Singularity Formation in the Inviscid Burgers Equation

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Abstract: We provide a lower bound for the blow up time of the H^2 norm of the entropy solutions of the inviscid Burgers equation in terms of the H^2 norm of the initial datum. This shows an interesting symmetry of the Burgers equation: the invariance of the space H^2 under the action of such nonlinear equation. The argument is based on a priori estimates of energy and stability type for the (viscous) Burgers equation.

Keywords: existence; uniqueness; stability; Burgers equation; Cauchy problem

MSC: 35G25; 35K55

1. Introduction

Consider the Cauchy problem for the inviscid Burgers equation:

$$\begin{cases} \partial_t u + u \partial_x u = 0, & 0 < t < T, \quad x \in \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}, \end{cases} \quad (1)$$

and on the initial datum assume

$$u_0 \in H^2(\mathbb{R}), \quad u_0 \neq 0. \quad (2)$$

Following the classical Kružkov approach [1], we use the following definition of solution.

Definition 1. A function $u : [0, T) \times \mathbb{R} \rightarrow \mathbb{R}$ is an entropy solution of (1) if

$$u \in L_{loc}^\infty((0, T) \times \mathbb{R})$$

and for every constant $c \in \mathbb{R}$ and every nonnegative test function $\varphi \in C^\infty((-\infty, T) \times \mathbb{R})$ with compact support

$$\int_0^T \int_{\mathbb{R}} \left(|u - c| \partial_t \varphi + \text{sign}(u - c) \frac{u^2 - c^2}{2} \partial_x \varphi \right) dt dx + \int_{\mathbb{R}} |u_0(x) - c| \varphi(0, x) dx \geq 0.$$

We know that, independently of the regularity assumptions on the initial datum, such as (2), the entropy solutions are unique and may develop a discontinuity in finite time (see [2–4]). On the other hand, the existence of smooth solutions of (1) is guaranteed for a short time by the Cauchy-Kovalevskaya Theorem [5]. Several papers on the development of singularities of nonlinear hyperbolic equations are available in the literature on this topic starting from the classical ones [6–8] and arriving at the more modern ones like [9–11] where tools of complex analysis are used and [12] where the time derivative is replaced by the Caputo one.

A very detailed blow-up analysis on the solutions of (1) can be carried using the characteristic lines of the equation for $\partial_x u$ that is

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$$\partial_{tx}^2 u + u \partial_x^2 u + (\partial_x u)^2 = 0.$$

Indeed, in ([2] Example 1.4), choosing

$$u_0(x) = \frac{1}{1+x^2},$$

the author shows that the characteristic lines cross at time $T = 8/\sqrt{27}$ causing the creation of a shock.

A more refined tool that can be used for the analysis of the geometric structure and the large-time behavior of the solution of (1) is the Hopf-Cole [13–15] transformation that turns the (inviscid) Burgers equation into the linear heat equation. It provides a simple geometric construction of the solution, and allows to conclude that

- the initial information $u_0(x_0)$ propagates along the characteristic line $x = u_0(x_0)tx_0$;
- the initial information that reaches a given location (t, x) depends on the global minimum of the function

$$F(t, x, y) = \frac{(x-y)^2}{2t} + \int_0^y u_0(\xi) d\xi;$$

- a shock wave in (1) is experienced when the minimum of F is attained in more than one location.

Here, we provide a lower bound for the maximal time T of existence of an H^2 solution of (1) in terms of the H^2 norm of the initial datum. This result shows also a symmetry of the Burgers equations: the invariance of the space H^2 under the action of that nonlinear equation in the time interval $[0, T]$. In addition, we also prove that as soon as the solution stays in H^2 it is also L^2 stable. It is interesting to note that, while the L^1 distance between entropy solutions of (1) is non-increasing, that is not the case for the L^2 distance. Our arguments are based on energy and stability estimates on the (viscous) Burgers equation.

The main result of this paper is the following theorem:

Theorem 1. Assume (2) and

$$T < \frac{1}{3} \log(A_0), \quad \text{or} \quad \|\partial_x u_0\|_{H^1(\mathbb{R})}^2 < \frac{1}{e^{3T} - 1}, \quad (3)$$

where

$$A_0 = \frac{\|\partial_x u_0\|_{H^1(\mathbb{R})}^2 + 1}{\|\partial_x u_0\|_{H^1(\mathbb{R})}^2}.$$

