symmetry

Article

A Fuzzy Approach to Support Evaluation of Fuzzy Cross Efficiency

Shun-Cheng Wu [1], Tim Lu [2,*] and Shiang-Tai Liu [3]

1 Department of Hotel Management, Vanung University, Zhongli, Taoyuan 320, Taiwan; scwu@vnu.edu.tw
2 Department of Marketing and Logistics, Vanung University, Zhongli, Taoyuan 320, Taiwan
3 Graduate School of Business and Management, Vanung University, Zhongli, Taoyuan 320, Taiwan; stliu@vnu.edu.tw
* Correspondence: timlu@vnu.edu.tw

Abstract: Cross-efficiency evaluation effectively distinguishes a set of decision-making units (DMUs) via self- and peer-evaluations. In constant returns to scale, this evaluation technique is usually applied for data envelopment analysis (DEA) models because negative efficiencies will not occur in this case. For situations of variable returns to scale, the negative cross-efficiencies may occur in this evaluation method. In the real world, the observations could be uncertain and difficult to measure precisely. The existing fuzzy cross-evaluation methods are restricted to production technologies with constant returns to scale. Generally, symmetry is a fundamental characteristic of binary relations used when modeling optimization problems. Additionally, the notion of symmetry appeared in many studies about uncertain theories employed in DEA problems, and this approach can be considered an engineering tool for supporting decision-making. This paper proposes a fuzzy cross-efficiency evaluation model with fuzzy observations under variable returns to scale. Since all possible weights of all DMUs are considered, a choice of weights is not required. Most importantly, negative cross-efficiencies are not produced. An example shows that this paper's fuzzy cross-efficiency evaluation method has discriminative power in ranking the DMUs when observations are fuzzy numbers.

Keywords: data envelopment analysis; cross-efficiency; fuzzy set; ranking

Citation: Wu, S.-C.; Lu, T.; Liu, S.-T. A Fuzzy Approach to Support Evaluation of Fuzzy Cross Efficiency. *Symmetry* **2021**, *13*, 882. https://doi.org/10.3390/sym13050882

Academic Editor: José Carlos R. Alcantud

Received: 17 April 2021
Accepted: 12 May 2021
Published: 16 May 2021

1. Introduction

Data envelopment analysis (DEA) is a data-oriented methodology for measuring the performance of a set of decision-making units (DMUs) that apply multiple inputs to produce multiple outputs. Because DEA requires very few assumptions, it brought the opportunity to other approaches because of the complex property of the relations between the inputs and outputs involved in DMUs [1]. As pointed out in [2], DEA models can provide new insights into activities that other methods have previously evaluated. Researchers have recognized that DEA is an easily used methodology for performance evaluations in many fields since it was first introduced in 1978. Thousands of applications and theories have been reported (see, e.g., [3–5]). According to the calculated efficiencies, the DMUs can be ranked. When DMUs represent the alternatives for a decision-making problem, the rankings of the DMUs can be used for selecting alternatives.

In traditional DEA, each DMU is evaluated against the remaining DMUs via a ratio of the sum of weighted outputs to the sum of weighted inputs, thereby distinguishing efficient units and inefficient ones. Due to the weight flexibility allowed in DEA, the obtained results may cause the situation in which many DMUs are measured as efficient, and the efficient units are impossible to be further distinguished. Cross-efficiency evaluation is an effective method to discriminate the efficient units. The evaluation method has been touted as a powerful extension of DEA since it was first proposed [6,7]. In performing the cross-evaluation, once the DMU selects a weighting approach to apply to all DMUs, the efficiency score provided to each DMU is set aside, forming a cross-efficiency matrix. Each DMU has its self-evaluation and the peer evaluations it has obtained from the other DMUs. We receive a DMU's final cross-efficiency through averaging across self- and peer-evaluations. A DMU has a high cross-efficiency, which indicates that it can make itself measure well and is considered more efficient by the majority of its peers.

Two main advantages of the cross-efficiency evaluation are often stated. First, it usually creates a unique ordering among the DMUs [7]. Second, it appears to eliminate unrealistic weights that the DMUs might use [8]. Since the seminal works of Sexton et al. [6] and Doyle and Green [7], many cross-efficiency models and applications have been reported in the literature. Örkcü et al. [9] and Wu et al. [10] provided good reviews on models. Kao and Liu [11] extended the calculation of cross efficiency to two-stage systems. Sexton et al. [6] and Doyle and Green [7] used an additional criterion for the selection of weights.

In constant returns to scale, Liang et al. [12] developed a game cross-efficiency that constitutes a Nash equilibrium point for DMUs. Lam [13] proposed applied a super-efficiency DEA model and a mixed-integer linear programming method to decide the suitable weight for calculating the cross-evaluation. Wang and Chin [14] investigated a neutral DEA model for cross-efficiency evaluation that could determine the associated input and output weights for each DMU. Some methods selected suitable weights from alternative solutions to avoid significant differences among the weights. Imposing lower bounds [15,16], applying ordered weighted averaging operators [17], and evaluating the robustness of the proposed methodology [18] are some examples. Additionally, Oral et al. [19] integrated both the first- and second-order voices of all DMUs to calculate the cross-efficiency. Al-Siyabi et al. [20] developed a mean-variance goal programming model for minimizing the risk of changing the DEA weights for DEA cross-efficiency evaluation. All of the studies mentioned above require a secondary goal in selecting the weights. Apart from the secondary goal approach, an alternative strategy is to consider all possible weights in the weight space to obtain an efficiency interval for the DMU being measured [21–24]. Different DMUs used different sets of weights in calculating the cross efficiency.

For variable returns to scale, Soares de Mello et al. [25] proposed the idea of producing positive efficiencies only for all DMUs by restricting the multiplier values in the model. Lim and Zhu [26] translated the coordinates to let negative efficiencies become positive. Kao and Liu [27] proposed a slacks-based measure to calculate the cross efficiency.

Cross-efficiency evaluation is usually applied for DEA models with the production technology of constant returns to scale because negative efficiencies never occur in this situation. However, negative efficiencies may happen for cases of variable returns to scale. Negative efficiencies are unreasonable and lead to difficulties in measuring the final cross-efficiency. Generally, symmetry is a fundamental characteristic of binary relations used when modeling optimization problems. Additionally, the notion of symmetry appeared in many studies about uncertain theories employed in DEA problems, and this approach can be considered an engineering tool for supporting decision-making and belongs to the scope of mathematics and symmetry. The fuzzy set theory is available in the existing studies to deal with uncertainties. In the literature, some studies are discussing fuzzy cross-efficiency evaluation in the fuzzy environment. Dotoli et al. [28] and Change and Wang [29] integrated the fuzzy DEA technique with the cross-efficiency method to evaluate DMUs, and the Doyle and Green method (1994) was applied for selecting the weights to calculate the fuzzy cross-efficiency. Ruiz and Sirvent [30] proposed a fuzzy cross-efficiency evaluation based on the possibility approach and developed the aggressive and benevolent formulations for determining the weights to calculate the fuzzy cross-efficiency. Liu and Lee [31] proposed a novel method that considered all possible weights of all the DMUs at the same time to calculate the fuzzy cross-efficiency directly, and the choice of weights is not required. However, these studies are all restricted to the assumptions of constant returns to scale, and the literature gives very little attention to the issue that the cross-efficiency evaluation with fuzzy data and the production technology of variable returns to scale.

This paper proposes a novel method to measure the fuzzy cross-efficiency scores for DMUs with fuzzy observations and variable returns to scale. Following the basic ideas of Yang et al. [21] and Soares de Mello [25], we construct a fuzzy cross-efficiency evaluation model. A set of secondary goals is added to tackle the problem of multiple optimal weights. Since all possible weights of all DMUs are considered, the choice of weights is not required. Most importantly, negative cross-efficiencies are not produced in the proposed model. In the literature, the α-level based approach is one of the most popular fuzzy DEA models for decision makers to set up their acceptable α levels [32,33]; we also use the α-level based approach to construct the fuzzy cross-efficiency model. A pair of mathematical programs is formulated to measure the fuzzy cross-efficiency. At a specific α-cut, solving this pair of programs produces the minimal and maximal cross-efficiencies. The final cross-efficiency of a DMU is obtained by averaging the derived minimal and maximal cross-efficiencies. With several α-cuts of the final cross-efficiencies, the ranking of DMUs is determined.

In the following, we first introduce the conventional cross-efficiency evaluation methodology. Then, based on the α-level based approach, we formulate a fuzzy cross-efficiency evaluation model with a technology of variable returns to scale. After that, an example is used to illustrate the idea proposed in this study. Finally, some conclusions of this work are presented.

