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Boundary Value Problems of Hadamard Fractional Differential Equations of Variable Order

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Abstract: A boundary value problem for Hadamard fractional differential equations of variable order is studied. Note the symmetry of a transformation of a system of differential equations is connected with the locally solvability which is the same as the existence of solutions. It leads to the necessity of obtaining existence criteria for a boundary value problem for Hadamard fractional differential equations of variable order. Also, the stability in the sense of Ulam–Hyers–Rassias is investigated. The results are obtained based on the Kuratowski measure of noncompactness. An example illustrates the validity of the observed results.

Keywords: derivatives and integrals of variable order; boundary value problem; measure of noncompactness; Ulam–Hyers–Rassias stability; Hadamard derivative

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1. Introduction

The idea of fractional calculus is to take real number powers or complex number powers of the differentiation and integration operators ([1,2]). Recently, fractional integration and differentiation of variable orders have also been studied (see, for example, [3–11]). A good review of variable-order (VO) fractional calculus and its practical applications in the context of scientific modeling is given in [12]; some physical discussion on VO fractional integral and derivative models is provided in [13].

One of the main problems in differential equations is the solvability. There are many different methods use to study the existence such as the Lie group symmetry ([14,15]). In this paper we use the integral equivalence to prove the existence result for the boundary value problem for the Hadamard fractional differential equation of variable order. The boundary value problems for various types of fractional differential equations have been set up and studied by many authors, such as for a nonlinear fractional differential equations of order $\alpha \in (2,3)$ ([16]), for ψ-Hilfer fractional derivatives on b-metric spaces ([17]), for two-point boundary value problems with singular differential equations of variable order ([9]) and for impulsive integro-differential equations with Riemann–Liouville boundary conditions ([18]).

While several research studies have been performed on investigating the existence of solutions to fractional constant-order problems, the existence of solutions to variable-order problems is rarely discussed in the literature and there have been very few studies on the stability of solutions; we refer to [9,11,19–23]. Therefore, investigating this interesting special research topic makes our results novel and worthy.

In addition, all results in this work show a great potential to be applied in various applications of multidisciplinary sciences. Further investigations on this open research problem could be also possible with the help of our original results in this research paper. In other words, in the future one could extend the proposed BVP to other complicated real mathematical fractional models by terms of newly-introduced operators with non-singular kernels.

Recently, the Hadamard-type operators originally introduced in [24] and later generalized to variable fractional order have been investigated in [25,26]. Some existence and Ulam stability properties were studied in [27]. Bai et al. [28] studied the following initial value problem for a Caputo–Hadamard differential equation of constant order $0 < u \leq 1$

$$\begin{cases} {}^{c}D_{0^+}^{u}x(t) = f(t, x(t), I_{0^+}^{u}x(t)), \ \ t \in [a, b], \ \ u \in]0, 1], \\ x(a) = x_a, \end{cases}$$

where $0 < a < b < \infty$ and ${}^{c}D_{0^+}^{u}$, $I_{0^+}^{u}$ stand for the Caputo–Hadamard derivative and Hadamard integral operators of constant order u, respectively, and f is a given continuous function, $x_a \in \mathbb{R}$.

In this paper we will study the boundary value problem (BVP) for the Hadamard fractional differential equation of variable order (VOHFDE)

$$\begin{cases} {}^{H}D_{1^+}^{u(t)}x(t) = f(t, x(t), {}^{H}I_{1^+}^{u(t)}x(t)), \ t \in J, \\ x(1) = x(T) = 0, \end{cases} \tag{1}$$

where $J = [1, T]$, $1 < T < \infty$, $u(t) : J \to (1, 2]$ is the variable order of the fractional derivatives, $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function and the left Hadamard fractional integral (HFI) of variable-order $u(t)$ for function $x(t)$ is (see, for example, [25,26])

$${}^{H}I_{1^+}^{u(t)}x(t) = \frac{1}{\Gamma(u(t))} \int_1^t (\ln \frac{t}{s})^{u(t)-1} \frac{x(s)}{s} ds, \ \ t \in J, \tag{2}$$

where $\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}dx$ is the Gamma function and the left Hadamard fractional derivative (HFD) of variable-order $u(t)$ for function $x(t)$ is (see, for example, [25,26])

$${}^{H}D_{1^+}^{u(t)}x(t) = \frac{t^2}{\Gamma(2 - u(t))} \frac{d^2}{dt^2} \int_1^t (\ln \frac{t}{s})^{1-u(t)} \frac{x(s)}{s} ds, \ \ t \in J. \tag{3}$$

The main purpose of this paper is the study of some qualitative properties of the solutions of BVPs for the VOHFDE (1). Initially some preliminary results about HFIs of variable order are provided. Further, based on the Kuratowski measure of noncompactness technique, an appropriate partition of the considered interval J is applied. The studied BVP for VOHFDE (1) is divided into appropriate BVPs for Hadamard fractional differential equations of constant orders. The existence result for (1) is proved.

The relevance between the concepts of stability and symmetry in differential equation is an old question studied by many authors for various types of differential equations such as for irreversible thermodynamics in [29], time-reversal symmetry in nonequilibrium statistical mechanics ([30]). The interplay of symmetries and stability in the analysis and control of nonlinear dynamical systems and networks is studied in [31]. Consequently, this question arises in differential equations with fractional derivatives as a topical problem. It is known that the stability of solutions is one of the most important properties of functional differential equations. In this paper we study the stability in the sense of Ulam–Hyers–Rassias of (1). An example is provided to illustrate the theoretical results.

2. Preliminaries

Denote by $C(J, \mathbb{R})$ the Banach space of continuous functions $\varkappa : J \to \mathbb{R}$ with the norm $\|\varkappa\| = sup\{|\varkappa(t)| : t \in J\}$.

For any number $\delta \in [0, 1]$ we define the set

$$C_\delta(J, \mathbb{R}) = \{h : h(.) \in C(J, \mathbb{R}), \ (\ln(.))^\delta h(.) \in C(J, \mathbb{R})\}.$$

Remark 1. *In (2) the variable order $u : J \to (1, 2]$ but the HFI could be defined for any $u : J \to (0, \infty)$.*

Remark 2. *In the case of a constant order u in Equations (2) and (3), the HFI and HFD coincide with the standard Hadamard derivative and standard Hadamard integral, respectively (see [2,25,26]).*

Remark 3. *Note that the semigroup property is satisfied for a standard Hadamard integral with constant orders but it is not fulfilled for the general case of variable orders* $u(t)$, $v(t)$, *i.e.,* ${}^{H}I_{1^+}^{u(t)}({}^{H}I_{1^+}^{v(t)})x(t) \neq {}^{H}I_{1^+}^{u(t)+v(t)}x(t)$.

Example 1. *Let* $J = [1,2]$ *and the function* $x(t) \equiv 1$ *for* $t \in J$*. Consider the following functions as orders of HFI:* $v(t) \equiv 2$ *and* $u(t) = t$ *for* $t \in J$*.*

Then for any $t \in J$ *we obtain for the HFI defined by (2)*

$$ {}^{H}I_{1^+}^{v(t)}h(t) = \frac{1}{\Gamma(1)}\int_1^t (\ln\frac{t}{s})^{1-1}\frac{1}{s}ds = \ln\, t, $$

$$ {}^{H}I_{1^+}^{u(t)}\left({}^{H}I_{1^+}^{v(t)}x(t)\right) = \frac{1}{\Gamma(t)}\int_1^t (\ln\frac{t}{s})^{t-1}\frac{\ln\, s}{s}ds, $$

and

$$ {}^{H}I_{1^+}^{u(t)+v(t)}x(t) = \frac{1}{\Gamma(t+1)}\int_1^t (\ln\frac{t}{s})^{t}\frac{1}{s}ds. $$

For $t = 1.5$ *we obtain*

$$ {}^{H}I_{1^+}^{u(t)}\left({}^{H}I_{1^+}^{v(t)}x(t)\right)|_{t=1.5} \simeq 0.027916 $$

and

$$ {}^{H}I_{1^+}^{u(t)+v(t)}x(t)|_{t=1.5} \simeq 0.0418739 $$

Therefore,the semigroup property is not satisfied for the general case of HFIs of variable orders.

Recall the following pivotal observation.

