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On Galilean Invariant and Energy Preserving BBM-Type Equations

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Abstract: We investigate a family of higher-order BENJAMIN–BONA–MAHONY-type equations, which appeared in the course of study towards finding a GALILEI-invariant, energy-preserving long wave equation. We perform local symmetry and conservation laws classification for this family of Partial Differential Equations (PDEs). The analysis reveals that this family includes a special equation which admits additional, higher-order local symmetries and conservation laws. We compute its solitary waves and simulate their collisions. The numerical simulations show that their collision is elastic, which is an indication of its $S-$integrability. This particular PDE turns out to be a rescaled version of the celebrated CAMASSA–HOLM equation, which confirms its integrability.

Keywords: nonlinear dispersive waves; BENJAMIN–BONA–MAHONY equation; conservation laws; symmetries

MSC: 76M60 (primary); 37K05; 35Q35 (secondary)

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1. Introduction

The famous BENJAMIN–BONA–MAHONY (BBM) equation was derived for the first time D. PEREGRINE [1], reintroduced later in [2]. In scaled variables, it is given by

$$u_t + u_x + u u_x - u_{xxt} = 0, \tag{1}$$

where subscripts $(-)_x$ and $(-)_t$ denote ∂_x and ∂_t—the partial derivative with respect to the spatial x and temporal t variables respectively. One of the main physical drawbacks of the BBM equation is the absence of the GALILEAN invariance. This question has already been raised in [3] and a remedy was proposed. It consisted in adding higher order terms that help recover this symmetry. However, this approach led to the simultaneous loss of another physically important property—the presence of the energy conservation law. Various investigations around these questions [4] led the authors of this communication to consider the following family of partial differential equations (PDEs):

$$u_t + u_x + u u_x - u_{xxt} - \alpha (u u_{xxx} + 2 u_x u_{xx}) = 0, \tag{2}$$

with $\alpha \in \mathbb{R}$. It is obvious that PDE (2) reduces to the BBM (1) when $\alpha = 0$. It is straightforward to see that the parameter α in Equation (2) cannot be removed by a rescaling or another simple equivalence transformation (see, e.g., [5]). Below, we study symmetries, conservation laws, and solitary wave solutions to family (2). The conservation laws of a variable coefficients BBM equation were studied in [6].

In particular, we show that additional structures and exceptional elastic collision-type numerical solution behavior arise for the PDE (2) corresponding to $\alpha = 1/3$:

$$u_t + u_x + u u_x - u_{xxt} - \frac{1}{3} u u_{xxx} - \frac{2}{3} u_x u_{xx} = 0. \quad (3)$$

We will refer to this integrable PDE as the eBBM$_{\frac{1}{3}}$ equation.

The present manuscript is organized as follows. In Section 2 the symmetries, HAMILTONIAN structure, and conservation laws to Equation (2) are studied for generic values of the parameter α. The potential symmetries are studied in Section 3. The classification of symmetries and conservation laws with respect to parameter α is done in Section 4. Finally, the solitary wave collisions are briefly discussed in Section 5. The main conclusions of this study are outlined in Section 6.

2. Symmetries, Conservation Laws, and Hamiltonian Structure of the α-Family

The point symmetries of the family of PDEs (2) holding for all α are given by the generators

$$\mathrm{X}_1 = \partial_t, \quad \mathrm{X}_2 = \partial_x, \quad (4)$$

$$\mathrm{X}_3 = (\alpha - 1) t \partial_t + \alpha t \partial_x + (1 + (1 - \alpha) u) \partial_u. \quad (5)$$