2. Background

Under the technology of variable returns to scale, the DEA model for measuring the efficiency of a DMU is referred to as the BCC model [34]. This model originated from the CCR model [35] by adding a constraint, which introduces an additional variable into the multiplier problems. This additional variable makes it possible to affect the returns-to-scale evaluation (increasing, constant and decreasing). Suppose that we have n DMUs, where every DMU j, $j = 1, \ldots, n$, produces the same s outputs in different amounts, Y_{rj} ($r = 1, \ldots, s$), using the same m inputs, X_{ij} ($i = 1, \ldots, m$) in different amounts. The BCC model for measuring the efficiency of DMU d has the input and output forms, which can be formulated as:

Output form

$$\begin{aligned} \frac{1}{E_{dd}} = \quad & \min \frac{\sum\limits_{i=1}^{m} v_{id} X_{id} + v_{0d}}{\sum\limits_{r=1}^{s} u_{rd} Y_{rd}} \\ & \text{s.t. } \Big(\sum_{i=1}^{m} v_{id} X_{ij} + v_{0d}\Big) - \sum_{r=1}^{s} u_{rd} Y_{rj} \geq 0,\ j = 1, \cdots, n, \\ & u_{rd},\ v_{id} \geq 0,\ r = 1, \ldots, s,\ i = 1, \ldots, m,\ v_{0d} \text{ free in sign.} \end{aligned} \tag{1}$$

Input form

$$\begin{aligned} E_{dd} = \quad & \max \frac{\sum\limits_{r=1}^{s} u_{rd} Y_{rd} - u_{0d}}{\sum\limits_{i=1}^{m} v_{id} X_{id}} \\ & \text{s.t. } \sum_{i=1}^{m} v_{id} X_{ij} - \Big(\sum_{r=1}^{s} u_{rd} Y_{rj} - u_{0d}\Big) \geq 0,\ j = 1, \ldots, n, \\ & u_{rd},\ v_{id} \geq 0,\ r = 1, \ldots, s,\ i = 1, \ldots, m,\ u_{0d} \text{ free in sign,} \end{aligned} \tag{2}$$

where u_{rd} and v_{id} are the weights selected by DMU d to calculate efficiencies. When $v_{0d} = 0$ in (1) and $u_{0d} = 0$ in (2), these two models reduce to the same CCR model [35], with a technology of constant returns to scale.

In DEA, each DMU selects the most favorable weights to measure its own efficiency. Since different DMUs use different weights, the efficiency scores among the DMUs may be incomparable. Using the weights selected by every DMU to calculate the efficiencies of all other DMUs is a way to make them comparable. Let v_{id}^{*} ($i = 1, \ldots, m$) and u_{rd}^{*} ($r = 1, \ldots,$

s) be an optimal solution of (2) for a given DMU d and E_{dj} denote the efficiency of DMU j calculated from the weights selected by DMU d. We then have the cross-efficiency

$$E_{dj} = \frac{\sum\limits_{r=1}^{s} u_{rd}^{*} Y_{rj} - u_{0d}}{\sum\limits_{i=1}^{m} v_{id}^{*} X_{ij}} \tag{3a}$$

We can repeat this process and use the weights selected by every DMU in calculating the efficiencies of all DMUs. The final cross-efficiency of DMU j is the average of E_{dj}, $d = 1, \ldots, n$, that is,

$$\overline{E}_j = \frac{1}{n}\sum_{d=1}^{n} E_{dj} \tag{3b}$$

The cross-efficiency score $\overline{E}_j$ gives a peer-evaluation of DMU j, and these derived values can thus be used for ranking DMUs. However, Equation (3a) has an unrestricted variable u_{0d} in the numerator. Therefore, we cannot ensure the non-negativity of the cross-efficiency when the derived weights v_{id}^{*} and u_{rd}^{*} are used to evaluate other DMUs in (3a). It should be noted that the negative efficiencies will not occur in the CCR model or the output form of the BCC model. The latter constitutes greater or equal type constraints in the model, which guarantees non-negative efficiency measures.

In Model (2), the negative efficiencies occur only if $\sum_{r=1}^{s} u_{rd} Y_{rj} - u_{0d} < 0$, $j = 1, \ldots, n$. To avoid the negative efficiencies, Soares de Mello et al. [25] proposed inserting a constraint for retaining $\sum_{r=1}^{s} u_{rd} Y_{rj} - u_{0d} \geq 0$, $j = 1, \ldots, n$. With this additional constraint, Model (2) becomes

$$\begin{aligned} E_{dd} = \quad & \max \frac{\sum\limits_{r=1}^{s} u_{rd} Y_{rd} - u_{0d}}{\sum\limits_{i=1}^{m} v_{id} X_{id}} \\ & \text{s.t.} \sum_{i=1}^{m} v_{id} X_{ij} - \Big(\sum_{r=1}^{s} u_{rd} Y_{rj} - u_{0d}\Big) \geq 0,\ j = 1, \ldots, n, \\ & \sum_{r=1}^{s} u_{rd} Y_{rj} - u_{0d} \geq 0,\ j = 1, \ldots, n, \\ & u_{rd},\ v_{id} \geq 0,\ r = 1, \ldots, s,\ i = 1, \ldots, m,\ u_{0d} \text{ free in sign,} \end{aligned} \tag{4}$$

Negative efficiencies will not occur in the approach of Soares de Mello et al. [25]. Presume that the observations of input and output items can be represented as convex fuzzy numbers $\widetilde{X}_{ij}$ and $\widetilde{Y}_{rj}$, respectively. Since the input and output data are fuzzy numbers, we should rewrite (4) in the following mathematical form to satisfy the fuzzy environment.

$$\begin{aligned} \widetilde{E}_{dd} = \quad & \max \sum_{r=1}^{s} u_{rd} \widetilde{Y}_{rd} - u_{0d} \\ & \text{s.t.} \sum_{i=1}^{m} v_{id} \widetilde{X}_{id} = 1, \\ & \sum_{i=1}^{m} v_{id} \widetilde{X}_{ij} - \Big(\sum_{r=1}^{s} u_{rd} \widetilde{Y}_{rj} - u_{0d}\Big) \geq 0,\ j = 1, \ldots, n, \\ & \sum_{r=1}^{s} u_{rd} \widetilde{Y}_{rj} - u_{0d} \geq 0,\ j = 1, \ldots, n, \\ & u_{rd},\ v_{id} \geq 0,\ r = 1, \ldots, s,\ i = 1, \ldots, m,\ u_{0d} \text{ free in sign.} \end{aligned} \tag{5}$$

Let $(X_{ij})_\alpha = [(X_{ij})_\alpha^L, (X_{ij})_\alpha^U]$, $(Y_{rj})_\alpha = [(Y_{rj})_\alpha^L, (Y_{rj})_\alpha^U]$, and $(E_{dd})_\alpha = [(E_{dd})_\alpha^L, (E_{dd})_\alpha^U]$ denote the α-level sets of $\widetilde{X}_{ij}$, $\widetilde{Y}_{rj}$, and $\widetilde{E}_{dd}$, respectively. In search for the minimal and maximal efficiencies for DMU d at a specified α-level under the assumption of variable returns to scale, Kao and Liu [36] provided the following formulations:

$$
\begin{aligned}
(E_{dd})_\alpha^L \quad & = \max \sum_{r=1}^{s} u_{rd}(Y_{rd})_\alpha^L - u_{0d} \\
& \text{s.t.} \sum_{i=1}^{m} v_{id}(X_{id})_\alpha^U = 1 \\
& \sum_{i=1}^{m} v_{id}(X_{id})_\alpha^U - (\sum_{r=1}^{s} u_{rd}(Y_{rd})_\alpha^L - u_{0d}) \geq 0 \\
& \sum_{i=1}^{m} v_{id}(X_{ij})_\alpha^L - (\sum_{r=1}^{s} u_{rd}(Y_{rj})_\alpha^U - u_{0d}) \geq 0, j = 1, \ldots, n, \ j \neq d \\
& \sum_{r=1}^{s} u_{rd}(Y_{rd})_\alpha^L - u_{0d} \geq 0 \\
& \sum_{r=1}^{s} u_{rd}(Y_{rj})_\alpha^U - u_{0d} \geq 0, j = 1, \ldots, n, \ j \neq d \\
& u_{rd}, \ v_{id} \geq 0, \ r = 1, \ldots, s, \ i = 1, \ldots, m, \ u_{0d} \text{ free in sign.}
\end{aligned} \tag{6}
$$

$$
\begin{aligned}
(E_{dd})_\alpha^U \quad & = \max \sum_{r=1}^{s} u_{rd}(Y_{rd})_\alpha^U \\
& \text{s.t.} \sum_{i=1}^{m} v_{id}(X_{id})_\alpha^L = 1 \\
& \sum_{i=1}^{m} v_{id}(X_{id})_\alpha^L - (\sum_{r=1}^{s} u_{rd}(Y_{rd})_\alpha^U - u_{0d}) \geq 0 \\
& \sum_{i=1}^{m} v_{id}(X_{ij})_\alpha^U - (\sum_{r=1}^{s} u_{rd}(Y_{rj})_\alpha^L - u_{0d}) \geq 0, j = 1, \ldots, n, \ j \neq d \\
& \sum_{r=1}^{s} u_{rd}(Y_{rd})_\alpha^U - u_{0d} \geq 0 \\
& \sum_{r=1}^{s} u_{rd}(Y_{rj})_\alpha^L - u_{0d} \geq 0, j = 1, \ldots, n, \ j \neq d \\
& u_{rd}, \ v_{id} \geq 0, \ r = 1, \ldots, s, \ i = 1, \ldots, m, \ u_{0d} \text{ free in sign.}
\end{aligned} \tag{7}
$$