Lemma 1. *([2]) Let* $\alpha_1,\ \alpha_2 > 0$, $a_1 > 1$, $h \in L^1(a_1,a_2)$, *and* ${}^{H}D_{a_1^+}^{\alpha_1}h \in L^1(a_1,a_2)$*. Then, the differential equation*

$$ {}^{H}D_{a_1^+}^{\alpha_1}h = 0 $$

has a solution

$$ h(t) = \omega_1(\ln\frac{t}{a_1})^{\alpha_1-1} + \omega_2(\ln\frac{t}{a_1})^{\alpha_1-2} + ... + \omega_n(\ln\frac{t}{a_1})^{\alpha_1-n}, $$

and

$$ {}^{H}I_{a_1^+}^{\alpha_1}({}^{H}D_{a_1^+}^{\alpha_1})h(t) = h(t) + \omega_1(\ln\frac{t}{a_1})^{\alpha_1-1} + \omega_2(\ln\frac{t}{a_1})^{\alpha_1-2} + ... + \omega_n(\ln\frac{t}{a_1})^{\alpha_1-n}, $$

with $n-1 < \alpha_1 \leq n$, $\omega_\ell \in \mathbb{R}$, $\ell = 1,2,...,n$*.*

Furthermore,

$$ {}^{H}D_{a_1^+}^{\alpha_1}({}^{H}I_{a_1^+}^{\alpha_1})h(t) = h(t), $$

and

$$ {}^{H}I_{a_1^+}^{\alpha_1}({}^{H}I_{a_1^+}^{\alpha_2})h(t) = {}^{H}I_{a_1^+}^{\alpha_2}({}^{H}I_{a_1^+}^{\alpha_1})h(t) = {}^{H}I_{a_1^+}^{\alpha_1+\alpha_2}h(t). $$

We will give some results about HFI defined by (2) which will be used later.

Lemma 2. *If* $u \in C(J,(1,2])$ *and there exists a number* $\delta \in [0,1]$ *such that* $h \in C_\delta(J,\mathbb{R})$*, then the variable-order fractional integral* ${}^{H}I_{1^+}^{u(t)}h(t)$ *exists for* $t \in J$*.*

Proof. The function $\Gamma(u(t))$ is continuous non-zero function on J. Denote $M_u = \max_{t\in J} |\frac{1}{\Gamma(u(t))}| > 0$ and $u^* = \max_{t\in J} |(u(t))|$. Thus, for $1 \le s \le t \le T$, we have

$$(\ln \frac{t}{s})^{u(t)-1} \le 1, \quad if \;\; 1 \le \frac{t}{s} \le e,$$

$$(\ln \frac{t}{s})^{u(t)-1} \le (\ln \frac{t}{s})^{u^*-1}, \; if \;\; \frac{t}{s} > e.$$

Then, for $1 \le \frac{t}{s} < +\infty$, we obtain

$$(\ln \frac{t}{s})^{u(t)-1} \le \max\{1, (\ln \frac{t}{s})^{u^*-1}\} = M^*.$$

For $t \in J$ by (2), applying that the function $(\ln(.))^{\delta}$ is an increasing function on J for any $\delta \in [0,1]$ we obtain

$$\begin{aligned} |{}^H I_{1^+}^{u(t)} h(t)| &= \frac{1}{\Gamma(u(t))} \int_1^t (\ln \frac{t}{s})^{u(t)-1} \frac{|h(s)|}{s} ds \\ &\le M_u \int_1^t (\ln \frac{t}{s})^{u(t)-1} (\ln\ s)^{-\delta} (\ln\ s)^{\delta} \frac{|h(s)|}{s} ds \\ &\le M_u M^* (\ln\ T)^{\delta} h^{\star} \int_1^t \frac{1}{s} (\ln s)^{-\delta} ds \le M_u M^* h^{\star} \frac{(\ln T)}{1-\delta} < \infty. \end{aligned}$$

where $h^{\star} = \max_{t\in J} |h(t)|$. It yields that the variable-order fractional integral ${}^H I_{1^+}^{u(t)} h(t)$ exists for any $t \in J$. $\square$

Lemma 3. *Let $u \in C(J, (1,2])$, then ${}^H I_{1^+}^{u(t)} h \in C(J, \mathbb{R})$ for $h \in C(J, \mathbb{R})$.*

Proof. For $t, t_0 \in (1, T]$, $t_0 \le t$ and $h \in C(J, \mathbb{R})$, using (2) and the substitutions $s = r(t-1)+1, ds = (t-1)dr$ and $s = r(t_0-1)+1, ds = (t_0-1)dr$, respectively, we obtain

$$\begin{aligned} &\Big| {}^H I_{1^+}^{u(t)} h(t) - {}^H I_{1^+}^{u(t_0)} h(t_0) \Big| \\ &= \Big| \int_0^1 \frac{1}{\Gamma(u(t))} \frac{(t-1)}{r(t-1)+1} (\ln \frac{t}{r(t-1)+1})^{u(t)-1} h(r(t-1)+1) dr \\ &\quad - \int_0^1 \frac{1}{\Gamma(u(t_0))} \frac{(t_0-1)}{r(t_0-1)+1} (\ln \frac{t_0}{r(t_0-1)+1})^{u(t_0)-1} h(r(t_0-1)+1) dr \Big|. \end{aligned} \tag{4}$$

Denote $h^{\star} = \max_{t\in J} |h(t)|$ and obtain the following upper bounds of the absolute values

$$\begin{aligned} &\Big| \int_0^1 \Big[\frac{1}{\Gamma(u(t))} \frac{(t-1)}{r(t-1)+1} (\ln \frac{t}{r(t-1)+1})^{u(t)-1} h(r(t-1)+1) \\ &\quad - \frac{1}{\Gamma(u(t))} \frac{(t_0-1)}{r(t_0-1)+1} (\ln \frac{t}{r(t-1)+1})^{u(t)-1} h(r(t-1)+1) \Big] dr \Big| \\ &\le h^{\star} \int_0^1 \frac{1}{\Gamma(u(t))} (\ln \frac{t}{r(t-1)+1})^{u(t)-1} \Big| \frac{(t-1)}{r(t-1)+1} - \frac{(t_0-1)}{r(t_0-1)+1} \Big| dr \end{aligned} \tag{5}$$

$$\begin{aligned} &\Big| \int_0^1 \Big[\frac{1}{\Gamma(u(t))} \frac{(t_0-1)}{r(t_0-1)+1} (\ln \frac{t_0}{r(t_0-1)+1})^{u(t_0)-1} h(r(t-1)+1) \\ &\quad - \frac{1}{\Gamma(u(t_0))} \frac{(t_0-1)}{r(t_0-1)+1} (\ln \frac{t_0}{r(t_0-1)+1})^{u(t_0)-1} h(r(t-1)+1) \Big] dr \Big| \\ &\le h^{\star} \int_0^1 \frac{(t_0-1)}{r(t_0-1)+1} (\ln \frac{t_0}{r(t_0-1)+1})^{u(t_0)-1} \Big| \frac{1}{\Gamma(u(t))} - \frac{1}{\Gamma(u(t_0))} \Big| dr, \end{aligned} \tag{6}$$

$$\begin{aligned}
&\Big|\int_0^1 \Big[\frac{1}{\Gamma(u(t_0))}\frac{(t_0-1)}{r(t_0-1)+1}(\ln\frac{t_0}{r(t_0-1)+1})^{u(t_0)-1}h(r(t-1)+1)\\
&-\frac{1}{\Gamma(u(t_0))}\frac{(t_0-1)}{r(t_0-1)+1}(\ln\frac{t_0}{r(t_0-1)+1})^{u(t_0)-1}h(r(t_0-1)+1)\Big]dr\Big|\\
&\leq \int_0^1 \frac{1}{\Gamma(u(t_0))}\frac{(t_0-1)}{r(t_0-1)+1}(\ln\frac{t_0}{r(t_0-1)+1})^{u(t_0)-1}\Big|h(r(t-1)+1)\\
&\quad -h(r(t_0-1)+1)\Big|dr,
\end{aligned} \tag{7}$$

From (4)–(7) taking into account the continuity of functions $\frac{(t-1)}{r(t-1)+1}$, $(\ln\frac{t}{r(t-1)+1})^{u(t)-1}$, $\frac{1}{\Gamma(u(t))}$, $h(t)$ on J, we obtain that the integral ${}^{H}I_{1^+}^{u(t)}h(t)$ is continuous at point t_0, and therefore, ${}^{H}I_{1^+}^{u(t)}h \in C(J,\mathbb{R})$ for $h \in C(J,\mathbb{R})$. □

Definition 1. *([5,10,11]) Let the set $I \subset \mathbb{R}$.*

- *The set I is called a generalized interval if it is either an interval or a point or the empty set.*
- *The finite set $\mathcal{P}$ of generalized intervals is called a partition of I if each x in I lies in exactly one of the generalized intervals E in $\mathcal{P}$.*
- *The function $g : I \to \mathbb{R}$ is called a piecewise constant with respect to partition $\mathcal{P}$ of I if for any $E \in \mathcal{P}$, g is constant on E.*

In addition, we will provide some necessary background information about the Kuratowski measure of noncompactness, defined in [32]). These results will be used later in the proof of the main result.