There, the unique entropy solution u of (1) in the sense of Definition 1 satisfies

$$\begin{aligned} u &\in H^1((0, T) \times \mathbb{R}) \cap L^\infty(0, T; H^2(\mathbb{R})) \cap W^{1,\infty}((0, T) \times \mathbb{R}), \\ \partial_t \partial_x u &\in L^2(\mathbb{R}), \quad \text{for every } 0 \leq t \leq T. \end{aligned} \quad (4)$$

Moreover, if u_1 and u_2 are the two entropy solutions of (1) obtained in correspondence of the initial data $u_{1,0}$ and $u_{2,0}$ satisfying (3), the following stability estimate holds

$$\|u_1(t, \cdot) - u_2(t, \cdot)\|_{L^2(\mathbb{R})}^2 \leq e^{Ct} \|u_{1,0} - u_{2,0}\|_{L^2(\mathbb{R})}^2, \quad (5)$$

for every $0 \leq t < T$, and for some positive constant C .

Our argument is based on a priori estimates on the smooth solution u_ε of the (viscous) Burgers equation of energy and stability type (see [16–18]):

$$\begin{cases} \partial_t u_\varepsilon + u_\varepsilon \partial_x u_\varepsilon = \varepsilon \partial_x^2 u_\varepsilon, & 0 < t < T, \quad x \in \mathbb{R}, \\ u_\varepsilon(0, x) = u_{\varepsilon,0}(x), & x \in \mathbb{R}, \end{cases} \quad (6)$$

where $0 < \varepsilon < 1$ and $u_{\varepsilon,0}$ is an analytic approximation of u_0 , such that

$$\|u_{\varepsilon,0}\|_{H^2(\mathbb{R})} \leq \|u_0\|_{H^2(\mathbb{R})}, \quad \|u_{\varepsilon,0}\|_{L^\infty(\mathbb{R})} \leq \|u_0\|_{L^\infty(\mathbb{R})}, \quad \varepsilon \left\| \partial_x^3 u_{\varepsilon,0} \right\|_{L^2(\mathbb{R})}^2 \leq C_0, \quad (7)$$

where C_0 is a positive constant independent of ε . Indeed we know that [1,19]

$$u_\varepsilon \rightarrow u \quad \text{a.e. and in every } L^p_{loc}((0, \infty) \times \mathbb{R}), \quad 1 \leq p < \infty \text{ as } \varepsilon \rightarrow 0. \quad (8)$$

Thanks to the uniqueness of the entropy solution u of (1) the limit is taken along all the family $\varepsilon \rightarrow 0$ and not only merely along a subsequence.

We obtain the regularity stated in (4), proving several energy estimates of u_ε and using (4). Those are also the key tool for the proof of the stability estimate (5).

The proof of Theorem 1 is given in the next section.

2. Proof of Theorem 1

In what follows, we denote with C all the positive constants independent on ε and t .

The key tool in our argument is a precise analysis on the blow-up of the H^1 norm of the solution u_ε of (6). For the sake of readability we state and prove an L^∞ and L^2 estimates on u_ε , that do not need assumption (3) (see [1]).

Lemma 1. Assume that (7) holds. For each $0 < \varepsilon < 1$, we have that

$$\|u_\varepsilon\|_{L^\infty((0,T) \times \mathbb{R})} \leq \|u_0\|_{L^\infty(\mathbb{R})}. \quad (9)$$

for every $0 \leq t \leq T$.

Proof. Let $0 \leq t \leq T$. Observe that the constant maps $\|u_0\|_{L^\infty(\mathbb{R})}$ and $-\|u_0\|_{L^\infty(\mathbb{R})}$ solve the equation in (6). Therefore, is consequence of the comparison principle for parabolic equation and of the fact that (see (7))

$$-\|u_0\|_{L^\infty(\mathbb{R})} \leq u_{0,\varepsilon} \leq \|u_0\|_{L^\infty(\mathbb{R})},$$

that is (9). $\square$

Directly multiplying (6) by u_ε we get the following L^2 estimate.

Lemma 2. Assume that (7) holds. We have that

$$\|u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2\varepsilon \int_0^t \|\partial_x u(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \leq \|u_0\|_{L^2(\mathbb{R})}^2, \quad (10)$$

for every $0 \leq t \leq T$.

Proof. Let $0 \leq t \leq T$. Multiplying (6) by $2u_\varepsilon$, an integration on $\mathbb{R}$ gives

$$\begin{aligned} \frac{d}{dt} \|u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 &= 2 \int_{\mathbb{R}} u \partial_t u \, dx \\ &= -2 \int_{\mathbb{R}} u_\varepsilon^2 \partial_x u_\varepsilon \, dx + 2\varepsilon \int_{\mathbb{R}} u_\varepsilon \partial_x^2 u_\varepsilon \, dx = -2\varepsilon \|\partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2. \end{aligned}$$

Therefore, we have

$$\frac{d}{dt} \|u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2\varepsilon \|\partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 = 0.$$

Integrating on $(0, t)$, thanks to (7), we have (10). $\square$

In order to prove the invariance of the space H^2 , we need to prove an a priori estimate on the H^2 -norm of u_ε on a time interval independent on ε . As a first step in that direction we prove the following estimate that holds for any $t \geq 0$ and does not require (3).

