebra with the Lie bracket of vector fields. Let $\langle \cdot : \cdot \rangle^{LC}$ be the symmetric bracket determined by the Levi-Civita connection associated with g.

Let us note that the relations between the symplectic structure, the almost complex structure, and the metric imply the following identities:

$$\sharp_g = J \circ \sharp_\omega, \quad X^\flat = -(JX)^\omega, \quad X^\omega = (JX)^\flat \tag{26}$$

for $X, Y \in \Gamma(TM)$. From this, we have

$$\mathrm{grad}\, f = JH_f$$

for $f \in C^\infty(M)$.

For any $X, Y \in \Gamma(TM)$, we define

$$(X : Y) = J\langle JX : JY\rangle^{LC}.$$

Since $\left(\mathfrak{X}^{\mathrm{Ham}}(M) : \mathfrak{X}^{\mathrm{Ham}}(M)\right) \subset \mathfrak{X}^{\mathrm{Ham}}(M)$ holds, the map $(\cdot : \cdot)$ introduces the structure of symmetric algebra into the space of Hamiltonian vector fields. From the general form of the Levi-Civita connection and identities (26), it follows that

$$\begin{aligned} (X : Y) &= -(J \circ \sharp_g)(\mathcal{L}^a_{JX}(JY)^\flat + \mathcal{L}^a_{JY}(JX)^\flat - d^a(g(X,Y)) \\ &= \sharp_\omega(\mathcal{L}^a_{JX}Y^\omega + \mathcal{L}^a_{JY}X^\omega - d^a(\omega(X,JY)) \end{aligned}$$

for $X, Y \in \Gamma(TM)$.

The symmetric bracket of smooth functions introduced at the beginning of this section by (24) is connected with the corresponding Hamiltonian vector fields. This relation is shown in the theorem below.

Theorem 13. *Let $f, h \in C^\infty(M)$. Then,*

$$(f,h) = \omega(H_h, JH_f). \tag{27}$$

Theorems 12 and 13 now lead to

$$(X : Y) = H_{\omega(Y,JX)}$$

for all $X, Y \in \mathfrak{X}^{\mathrm{Ham}}(M)$. The obtained property of the symmetric bracket of Hamiltonian vector fields is a symmetric analogy of the properties of the Lie bracket of such fields because

$$[X,Y] = H_{\omega(Y,X)}$$

for all $X, Y \in \mathfrak{X}^{\mathrm{Ham}}(M)$.

9. Concluding Remarks

The result obtained in the last section generalizes the result for Riemannian manifolds and the Levi-Civita connections [26] and shows that the properties analogous to those for the Hamiltonian vector fields on Poisson manifolds naturally occur for gradients and symmetric brackets determined by some connections. According to Theorem 2, it is important to assume that a given connection compatible with the pseudometric has a totally skew-symmetric torsion with respect to the metric. However, the symmetric brackets of such connection are completely determined by the bracket for the Levi-Civita connection. Examples show that the symmetric bracket associated with a metric connection and brackets for the dual connections with respect to a given nondegenerate bilinear form have an important role in the construction of linear connections on various geometric structures related to skew-symmetric algebroids.

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