Proposition 1. *([32,33]). Let X be a Banach space and $D,\ D_1,\ D_2$ are bounded subsets of X, then*

1. $\zeta(D) = 0 \Longleftrightarrow D$ *is relatively compact* .
2. $\zeta(\phi) = 0$.
3. $\zeta(D) = \zeta(\overline{D}) = \zeta(convD)$.
4. $D_1 \subset D_2 \Longrightarrow \zeta(D_1) \leq \zeta(D_2)$.
5. $\zeta(D_1 + D_2) \leq \zeta(D_1) + \zeta(D_2)$.
6. $\zeta(\lambda D) = |\lambda|\zeta(D), \lambda \in \mathbb{R}$.
7. $\zeta(D_1 \cup D_2) = Max\{\zeta(D_1), \zeta(D_2)\}$.
8. $\zeta(D_1 \cap D_2) = Min\{\zeta(D_1), \zeta(D_2)\}$.
9. $\zeta(D + x_0) = \zeta(D)$ *for any* $x_0 \in X$.

Lemma 4. *([34]) If $U \subset C(J,\mathbb{R})$ is an equicontinuous and bounded set, then*

(i) the function $\zeta(U(t))$ is continuous for $t \in J$, and $\widehat{\zeta}(U) = \sup_{t\in J}\zeta(U(t))$.

(ii) $\zeta\Big(\int_0^T x(\theta)d\theta : x \in U\Big) \leq \int_0^T \zeta(U(\theta))d\theta$,

where for any $s \in J$ it is defined $U(s) = \{x(s) :\ x \in U\}$.

Theorem 1. *([32]) Let Λ be a nonempty, closed, bounded and convex subset of a Banach space X and $\digamma : \Lambda \longrightarrow \Lambda$ is a continuous operator satisfying*

$$\zeta(\digamma(S)) \leq k\zeta(S),\ for\ any\ (S \neq \emptyset) \subset \Lambda,\ k \in [0,1),$$

i.e., $\digamma$ denotes $k -$ set contractions.

Then, $\digamma$ has at least one fixed point in Λ.

Following the ideas in [27] we will give a definition for Hyers–Ulam–Rassias stability of the BVP for VOHFDE (1) .

Definition 2. *Let the function $\vartheta \in C(J, \mathbb{R}_+)$. The BVP for VOHFDE (1) is Hyers–Ulam–Rassias stable with respect to ϑ (UHR) if there exists a constant $c_f > 0$, such that for any $\epsilon > 0$ and for every $z \in C(J, \mathbb{R})$ such that*

$$|{}^H D_{1^+}^{u(t)} z(t) - f(t, z(t), I_{1^+}^{u(t)} z(t))| \le \epsilon \vartheta(t), \ t \in J, \tag{8}$$

there exists a solution $x \in C(J, \mathbb{R})$ of the BVP for VOHFDE (1) with

$$|z(t) - x(t)| \le c_f \epsilon \vartheta(t), \ t \in J.$$

3. Existence of Solutions of BVP for VOHFDE

Let us introduce the following assumption:

Hypothesis 1 (H1). *Let $n \in \mathbb{N}$ be an integer and the finite sequence of points $\{T_k\}_{k=0}^n$ be given such that $1 = T_0 < T_k < T_n = T$, $k = 1, \dots, n-1$. Denote $J_k := (T_{k-1}, T_k]$, $k = 1, 2, \dots, n$. Then $\mathcal{P} = \cup_{k=1}^n J_k$ is a partition of the interval J.*

For each $l = 1, 2, \dots, n$, the symbol $E_l = C(J_l, \mathbb{R})$, indicates the Banach space of continuous functions $x : J_l \to \mathbb{R}$ equipped with the norm $\|x\|_{E_l} = \sup_{t \in J_l} |x(t)|$.

Let $u(t) : J \to (1, 2]$ be a piecewise constant function with respect to $\mathcal{P}$, i.e., $u(t) = \sum_{l=1}^n u_l I_l(t)$, where $1 < u_l \le 2$ are constants and I_l is the indicator of the interval J_l, $l = 1, 2, \dots, n$:

$$I_l(t) = \begin{cases} 1, & for\ t \in J_l, \\ 0, & elsewhere. \end{cases}$$

Then, for any $t \in J_l$, $l = 1, 2, \dots, n$, the left Hadamard fractional derivative of variable-order $u(t)$ for function $x(t) \in C(J, \mathbb{R})$, defined by (3), could be presented as a sum of left Hadamard fractional derivatives of constant orders u_k, $k = 1, 2, \dots, l$

$$\begin{aligned} {}^H D_{1^+}^{u(t)} x(t) &= \frac{t^2}{\Gamma(2 - u(t))} \frac{d^2}{dt^2} \int_1^t (\ln \frac{t}{s})^{1-u(t)} \frac{x(s)}{s} ds \\ &= \frac{t^2}{\Gamma(2 - u(t))} \Big(\sum_{k=1}^{l-1} \frac{d^2}{dt^2} \int_{T_{k-1}}^{T_k} (\ln \frac{t}{s})^{1-u_k} \frac{x(s)}{s} ds \\ &\quad + \frac{d^2}{dt^2} \int_{T_{l-1}}^t (\ln \frac{t}{s})^{1-u_\ell} \frac{x(s)}{s} ds \Big). \end{aligned} \tag{9}$$

Thus, the BVP for VOHFDE (1) can be written for any $t \in J_l$, $l = 1, 2, \dots, n$ in the form

$$\begin{aligned} &\frac{t^2}{\Gamma(2 - u(t))} \Big(\sum_{k=1}^{l-1} \frac{d^2}{dt^2} \int_{T_{k-1}}^{T_k} (\ln \frac{t}{s})^{1-u_k} \frac{x(s)}{s} ds + \frac{d^2}{dt^2} \int_{T_{l-1}}^t (\ln \frac{t}{s})^{1-u_\ell} \frac{x(s)}{s} ds \Big) \\ &= f(t, x(t), {}^H I_{1^+}^{u(t)} x(t)). \end{aligned} \tag{10}$$

Let the function $\tilde{x} \in C(J_\ell, \mathbb{R})$ be such that $\tilde{x}(t) \equiv 0$ on $t \in [1, T_{\ell-1}]$ and it solves integral Equation (10). Then (10) is reduced to

$${}^H D_{T_{\ell-1}^+}^{u_\ell} \tilde{x}(t) = f(t, \tilde{x}(t), {}^H I_{T_{\ell-1}^+}^{u_\ell} \tilde{x}(t)), \ \ t \in J_\ell.$$

Taking into account the above for any $\ell = 1, 2, \dots, n$, we consider the following auxiliary BVP for Hadamard fractional differential equations of constant order

$$\begin{cases} {}^H D_{T_{\ell-1}^+}^{u_\ell} y(t) = f(t, y(t), {}^H I_{T_{\ell-1}^+}^{u_\ell} y(t)), & t \in J_\ell \\ y(T_{\ell-1}) = 0, \ \ y(T_\ell) = 0. \end{cases} \tag{11}$$

Lemma 5. *Let $\ell \in \{1,2\dots,n\}$ be a natural number , $f \in C(J_\ell \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and there exists a number $\delta \in (0,1)$ such that $(\ln t)^\delta f \in C(J_\ell \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$.*

Then the function $y_\ell \in E_\ell$ is a solution of (11) if and only if y_ℓ solves the integral equation

$$\begin{aligned} y(t) \quad &= \quad -(\ln \frac{T_\ell}{T_{\ell-1}})^{1-u_\ell}(\ln \frac{t}{T_{\ell-1}})^{u_\ell-1}\Big({}^H I^{u_\ell}_{T^+_{\ell-1}} f(t, y(t), {}^H I^{u_\ell}_{T^+_{\ell-1}} y(t))\Big)|_{t=T_\ell} \\ &\quad + {}^H I^{u_\ell}_{T^+_{\ell-1}} f(t, y(t), {}^H I^{u_\ell}_{T^+_{\ell-1}} y(t)). \end{aligned} \tag{12}$$

Proof. Let $y_\ell \in E_\ell$ be a solution of BVP (11). Employing the operator ${}^H I^{u_\ell}_{T^+_{\ell-1}}$ to both sides of (11), we find (see Lemma 1)

$$y_\ell(t) = \omega_1(\ln \frac{t}{T_{\ell-1}})^{u_\ell-1} + \omega_2(\ln \frac{t}{T_{\ell-1}})^{u_\ell-2} + {}^H I^{u_\ell}_{T^+_{\ell-1}} f(t, y_\ell(t), {}^H I^{u_\ell}_{T^+_{\ell-1}} y_\ell(t)), \quad t \in J_\ell,$$

where ω_1, ω_2 are constants.

From the conditions of function f and the boundary condition $y(T_{\ell-1}) = 0$, we conclude that $\omega_2 = 0$.