The detailed explanations of (6) and (7) were described by Kao and Liu [36].

Soares de Mello et al. [25] did not use the secondary goals to tackle multiple optimal weights and is restricted to the aggressive formulation. Yang et al. [21] considered the secondary goals and the aggressive and benevolent formulations. In their approach, all the possible weight sets in the weight space are considered, and the following two models measure the minimal and maximal cross-efficiency cross-efficiencies:

$$
\begin{aligned}
E_{dj}^{A(CCR)} \quad = \quad & \min \sum_{r=1}^{s} u_{rd} Y_{rd} \\
& \text{s.t.} \sum_{i=1}^{m} v_{id} X_{id} = 1, \\
& \sum_{r=1}^{s} u_{rd} Y_{rd} - E_{dd} \sum_{i=1}^{m} v_{id} X_{id} = 0, \\
& \sum_{i=1}^{m} v_{id} X_{ij} - \sum_{r=1}^{s} u_{rd} Y_{rj} \geq 0, \ j = 1, \ldots, \ n, \ j \neq d \\
& u_{rd}, \ v_{id} \geq 0, \ r = 1, \ldots, s, \ i = 1, \ldots, m.
\end{aligned} \tag{8}
$$

$$
\begin{aligned}
E_{dj}^{B(CCR)} \quad = \quad & \max \sum_{r=1}^{s} u_{rd} Y_{rd} \\
& \text{s.t.} \sum_{i=1}^{m} v_{id} X_{id} = 1, \\
& \sum_{r=1}^{s} u_{rd} Y_{rd} - E_{dd} \sum_{i=1}^{m} v_{id} X_{id} = 0, \\
& \sum_{i=1}^{m} v_{id} X_{ij} - \sum_{r=1}^{s} u_{rd} Y_{rj} \geq 0, \ j = 1, \ldots, \ n, \ j \neq d \\
& u_{rd}, \ v_{id} \geq 0, \ r = 1, \ldots, s, \ i = 1, \ldots, m.
\end{aligned} \tag{9}
$$

The first constraint $\sum_{r=1}^{s} u_{rd}Y_{rd} - E_{dd}\sum_{i=1}^{m} v_{id}X_{id} = 0$ is to maintain the efficiency score E_{dd} of DMU d at its current level. Yang et al. [21] developed their method under the assumption of constant returns to scale. We need to modify their approach to the input-oriented form of the BCC model, and, similar to the idea of Soares de Mello et al. [25], a constraint $\sum_{r=1}^{s} u_{rd}Y_{rj} - u_{0d} \geq 0$, $j = 1, \ldots, n$, is added to avoid the occurrence of negative cross-efficiencies. In other words, Models (8) and (9) are rewritten in the following mathematical forms:

$$
\begin{aligned}
E_{dj}^{A(BCC)} = \quad & \min \sum_{r=1}^{s} u_{rd}Y_{rd} - u_{0d} \\
& \text{s.t.} \sum_{i=1}^{m} v_{id}X_{id} = 1, \\
& (\sum_{r=1}^{s} u_{rd}Y_{rd} - u_{0d}) - E_{dd}\sum_{i=1}^{m} v_{id}X_{id} = 0, \\
& \sum_{i=1}^{m} v_{id}X_{ij} - (\sum_{r=1}^{s} u_{rd}Y_{rj} - u_{0d}) \geq 0,\ j = 1, \ldots, n, j \neq d \\
& \sum_{r=1}^{s} u_{rd}Y_{rj} - u_{0d} \geq 0,\ j = 1, \ldots, n, \\
& u_{rd},\ v_{id} \geq 0,\ r = 1, \ldots, s,\ i = 1, \ldots, m,\ u_{0d} \text{ free in sign.}
\end{aligned}
\tag{10}
$$

$$
\begin{aligned}
E_{dj}^{B(BCC)} \quad & = \max \sum_{r=1}^{s} u_{rd}Y_{rd} - u_{0d} \\
& \text{s.t.} \sum_{i=1}^{m} v_{id}X_{id} = 1, \\
& (\sum_{r=1}^{s} u_{rd}Y_{rd} - u_{0d}) - E_{dd}\sum_{i=1}^{m} v_{id}X_{id} = 0, \\
& \sum_{i=1}^{m} v_{id}X_{ij} - (\sum_{r=1}^{s} u_{rd}Y_{rj} - u_{0d}) \geq 0,\ j = 1, \ldots, n, j \neq d \\
& \sum_{r=1}^{s} u_{rd}Y_{rj} - u_{0d} \geq 0,\ j = 1, \ldots, n, \\
& u_{rd},\ v_{id} \geq 0,\ r = 1, \ldots, s,\ i = 1, \ldots, m,\ u_{0d} \text{ free in sign.}
\end{aligned}
\tag{11}
$$

Although Models (10) and (11) avoid negative efficiencies, they can only deal with the crisp observations. When the observations are fuzzy numbers, we cannot apply these two models to calculate the cross-efficiency and need to develop a novel solution procedure to solve this issue. In the next section, we create a fuzzy BCC cross-efficiency evaluation model, where the observations are represented as fuzzy numbers, to measure the fuzzy cross-efficiency, with a technology of variable returns to scale.

3. Fuzzy BCC Cross-Efficiency Evaluation

Most studies investigated fuzzy cross-efficiency evaluation methods under the cases of returns to scale. Unlike these studies, we propose a novel approach to measure fuzzy cross-efficiency scores with variable returns to scale. All possible weights of all DMUs are considered in the proposed method, and the choice of weights is not required. Most importantly, negative cross-efficiencies are not produced in the model under the fuzzy environment.

Similar to Model (4), the mathematical forms of Models (10) and (11) with fuzzy observations can be reformulated, respectively, as:

$$
\begin{aligned}
\widetilde{E}_{dj}^{A(BCC)} \quad & = \min \sum_{r=1}^{s} u_{rd} \widetilde{Y}_{rd} - u_{0d} \\
& \text{s.t.} \sum_{i=1}^{m} v_{id} \widetilde{X}_{id} = 1 \\
& (\sum_{r=1}^{s} u_{rd} \widetilde{Y}_{rd} - u_{0d}) - E_{dd} \sum_{i=1}^{m} v_{id} \widetilde{X}_{id} = 0 \\
& \sum_{i=1}^{m} v_{id} \widetilde{X}_{ij} - (\sum_{r=1}^{s} u_{rd} \widetilde{Y}_{rj} - u_{0d}) \geq 0, j = 1, \ldots, n, \ j \neq d \\
& \sum_{r=1}^{s} u_{rd} \widetilde{Y}_{rj} - u_{0d} \geq 0, \ j = 1, \ldots, n \\
& u_{rd}, \ v_{id} \geq 0, \ r = 1, \ldots, s, \ i = 1, \ldots, m, \ u_{0d} \text{ free in sign.}
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
\widetilde{E}_{dj}^{B(BCC)} \quad & = \max \sum_{r=1}^{s} u_{rd} \widetilde{Y}_{rd} - u_{0d} \\
& \text{s.t.} \sum_{i=1}^{m} v_{id} \widetilde{X}_{id} = 1 \\
& (\sum_{r=1}^{s} u_{rd} \widetilde{Y}_{rd} - u_{0d}) - E_{dd} \sum_{i=1}^{m} v_{id} \widetilde{X}_{id} = 0 \\
& \sum_{i=1}^{m} v_{id} \widetilde{X}_{ij} - (\sum_{r=1}^{s} u_{rd} \widetilde{Y}_{rj} - u_{0d}) \geq 0, j = 1, \ldots, n, \ j \neq d \\
& \sum_{r=1}^{s} u_{rd} \widetilde{Y}_{rj} - u_{0d} \geq 0, \ j = 1, \ldots, n \\
& u_{rd}, \ v_{id} \geq 0, \ r = 1, \ldots, s, \ i = 1, \ldots, m, \ u_{0d} \text{ free in sign.}
\end{aligned}
\tag{13}
$$

Hatami-Marbini et al. [32] indicated that the α-level based approach is the most adopted method for investigating fuzzy DEA models. Therefore, we also employ this popular approach to develop the fuzzy BCC cross-efficiency evaluation model.