Lemma 3. Assume that (7) holds. We have that

$$\frac{\|\partial_x u_\varepsilon(t, \cdot)\|_{H^1(\mathbb{R})}^2}{\|\partial_x u_\varepsilon(t, \cdot)\|_{H^1(\mathbb{R})}^2 + 1} \leq \frac{\|\partial_x u_0\|_{H^1(\mathbb{R})}^2 e^{3t}}{\|\partial_x u_{\varepsilon,0}\|_{H^1(\mathbb{R})}^2 + 1}, \quad (11)$$

for every $t \geq 0$.

Proof. Multiplying (6) by $-2\partial_x^2 u_\varepsilon + 2\partial_x^4 u_\varepsilon$, an integration on $\mathbb{R}$ gives

$$\begin{aligned} & \frac{d}{dt} \left(\|\partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \|\partial_x^2 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \right) \\ &= -2 \int_{\mathbb{R}} \partial_x^2 u_\varepsilon \partial_t u_\varepsilon dx + 2 \int_{\mathbb{R}} \partial_x^4 u_\varepsilon \partial_t u_\varepsilon dx \\ &= 2 \int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon \partial_x^2 u_\varepsilon dx - 2 \int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon \partial_x^4 u_\varepsilon dx - 2\varepsilon \|\partial_x^2 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2\varepsilon \int_{\mathbb{R}} \partial_x^2 u_\varepsilon \partial_x^4 u_\varepsilon dx \\ &= - \int_{\mathbb{R}} (\partial_x u_\varepsilon)^3 dx + 2 \int_{\mathbb{R}} (\partial_x u_\varepsilon)^2 \partial_x^3 u_\varepsilon dx + 2 \int_{\mathbb{R}} u_\varepsilon \partial_x^2 u_\varepsilon \partial_x^3 u_\varepsilon dx \\ &\quad - 2\varepsilon \|\partial_x^2 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 - 2\varepsilon \|\partial_x^3 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\ &= - \int_{\mathbb{R}} (\partial_x u_\varepsilon)^3 dx - 5 \int_{\mathbb{R}} \partial_x u_\varepsilon (\partial_x^2 u_\varepsilon)^2 dx - 2\varepsilon \|\partial_x^2 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 - 2\varepsilon \|\partial_x^3 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2. \end{aligned}$$

Therefore, we have that

$$\begin{aligned} & \frac{d}{dt} \left(\|\partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \|\partial_x^2 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \right) \\ &\quad + 2\varepsilon \|\partial_x^2 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2\varepsilon \|\partial_x^3 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\ &= - \int_{\mathbb{R}} (\partial_x u_\varepsilon)^3 dx - 5 \int_{\mathbb{R}} \partial_x u_\varepsilon (\partial_x^2 u_\varepsilon)^2 dx \\ &\leq \|\partial_x u_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R})} \|\partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 5 \|\partial_x u_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R})} \|\partial_x^2 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\ &\leq 6 \|\partial_x u_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R})} \left(\|\partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \|\partial_x^2 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \right). \end{aligned} \quad (12)$$

Define the following function

$$X_\varepsilon(t) := \|\partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \|\partial_x^2 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 = \|\partial_x u_\varepsilon(t, \cdot)\|_{H^1(\mathbb{R})}^2. \quad (13)$$

Therefore, by (12),

$$\frac{dX_\varepsilon(t)}{dt} + 2\varepsilon \|\partial_x^2 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2\varepsilon \|\partial_x^3 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \leq 6 \|\partial_x u_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R})} X_\varepsilon(t). \quad (14)$$

Due to the Young inequality,

$$6\|\partial_x u_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R})} X_\varepsilon(t) \leq 3\|\partial_x u_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R})}^2 + 3X_\varepsilon^2(t).$$

It follows from (14) that

$$\frac{dX_\varepsilon(t)}{dt} + 2\varepsilon\|\partial_x^2 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2\varepsilon\|\partial_x^3 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \leq 3\|\partial_x u_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R})}^2 + 3X_\varepsilon^2(t). \quad (15)$$