In particular, when $\alpha = 0$, the generator X_3 yields the scaling symmetry group holding for the BBM Equation (1), and when $\alpha = 1$, X_3 yields the GALILEI group. The latter case is the only GALILEI-invariant representative of the PDE family (2). A direct computation shows that there are no higher-order symmetries for any member of the family (2) depending on derivatives of u up to the second-order, i.e., with generators of the form

$$\hat{\mathrm{X}} = \zeta(x, t, u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}) \partial_u,$$

except the point symmetries (4) and (5). Moreover, one can show there are no symmetries holding for an arbitrary α with components depending on x, t, u, and x-derivatives of u up to order five. (It will be shown below that the eBBM$_{\frac{1}{3}}$ Equation (3) does admit a higher-order symmetry). In the case of an arbitrary α, all PDEs in the α-family (2) admit local conservation laws with multipliers

$$\Lambda_1 = 1, \quad \Lambda_2 = u, \quad \Lambda_3 = u^2 - 2 u_{xt} - \alpha (u_x^2 + 2 u u_{xx}), \quad (6)$$

corresponding respectively to the conservation of the momentum and the energy:

$$\mathscr{M}(t) = \int_a^b (u - u_{xx}) \, \mathrm{d}x, \quad \mathscr{E}(t) = \int_a^b \frac{1}{2} (u^2 + u_x^2) \, \mathrm{d}x,$$

as well as the α-dependent conserved quantity

$$\mathscr{N}_\alpha(t) = \int_a^b \left(\frac{1}{3} u^3 + (\alpha u - 1) u_x^2 \right) \mathrm{d}x.$$

The latter, in a linear combination with energy $\mathscr{E}(t)$, yields a conserved quantity

$$\mathscr{H}_\alpha(t) = \frac{1}{2} \int_a^b \left(u^2 + \frac{1}{3} u^3 + \alpha u u_x^2 \right) \mathrm{d}x,$$

which defines a HAMILTONIAN for the whole α-family (2). Indeed, the PDEs (2) can be written in the HAMILTONIAN form

$$u_t = \mathbb{J} \frac{\delta \mathscr{H}_\alpha}{\delta u},$$

with the co-symplectic operator $\mathbb{J}$ shared with the original BBM Equation (1):

$$\mathbb{J} := (\mathbb{1} - \partial_x^2)^{-1} \cdot (-\partial_x).$$

3. The Potential α-Family and Its Lagrangian Structure

It is possible to show that similarly to the KORTEWEG–DE VRIES (KdV) and the BBM equations, the PDEs of the α-family (2), as it stands, does not have a self-adjoint FRÉCHET derivative, and hence does not directly arise from a variational principle for any LAGRANGIAN functional. This can also be seen from the obvious differences between conservation law multipliers (6) and the evolutionary forms of symmetries (4) and (5), which would contradict the first NOETHER's theorem (cf. [7]).

However, the α-family (2) has a potential LAGRANGIAN formulation. In fact, upon the introduction of the potential variable $w = w(x, t)$ defined by

$$u \overset{\text{def}}{:=} w_x, \tag{7}$$

the α-family (2) is written in the potential form

$$w_{tx} + w_{xx} + w_x w_{xx} - w_{xxxt} - \alpha(w_x w_{xxxx} + 2w_{xx}w_{xxx}) = 0, \tag{8}$$

which, for every α, is nonlocally related to the corresponding PDE (2) (see, e.g., [8]). One can show that the linearization of Equation (8) is self-adjoint. The homotopy formula ([7] Theorem 5.92) can be used to construct the LAGRANGIAN, which up to equivalence is readily found to be

$$\mathscr{L}[w] \overset{\text{def}}{:=} -\frac{1}{2}\int_{\mathbb{R}}\left(\frac{1}{3}w_x^3 + w_x^2 + w_x w_t + w_{xx}w_{xt} + \alpha w_x w_{xx}^2\right)\mathrm{d}x. \tag{9}$$

It is straightforward to check that the EULER–LAGRANGE equation $\delta\mathscr{L}/\delta w = 0$ yields the potential Equation (8) for each value of the parameter α.