From the boundary condition $y(T_\ell) = 0$ we obtain

$$\omega_1 = -(\ln \frac{T_\ell}{T_{\ell-1}})^{1-u_\ell}\, {}^H I^{u_\ell}_{T^+_{\ell-1}} f(T_\ell, x(T_\ell), {}^H I^{u_\ell}_{T^+_{\ell-1}} x(T_\ell))$$

Then, we find y_ℓ solves integral Equation (12).

Conversely, let $y_\ell \in E_\ell$ be a solution of integral Equation (12). Regarding the continuity of function $(\ln t)^\delta f$, we deduce that y_ℓ is the solution of BVP (11). □

We will prove the existence result for the BVP for Hadamard fractional differential equations of constant order (11). This result is based on Theorem 1.

Theorem 2. *Let the conditions of Lemma 5 be satisfied and there exist constants $K, L > 0$ such that*

$$(\ln t)^\delta |f(t, x_1, y_1) - f(t, x_2, y_2)| \le K|x_1 - x_2| + L|y_1 - y_2|, \quad x_i, y_i, \in \mathbb{R}, \ i = 1,2, \ t \in J_\ell$$

and the inequality

$$\mu < \frac{1}{2}, \tag{13}$$

holds, where

$$\mu = \frac{(\ln T_\ell)^{1-\delta} - (\ln T_{\ell-1})^{1-\delta}}{(1-\delta)\Gamma(u_\ell)}(\ln \frac{T_\ell}{T_{\ell-1}})^{u_\ell-1}\Big(K + \frac{L}{\Gamma(u_\ell+1)}(\ln \frac{T_\ell}{T_{\ell-1}})^{u_\ell}\Big).$$

Then, the BVP for Hadamard fractional differential equations of constant order (11) possesses at least one solution in E_ℓ.

Remark 4. *According to the remark on page 20 of [35], we can easily show that the inequality (13) and the following inequality*

$$\zeta((\ln t)^\delta f(t, B_1, B_2)) \le K\zeta(B_1) + L\zeta(B_2)$$

are equivalent for any bounded sets $B_1, B_2 \subset \mathbb{R}$ and for each $t \in J_\ell$.

Proof. For any function $x \in E_\ell$ we define the operator

$$\begin{aligned}
&Wx(t)\\
&= -\frac{1}{\Gamma(u_\ell)}(\ln\frac{T_\ell}{T_{\ell-1}})^{1-u_\ell}(\ln\frac{t}{T_{\ell-1}})^{u_\ell-1}\int_{T_{\ell-1}}^{T_\ell}(\ln\frac{T_\ell}{s})^{u_\ell-1}\frac{f(s,x(s),^H I^{u_\ell}_{T^+_{\ell-1}}x(s))}{s}ds \\
&+\frac{1}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t}(\ln\frac{t}{s})^{u_\ell-1}\frac{f(s,x(s),^H I^{u_\ell}_{T^+_{\ell-1}}x(s))}{s}ds.
\end{aligned} \tag{14}$$

It follows from the properties of fractional integrals and from the continuity of function $(\ln t)^\delta f$ that the operator $W : E_\ell \to E_\ell$ defined by (14) is well defined.

Let $R_\ell = \frac{\frac{2f^\star}{\Gamma(u_\ell+1)}(\ln\frac{T_\ell}{T_{\ell-1}})^{u_\ell}}{1-2\mu}$ with $f^\star = \sup_{t\in J_\ell}|f(t,0,0)|$.
Consider the set

$$B_{R_\ell} = \{x \in E_\ell,\ \|x\|_{E_\ell} \le R_\ell\}.$$

For all $\ell \in \{1,2,...,n\}$ the ball B_{R_ℓ} is a nonempty, bounded, closed convex subset of E_ℓ.

Now, we check the assumption of the Theorem 1 for the operator W. We shall prove it in four steps.

Step 1: Claim: $W(B_{R_\ell}) \subseteq B_{R_\ell}$.
For $x \in B_{R_\ell}$ and by the conditions of function f, we obtain

$$\begin{aligned}
&|Wx(t)|\\
&\le \frac{1}{\Gamma(u_\ell)}(\ln\frac{T_\ell}{T_{\ell-1}})^{1-u_\ell}(\ln\frac{t}{T_{\ell-1}})^{u_\ell-1}\int_{T_{\ell-1}}^{T_\ell}(\ln\frac{T_\ell}{s})^{u_\ell-1}|f(s,x(s),^H I^{u_\ell}_{T^+_{\ell-1}}x(s))|\frac{ds}{s}\\
&\quad+\frac{1}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t}(\ln\frac{t}{s})^{u_\ell-1}|f(s,x(s),^H I^{u_\ell}_{T^+_{\ell-1}}x(s))|\frac{ds}{s}\\
&\le \frac{2}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{T_\ell}(\ln\frac{T_\ell}{s})^{u_\ell-1}|f(s,x(s),^H I^{u_\ell}_{T^+_{\ell-1}}x(s))|\frac{ds}{s}\\
&\le \frac{2}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{T_\ell}(\ln\frac{T_\ell}{s})^{u_\ell-1}|f(s,x(s),^H I^{u_\ell}_{T^+_{\ell-1}}x(s))-f(s,0,0)|\frac{ds}{s}\\
&\quad+\frac{2}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{T_\ell}(\ln\frac{T_\ell}{s})^{u_\ell-1}|f(s,0,0)|\frac{ds}{s}\\
&\le \frac{2}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{T_\ell}(\ln\frac{T_\ell}{s})^{u_\ell-1}(\ln s)^{-\delta}\Big(K|x(s)|+L|^H I^{u_\ell}_{T^+_{\ell-1}}x(s)|\Big)\frac{ds}{s}\\
&\quad+\frac{2f^\star}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{T_\ell}(\ln\frac{T_\ell}{s})^{u_\ell-1}\frac{ds}{s}\\
&\le \frac{2}{\Gamma(u_\ell)}(\ln\frac{T_\ell}{T_{\ell-1}})^{u_\ell-1}\Big(K\|x\|_{E_\ell}+L\|^H I^{u_\ell}_{T^+_{\ell-1}}x\|_{E_\ell}\Big)\int_{T_{\ell-1}}^{T_\ell}(\ln s)^{-\delta}\frac{ds}{s}\\
&\quad+\frac{2f^\star}{\Gamma(u_\ell+1)}(\ln\frac{T_\ell}{T_{\ell-1}})^{u_\ell}\\
&\le \frac{2}{\Gamma(u_\ell)}(\ln\frac{T_\ell}{T_{\ell-1}})^{u_\ell-1}\Big(K+\frac{L}{\Gamma(u_\ell+1)}(\ln\frac{T_\ell}{T_{\ell-1}})^{u_\ell}\Big)\|x\|_{E_\ell}\int_{T_{\ell-1}}^{T_\ell}(\ln s)^{-\delta}\frac{ds}{s}\\
&\quad+\frac{2f^\star}{\Gamma(u_\ell+1)}(\ln\frac{T_\ell}{T_{\ell-1}})^{u_\ell}\\
&\le 2\mu R_\ell+\frac{2f^\star}{\Gamma(u_\ell+1)}(\ln\frac{T_\ell}{T_{\ell-1}})^{u_\ell}\\
&= R_\ell,
\end{aligned}$$

which means that $W(B_{R_\ell}) \subseteq B_{R_\ell}$.

*Step 2:*Claim: W is continuous.

Let $x_k \in E_\ell,\ k = 1, 2, \ldots$. Assume the sequence $\{x_k\}_{k=1}^{\infty}$ is convergent to $x \in E_\ell$. Then for any $k = 1, 2, \ldots$ we have