Thanks to the Hölder inequality,

$$\begin{aligned} (\partial_x u_\varepsilon(t, x))^2 &= 2 \int_{-\infty}^x \partial_x u_\varepsilon \partial_x^2 u_\varepsilon dy \leq 2 \int_{\mathbb{R}} |\partial_x u_\varepsilon| |\partial_x^2 u_\varepsilon| dx \\ &\leq 2\|\partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})} \|\partial_x^2 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}. \end{aligned}$$

Hence,

$$\|\partial_x u_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R})}^2 \leq 2\|\partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})} \|\partial_x^2 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}.$$

Due to (13) and the Young inequality,

$$\|\partial_x u_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R})}^2 \leq \|\partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \|\partial_x^2 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 = X_\varepsilon(t). \quad (16)$$

Consequently, by (15),

$$\frac{dX_\varepsilon(t)}{dt} + 2\varepsilon\|\partial_x^2 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2\varepsilon\|\partial_x^3 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \leq 3X_\varepsilon(t)(X_\varepsilon(t) + 1).$$

Then,

$$\frac{dX_\varepsilon(t)}{dt} \leq 3X_\varepsilon(t)(X_\varepsilon(t) + 1), \quad (17)$$

by (13),

$$\frac{1}{X_\varepsilon(t)(X_\varepsilon(t) + 1)} \frac{dX_\varepsilon(t)}{dt} \leq 3.$$

Integrating on $(0, t)$, we get

$$\log\left(\frac{X_\varepsilon(t)}{X_\varepsilon(t) + 1}\right) - \log\left(\frac{X_{0,\varepsilon}}{X_{0,\varepsilon} + 1}\right) \leq 3t.$$

Hence, thanks to (13),

$$\log\left(\frac{\|\partial_x u_\varepsilon(t, \cdot)\|_{H^1(\mathbb{R})}^2}{\|\partial_x u_\varepsilon(t, \cdot)\|_{H^1(\mathbb{R})}^2 + 1}\right) \leq \log\left(\frac{\|\partial_x u_{0,\varepsilon}\|_{H^1(\mathbb{R})}^2}{\|\partial_x u_{0,\varepsilon}\|_{H^1(\mathbb{R})}^2 + 1}\right) + 3t,$$

which gives

$$\frac{\|\partial_x u_\varepsilon(t, \cdot)\|_{H^1(\mathbb{R})}^2}{\|\partial_x u_\varepsilon(t, \cdot)\|_{H^1(\mathbb{R})}^2 + 1} \leq \frac{\|\partial_x u_{\varepsilon,0}\|_{H^1(\mathbb{R})}^2 e^{3t}}{\|\partial_x u_{\varepsilon,0}\|_{H^1(\mathbb{R})}^2 + 1}, \quad (18)$$

Using (7) in (18) we gain (11). $\square$

Using the assumption (3) on the time T , we are able to deduce for the previous general estimate the boundedness of the family $\{u_\varepsilon\}_\varepsilon$ in the spaces $L^\infty(0, T; H^2(\mathbb{R}))$ and $L^\infty(0, T; W^{1,\infty}(\mathbb{R}))$. This is the core of our argument because it allows us to find that the entropy solution u of (1) lives in the same spaces.

Lemma 4. Assume that (3) and (7) hold. We have that

$$\frac{-1}{\|\partial_x u_\varepsilon(t, \cdot)\|_{H^1(\mathbb{R})}^2 + 1} \leq \frac{-1}{\|\partial_x u_{\varepsilon,0}\|_{H^1(\mathbb{R})}^2 + 1} + \frac{\|\partial_x u_0\|_{H^1(\mathbb{R})}^2}{\|\partial_x u_{\varepsilon,0}\|_{H^1(\mathbb{R})}^2 + 1} (e^{3t} - 1), \quad (19)$$

$$\|\partial_x u_\varepsilon(t, \cdot)\|_{H^1(\mathbb{R})}^2 \leq \frac{\|\partial_x u_0\|_{H^1(\mathbb{R})}^2 e^{3T}}{\left[1 - \|\partial_x u_0\|_{H^1(\mathbb{R})}^2 (e^{3T} - 1)\right]}, \quad (20)$$

for every $0 \leq t \leq T$. In particular, we have

$$\|\partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})} \leq C, \quad (21)$$

$$\|\partial_x^2 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})} \leq C, \quad (22)$$

$$\|\partial_x u_\varepsilon\|_{L^\infty((0,T) \times \mathbb{R})} \leq C, \quad (23)$$

for every $0 \leq t \leq T$ and some constant $C > 0$, that depends on u_0 and T but not on ε .