It is of interest to briefly consider symmetries and conservation laws of the potential α-family (8). The first reason for that is that the potential α-family Equation (8) are nonlocally related to the PDEs (2), and hence the local symmetries and conservation laws of (2) vs. (8) can differ (cf. [8]). Secondly, since the PDEs (8) arise from a variational principle, the first NOETHER's theorem will provide an evident relation between their local symmetries and conservation laws. Seeking local symmetries of the potential α-family Equation (8) in evolutionary form

$$\hat{\mathrm{X}} = \zeta(x, t, w, w_x, w_{xx}, w_{xxt}, w_{xxx})\partial_w, \tag{10}$$

holding for all $\alpha \in \mathbb{R}$, we find the admitted symmetry generator components:

$$\begin{aligned}
\zeta_1 &= w_t, && (11)\\
\zeta_2 &= w_x, &&\\
\zeta_3 &= x - (\alpha-1)(w + tw_t) - \alpha t w_x, && (12)\\
\zeta_4 &= 2(x - tw_x) + (\alpha-1)^2 t(w_{xx}^2 + 2w_x w_{xxx}) && (13)\\
&\quad + (\alpha-1)\big(2w + t(2w_{xxt} + w_{xx}^2 + 2w_x w_{xxx} - w_x^2)\big), &&\\
\zeta_5 &= w_x^2 - 2w_{xxt} - \alpha(w_{xx}^2 + 2w_x w_{xxx}), && (14)\\
\zeta_F &= F(t). && (15)
\end{aligned}$$

The generators (10) with ζ_1 and ζ_2 correspond to the time and space translation point symmetries $\mathrm{X}_1 = \partial_t$, $\mathrm{X}_2 = \partial_x$ of the $\alpha-$family (2). The local symmetry with ζ_3 (12) corresponds to the point symmetry $\mathrm{X}_3 = \alpha t\partial_x + (\alpha-1)t\partial_t + (x - (\alpha-1)w)\partial_w$ of (2), which includes the GALILEI group only when $\alpha = 1$. The generator with ζ_4 (13) is generally a higher-order symmetry generator, which also degenerates into the GALILEI group $\mathrm{X}_4 = t\partial_x + \partial_w$ when $\alpha = 1$. For $\alpha \neq 1$, the generator ζ_4 explicitly involves the potential variable w, and hence is a nonlocal higher-order symmetry generator of the corresponding PDE from the α-family (2) (cf. [8]). The local generator $\hat{\mathrm{X}}_5$ with symmetry component ζ_5 (14) is a higher-order symmetry of the potential equation for each α. The local symmetry generator with ζ_F (15) admitted by the potential family (8) is its point symmetry $w \leftarrow w + F(t)$, with no projection on the space of variables (x, t, u) of Equation (2), related to the non-uniqueness of the definition (7) of the potential up to an arbitrary function of time.

Since the potential family (8) arises from a LAGRANGIAN, by NOETHER's theorem, its variational symmetries yield conservation law multipliers. A brief computation yields

$$\Lambda_1 = \zeta_1, \qquad \Lambda_2 = \zeta_2, \qquad \Lambda_5 = \zeta_5, \qquad \Lambda_F = \zeta_F$$

which are local conservation law multipliers for all PDEs in the potential α-family (8), whereas the local symmetries with ζ_3 and ζ_4 are not variational, and thus have no corresponding conservation laws. The multiplier Λ_F corresponds to a conservation law

$$\mathcal{D}_t\left(F(t)(w_x - w_{xxx})\right) + \mathcal{D}_x\Big(-F'(t)(w - w_{xx}) + F(t)\Big(w_x(1 - \alpha w_{xxx}) + \frac{1}{2}(w_x^2 - \alpha w_{xx}^2)\Big)\Big) = 0, \tag{16}$$

holding for an arbitrary function $F(t)$, which is a nonlocal conservation law of the α-family (2) when $F(t) \neq 1$. Indeed, the spatial flux in (16) is explicitly dependent on the potential $w = \int u\,\mathrm{d}x$ itself, and hence is not equivalent to any local expression in terms of u. The conserved density in (16),

$$\rho_F \stackrel{\text{def}}{:=} F(t)(w_x - w_{xxx}) \equiv F(t)(u - u_{xx})$$

is, however, a local quantity in terms of the dependent variable u of (8).