$$\begin{aligned}
& |Wx_k(t) - Wx(t)| \\
\leq\ & \tfrac{1}{\Gamma(u_\ell)}(\ln \tfrac{T_\ell}{T_{\ell-1}})^{1-u_\ell}(\ln \tfrac{t}{T_{\ell-1}})^{u_\ell-1}\textstyle\int_{T_{\ell-1}}^{T_\ell}(\ln \tfrac{T_\ell}{s})^{u_\ell-1} \\
& |f(s, x_k(s), {}^H I^{u_\ell}_{T^+_{\ell-1}} x_k(s)) - f(s, x(s), {}^H I^{u_\ell}_{T^+_{\ell-1}} x(s))|\tfrac{ds}{s} \\
+\ & \tfrac{1}{\Gamma(u_\ell)}\textstyle\int_{T_{\ell-1}}^{t}(\ln \tfrac{t}{s})^{u_\ell-1}|f(s, x_k(s), {}^H I^{u_\ell}_{T^+_{\ell-1}} x_k(s)) - f(s, x(s), {}^H I^{u_\ell}_{T^+_{\ell-1}} x(s))|\tfrac{ds}{s} \\
\leq\ & \tfrac{2}{\Gamma(u_\ell)}(\ln \tfrac{t}{T_{\ell-1}})^{u_\ell-1}\textstyle\int_{T_{\ell-1}}^{T_\ell}|f(s, x_k(s), {}^H I^{u_\ell}_{T^+_{\ell-1}} x_k(s)) - f(s, x(s), {}^H I^{u_\ell}_{T^+_{\ell-1}} x(s))|\tfrac{ds}{s} \\
\leq\ & \tfrac{2}{\Gamma(u_\ell)}(\ln \tfrac{t}{T_{\ell-1}})^{u_\ell-1}\textstyle\int_{T_{\ell-1}}^{T_\ell}(\ln s)^{-\delta}\Big(K|x_k(s) - x(s)| + L\ {}^H I^{u_\ell}_{T^+_{\ell-1}}|x_k(s) - x(s)|\Big)\tfrac{ds}{s} \\
\leq\ & \tfrac{2}{\Gamma(u_\ell)}\Big(K\|x_k - x\|_{E_\ell} + L\ \|{}^H I^{u_\ell}_{T^+_{\ell-1}}(x_k - x)\|_{E_\ell}\Big)(\ln \tfrac{t}{T_{\ell-1}})^{u_\ell-1}\textstyle\int_{T_{\ell-1}}^{T_\ell}(\ln s)^{-\delta}\tfrac{ds}{s} \\
\leq\ & \tfrac{2[(\ln T_\ell)^{1-\delta} - (\ln T_{\ell-1})^{1-\delta}]}{(1-\delta)\Gamma(u_\ell)}(\ln \tfrac{T_\ell}{T_{\ell-1}})^{u_\ell-1}\Big(K + \tfrac{L}{\Gamma(u_\ell+1)}(\ln \tfrac{T_\ell}{T_{\ell-1}})^{u_\ell}\Big)\|x_k - x\|_{E_\ell},
\end{aligned}$$

i.e., we obtain

$$\|Wx_k - Wx\|_{E_\ell} \to 0 \ \ as \ \ k \to \infty.$$

Ergo, the operator W is continuous on E_ℓ.

Step 3: Claim: W is bounded and equicontinuous.

By Step 1 for any $x \in B_{R_\ell}$ we have $\|W(x)\|_{E_\ell} \leq R_\ell$, i.e., $W(B_{R_\ell})$ is bounded. We will prove $W(B_{R_\ell})$ is equicontinuous.

Let $t_1, t_2 \in J_\ell,\ t_1 < t_2$ and $x \in B_{R_\ell}$. Then we have

$$\begin{aligned}
& |Wx(t_2) - Wx(t_1)| \\
& \leq \frac{1}{\Gamma(u_\ell)}(\ln \frac{T_\ell}{T_{\ell-1}})^{1-u_\ell}\Big((\ln \frac{t_2}{T_{\ell-1}})^{u_\ell-1} - (\ln \frac{t_1}{T_{\ell-1}})^{u_\ell-1}\Big) \\
& \times \int_{T_{\ell-1}}^{T_\ell}(\ln \frac{T_\ell}{s})^{u_\ell-1}|f(s, x(s), {}^H I^{u_\ell}_{T^+_{\ell-1}} x(s))|\frac{ds}{s} \\
& + \frac{1}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t_1}\Big((\ln \frac{t_2}{s})^{u_\ell-1} - (\ln \frac{t_1}{s})^{u_\ell-1}\Big)|f(s, x(s), {}^H I^{u_\ell}_{T^+_{\ell-1}} x(s))|\frac{ds}{s} \\
& + \frac{1}{\Gamma(u_\ell)}\int_{t_1}^{t_2}(\ln \frac{t_2}{s})^{u_\ell-1}|f(s, x(s), {}^H I^{u_\ell}_{T^+_{\ell-1}} x(s))|\frac{ds}{s}
\end{aligned}$$

or

$$
\begin{aligned}
&|Wx(t_2) - Wx(t_1)| \\
&\le \frac{1}{\Gamma(u_\ell)}(\ln\frac{T_\ell}{T_{\ell-1}})^{1-u_\ell}\Big((\ln\frac{t_2}{T_{\ell-1}})^{u_\ell-1} - (\ln\frac{t_1}{T_{\ell-1}})^{u_\ell-1}\Big) \\
&\quad\times\int_{T_{\ell-1}}^{T_\ell}(\ln\frac{T_\ell}{s})^{u_\ell-1}(\ln s)^{-\delta}(K|x(s)| + L\,{}^H I^{u_\ell}_{T^+_{\ell-1}}|x(s)|)\frac{ds}{s} \\
&\quad+\frac{f^\star}{\Gamma(u_\ell)}(\ln\frac{T_\ell}{T_{\ell-1}})^{1-u_\ell}\Big((\ln\frac{t_2}{T_{\ell-1}})^{u_\ell-1} - (\ln\frac{t_1}{T_{\ell-1}})^{u_\ell-1}\Big)\int_{T_{\ell-1}}^{T_\ell}(\ln\frac{T_\ell}{s})^{u_\ell-1}\frac{ds}{s} \\
&\quad+\frac{1}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t_1}(\ln\frac{t_2}{t_1})^{u_\ell-1}(\ln s)^{-\delta}(K|x(s)| + L\,{}^H I^{u_\ell}_{T^+_{\ell-1}}|x(s)|)\frac{ds}{s} \\
&\quad+\frac{f^\star}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t_1}(\ln\frac{t_2}{t_1})^{u_\ell-1}\frac{ds}{s} + \frac{f^\star}{\Gamma(u_\ell)}\int_{t_1}^{t_2}(\ln\frac{t_2}{s})^{u_\ell-1}\frac{ds}{s} \\
&\quad+\frac{1}{\Gamma(u_\ell)}\int_{t_1}^{t_2}(\ln\frac{t_2}{s})^{u_\ell-1}(\ln s)^{-\delta}(K|x(s)| + L\,{}^H I^{u_\ell}_{T^+_{\ell-1}}|x(s)|)\frac{ds}{s} \\
&\le \frac{1}{\Gamma(u_\ell)}(\ln\frac{T_\ell}{T_{\ell-1}})^{1-u_\ell}(\ln\frac{T_\ell}{T_{\ell-1}})^{u_\ell-1}\Big((\ln\frac{t_2}{T_{\ell-1}})^{u_\ell-1} - (\ln\frac{t_1}{T_{\ell-1}})^{u_\ell-1}\Big) \\
&\quad\times\Big(K\|x\|_{E_\ell} + L\|{}^H I^{u_\ell}_{T^+_{\ell-1}}x\|_{E_\ell}\Big) \\
&\quad\times\int_{T_{\ell-1}}^{T_\ell}(\ln s)^{-\delta}\frac{ds}{s} + \frac{f^\star}{\Gamma(u_\ell+1)}(\ln\frac{T_\ell}{T_{\ell-1}})^{1-u_\ell}\Big((\ln\frac{t_2}{T_{\ell-1}})^{u_\ell-1} \\
&\quad-(\ln\frac{t_1}{T_{\ell-1}})^{u_\ell-1}\Big)(\ln\frac{T_\ell}{T_{\ell-1}})^{u_\ell} \\
&\quad+\frac{(\ln t_2)^{1-\delta} - (\ln t_1)^{1-\delta}}{(1-\delta)\Gamma(u_\ell)}(\ln\frac{t_2}{t_1})^{u_\ell-1}\Big(K\|x\|_{E_\ell} + L\|{}^H I^{u_\ell}_{T^+_{\ell-1}}x\|_{E_\ell}\Big) \\
&\quad+\frac{f^\star}{\Gamma(u_\ell)}(\ln\frac{t_2}{t_1})^{u_\ell-1}(\ln t_1 - \ln T_{\ell-1}) \\
&\quad+\frac{(\ln t_2)^{1-\delta} - (\ln t_1)^{1-\delta}}{(1-\delta)\Gamma(u_\ell)}(\ln\frac{t_2}{t_1})^{u_\ell-1}\Big(K\|x\|_{E_\ell} + L\|{}^H I^{u_\ell}_{T^+_{\ell-1}}x\|_{E_\ell}\Big) \\
&\quad+\frac{f^\star}{\Gamma(u_\ell+1)}(\ln\frac{t_2}{t_1})^{u_\ell}.
\end{aligned}
$$

Therefore,we obtain

$$
\begin{aligned}
&|Wx(t_2) - Wx(t_1)| \\
&\le \Big[\frac{(\ln T_\ell)^{1-\delta} - (\ln T_{\ell-1})^{1-\delta}}{(1-\delta)\Gamma(u_\ell)}\Big(K + \frac{L}{\Gamma(u_\ell+1)}(\ln\frac{T_\ell}{T_{\ell-1}})^{u_\ell}\Big)\|x\|_{E_\ell} + \frac{f^\star}{\Gamma(u_\ell+1)}(\ln\frac{T_\ell}{T_{\ell-1}})\Big] \\
&\times\Big((\ln\frac{t_2}{T_{\ell-1}})^{u_\ell-1} - (\ln\frac{t_1}{T_{\ell-1}})^{u_\ell-1}\Big) \\
&\;+\Big[\frac{2((\ln t_2)^{1-\delta} - (\ln t_1)^{1-\delta})}{(1-\delta)\Gamma(u_\ell)}\Big(K + \frac{L}{\Gamma(u_\ell+1)}(\ln\frac{T_\ell}{T_{\ell-1}})^{u_\ell}\Big)\|x\|_{E_\ell} \\
&\;+\frac{f^\star}{\Gamma(u_\ell)}(\ln t_1 - \ln T_{\ell-1})\Big](\ln\frac{t_2}{t_1})^{u_\ell-1} + \frac{f^\star}{\Gamma(u_\ell+1)}(\ln\frac{t_2}{t_1})^{u_\ell}.
\end{aligned}
\tag{15}
$$

Hence $|Wx(t_2) - Wx(t_1)| \to 0$ as $|t_2 - t_1| \to 0$. This implies that $W(B_{R_\ell})$ is equicontinuous.