Since Models (12) and (13) are to find the minimal and maximal cross-efficiency scores for every DMU with fuzzy observations through the aggressive and benevolent formulations, using the α-level based approach, we can formulate these two models as:

$$
(E_{dj}^{A(BCC)})_{\alpha}^{L} = \min_{\substack{(E_{dd})_{\alpha}^{L} \leq e_{dd} \leq (E_{dd})_{\alpha}^{U} \\ (X_{ij})_{\alpha}^{L} \leq x_{ij} \leq (X_{ij})_{\alpha}^{U} \\ (Y_{rj})_{\alpha}^{L} \leq y_{rj} \leq (Y_{rj})_{\alpha}^{U} \\ \forall i, j, r}} E_{dj}^{A}(\mathbf{x,y,e})
\tag{14}
$$

$$
(E_{dj}^{B(BCC)})_{\alpha}^{U} = \max_{\substack{(E_{dd})_{\alpha}^{L} \leq e_{dd} \leq (E_{dd})_{\alpha}^{U} \\ (X_{ij})_{\alpha}^{L} \leq x_{ij} \leq (X_{ij})_{\alpha}^{U} \\ (Y_{rj})_{\alpha}^{L} \leq y_{rj} \leq (Y_{rj})_{\alpha}^{U} \\ \forall i, j, r}} E_{dj}^{B}(\mathbf{x,y,e})
\tag{15}
$$

where $E_{dj}^{A}(\mathbf{x,y,e})$ and $E_{dj}^{B}(\mathbf{x,y,e})$ are defined in Models (7) and (8), respectively.

According to the mathematical forms of (14) and (15), we can transform these two models into (16) and (17), the two-level mathematical programs, respectively:

$$\left(E_{dj}^{B(BCC)}\right)_{\alpha}^{L} = \begin{array}{c} \min \\ (E_{dd})_{\alpha}^{L} \leq e_{dd} \leq (E_{dd})_{\alpha}^{U} \\ (X_{ij})_{\alpha}^{L} \leq x_{ij} \leq (X_{ij})_{\alpha}^{U} \\ (Y_{rj})_{\alpha}^{L} \leq y_{rj} \leq (Y_{rj})_{\alpha}^{U} \\ \forall i, j, r \end{array} \left\{ \begin{array}{l} \max \sum_{r=1}^{s} u_{rd} Y_{rd} - u_{0d} \\ \text{s.t.} \\ \sum_{i=1}^{m} v_{id} x_{id} = 1 \\ (\sum_{r=1}^{s} u_{rd} y_{rd} - u_{0d}) - e_{dd} \sum_{i=1}^{m} v_{id} x_{id} = 0 \\ \sum_{i=1}^{m} v_{id} x_{ij} - (\sum_{r=1}^{s} u_{rd} y_{rj} - u_{0d}) \geq 0,\ j = 1, \ldots, n,\ j \neq d \\ \sum_{r=1}^{s} u_{rd} Y_{rj} - u_{0d} \geq 0,\ j = 1, \ldots, n \\ u_{rd},\ v_{id} \geq 0, r = 1, \ldots, s,\ i = 1, \ldots, m, \\ u_{0d} \text{ free in sign.} \end{array} \right. \tag{16}$$

.

$$\left(E_{dj}^{B(BCC)}\right)_{\alpha}^{U} = \begin{array}{c} \max \\ (E_{dd})_{\alpha}^{L} \leq e_{dd} \leq (E_{dd})_{\alpha}^{U} \\ (X_{ij})_{\alpha}^{L} \leq x_{ij} \leq (X_{ij})_{\alpha}^{U} \\ (Y_{rj})_{\alpha}^{L} \leq y_{rj} \leq (Y_{rj})_{\alpha}^{U} \\ \forall i, j, r \end{array} \left\{ \begin{array}{l} \max \sum_{r=1}^{s} u_{rd} Y_{rd} - u_{0d} \\ \text{s.t.} \\ \sum_{i=1}^{m} v_{id} x_{id} = 1 \\ (\sum_{r=1}^{s} u_{rd} y_{rd} - u_{0d}) - e_{dd} \sum_{i=1}^{m} v_{id} x_{id} = 0 \\ \sum_{i=1}^{m} v_{id} x_{ij} - (\sum_{r=1}^{s} u_{rd} y_{rj} - u_{0d}) \geq 0,\ j = 1, \ldots, n,\ j \neq d \\ \sum_{r=1}^{s} u_{rd} Y_{rj} - u_{0d} \geq 0,\ j = 1, \ldots, n \\ u_{rd},\ v_{id} \geq 0, r = 1, \ldots, s,\ i = 1, \ldots, m, \\ u_{0d} \text{ free in sign.} \end{array} \right. \tag{17}$$

.

In Model (16), the inner-level and outer-level programs have the same minimum operation, and we can combine these two programs into a one-level mathematical program. The minimum operation is the overall objective function of this combined one-level program, and the constraints at the two different levels are regarded as the overall constraints. Similarly, the maximum operation is the final objective function of Model (17), and the associated constraints are integrated as the overall constraints. Now, we can reduce Models (16) and (17) to the following mathematical programs:

$$\begin{array}{rl} \left(E_{dj}^{A(BCC)}\right)_{\alpha}^{L} & = \min \sum_{r=1}^{s} u_{rd} y_{rd} - u_{0d} \\ & \text{s.t.} \sum_{i=1}^{m} v_{id} x_{id} = 1 \\ & (\sum_{r=1}^{s} u_{rd} y_{rd} - u_{0d}) - e_{dd} \sum_{i=1}^{m} v_{id} x_{id} = 0 \\ & \sum_{i=1}^{m} v_{id} x_{ij} - (\sum_{r=1}^{s} u_{rd} y_{rj} - u_{0d}) \geq 0, j = 1,\ \ldots,\ n,\ j \neq d \\ & \sum_{r=1}^{s} u_{rd} y_{rj} - u_{0d} \geq 0,\ j = 1,\ \ldots,\ n \\ & (E_{dd})_{\alpha}^{L} \leq e_{dd} \leq (E_{dd})_{\alpha}^{U} \\ & (X_{ij})_{\alpha}^{L} \leq x_{ij} \leq (X_{ij})_{\alpha}^{U},\ i = 1,\ \ldots,\ m,\ j = 1,\ \ldots,\ n \\ & (Y_{rj})_{\alpha}^{L} \leq y_{rj} \leq (Y_{rj})_{\alpha}^{U},\ r = 1,\ \ldots,\ s,\ j = 1,\ \ldots,\ n \\ & u_{rd},\ v_{id} \geq 0,\ r = 1,\ \ldots,\ s,\ i = 1,\ \ldots,\ m,\ u_{0d} \text{ free in sign,} \end{array} \tag{18}$$

$$
\begin{aligned}
\left(E_{dj}^{B(BCC)}\right)_{\alpha}^{U} \quad & = \max \sum_{r=1}^{s} u_{rd} y_{rd} - u_{0d} \\
& \text{s.t. } \sum_{i=1}^{m} v_{id} x_{id} = 1 \\
& \left(\sum_{r=1}^{s} u_{rd} y_{rd} - u_{0d}\right) - e_{dd} \sum_{i=1}^{m} v_{id} x_{id} = 0 \\
& \sum_{i=1}^{m} v_{id} x_{ij} - \left(\sum_{r=1}^{s} u_{rd} y_{rj} - u_{0d}\right) \geq 0,\ j = 1, \ldots, n,\ j \neq d \\
& \sum_{r=1}^{s} u_{rd} y_{rj} - u_{0d} \geq 0,\ j = 1, \ldots, n \\
& (E_{dd})_{\alpha}^{L} \leq e_{dd} \leq (E_{dd})_{\alpha}^{U}, \\
& (X_{ij})_{\alpha}^{L} \leq x_{ij} \leq (X_{ij})_{\alpha}^{U},\ i = 1, \ldots, m,\ j = 1, \ldots, n \\
& (Y_{rj})_{\alpha}^{L} \leq y_{rj} \leq (Y_{rj})_{\alpha}^{U},\ r = 1, \ldots, s,\ j = 1, \ldots, n \\
& u_{rd},\ v_{id} \geq 0,\ r = 1, \ldots, s,\ i = 1, \ldots, m,\ u_{0d} \text{ free in sign,}
\end{aligned} \tag{19}
$$