Proof. We begin by observing that, arguing as in Lemma 3, we have (17). Therefore, thanks to (11), (13) and (17),

$$\begin{aligned} \frac{dX_\varepsilon(t)}{dt} &\leq \frac{3}{6} X_\varepsilon(t) (X_\varepsilon(t) + 1) \\ &= \frac{3X_\varepsilon(t)}{X_\varepsilon(t) + 1} (X_\varepsilon(t) + 1)^2 \leq \frac{3\|\partial_x u_0\|_{H^1(\mathbb{R})}^2 e^{3t}}{\|\partial_x u_{\varepsilon,0}\|_{H^1(\mathbb{R})}^2 + 1} (X_\varepsilon(t) + 1)^2. \end{aligned}$$

Therefore,

$$\frac{1}{(X_\varepsilon(t) + 1)^2} \frac{dX_\varepsilon(t)}{dt} \leq \frac{3\|\partial_x u_0\|_{H^1(\mathbb{R})}^2 e^{3t}}{\|\partial_x u_{\varepsilon,0}\|_{H^1(\mathbb{R})}^2 + 1}.$$

Integrating on $(0, t)$, we have that

$$\frac{-1}{X_\varepsilon(t) + 1} + \frac{1}{X_{\varepsilon,0} + 1} \leq \frac{\|\partial_x u_0\|_{H^1(\mathbb{R})}^2}{\|\partial_x u_{\varepsilon,0}\|_{H^1(\mathbb{R})}^2 + 1} (e^{3t} - 1).$$

Thanks to (13), we get

$$\frac{-1}{\|\partial_x u_\varepsilon(t, \cdot)\|_{H^1(\mathbb{R})}^2 + 1} \leq \frac{-1}{\|\partial_x u_{\varepsilon,0}\|_{H^1(\mathbb{R})}^2 + 1} + \frac{\|\partial_x u_0\|_{H^1(\mathbb{R})}^2}{\|\partial_x u_{\varepsilon,0}\|_{H^1(\mathbb{R})}^2 + 1} (e^{3t} - 1),$$

which gives (19).

We conclude by proving (20). Thanks to (19), we have

$$\frac{-1}{\|\partial_x u_\varepsilon(t, \cdot)\|_{H^1(\mathbb{R})}^2 + 1} \leq \frac{-1}{\|\partial_x u_{\varepsilon,0}\|_{H^1(\mathbb{R})}^2 + 1} + \frac{\|\partial_x u_0\|_{H^1(\mathbb{R})}^2}{\|\partial_x u_{\varepsilon,0}\|_{H^1(\mathbb{R})}^2 + 1} (e^{3T} - 1).$$

Consequently, we obtain that

$$\frac{\left[1 - \|\partial_x u_0\|_{H^1(\mathbb{R})}^2 (e^{3T} - 1)\right]}{\|\partial_x u_{\varepsilon,0}\|_{H^1(\mathbb{R})}^2 + 1} \leq \frac{1}{\|\partial_x u_\varepsilon(t, \cdot)\|_{H^1(\mathbb{R})}^2 + 1}. \quad (24)$$

Multiplying (24) by $\|\partial_x u_\varepsilon(t, \cdot)\|_{H^1(\mathbb{R})}^2$, we have that

$$\frac{\left[1 - \|\partial_x u_0\|_{H^1(\mathbb{R})}^2 (e^{3T} - 1)\right] \|\partial_x u_\varepsilon(t, \cdot)\|_{H^1(\mathbb{R})}^2}{\|\partial_x u_{\varepsilon,0}\|_{H^1(\mathbb{R})}^2 + 1} \leq \frac{\|\partial_x u_\varepsilon(t, \cdot)\|_{H^1(\mathbb{R})}^2}{\|\partial_x u_\varepsilon(t, \cdot)\|_{H^1(\mathbb{R})}^2 + 1}. \quad (25)$$

Observe that by (11), we get

$$\frac{\|\partial_x u_\varepsilon(t, \cdot)\|_{H^1(\mathbb{R})}^2}{\|\partial_x u_\varepsilon(t, \cdot)\|_{H^1(\mathbb{R})}^2 + 1} \leq \frac{\|\partial_x u_0\|_{H^1(\mathbb{R})}^2 e^{3T}}{\|\partial_x u_{\varepsilon,0}\|_{H^1(\mathbb{R})}^2 + 1}. \quad (26)$$