*Step 4:*Claim: W is k-set contraction.

For $U \in B_{R_\ell}$, $t \in J_\ell$, we obtain,

$$
\begin{aligned}
\zeta(W(U)(t)) &= \zeta((Wy)(t), y \in U) \\
&\le \Big\{\tfrac{1}{\Gamma(u_\ell)}(\ln\tfrac{T_\ell}{T_{\ell-1}})^{1-u_\ell}(\ln\tfrac{t}{T_{\ell-1}})^{u_\ell-1}\int_{T_{\ell-1}}^{T_\ell}(\ln\tfrac{T_\ell}{s})^{u_\ell-1}\zeta f(s, x(s), {}^H I^{u_\ell}_{T^+_{\ell-1}}x(s))ds \\
&\quad+\tfrac{1}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t}(\ln\tfrac{t}{s})^{u_\ell-1}\zeta f(s, x(s), {}^H I^{u_\ell}_{T^+_{\ell-1}}x(s))ds,\ y \in U\Big\}.
\end{aligned}
$$

Then Remark 4 implies that, for each $s \in J_\ell$,

$$\begin{aligned}
\zeta(W(U)(t)) \leq & \Big\{\tfrac{1}{\Gamma(u_\ell)}(\ln \tfrac{T_\ell}{T_{\ell-1}})^{1-u_\ell}(\ln \tfrac{t}{T_{\ell-1}})^{u_\ell-1}\int_{T_{\ell-1}}^{T_\ell}(\ln \tfrac{T_\ell}{s})^{u_\ell-1} \\
& \times\Big[K\widehat{\zeta}(U)s^{-\delta}+\tfrac{L}{\Gamma(u_\ell+1)}(\ln \tfrac{T_\ell}{T_{\ell-1}})^{u_\ell}\,\widehat{\zeta}(U)s^{-\delta}\Big]ds \\
& +\tfrac{1}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t}(\ln \tfrac{t}{s})^{u_\ell-1}\Big[K\widehat{\zeta}(U)s^{-\delta}+\tfrac{L}{\Gamma(u_\ell+1)}(\ln \tfrac{T_\ell}{T_{\ell-1}})^{u_\ell}\,\widehat{\zeta}(U)s^{-\delta}\Big]ds,\ y\in U\Big\}. \\
\leq & \Big\{\tfrac{1}{\Gamma(u_\ell)}(\ln \tfrac{T_\ell}{T_{\ell-1}})^{u_\ell-1}\int_{T_{\ell-1}}^{T_\ell}\Big[K\widehat{\zeta}(U)s^{-\delta}+\tfrac{L}{\Gamma(u_\ell+1)}(\ln \tfrac{T_\ell}{T_{\ell-1}})^{u_\ell}\,\widehat{\zeta}(U)s^{-\delta}\Big]ds \\
& +\tfrac{1}{\Gamma(u_\ell)}(\ln \tfrac{T_\ell}{T_{\ell-1}})^{u_\ell-1}\int_{T_{\ell-1}}^{t}\Big[K\widehat{\zeta}(U)s^{-\delta}+\tfrac{L}{\Gamma(u_\ell+1)}(\ln \tfrac{T_\ell}{T_{\ell-1}})^{u_\ell}\,\widehat{\zeta}(U)s^{-\delta}\Big]ds,\ y\in U\Big\}. \\
\leq & \tfrac{(\ln T_\ell)^{1-\delta}-(\ln T_{\ell-1})^{1-\delta}}{(1-\delta)\Gamma(u_\ell)}(\ln \tfrac{T_\ell}{T_{\ell-1}})^{u_\ell-1}\Big[K+\tfrac{L}{\Gamma(u_\ell+1)}(\ln \tfrac{T_\ell}{T_{\ell-1}})^{u_\ell}\Big]\widehat{\zeta}(U) \\
& +\tfrac{(\ln t)^{1-\delta}-(\ln T_{\ell-1})^{1-\delta}}{(1-\delta)\Gamma(u_\ell)}(\ln \tfrac{T_\ell}{T_{\ell-1}})^{u_\ell-1}\Big[K+\tfrac{L}{\Gamma(u_\ell+1)}(\ln \tfrac{T_\ell}{T_{\ell-1}})^{u_\ell}\Big]\widehat{\zeta}(U) \\
\leq & \tfrac{2[(\ln T_\ell)^{1-\delta}-(\ln T_{\ell-1})^{1-\delta}]}{(1-\delta)\Gamma(u_\ell)}(\ln \tfrac{T_\ell}{T_{\ell-1}})^{u_\ell-1}\Big(K+\tfrac{L}{\Gamma(u_\ell+1)}(\ln \tfrac{T_\ell}{T_{\ell-1}})^{u_\ell}\Big)\widehat{\zeta}(U)
\end{aligned}$$

Thus

$$\widehat{\zeta}(WU) \leq \frac{2[(\ln T_\ell)^{1-\delta}-(\ln T_{\ell-1})^{1-\delta}]}{(1-\delta)\Gamma(u_\ell)}(\ln \frac{T_\ell}{T_{\ell-1}})^{u_\ell-1}\Big(K+\frac{L}{\Gamma(u_\ell+1)}(\ln \frac{T_\ell}{T_{\ell-1}})^{u_\ell}\Big)\widehat{\zeta}(U).$$

From inequality (13) it follows that W is a k-set contraction.

Therefore, all conditions of Theorem 1 are fulfilled and thus there exists $\widetilde{x}_\ell \in B_{R_\ell}$, such that $W(\widetilde{x}_\ell) = \widetilde{x}_\ell$, which is a solution of the BVP for Hadamard fractional differential equations of constant order (11). Since $B_{R_\ell} \subset E_\ell$ the claim of Theorem 2 is proved. □

Now, we will prove the existence result for the BVP for VOHFDE (1).

Introduce the following assumption:

Hypothesis 2 (H2). *Let $f \in C(J\times\mathbb{R}\times\mathbb{R},\mathbb{R})$ and there exists a number $\delta \in (0,1)$ such that $(\ln t)^\delta f \in C(J\times\mathbb{R}\times\mathbb{R},\mathbb{R})$ and there exist constants $K,\ L > 0$ such that*

$$(\ln t)^\delta |f(t,x_1,y_1) - f(t,x_2,y_2)| \leq K|x_1-x_2| + L|y_1-y_2|$$

for any $x_1,\ x_2,\ y_1,\ y_2, \in \mathbb{R}$ and $t \in J$.

Theorem 3. *Let the conditions (H1), (H2) and inequality (13) be satisfied for all $\ell \in \{1,2,\dots,n\}$.*

Then, the BVP for VOHFDE (1) possesses at least one solution in $C(J,\mathbb{R})$.

Proof. For any $\ell \in \{1,2,\dots,n\}$, according to Theorem 2 the BVP for Hadamard fractional differential equations of constant order (11) possesses at least one solution $\widetilde{x}_\ell \in E_\ell$.

For any $\ell \in \{1,2,\dots,n\}$ we define the function

$$x_\ell = \begin{cases} 0, & t \in [1,T_{\ell-1}], \\ \widetilde{x}_\ell, & t \in J_\ell. \end{cases} \tag{16}$$

Thus, the function $x_\ell \in C([1,T_\ell],\mathbb{R})$ solves the integral Equation (10) for $t \in J_\ell$, which means that $x_\ell(1) = 0,\ x_\ell(T_\ell) = \widetilde{x}_\ell(T_\ell) = 0$ and solves (10) for $t \in J_\ell,\ \ \ell \in \{1,2,\dots,n\}$.