An additional important conservation law arises for the potential $\alpha-$family (8). In fact, the left-hand side of each Equation (8) is a total $x-$derivative:

$$\mathcal{D}_x\left(w_t + w_x + \frac{1}{2}w_x^2 - w_{xxt} - \alpha\left(\frac{1}{2}w_{xx}^2 + w_x w_{xxx}\right)\right) = 0,$$

which can be written as

$$w_t + w_x + \frac{1}{2}w_x^2 - w_{xxt} - \alpha\left(\frac{1}{2}w_{xx}^2 + w_x w_{xxx}\right) = C(t), \tag{17}$$

for an arbitrary function $C(t)$.

In addition to the potential form (8), other potential systems non-locally related to the α-family (2) can be constructed, using, for example, the linearly independent conservation law multipliers (6), to obtain independent singlet, couplet, and triplet potential systems [8,9]. Their study may lead to new analytical results of α-family (2), including the original BBM Equation (1).

4. Symmetry and Conservation Law Classifications for the α-Family

Since the PDE family (2) and its potential form (8) involve a parameter α, its properties can be classified according to α (see e.g., [8] and references therein). In Section 2, symmetries and conservation laws that arise for an arbitrary α were listed; we now consider symmetry and conservation law classifications. First, classifying local conservation laws of the α-family (2) with third-order multipliers, the following cases are distinguished:

1. In the general case with arbitrary α, as reported above, only three common local conservation laws arise, with multipliers (6);
2. An additional local conservation law:

$$\mathcal{D}_t\left(\left(t - x + \frac{1}{2}tu\right)(u - u_{xx})\right) + \mathcal{D}_x\left(\frac{1}{3}tu^3 - \left(\frac{1}{2}x + t(u_{xx} - 1)\right)u^2 + \left((t - x)(u_{xx} - 1) + \frac{1}{2}(u_x + tu_{tx})\right)u + \frac{1}{2}(2 + tu_t + (x - t)u_x)u_x\right) = 0 \quad (18)$$

with multiplier

$$\Lambda_4 = x - t(u + 1)$$

arises when $\alpha = 1$; this is the case of the so-called extended BBM (eBBM) equation (see [4]);

3. The case when $\alpha = \frac{1}{3}$, that is, the eBBM$_{\frac{1}{3}}$ Equation (3), turns out to be the only other special case when additional conservation laws arise.

In terms of local conservation laws in the specified multiplier ansatz, the eBBM$_{\frac{1}{3}}$ Equation (3) can be shown to admit and three additional local conservation law multipliers

$$\Lambda_4^{\left(\frac{1}{3}\right)} = 5u^3 + (9 - 4u_{xx})u^2 - (u_x^2 + 6u_{xt})u + 6(u_x u_t + 3u_{tt}), \quad (19)$$

$$\Lambda_5^{\left(\frac{1}{3}\right)} = (2(u - u_{xx}) + 3)^{-1/2}, \quad (20)$$

$$\Lambda_6^{\left(\frac{1}{3}\right)} = \frac{u_{xxxx} - \frac{1}{2}(2u + 3)}{\left(2(u - u_{xx}) + 3\right)^{5/2}} + \frac{5}{2}\frac{u_{xxx}^2 + u_x(u_x - 2u_{xxx})}{\left(2(u - u_{xx}) + 3\right)^{7/2}}. \quad (21)$$

The local conservation law with the multipliers (19)–(21) have the form:

$$\mathcal{D}_t\,\mathscr{T}_i[u] + \mathcal{D}_x\,\mathscr{F}_i[u] = 0, \quad i = 4, 5, 6,$$

with the corresponding conserved quantities

$$\mathscr{Q}_i^{\left(\frac{1}{3}\right)}(t) \overset{\text{def}}{:=} \int_a^b \mathscr{T}_i[u]\,\mathrm{d}x.$$