It follows from (25) and (26) that

$$\frac{\left[1 - \|\partial_x u_0\|_{H^1(\mathbb{R})}^2 (e^{3T} - 1)\right] \|\partial_x u_\varepsilon(t, \cdot)\|_{H^1(\mathbb{R})}^2}{\|\partial_x u_{\varepsilon,0}\|_{H^1(\mathbb{R})}^2 + 1} \leq \frac{\|\partial_x u_0\|_{H^1(\mathbb{R})}^2 e^{3T}}{\|\partial_x u_{\varepsilon,0}\|_{H^1(\mathbb{R})}^2 + 1},$$

that is

$$\left[1 - \|\partial_x u_0\|_{H^1(\mathbb{R})}^2 (e^{3T} - 1)\right] \|\partial_x u_\varepsilon(t, \cdot)\|_{H^1(\mathbb{R})}^2 \leq \|\partial_x u_0\|_{H^1(\mathbb{R})}^2 e^{3T}. \quad (27)$$

Thanks to (3), we have that

$$1 - \|\partial_x u_0\|_{H^1(\mathbb{R})}^2 (e^{3T} - 1) > 0. \quad (28)$$

Therefore, by (27) and (28), we get

$$\|\partial_x u_\varepsilon(t, \cdot)\|_{H^1(\mathbb{R})}^2 \leq \frac{\|\partial_x u_0\|_{H^1(\mathbb{R})}^2 e^{3T}}{\left[1 - \|\partial_x u_0\|_{H^1(\mathbb{R})}^2 (e^{3T} - 1)\right]},$$

which gives (20).

Finally, (21)–(23) follows from (13), (16) and (20), respectively. $\square$

In order to prove the L^2 -regularity of $\partial_t u$ we need the following energy estimate on the H^1 norm of u_ε .

Lemma 5. Assume that (3) and (7) hold. We have that

$$\|\partial_x u_\varepsilon(t, \cdot)\|_{H^1(\mathbb{R})}^2 + 2\varepsilon \int_0^t \left\| \partial_x^2 u_\varepsilon(s, \cdot) \right\|_{H^1(\mathbb{R})}^2 ds \leq C, \quad (29)$$

for every $0 \leq t \leq T$ and some constant $C > 0$, that depends on u_0 and T but not on ε .

Proof. Let $0 \leq t \leq T$, where T is defined in (3). Arguing as in Lemma 3, we (12). Therefore, by (12), (21)–(23), we get

$$\begin{aligned} & \frac{d}{dt} \left(\|\partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \left\| \partial_x^2 u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \right) \\ & + 2\varepsilon \left\| \partial_x^2 u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2\varepsilon \left\| \partial_x^3 u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \\ & \leq 6 \|\partial_x u_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R})} \left(\|\partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \left\| \partial_x^2 u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \right) \leq C. \end{aligned}$$

Integrating on $(0, t)$, by (7), we have

$$\|\partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \left\| \partial_x^2 u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^2$$

$$\begin{aligned}
& + 2\varepsilon \int_0^t \left\| \partial_x^2 u_\varepsilon(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds + 2\varepsilon \int_0^t \left\| \partial_x^3 u_\varepsilon(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds \\
& \leq \|u_0\|_{H^2(\mathbb{R})} + Ct \leq C,
\end{aligned}$$

which gives (29). $\square$

We complete the proof of the H^1 regularity of u proving the following bound L^2 on $\partial_t u_\varepsilon$.

Lemma 6. Assume that (3) and (7) hold. We have that

$$\|\partial_t u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \leq C, \quad (30)$$

for every $0 \leq t \leq T$ and some constant $C > 0$, that depends on u_0 and T but not on ε .

Proof. Let $0 \leq t \leq T$. We begin by observing that, by (6),

$$(\partial_t u_\varepsilon)^2 = \left(-u_\varepsilon \partial_x u_\varepsilon + \varepsilon \partial_x^2 u_\varepsilon \right)^2 = u_\varepsilon^2 (\partial_x u_\varepsilon)^2 - 2\varepsilon u_\varepsilon \partial_x u_\varepsilon \partial_x^2 u_\varepsilon + \varepsilon^2 (\partial_x^2 u_\varepsilon)^2. \quad (31)$$

Since

$$-2\varepsilon \int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon \partial_x^2 u_\varepsilon dx = \varepsilon \int_{\mathbb{R}} (\partial_x u_\varepsilon)^3 dx,$$

integrating (31) on $\mathbb{R}$, we have

$$\|\partial_t u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 = \int_{\mathbb{R}} u_\varepsilon^2 (\partial_x u_\varepsilon)^2 dx + \varepsilon \int_{\mathbb{R}} (\partial_x u_\varepsilon)^3 dx + \varepsilon^2 \left\| \partial_x^2 u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^2. \quad (32)$$