Then the function

$$x(t) = \begin{cases} x_1(t), & t \in J_1, \\ x_2(t), & t \in J_2, \\ \dots\dots\dots\dots\dots & \\ x_n(t), & t \in J_n = [1,T] \end{cases}$$

is a solution of the BVP for VOHFDE (1) in $C(J,\mathbb{R})$. □

4. Ulam–Hyers–Rassias Stability of VOHFDE

We introduce the following assumption:

Hypothesis 3 (H3). *The function $\vartheta \in C(J, \mathbb{R}_+)$ is increasing and there exists $\lambda_\vartheta > 0$ such that*

$$ {}^H I^{u_\ell}_{T_{\ell-1}^+} \vartheta(t) \leq \lambda_\vartheta \, \vartheta(t), \quad \text{for } t \in J_\ell, \quad \ell = 1, 2, \dots, n. $$

Theorem 4. *Let the conditions (H1), (H2), (H3) and inequality (13) be satisfied.*
Then, the BVP for VOHFDE (1) is UHR stable with respect to ϑ.

Proof. Let $\epsilon > 0$ be an arbitrary number and the function $z(t)$ from $C(J, \mathbb{R})$ satisfy inequality (8).

For any $\ell \in \{1, 2, \dots, n\}$ we define the functions $z_1(t) \equiv z(t)$, $t \in [1, T_1]$ and for $\ell = 2, 3, \dots, n$:

$$ z_\ell(t) = \begin{cases} 0, & t \in [1, T_{\ell-1}], \\ z(t), & t \in J_\ell. \end{cases} $$

For any$\ell \in \{1, 2, \dots, n\}$ according to equality (3) for $t \in J_\ell$ we obtain

$$ {}^H D^{u(t)}_{1^+} z_\ell(t) = \frac{t^2}{\Gamma(2 - u(t))} \frac{d^2}{dt^2} \int_{T_{\ell-1}}^t (\ln \frac{t}{s})^{1-u_\ell} \frac{z(s)}{s} ds. $$

We take the HFI ${}^H I^{u_\ell}_{T_{\ell-1}^+}$ of both sides of the inequality (8), apply (H3) and obtain

$$ \begin{aligned} \Big| z_\ell(t) \quad &+ \quad \tfrac{1}{\Gamma(u_\ell)} (\ln \tfrac{T_\ell}{T_{\ell-1}})^{1-u_\ell} (\ln \tfrac{t}{T_{\ell-1}})^{u_\ell - 1} \int_{T_{\ell-1}}^{T_\ell} (\ln \tfrac{T_\ell}{s})^{u_\ell - 1} \frac{f(s, z_\ell(s), {}^H I^{u_\ell}_{T_{\ell-1}^+} z_\ell(s))}{s} ds \\ &- \quad \tfrac{1}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^t (\ln \tfrac{t}{s})^{u_\ell - 1} \frac{f(s, z_\ell(s), {}^H I^{u_\ell}_{T_{\ell-1}^+} z_\ell(s))}{s} ds \Big| \\ &\leq \quad \epsilon \, {}^H I^{u_\ell}_{T_{\ell-1}^+} \vartheta(t) \leq \epsilon \, \lambda_\vartheta \, \vartheta(t). \end{aligned} $$

According to Theorem 3, the BVP for VOHFDE (1) has a solution $x \in C(J, \mathbb{R})$ defined by $x(t) = x_\ell(t)$ for $t \in J_\ell$, $\ell = 1, 2, \dots, n$, where

$$ x_\ell = \begin{cases} 0, & t \in [1, T_{\ell-1}], \\ \widetilde{x}_\ell, & t \in J_\ell. \end{cases} \tag{17} $$

and $\widetilde{x}_\ell \in E_\ell$ is a solution of (11). According to Lemma 5 the integral equation

$$ \begin{aligned} \widetilde{x}_\ell(t) \quad = \quad & -(\ln \frac{T_\ell}{T_{\ell-1}})^{1-u_\ell} (\ln \frac{t}{T_{\ell-1}})^{u_\ell - 1} \Big({}^H I^{u_\ell}_{T_{\ell-1}^+} f(t, \widetilde{x}_\ell(t), {}^H I^{u_\ell}_{T_{\ell-1}^+} \widetilde{x}_\ell(t)) \Big)|_{t=T_\ell} \\ & + {}^H I^{u_\ell}_{T_{\ell-1}^+} f(t, \widetilde{x}_\ell(t), {}^H I^{u_\ell}_{T_{\ell-1}^+} \widetilde{x}_\ell(t)), \quad t \in J_\ell. \end{aligned} \tag{18} $$

holds.

Let $t \in J_\ell$ where $\ell \in \{1, 2, \dots, n\}$. Then by Equations (13) and (14) we obtain

$$
\begin{aligned}
& |z(t)-x(t)| = |z(t)-x_\ell(t)| = |z_\ell(t)-\widetilde{x}_\ell(t)| \\
\leq\ & \Bigg| z_\ell(t) + \frac{1}{\Gamma(u_\ell)}(\ln\frac{T_\ell}{T_{\ell-1}})^{1-u_\ell}(\ln\frac{t}{T_{\ell-1}})^{u_\ell-1} \\
\times\ & \int_{T_{\ell-1}}^{T_\ell}(\ln\frac{T_\ell}{s})^{u_\ell-1}\frac{f(s,z_\ell(s),{}^H I^{u_\ell}_{T^+_{\ell-1}} z_\ell(s))}{s}ds \\
& -\frac{1}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t}(\ln\frac{t}{s})^{u_\ell-1}\frac{f(s,z_\ell(s),{}^H I^{u_\ell}_{T^+_{\ell-1}} z_\ell(s))}{s}ds\Bigg| \\
+\ & \frac{1}{\Gamma(u_\ell)}(\ln\frac{T_\ell}{T_{\ell-1}})^{1-u_\ell}(\ln\frac{t}{T_{\ell-1}})^{u_\ell-1} \\
& \times\int_{T_{\ell-1}}^{T_\ell}\frac{1}{s}(\ln\frac{T_\ell}{s})^{u_\ell-1}\Big|f(s,z_\ell(s),{}^H I^{u_\ell}_{T^+_{\ell-1}} z_\ell(s)) - f(s,\widetilde{x}_\ell(s),{}^H I^{u_\ell}_{T^+_{\ell-1}}\widetilde{x}_\ell(s))\Big|ds \\
& +\frac{1}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t}\frac{1}{s}(\ln\frac{t}{s})^{u_\ell-1}\Big|f(s,z_\ell(s),{}^H I^{u_\ell}_{T^+_{\ell-1}} z_\ell(s)) - f(s,\widetilde{x}_\ell(s),{}^H I^{u_\ell}_{T^+_{\ell-1}}\widetilde{x}_\ell(s))\Big|ds \\
& \times\int_{T_{\ell-1}}^{T_\ell}(\ln\frac{T_\ell}{s})^{u_\ell-1}\frac{1}{s}(\ln s)^{-\delta}(K|z_\ell(s)-\widetilde{x}_\ell(s)| + L\,{}^H I^{u_\ell}_{T^+_{\ell-1}}|z_\ell(s)-\widetilde{x}_\ell(s)|)ds \\
+\ & \frac{1}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t}(\ln\frac{t}{s})^{u_\ell-1}\frac{1}{s}(\ln s)^{-\delta}(K|z_\ell(s)-\widetilde{x}_\ell(s)| + L\,{}^H I^{u_\ell}_{T^+_{\ell-1}}|z_\ell(s)-\widetilde{x}_\ell(s)|)ds \\
\leq\ & \lambda_\vartheta\,\epsilon\,\vartheta(t) + \frac{1}{\Gamma(u_\ell)}(\ln\frac{t}{T_{\ell-1}})^{u_\ell-1}\int_{T_{\ell-1}}^{T_\ell}\frac{1}{s}(\ln s)^{-\delta}(K|z_\ell(s)-\widetilde{x}_\ell(s)| \\
& +L\,{}^H I^{u_\ell}_{T^+_{\ell-1}}|z_\ell(s)-\widetilde{x}_\ell(s)|)ds \\
+\ & \frac{1}{\Gamma(u_\ell)}(\ln\frac{t}{T_{\ell-1}})^{u_\ell-1}\int_{T_{\ell-1}}^{t}\frac{1}{s}(\ln s)^{-\delta}(K|z_\ell(s)-\widetilde{x}_\ell(s)| + L\,{}^H I^{u_\ell}_{T^+_{\ell-1}}|z_\ell(s)-\widetilde{x}_\ell(s)|)ds \\
\leq\ & \lambda_\vartheta\,\epsilon\,\vartheta(t) + \frac{(\ln T_\ell)^{1-\delta}-(\ln T_{\ell-1})^{1-\delta}}{(1-\delta)\Gamma(u_\ell)}(\ln\frac{t}{T_{\ell-1}})^{u_\ell-1}(K\|z_\ell-\widetilde{x}_\ell\|_{E_\ell} + L\,{}^H I^{u_\ell}_{T^+_{\ell-1}}\|z_\ell-\widetilde{x}_\ell\|_{E_\ell}) \\
+\ & \frac{(\ln t)^{1-\delta}-(\ln T_{\ell-1})^{1-\delta}}{(1-\delta)\Gamma(u_\ell)}(\ln\frac{t}{T_{\ell-1}})^{u_\ell-1}(K\|z_\ell-\widetilde{x}_\ell\|_{E_\ell} + L\,{}^H I^{u_\ell}_{T^+_{\ell-1}}\|z_\ell-\widetilde{x}_\ell\|_{E_\ell}) \\
\leq\ & \lambda_\vartheta\,\epsilon\,\vartheta(t) + \mu\|z-x\|_J.
\end{aligned}
$$