The conserved densities for the respective conservation laws are given by

$$\mathscr{T}_4[u] \overset{\text{def}}{:=} 5u^4 + 12u^3 + (26u_x^2 + 4u_{xx}^2)u^2 + (u_x^2 u_{xx} + 24u_x^2 + 36u_{tt})u - 36u_{tt}u_{xx},$$

$$\mathscr{T}_5[u] \overset{\text{def}}{:=} \sqrt{2(u - u_{xx}) + 3},$$

$$\mathscr{T}_6[u] \overset{\text{def}}{:=} (2(u - u_{xx}) + 3)^{-5/2}\cdot\left(2\left(2(u - u_{xx}) + 3\right)^2 + (2u + 3)\left(2(u - u_{xx}) + 3\right) + 3(u_{xxx}^2 - u_x^2)\right).$$

It is also possible to show by a direct verification that the exceptional eBBM$_{\frac{1}{3}}$ Equation (3) admits a third-order local evolutionary symmetry generator:

$$\hat{X}_4 = \frac{u_x - u_{xxx}}{(2(u - u_{xx}) + 3)^{3/2}}\,\partial_u. \quad (22)$$

The existence of this higher-order symmetry and the above higher-order conservation laws point at the possibility of integrability of the eBBM $_{\frac{1}{3}}$ Equation (3), in particular, in the light of the FOKAS conjecture in [10], and the similarities between the eBBM $_{\frac{1}{3}}$ and the integrable short pulse equation [4].

5. Solitary Waves Collision

Motivated by the exceptional symmetry and conservation law properties of eBBM $_{\frac{1}{3}}$ Equation (3), we decided to study the interaction of solitary waves in this equation. For the sake of comparison, we shall also perform the same simulations in the eBBM equation given by (2) with $\alpha = 1$. This choice of α is also interesting because it is a GALILEI-invariant model. For extended numerical comparisons, see [4].

First of all, the solitary waves to Equation (2) have to be computed. For this purpose we employ the PETVIASHVILI iterative method [11–13]. Then, in order to simulate the α-family dynamics, we employ a FOURIER-type pseudo-spectral discretization with adaptive higher-order embedded RUNGE–KUTTA temporal discretization [14]. We employ 16,384 FOURIER modes. Since we deal with unidirectional models, we simulate the overtaking collision of two solitary waves. For this purpose we generate two right-going solitary wave profiles, corresponding to propagation velocities $c^+ = 1.4$ and $c^- = 1.1$. Then, they are placed in a periodic computational domain $[-140, 140]$. The larger (and faster) solitary wave is initially placed at $x = -50$ and the smaller one at $x = 50$. The initial condition is depicted in Figure 1. Then, this initial configuration is propagated by both models from $t = 0$ to $t = 500$. The final simulation time is chosen so that the first overtaking collision takes place and solitary waves have enough time to separate again. A zoom on the numerical solution at $t = 500$ is shown in Figure 2. We can see that the interaction in the eBBM $_{\frac{1}{3}}$ Equation (3) is elastic, while the choice of $\alpha = 1$ (the energy-preserving GALILEI-invariant eBBM equation) yields an inelastic collision.