Since $0 < \varepsilon < 1$, due to (9), (21) and (22),

$$\begin{aligned}
\int_{\mathbb{R}} u_\varepsilon^2 (\partial_x u_\varepsilon)^2 dx & \leq \|u_\varepsilon\|_{L^\infty((0,T) \times \mathbb{R})}^2 \|\partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \leq C, \\
\varepsilon \int_{\mathbb{R}} |\partial_x u_\varepsilon|^3 dx & \leq \varepsilon \|\partial_x u_\varepsilon(t, \cdot)\|_{L^\infty((0,T) \times \mathbb{R})} \|\partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \leq C, \\
\varepsilon^2 \left\| \partial_x^2 u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 & \leq C.
\end{aligned}$$

Hence, by (32),

$$\|\partial_t u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})} \leq C,$$

which gives (30). $\square$

In order to prove the L^2 -regularity of $\partial_{tx}^2 u$, we need the following energy estimate on the H^1 norm of u_ε .

Lemma 7. Assume that (3) and (7) hold. We have that

$$\varepsilon \left\| \partial_x^3 u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2\varepsilon^2 \int_0^t \left\| \partial_x^4 u_\varepsilon(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds \leq C, \quad (33)$$

for every $0 \leq t \leq T$ and some constant $C > 0$, that depends on u_0 and T but not on ε .

Proof. Let $0 \leq t \leq T$. Multiplying (6), by $-2\varepsilon \partial_x^6 u_\varepsilon$, we have that

$$\begin{aligned}
-2\varepsilon \int_{\mathbb{R}} \partial_x^6 u_\varepsilon \partial_t u_\varepsilon dx & = \varepsilon \frac{d}{dt} \left\| \partial_x^3 u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \\
& = 2\varepsilon \int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon \partial_x^6 u_\varepsilon dx + 2\varepsilon^2 \int_{\mathbb{R}} \partial_x^2 u_\varepsilon \partial_x^6 u_\varepsilon dx \\
& = -2\varepsilon \int_{\mathbb{R}} (\partial_x u_\varepsilon)^2 \partial_x^5 u_\varepsilon dx - 2\varepsilon \int_{\mathbb{R}} u_\varepsilon \partial_x^2 u_\varepsilon \partial_x^5 u_\varepsilon dx - 2\varepsilon^2 \int_{\mathbb{R}} \partial_x^3 u_\varepsilon \partial_x^5 u_\varepsilon dx
\end{aligned}$$

$$\begin{aligned}
&= 6\varepsilon \int_{\mathbb{R}} \partial_x u_\varepsilon \partial_x^2 u_\varepsilon \partial_x^4 u_\varepsilon dx + 2\varepsilon \int_{\mathbb{R}} u_\varepsilon \partial_x^3 u_\varepsilon \partial_x^4 u_\varepsilon dx - 2\varepsilon^2 \left\| \partial_x^4 u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \\
&= -7\varepsilon \int_{\mathbb{R}} \partial_x u_\varepsilon (\partial_x^3 u_\varepsilon)^2 dx + 2\varepsilon^2 \left\| \partial_x^4 u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^2.
\end{aligned}$$

Therefore, we have that

$$\varepsilon \frac{d}{dt} \left\| \partial_x^3 u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2\varepsilon^2 \left\| \partial_x^4 u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 = -7\varepsilon \int_{\mathbb{R}} \partial_x u_\varepsilon (\partial_x^3 u_\varepsilon)^2 dx. \quad (34)$$

Due to (23),

$$7\varepsilon \int_{\mathbb{R}} |\partial_x u_\varepsilon| (\partial_x^3 u_\varepsilon)^2 dx \leq \varepsilon \|\partial_x u_\varepsilon\|_{L^\infty((0,T) \times \mathbb{R})} \leq \varepsilon C \left\| \partial_x^3 u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^2.$$

Consequently, by (34),

$$\varepsilon \frac{d}{dt} \left\| \partial_x^3 u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2\varepsilon^2 \left\| \partial_x^4 u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \leq \varepsilon C \left\| \partial_x^3 u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^2.$$

Integrating on $(0, t)$, by (7) and (29), we have that

$$\varepsilon \left\| \partial_x^3 u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2\varepsilon^2 \int_0^t \left\| \partial_x^4 u_\varepsilon(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds \leq C_0 + C\varepsilon \int_0^t \left\| \partial_x^3 u_\varepsilon(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds \leq C,$$

which gives (33). $\square$

In the following final lemma we prove an L^2 bound on the mixed derivative $\partial_t \partial_x u_\varepsilon$, that implies an L^∞ bound on the time derivative $\partial_t u_\varepsilon$.