In (18) and (19), there are nonlinear terms $u_{rd}y_{rj}$ and $v_{id}x_{ij}$. The variable transformation technique is used to transform the nonlinear terms into linear ones to simplify these two models. We let $\hat{x}_{ij} = v_{id}x_{ij}$, $\forall i, j$, and $\hat{y}_{rj} = u_{rd}y_{rj}$, $\forall r, j$, and multiply the constraints $(X_{ij})_{\alpha}^{L} \leq x_{ij} \leq (X_{ij})_{\alpha}^{U}$ and $(Y_{rj})_{\alpha}^{L} \leq y_{rj} \leq (Y_{rj})_{\alpha}^{U}$ by v_{id} and u_{rd}, respectively, and we have $v_{id}(X_{ij})_{\alpha}^{L} \leq \hat{x}_{ij} \leq v_{id}(X_{ij})_{\alpha}^{U}$, $\forall i, j$, and $u_{rd}(Y_{rj})_{\alpha}^{L} \leq\leq \hat{y}_{rj} \leq u_{rd}(Y_{rj})_{\alpha}^{U}$, $\forall r, j$. This transformation makes Models (18) and (19) become

$$
\begin{aligned}
\left(E_{dj}^{A(BCC)}\right)_{\alpha}^{L} \quad & = \min \sum_{r=1}^{s} \hat{y}_{rd} - u_{0d} \\
& \text{s.t. } \sum_{i=1}^{m} \hat{x}_{id} = 1 \\
& \left(\sum_{r=1}^{s} \hat{y}_{rd} - u_{0d}\right) - e_{dd} \sum_{i=1}^{m} \hat{x}_{id} = 0 \\
& \sum_{i=1}^{m} \hat{x}_{ij} - \left(\sum_{r=1}^{s} \hat{y}_{rj} - u_{0d}\right) \geq 0,\ j = 1, \ldots, n,\ j \neq d \\
& \sum_{r=1}^{s} \hat{y}_{rj} - u_{0d} \geq 0,\ j = 1, \ldots, n \\
& (E_{dd})_{\alpha}^{L} \leq e_{dd} \leq (E_{dd})_{\alpha}^{U} \\
& v_{id}(X_{ij})_{\alpha}^{L} \leq \hat{x}_{ij} \leq v_{id}(X_{ij})_{\alpha}^{U},\ i = 1, \ldots, m,\ j = 1, \ldots, n \\
& u_{rd}(Y_{rj})_{\alpha}^{L} \leq \hat{y}_{rj} \leq u_{rd}(Y_{rj})_{\alpha}^{U},\ r = 1, \ldots, s,\ j = 1, \ldots, n \\
& u_{rd},\ v_{id} \geq 0,\ r = 1, \ldots, s,\ i = 1, \ldots, m,\ u_{0d} \text{ free in sign,}
\end{aligned} \tag{20}
$$

$$
\begin{aligned}
\left(E_{dj}^{B(BCC)}\right)_{\alpha}^{U} \quad & = \max \sum_{r=1}^{s} u_{rd} y_{rd} - u_{0d} \\
& \text{s.t. } \sum_{i=1}^{m} \hat{x}_{id} = 1 \\
& \left(\sum_{r=1}^{s} \hat{y}_{rd} - u_{0d}\right) - e_{dd} \sum_{i=1}^{m} \hat{x}_{id} = 0 \\
& \sum_{i=1}^{m} \hat{x}_{ij} - \left(\sum_{r=1}^{s} \hat{y}_{rj} - u_{0d}\right) \geq 0,\ j = 1, \ldots, n,\ j \neq d \\
& \sum_{r=1}^{s} \hat{y}_{rj} - u_{0d} \geq 0,\ j = 1, \ldots, n \\
& (E_{dd})_{\alpha}^{L} \leq e_{dd} \leq (E_{dd})_{\alpha}^{U} \\
& v_{id}(X_{ij})_{\alpha}^{L} \leq \hat{x}_{ij} \leq v_{id}(X_{ij})_{\alpha}^{U},\ i = 1, \ldots, m,\ j = 1, \ldots, n \\
& u_{rd}(Y_{rj})_{\alpha}^{L} \leq \hat{y}_{rj} \leq u_{rd}(Y_{rj})_{\alpha}^{U},\ r = 1, \ldots, s,\ j = 1, \ldots, n \\
& u_{rd},\ v_{id} \geq 0,\ r = 1, \ldots, s,\ i = 1, \ldots, m,\ u_{0d} \text{ free in sign,}
\end{aligned} \tag{21}
$$

The nonlinear term $e_{dd}\sum_{i=1}^{m} \hat{x}_{id}$ in the second constraint of (20) and (21) cannot be linearized further. Fortunately, the term $e_{dd}\sum_{i=1}^{m} \hat{x}_{id}$ is bounded between $\sum_{i=1}^{m} v_{id}(X_{id})_{\alpha}^{L}(E_{dd})_{\alpha}^{L}$ and $\sum_{i=1}^{m} v_{id}(X_{id})_{\alpha}^{U}(E_{dd})_{\alpha}^{U}$. Since the nonlinearity of the second constraint is not strong in this case, any commercial software can solve this problem easily. At a specific α-level, the lower bound $\left(E_{dj}^{A(BCC)}\right)_{\alpha}^{L}$ and upper bound $\left(E_{dj}^{B(BCC)}\right)_{\alpha}^{U}$ of the cross-efficiency can be derived by solving (20) and (21), respectively. Similar to Equation (3b), we can obtain the final fuzzy cross-efficiency score via (22).

$$(E_j)_{\alpha} = [(E_j)_{\alpha}^{L}, (E_j)_{\alpha}^{U}] = \frac{1}{n}\sum_{d=1}^{n}[\left(E_{dj}^{A(BCC)}\right)_{\alpha}^{L}, \left(E_{dj}^{B(BCC)}\right)_{\alpha}^{U}] \tag{22}$$

For ranking the derived fuzzy cross-efficiencies, a fuzzy number ranking method is required to discriminate the DMUs. Some strategies for ranking fuzzy numbers [37–40] are introduced in the literature. Among these approaches, the Chen and Klein [37] method does not require the exact membership functions of the fuzzy numbers, and this method is suitable for ranking the fuzzy cross-efficiency scores. Chen and Klein [37] devised the following index for ranking fuzzy numbers:

$$I(\widetilde{E}_j) = \sum_{p=0}^{\infty}((E_j)_{\alpha_p}^{U} - \beta) / \left[\sum_{p=0}^{\infty}((E_j)_{\alpha_p}^{U} - \beta) - \sum_{p=0}^{\infty}((E_j)_{\alpha_p}^{L} - \gamma)\right] \tag{23}$$

where $\beta = \min_{j,\,p}\{(E_j)_{\alpha_p}^{L}\}$ and $\gamma = \max_{j,\,p}\{(E_j)_{\alpha_p}^{U}\}$. Chen and Klein [37] suggested that three or four α-levels are sufficient to discriminate the differences. However, from (23), more α-levels should be used to derive the ranking indices more accurately. Additionally, the larger is the value of the ranking index $I(\widetilde{E}_j)$, the larger is the fuzzy number. Based on the ranking index $I(\widetilde{E}_j)$, the rankings of DMUs are determined.

4. An Example

In the real-world applications, some data are uncertain and need to be estimated. For example, Liu [41] used the cost and floor space requirements as the inputs and the improvements in the qualitative factor, work-in-process (WIP), numbers of tardy jobs, and yield as the outputs to evaluate the performance of flexible manufacturing system (FMS) alternatives. The cost includes purchasing cost and the estimated operation and maintenance costs of an FMS alternative. Since the maintenance cost relies on the robot operating appropriately, they are not known beforehand. The item of cost is treated as an uncertain number. Because of the lack of sufficient historical cost data, estimating the probability distribution is not easy for this item. In the case of the absence of data, one may ask experts for their subjective estimates of the data. The triangular fuzzy number of the cost item can be established from the pessimistic, optimistic, and most likely estimates. The fuzzy input and output data are denoted as (a, b, c), where a, b, and c represent the pessimistic, most likely, and optimistic estimates of a fuzzy number, respectively.