Then,

$$\|z-x\|_J\Big(1-\mu\Big) \quad\leq\quad \lambda_\vartheta\,\epsilon\,\vartheta(t)$$

or for any $t \in J$

$$|z(t)-x(t) \leq \|z-x\|_J \leq \frac{\lambda_\vartheta}{1-\mu}\epsilon\vartheta(t).$$

Therefore,the BVP for VOHFDE (1) is **UHR** stable with respect to ϑ. □

5. Example

Let $J := [1,e]$, $T_0 = 1,\ T_1 = 2,\ T_2 = e$. Consider the scalar BVP for VOHFDE

$$
\begin{cases}
D^{u(t)}_{1^+}x(t) = \frac{7}{5\sqrt{\pi}}(\ln t)^{u(t)} + \frac{(\ln t)^{-\frac{1}{3}}}{t+3}x(t) + \frac{\sqrt{\pi}\ln t}{t^2+1}\,{}^H I^{u(t)}_{1^+}x(t), \quad t \in J, \\
x(1) = 0,\ \ x(e) = 0.
\end{cases}
\tag{19}
$$

where

$$
u(t) = \begin{cases} 1.3, & t \in J_1 := [1,2], \\ 1.7, & t \in J_2 :=]2,e]. \end{cases}
\tag{20}
$$

Denote

$$f(t,x,y) = \frac{7}{5\sqrt{\pi}}(\ln t)^{u(t)} + \frac{(\ln t)^{-\frac{1}{3}}}{t+3}x + \frac{\sqrt{\pi}\ln t}{t^2+1}y,\ (t,x,y) \in [1,e]\times\mathbb{R}\times\mathbb{R}.$$

Then,we have

$$\begin{aligned} &(\ln t)^{\frac{1}{3}}\Big|f(t,x_1,y_1) - f(t,x_2,y_2)\Big| \\ &= \Big|\frac{1}{t+3}x_1 + \frac{\sqrt{\pi}(\ln t)^{\frac{4}{3}}}{t^2+1}y_1 - \frac{1}{t+3}x_2 - \frac{\sqrt{\pi}(\ln t)^{\frac{4}{3}}}{t^2+1}y_2\Big| \\ \leq\ & \frac{1}{t+3}|x_1 - x_2| + \frac{\sqrt{\pi}(\ln t)^{\frac{4}{3}}}{t^2+1}|y_1 - y_2| \leq \frac{1}{4}|x_1 - x_2| + \frac{\sqrt{\pi}}{2}|y_1 - y_2|. \end{aligned}$$

Thus, assumption (H2) holds with $\delta = \frac{1}{3}, K = \frac{1}{4}, L = \frac{\sqrt{\pi}}{2}$.

By (20), according to (11) we consider two auxiliary BVPs for Hadamard fractional differential equations of constant order

$$\begin{cases} D_{1^+}^{1.3}x(t) = \frac{7}{5\sqrt{\pi}}(\ln t)^{1.3} + \frac{(\ln t)^{-\frac{1}{3}}}{t+3}x(t) + \frac{\sqrt{\pi}\ln t}{t^2+1}\,{}^{H}I_{1^+}^{1.3}x(t), \quad t \in J_1, \\ x(1) = 0,\ \ x(2) = 0. \end{cases} \tag{21}$$

and

$$\begin{cases} D_{2^+}^{1.7}x(t) = \frac{7}{5\sqrt{\pi}}(\ln t)^{1.7} + \frac{(\ln t)^{-\frac{1}{3}}}{t+3}x(t) + \frac{\sqrt{\pi}\ln t}{t^2+1}\,{}^{H}I_{1^+}^{1.7}x(t), \quad t \in J_2, \\ x(2) = 0,\ \ x(e) = 0. \end{cases} \tag{22}$$

Next,we prove that the condition (13) is fulfilled for $\ell = 1$. Indeed,

$$\begin{aligned} &\frac{(\ln T_1)^{1-\delta} - (\ln T_0)^{1-\delta}}{(1-\delta)\Gamma(u_1)}(\ln\frac{T_1}{T_0})^{u_1-1}\Big(K + \frac{L}{\Gamma(u_1+1)}(\ln\frac{T_1}{T_0})^{u_1}\Big) \\ =\ & \frac{(\ln 2)^{\frac{2}{3}}}{(\frac{2}{3})\Gamma(1.3)}(\ln 2)^{0.3}\Big(\frac{1}{4} + \frac{\frac{\sqrt{\pi}}{2}}{\Gamma(2.3)}(\ln 2)^{1.3}\Big) \simeq 0.3809 < \frac{1}{2}. \end{aligned}$$

Let$\vartheta(t) = (\ln t)^{\frac{1}{2}}$. Then, we obtain

$$\begin{aligned} {}^{H}I_{1^+}^{u_1}\vartheta(t) &= \frac{1}{\Gamma(1.3)}\int_1^t(\ln\frac{t}{s})^{1.3-1}\frac{(\ln s)^{\frac{1}{2}}}{s}ds \\ &\leq \frac{1}{\Gamma(1.3)}\int_1^t(\ln\frac{t}{s})^{0.3}\frac{1}{s}ds \leq \frac{0.75}{\Gamma(2.3)}(\ln t)^{\frac{1}{2}} = \lambda_\vartheta\,\vartheta(t), \end{aligned}$$

where $\lambda_\vartheta = \frac{0.75}{\Gamma(2.3)}$. Thus, condition (H3) is satisfied.

By Theorem 2, the BVP (21) has a solution $\widetilde{x}_1 \in E_1$.

We prove that the condition (13) is fulfilled for $\ell = 2$. Indeed,

$$\begin{aligned} &\frac{(\ln T_2)^{1-\delta} - (\ln T_1)^{1-\delta}}{(1-\delta)\Gamma(u_2)}(\ln\frac{T_2}{T_1})^{u_2-1}\Big(K + \frac{L}{\Gamma(u_2+1)}(\ln\frac{T_2}{T_1})^{u_2}\Big) \\ =\ & \frac{1-(\ln 2)^{\frac{2}{3}}}{(\frac{2}{3})\Gamma(1.7)}(\ln\frac{e}{2})^{0.7}\Big(\frac{1}{4} + \frac{\frac{\sqrt{\pi}}{2}}{\Gamma(2.7)}(\ln\frac{e}{2})^{1.7}\Big) \simeq 0.0242 < \frac{1}{2}. \end{aligned}$$

Accordingly the condition (13) is achieved. We obtain

$$
\begin{aligned}
{}^{H}I_{2^+}^{u_2}\vartheta(t) &= \frac{1}{\Gamma(1.7)}\int_2^t (\ln\frac{t}{s})^{1.7-1}\frac{(\ln s)^{\frac{1}{2}}}{s}ds \\
&\le \frac{1}{\Gamma(1.7)}\int_2^t (\ln\frac{t}{s})^{0.7}\frac{1}{s}ds \le \frac{1}{\Gamma(2.7)}(\ln t)^{\frac{1}{2}} = \lambda_\vartheta\, \vartheta(t).
\end{aligned}
$$

where $\lambda_\vartheta = \frac{1}{\Gamma(2.7)}$. Thus condition (H3) is fulfilled.

According to Theorem 2, the BVP (22) possesses a solution $\widetilde{x}_2 \in E_2$.

Thus, according to Theorem 3 the BVP (19) has a solution

$$
x(t) = \begin{cases} \widetilde{x}_1(t), & t \in J_1, \\ x_2(t), & t \in J_2, \end{cases}
$$

where

$$
x_2(t) = \begin{cases} 0, & t \in J_1 \\ \widetilde{x}_2(t), & t \in J_2. \end{cases}
$$

According to Theorem (4), the BVP for VOHFDE (19) is UHR stable with respect to ϑ.

6. Conclusions

Our proposed multiterm BVP has been successfully investigated in this work via the Kuratowski measure of noncompactness technique to prove the existence of solutions to VOHFDEs. A numerical example was given at the end to support and validate the potentiality of all our obtained results. Therefore, all results in this work show a great potential to be applied in various applications of multidisciplinary sciences. Further investigations on this open research problem could be also possible with the help of our original results in this research paper. In other words, in the future one could extend the proposed BVP to other complicated real mathematical fractional models by terms of newly-introduced operators with non-singular kernels.

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