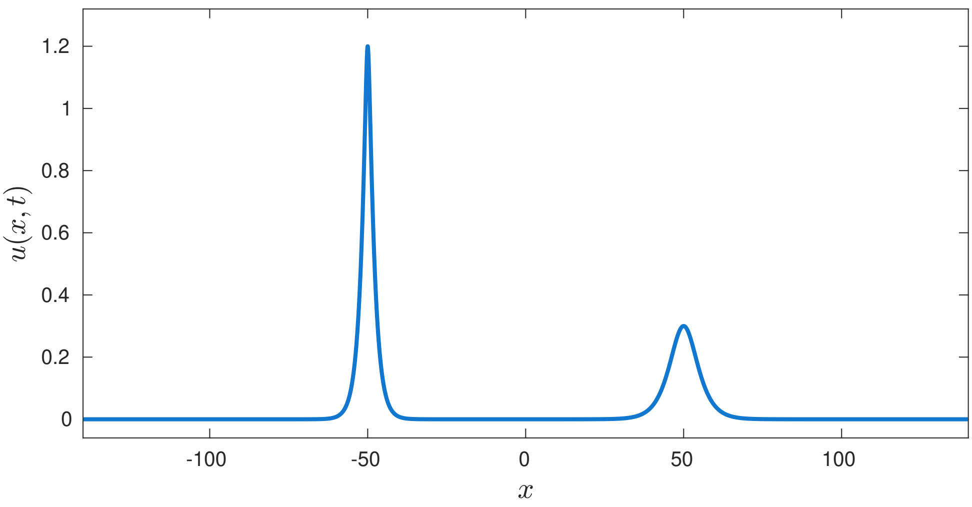
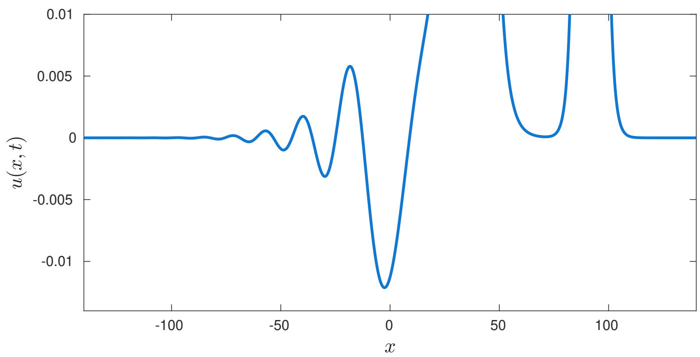
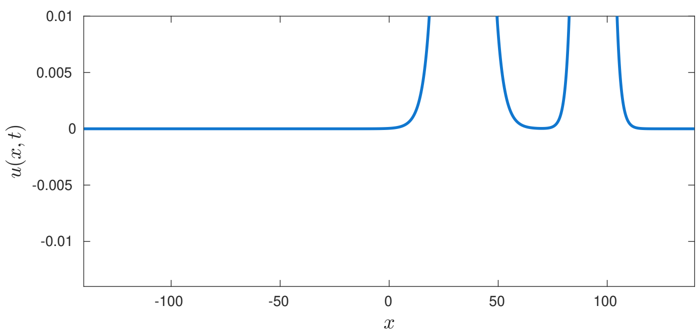


Figure 1. The initial condition ($t = 0$) for the overtaking collision.



(**a**) eBBM.



(**b**) eBBM$_{\frac{1}{3}}$.

Figure 2. Zoom on the numerical solution at $t = 500$ after the overtaking collision. The left panel shows the collision result in the eBBM Equation (2) with $\alpha = 1$ and the right panel corresponds to Equation (3).

6. Discussion and Conclusions

In this communication, a family of PDEs (2) with one free parameter was considered. This family was inspired by our quest for GALILEI-invariant and energy-preserving higher order analogs of the classical BBM equation [2]. The detailed symmetry and conservation law analysis allowed us to extract one particular member (3) of this family, which admited a higher-order local symmetry (22) and local conservation laws with higher-order multipliers (19)–(21). Moreover, our spectrally accurate numerical simulations showed that the solitary wave interaction in Equation (3) was elastic in contrast to all other values of the free parameter α. This fact allowed us to formulate the conjecture of the integrability of eBBM$_{\frac{1}{3}}$ Equation (3). This conjecture turned out to be true because a close inspection of Equation (3) showed that it was a rescaled version of the celebrated CAMASSA–HOLM (CH) equation [15,16]. Indeed, the eBBM$_{\frac{1}{3}}$ equation was related to the CH by a local scaling transformation. For example, a time scaling $t = 3\,\tau$ and the change of notation $\tau \rightarrow t$ maps the PDE (3) into the CH equation with parameter $\kappa = \frac{3}{2}$:

$$u_t + 3u_x + 3\,u\,u_x - 2\,u_x\,u_{xx} - u_{xxt} - u\,u_{xxx} = 0\,.$$

As a result, the present study can be considered as a symmetry-based derivation of a famous PDE inspired by physically sound modeling of the unidirectional nonlinear wave propagation.

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Abbreviations

The following abbreviations are used in this manuscript:

BBM	BENJAMIN–BONA–MAHONY
CH	CAMASSA–HOLM
eBBM	extended BENJAMIN–BONA–MAHONY
KdV	KORTEWEG–DE VRIES
PDE	Partial Differential Equation

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