To show the generality of the proposed method, we should have fuzzy and crisp input/output observations due to crisp observations can be treated as degenerated fuzzy numbers. In this example, we use the datasets of Liu [41], as listed in Table 1, to calculate the fuzzy BCC cross-efficiencies of the FMS alternatives. Ruiz et al. [30] and Liu and Lee [31] also used the datasets of Liu [41] to illustrate their ideas for the measurement of CCR cross-efficiency scores.

Before measuring the fuzzy BCC cross-efficiencies, we should calculate the fuzzy BCC efficiency scores of the FMS alternatives first. In this regard, Models (6) and (7) are used to calculate the lower bound $(E_{dd})_{\alpha}^{L}$ and upper bound $(E_{dd})_{\alpha}^{U}$ of the fuzzy BCC efficiency scores. Enumerating the calculation process for all FMS alternatives at α = 0.0, 0.1, . . . , 1.0, we obtain the fuzzy BCC efficiencies at eleven different α-levels, with the results shown in Table 2. Since the fuzzy BCC efficiency scores vary in ranges, the different values of the α-level show the various intervals of the efficiencies. Moreover, the smaller is the α-level, the wider is the interval. Especially α-level = 0 offers the broadest range that the BCC efficiency will appear, while α-level = 1.0 gives the most likely efficiency. For example, while the BCC efficiency of FMS 8 in Table 2 is fuzzy, its value cannot exceed 1.000 or fall below 0.859. At α-level = 1, this FMS's most likely efficiency value is 0.989. Among the twelve FMS alternatives, four FMS alternatives are being measured as BCC-efficient at all α-levels, namely FMS 2, 5, 6, and 9, although some of the input and output data are fuzzy, and their ranks are indistinguishable.

Table 1. Data used to measure the fuzzy cross-efficiency scores of FMS alternatives.

FMS	Inputs		Outputs			
	Cost	Space	Qualitative	WIP	No. of Tardy	Yield
1	(16.17, 17.02, 17.87)	5	42	(43.0, 45.3, 47.6)	(13.5, 14.2, 14.9)	(28.6, 30.1, 31.6)
2	(15.64, 16.46, 17.28)	4.5	39	(38.1, 40.1, 42.1)	(12.4, 13.0, 13.7)	(28.3, 29.8, 31.3)
3	(11.17, 11.76, 12.35)	6	26	(37.6, 39.6, 41.6)	(13.1, 13.8, 14.5)	(23.3, 24.5, 25.7)
4	(9.99, 10.52, 11.05)	4	22	(34.2, 36.0, 37.8)	(10.7, 11.3, 11.9)	(23.8, 25.0, 26.3)
5	(9.03, 9.50, 9.98)	3.8	21	(32.5, 34.2, 35.9)	(11.4, 12.0, 12.6)	(19.4, 20.4, 21.4)
6	(4.55, 4.79, 5.03)	5.4	10	(19.1, 20.1, 21.1)	(4.8, 5.0, 5.3)	(15.7, 16.5, 17.3)
7	(5.90, 6.21, 6.52)	6.2	14	(25.2, 26.5, 27.8)	(6.7, 7.0, 7.4)	(18.7, 19.7, 20.7)
8	(10.56, 11.12, 11.68)	6	25	(34.1, 35.9, 37.7)	(8.6, 9.0, 9.5)	(23.5, 24.7, 25.9)
9	(3.49, 3.67, 3.85)	8	4	(16.5, 17.4, 18.3)	(0.1, 0.1, 0.1)	(17.2, 18.1, 19.0)
10	(8.48, 8.93, 9.38)	7	16	(32.6, 34.3, 36.0)	(6.2, 6.5, 6.8)	(19.6, 20.6, 21.6)
11	(16.85, 17.74, 18.63)	7.1	43	(43.3, 45.6, 47.9)	(13.3, 14.0, 14.7)	(29.5, 31.1, 32.7)
12	(14.11, 14.85, 15.59)	6.2	27	(36.8, 38.7, 40.6)	(13.1, 13.8, 14.5)	(24.1, 25.4, 26.7)

Table 2. The bounds of the fuzzy BCC efficiency scores for FMS alternatives.

FMS		α = 0.0	α = 0.1	α = 0.2	α = 0.3	α = 0.4	α = 0.5	α = 0.6	α = 0.7	α = 0.8	α = 0.9	α = 1.0
1	L	0.977	0.977	0.977	0.985	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	U	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
2	L	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	U	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
3	L	0.868	0.880	0.892	0.911	0.929	0.949	0.968	0.993	1.000	1.000	1.000
	U	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
4	L	0.991	0.996	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	U	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
5	L	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	U	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
6	L	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	U	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
7	L	0.930	0.943	0.958	0.974	0.991	1.000	1.000	1.000	1.000	1.000	1.000
	U	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
8	L	0.859	0.867	0.876	0.884	0.893	0.902	0.915	0.931	0.949	0.969	0.989
	U	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.989
9	L	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	U	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
10	L	0.824	0.846	0.868	0.891	0.914	0.938	0.962	0.987	1.000	1.000	1.000
	U	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
11	L	0.891	0.900	0.909	0.918	0.927	0.937	0.946	0.956	0.968	0.995	1.000
	U	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
12	L	0.708	0.717	0.727	0.739	0.753	0.767	0.782	0.798	0.813	0.829	0.845
	U	1.000	1.000	1.000	1.000	0.974	0.948	0.923	0.900	0.878	0.862	0.845

After obtaining the fuzzy BCC efficiencies, every FMS alternative can apply (20) and (21) to calculate the associated fuzzy cross-efficiency scores. At a specific α-level, we can enumerate this process for all FMS alternatives to obtain the cross-efficiency score for FMS alternative j. Table 3 shows the lower bound and upper bound of the fuzzy cross efficiencies of the twelve FMS alternative, with the calculation results at α = 0.0, 0.1, . . . , 1. The last two rows are the lower bound and upper bound of the final cross-efficiencies of the twelve FMS alternative by applying (22). We can find that the efficiencies in the self-evaluation are greater than or equal to the corresponding efficiencies in the cross-evaluation. The obtained results coincide with the study of Liu and Lee [31], and this is because that the self-evaluation measured the efficiency scores from the most favorable weights of itself, while the cross-evaluation calculated the efficiency scores from the weights determined by all the other DMUs.

Table 3. The fuzzy cross-efficiency at eleven α-levels for FMS alternatives.

FMS		$\alpha = 0.0$	$\alpha = 0.1$	$\alpha = 0.2$	$\alpha = 0.3$	$\alpha = 0.4$	$\alpha = 0.5$	$\alpha = 0.6$	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1.0$
1	L	0.339	0.348	0.457	0.515	0.584	1.000	0.742	0.448	0.366	0.506	0.314
	U	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
2	L	0.342	0.353	0.461	0.522	0.591	1.000	0.752	0.454	0.370	0.512	0.317
	U	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
3	L	0.346	0.358	0.464	0.529	0.597	1.000	0.757	0.460	0.374	0.519	0.319
	U	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
4	L	0.349	0.363	0.468	0.537	0.602	1.000	0.763	0.467	0.378	0.525	0.322
	U	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
5	L	0.352	0.366	0.472	0.544	0.607	1.000	0.768	0.473	0.382	0.532	0.325
	U	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
6	L	0.355	0.370	0.476	0.552	0.612	1.000	0.774	0.479	0.386	0.539	0.328
	U	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
7	L	0.358	0.373	0.480	0.560	0.617	1.000	0.779	0.486	0.390	0.546	0.331
	U	1.000	1.000	0.993	1.000	1.000	1.000	1.000	0.999	1.000	0.986	1.000
8	L	0.361	0.377	0.484	0.566	0.623	1.000	0.785	0.493	0.394	0.553	0.333
	U	1.000	1.000	0.975	1.000	1.000	1.000	1.000	0.988	1.000	0.971	0.995
9	L	0.365	0.380	0.488	0.571	0.628	1.000	0.790	0.499	0.398	0.560	0.336
	U	1.000	1.000	0.960	1.000	1.000	1.000	1.000	0.977	1.000	0.956	0.986
10	L	0.368	0.383	0.492	0.576	0.634	1.000	0.796	0.506	0.403	0.567	0.339
	U	1.000	1.000	0.951	1.000	1.000	1.000	1.000	0.966	1.000	0.941	0.977
11	L	0.371	0.387	0.496	0.581	0.639	1.000	0.801	0.513	0.407	0.575	0.342
	U	1.000	0.990	0.942	1.000	1.000	1.000	1.000	0.955	1.000	0.926	0.967
12	L	0.339	0.348	0.457	0.515	0.584	1.000	0.742	0.448	0.366	0.506	0.314
	U	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
Final	L	0.499	0.504	0.509	0.513	0.517	0.522	0.526	0.530	0.535	0.539	0.543
	U	0.993	0.992	0.990	0.989	0.988	0.986	0.984	0.979	0.974	0.969	0.964

To discriminate among the twelve FMS alternatives, we employ Equation (23) to derive the ranking indices of the alternatives, and the number of α-cuts is set to eleven. The calculation results are listed in Table 4 under the headings of "Index". Based on the derived ranking indices, the rankings of FMS alternatives are determined and shown in the last column of Table 4. FMS 7 is the preferred one.

Table 4. The bounds of the final fuzzy cross-efficiency scores and their ranks of FMS alternatives.

FMS		$\alpha = 0.0$	$\alpha = 0.1$	$\alpha = 0.2$	$\alpha = 0.3$	$\alpha = 0.4$	$\alpha = 0.5$	$\alpha = 0.6$	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1.0$	Index	Rank
1	L	0.362	0.373	0.383	0.399	0.424	0.434	0.443	0.453	0.461	0.466	0.472	0.527	10
	U	1.000	0.917	0.917	1.000	0.998	0.996	0.993	0.989	0.986	0.981	0.974		
2	L	0.342	0.352	0.362	0.373	0.382	0.392	0.401	0.409	0.416	0.423	0.428	0.504	12
	U	0.993	0.990	0.987	0.981	0.972	0.962	0.950	0.937	0.925	0.914	0.904		
3	L	0.384	0.403	0.424	0.450	0.477	0.510	0.544	0.585	0.610	0.627	0.644	0.572	5
	U	1.000	1.000	1.000	1.000	0.998	0.996	0.994	0.974	0.964	0.958	0.950		
4	L	0.414	0.431	0.449	0.462	0.475	0.486	0.497	0.505	0.512	0.519	0.527	0.559	8
	U	1.000	1.000	1.000	1.000	0.998	0.996	0.994	0.992	0.990	0.988	0.986		
5	L	0.463	0.469	0.475	0.480	0.486	0.491	0.497	0.502	0.508	0.513	0.519	0.563	6
	U	1.000	1.000	1.000	0.997	0.995	0.992	0.989	0.986	0.984	0.981	0.976		
6	L	0.499	0.504	0.509	0.513	0.517	0.522	0.526	0.530	0.535	0.539	0.543	0.575	4
	U	0.993	0.992	0.990	0.989	0.988	0.986	0.984	0.979	0.974	0.969	0.964		
7	L	0.471	0.498	0.530	0.563	0.598	0.624	0.637	0.651	0.664	0.678	0.693	0.622	1
	U	1.000	1.000	0.999	0.997	0.995	0.993	0.991	0.989	0.988	0.987	0.984		
8	L	0.402	0.423	0.446	0.469	0.494	0.521	0.553	0.596	0.665	0.781	0.917	0.600	2
	U	0.917	1.000	1.000	1.000	0.997	0.995	0.993	0.990	0.988	0.979	0.917		

Table 4. *Cont.*

FMS		α = 0.0	α = 0.1	α = 0.2	α = 0.3	α = 0.4	α = 0.5	α = 0.6	α = 0.7	α = 0.8	α = 0.9	α = 1.0	Index	Rank
9	L	0.335	0.342	0.348	0.353	0.359	0.364	0.369	0.375	0.381	0.386	0.392	0.506	11
	U	1.000	0.998	0.996	0.993	0.991	0.989	0.987	0.984	0.978	0.971	0.957		
10	L	0.426	0.451	0.478	0.505	0.534	0.568	0.606	0.647	0.676	0.693	0.710	0.599	3
	U	1.000	1.000	1.000	0.999	0.996	0.993	0.986	0.964	0.944	0.932	0.920		
11	L	0.373	0.393	0.413	0.434	0.456	0.478	0.501	0.523	0.550	0.595	0.619	0.558	9
	U	1.000	1.000	1.000	1.000	0.998	0.996	0.994	0.992	0.987	0.956	0.927		
12	L	0.358	0.376	0.395	0.373	0.447	0.478	0.509	0.543	0.600	0.670	0.782	0.559	7
	U	1.000	1.000	1.000	1.000	0.998	0.996	0.994	0.986	0.958	0.909	0.782		

5. Conclusions

In the literature, the cross-efficiency evaluation is proposed to improve the discriminative power among DMUs. This evaluation method is mostly used for DEA models under the assumption of constant returns to scale. However, due to the negative efficiencies in the evaluation process, the cross-evaluation for input-oriented BCC models receives little attention in the literature. In particular, when observations are uncertain and expressed by fuzzy numbers, the problem becomes a cross-efficiency evaluation with variable returns to scale and fuzzy observations. Nevertheless, existing fuzzy cross-evaluation methods are restricted to production technologies with constant returns to scale. This paper develops an evaluation method to calculate the fuzzy cross-efficiencies for the input-oriented BCC model with fuzzy observations.

In the proposed model, a set of secondary goals is added to tackle multiple optimal weights in calculating the cross-efficiency. Moreover, since this model considers all possible weights of all DMUs in the weight space, a choice of weights is not required. The most significant merit is that this model does not produce negative efficiencies, making it appropriate for variable returns to scale. A pair of two-level mathematical programs is developed to calculate the fuzzy cross-efficiency. The lower bound and upper bound of the fuzzy cross-efficiency can be easily derived by solving this pair of mathematical programs. The proposed method is applied to a case involving selecting the most efficient FMS system for production. The calculated results help identify the top-ranked FMS system in the fuzzy environment.

With the approach developed in this paper, we can measure the fuzzy cross-efficiency scores for cases of variable returns to scale with fuzzy observation and apply to different fields of engineering applications (e.g., [42–47]) for decision-making. In this paper, the inputs and outputs are treated as triangular fuzzy numbers. Nevertheless, any convex fuzzy number can be applied in this method since convex fuzzy numbers can be represented as forms of α-level sets.

Author Contributions: Conceptualization, S.-C.W.; Formal analysis, S.-C.W. and T.L.; Methodology, S.-C.W. and T.L.; Resources, S.-T.L.; Software, S.-T.L.; Writing—original draft, T.L.; Funding acquisition, S.-T.L. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the Ministry of Science and Technology of Taiwan, under grant No. MOST 109-2410-H-238-003.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: The utilized data are available under request.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Cooper, W.W.; Seiford, L.M.; Zhu, J. Data Envelopment Analysis: History, Models, and Interpretations. In *Handbook on Data Envelopment Analysis. International Series in Operations Research & Management Science*; Cooper, W.W., Seiford, L.M., Zhu, J., Eds.; Springer: Boston, MA, USA, 2011.
2. Cooper, W.W.; Seiford, L.M.; Tone, K. *Data Envelopment Analysis: A Comprehensive Text with Models, Applications, References and DEA-Solver Software*; Kluwer Academic Publishers: Boston, MA, USA, 2000.
3. Liu, J.S.; Lu, L.Y.Y.; Lu, W.M.; Lin, B.J.Y. A survey of DEA applications. *Omega* **2013**, *41*, 893–902. [CrossRef]
4. Kao, C. Network data envelopment analysis: A review. *Eur. J. Oper. Res.* **2014**, *239*, 1–16. [CrossRef]
5. Emrouznejad, A.; Yang, G. A survey and analysis of the first 40 years of scholarly literature in DEA: 1978–2016. *Socio-Econ. Plan. Sci.* **2018**, *61*, 1–5. [CrossRef]
6. Sexton, T.R.; Silkman, R.H.; Hogan, A.J. Data envelopment analysis: Critique and extensions. In *Measuring Efficiency: An Assessment of Data Envelopment Analysis*; Silkman, R.H., Ed.; Jossey-Bass: San Francisco, CA, USA, 1986.
7. Doyle, J.R.; Green, R.H. Efficiency and cross efficiency in DEA: Derivations, meanings and uses. *J. Oper. Res. Soc.* **1994**, *45*, 567–578. [CrossRef]
8. Anderson, T.R.; Hollingsworth, K.B.; Inman, L.B. The fixed weighting nature of a cross-evaluation model. *J. Prod. Anal.* **2002**, *17*, 249–255. [CrossRef]
9. Örkcü, H.H.; Ünsal, M.G.; Bal, H. A modification of a mixed integer linear programming (MILP) model to avoid the computational complexity. *Ann. Oper. Res.* **2015**, *235*, 599–623. [CrossRef]
10. Wu, J.; Chu, J.; Sun, J.; Zhu, Q.; Liang, L. Extended secondary goal models for weights selection in DEA cross-efficiency evaluation. *Comput. Ind. Eng.* **2016**, *93*, 143–151. [CrossRef]
11. Kao, C.; Liu, S.T. Cross efficiency measurement and decomposition in two basic network systems. *Omega* **2019**, *83*, 70–79. [CrossRef]
12. Liang, L.; Wu, J.; Cook, W.D.; Zhu, J. The DEA game cross efficiency model and its Nash equilibrium. *Oper. Res.* **2008**, *56*, 1278–1288. [CrossRef]
13. Lam, K.F. In the determination weight sets to compute cross-efficiency ratios in DEA. *J. Oper. Res. Soc.* **2010**, *61*, 134–143. [CrossRef]
14. Wang, Y.M.; Chin, K.S. A neural DEA model for cross-efficiency evaluation and its extension. *Expert Syst. Appl.* **2010**, *37*, 3666–3675. [CrossRef]
15. Ramón, N.; Ruiz, J.L.; Sirvent, I. Reducing differences between profiles of weights: A "peer restricted" cross-efficiency evaluation. *Omega* **2011**, *39*, 634–641. [CrossRef]
16. Wang, Y.M.; Chin, K.S.; Wang, S. DEA models for minimizing weight disparity in cross-efficiency evaluation. *J. Oper. Res. Soc.* **2012**, *63*, 1079–1088. [CrossRef]
17. Wang, Y.M.; Chin, K.S. The use of OWA operator weights for cross-efficiency aggregation. *Omega* **2011**, *39*, 493–503. [CrossRef]
18. Oukil, A. Ranking via composite weighting schemes under a DEA cross-evaluation framework. *Comput. Ind. Eng.* **2018**, *117*, 217–224. [CrossRef]
19. Oral, M.; Amin, G.R.; Oukil, A. Cross-efficiency in DEA: A maximum resonated appreciative model. *Measurement* **2015**, *63*, 159–167. [CrossRef]
20. Al-Siyabi, M.; Amin, G.R.; Bose, S.; Al-Masroori, H. Peer-judgment risk minimization using DEA cross-evaluation with an application in fishery. *Ann. Oper. Res.* **2019**, *274*, 39–55. [CrossRef]
21. Yang, F.; Ang, S.; Xia, Q.; Yang, C. Ranking DMUs by using interval DEA cross efficiency matrix with acceptability analysis. *Eur. J. Oper. Res.* **2012**, *223*, 483–488. [CrossRef]
22. Alcaraz, J.; Ramón, N.; Ruiz, J.L.; Sirvent, I. Ranking ranges in cross-efficiency evaluations. *Eur. J. Oper. Res.* **2013**, *226*, 516–521. [CrossRef]
23. Ramón, N.; Ruiz, J.L.; Sirvent, I. Dominance relations and ranking of units by using interval number ordering with cross-efficiency intervals. *J. Oper. Res. Soc.* **2014**, *65*, 1336–1343. [CrossRef]
24. Liu, S.T. A DEA ranking method based on cross-efficiency intervals and signal-to-noise ratio. *Ann. Oper. Res.* **2018**, *261*, 207–232. [CrossRef]
25. Soares de Mello, J.C.C.B.; Meza, L.A.; Quintanilha da Silveira, J.; Gomes, E.G. About negative efficiencies in Cross Evaluation BCC input oriented model. *Eur. J. Oper. Res.* **2013**, *229*, 732–737. [CrossRef]
26. Lim, S.; Zhu, J. DEA cross-efficiency evaluation under variable returns to scale. *J. Oper. Res. Soc.* **2015**, *66*, 476–487. [CrossRef]
27. Kao, C.; Liu, S.T. A slacks-based measure model for calculating cross efficiency in data envelopment analysis. *Omega* **2020**, *95*, 102192. [CrossRef]
28. Dotoli, M.; Epicoco, N.; Falagario, M.; Sciancalepore, F. A cross-efficiency fuzzy data envelopment analysis technique for performance evaluation of decision making units under uncertainty. *Comput. Ind. Eng.* **2015**, *79*, 103–114. [CrossRef]
29. Chen, L.; Wang, Y.M. Data envelopment analysis cross-efficiency model in fuzzy environments. *J. Intell. Fuzzy Syst.* **2016**, *30*, 2601–2609. [CrossRef]
30. Ruiz, J.L.; Sirvent, I. Fuzzy cross-efficiency evaluation: A possibility approach. *Fuzzy Optim. Decis. Mak.* **2017**, *16*, 111–126. [CrossRef]

31. Liu, S.T.; Lee, Y.C. Fuzzy measures for fuzzy cross efficiency in data envelopment analysis. *Ann. Oper. Res.* **2021**, *300*, 369–398. [CrossRef]
32. Hatami-Marbini, A.; Emrouznejad, A.; Tavana, M. A taxonomy and review of the fuzzy data envelopment analysis literature: Two decades in the making. *Eur. J. Oper. Res.* **2011**, *214*, 457–472. [CrossRef]
33. Emrouznejad, A.; Tavana, M. *Performance Measurement with Fuzzy Data Envelopment Analysis*; Springer: Heidelberg, Germany, 2014.
34. Banker, R.D.; Charnes, A.; Cooper, W.W. Some models for estimating technical and scale inefficiencies in data envelopment analysis. *Manag. Sci.* **1984**, *30*, 1078–1092. [CrossRef]
35. Charnes, A.; Cooper, W.W.; Rhodes, E. Measuring the efficiency of decision making units. *Eur. J. Oper. Res.* **1978**, *2*, 429–444. [CrossRef]
36. Kao, C.; Liu, S.T. Fuzzy efficiency measures in data envelopment analysis. *Fuzzy Sets Syst.* **2000**, *113*, 427–437. [CrossRef]
37. Chen, C.B.; Klein, C.M. A simple approach to ranking a group of aggregated utilities. *IEEE Trans. Syst. ManCybern. Part B* **1997**, *27*, 6–35.
38. Chu, T.C.; Tsao, C.T. Ranking fuzzy numbers with an area between the centroid point and original point. *Comput. Math. Appl.* **2002**, *43*, 111–117. [CrossRef]
39. Abbasbandy, S.; Hahhari, T. A new approach for ranking of trapezoidal fuzzy numbers. *Comput. Math. Appl.* **2009**, *57*, 413–419. [CrossRef]
40. Boulmakou1, A.; Laarabi, M.H.; Sacile, R.; Garbolino, E. An original approach to ranking fuzzy numbers by inclusion index and Biset Encoding. *Fuzzy Optim. Decis. Mak.* **2017**, *16*, 23–49. [CrossRef]
41. Liu, S.T. A fuzzy DEA/AR approach to the selection of flexible manufacturing systems. *Comput. Ind. Eng.* **2008**, *54*, 66–76. [CrossRef]
42. Jiang, H.; He, M.; Xi, Y.; Zeng, J. Machine-Learning-Based User Position Prediction and Behavior Analysis for Location Services. *Information* **2021**, *12*, 180. [CrossRef]
43. Li, X.; Huang, Y.; Heng, W.; Wu, J. Machine Learning-Inspired Hybrid Precoding for mmWave MU-MIMO Systems with Domestic Switch Network. *Sensors* **2021**, *21*, 3019. [CrossRef] [PubMed]
44. Salvati, L.; d'Amore, M.; Fiorentino, A.; Pellegrino, A.; Sena, P.; Villecco, F. On-Road Detection of Driver Fatigue and Drowsiness during Medium-Distance Journeys. *Entropy* **2021**, *23*, 135. [CrossRef]
45. Wang, C.-N.; Nguyen, H.-P.; Chang, C.-W. Environmental Efficiency Evaluation in the Top Asian Economies: An Application of DEA. *Mathematics* **2021**, *9*, 889. [CrossRef]
46. Xing, J.; Song, W.; Villecco, F. Generalized Cauchy Process: Difference Iterative Forecasting Model. *Fractal Fract.* **2021**, *5*, 38. [CrossRef]
47. Yang, H.; Wei, Y.; Zhang, W.; Ai, Y.; Ye, Z.; Wang, L. Development of Piezoelectric Energy Harvester System through Optimizing Multiple Structural Parameters. *Sensors* **2021**, *21*, 2876. [CrossRef]