Lemma 8. Assume that (3) and (7) hold. We have that

$$\|\partial_t \partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})} \leq C, \quad (35)$$

$$\|\partial_t u_\varepsilon\|_{L^\infty((0,T) \times \mathbb{R})} \leq C, \quad (36)$$

for every $0 \leq t \leq T$ and some constant $C > 0$, that depends on u_0 and T but not on ε .

Proof. Let $0 \leq t \leq T$, where T is defined in (3). Multiplying (6) by $-2\partial_t \partial_x^2 u_\varepsilon$, we have that

$$-2 \int_{\mathbb{R}} \partial_t \partial_x^2 u_\varepsilon \partial_t u_\varepsilon dx = 2 \int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon \partial_t \partial_x^2 u_\varepsilon dx - 2\varepsilon \int_{\mathbb{R}} \partial_x^2 u_\varepsilon \partial_t \partial_x^2 u_\varepsilon dx. \quad (37)$$

Since

$$\begin{aligned}
-2 \int_{\mathbb{R}} \partial_t \partial_x^2 u_\varepsilon \partial_t u_\varepsilon dx &= 2 \left\| \partial_t \partial_x u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^2, \\
2 \int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon \partial_t \partial_x^2 u_\varepsilon dx &= -2 \int_{\mathbb{R}} (\partial_x u_\varepsilon)^2 \partial_t \partial_x u_\varepsilon dx - 2 \int_{\mathbb{R}} u_\varepsilon \partial_x^2 u_\varepsilon \partial_t \partial_x u_\varepsilon dx, \\
2\varepsilon \int_{\mathbb{R}} \partial_x^2 u_\varepsilon \partial_t \partial_x^2 u_\varepsilon dx &= -2\varepsilon \int_{\mathbb{R}} \partial_x^3 u_\varepsilon \partial_t \partial_x u_\varepsilon dx,
\end{aligned}$$

an integration of (37) on $\mathbb{R}$ gives

$$\begin{aligned}
2 \left\| \partial_t \partial_x u_\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 &= -2 \int_{\mathbb{R}} (\partial_x u_\varepsilon)^2 \partial_t \partial_x u_\varepsilon dx - 2 \int_{\mathbb{R}} u_\varepsilon \partial_x^2 u_\varepsilon \partial_t \partial_x u_\varepsilon dx \\
&\quad - 2\varepsilon \int_{\mathbb{R}} \partial_x^3 u_\varepsilon \partial_t \partial_x u_\varepsilon dx.
\end{aligned} \quad (38)$$

Since $0 < \varepsilon < 1$, thanks to (9), (21)–(23), (33) and the Young inequality,

$$\begin{aligned}
 2 \int_{\mathbb{R}} (\partial_x u_\varepsilon)^2 |\partial_t \partial_x u_\varepsilon| dx &\leq 2 \|\partial_x u_\varepsilon\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} |\partial_x u_\varepsilon| |\partial_t \partial_x u_\varepsilon| dx \\
 &\leq 2C \int_{\mathbb{R}} |\partial_x u_\varepsilon| |\partial_t \partial_x u_\varepsilon| dx \leq \int_{\mathbb{R}} |2C \partial_x u_\varepsilon| |\partial_t \partial_x u_\varepsilon| dx \\
 &\leq C \|\partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \frac{1}{2} \|\partial_t \partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\
 &\leq C + \frac{1}{2} \|\partial_t \partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2, \\
 2 \int_{\mathbb{R}} |u_\varepsilon| |\partial_x^2 u_\varepsilon| |\partial_t \partial_x u_\varepsilon| dx &\leq 2 \|u_\varepsilon\|_{L^\infty((0,T) \times \mathbb{R})} \int_{\mathbb{R}} |\partial_x^2 u_\varepsilon| |\partial_t \partial_x u_\varepsilon| dx \\
 &\leq 2 \|u_0\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} |\partial_x^2 u_\varepsilon| |\partial_t \partial_x u_\varepsilon| dx = \int_{\mathbb{R}} |2 \|u_0\|_{L^\infty(\mathbb{R})} \partial_x^2 u_\varepsilon| |\partial_t \partial_x u_\varepsilon| dx \\
 &\leq 2 \|u_0\|_{L^\infty(\mathbb{R})}^2 \|\partial_x^2 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \frac{1}{2} \|\partial_t \partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\
 &\leq C + \frac{1}{2} \|\partial_t \partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2, \\
 2|\varepsilon| \int_{\mathbb{R}} |\partial_x^3 u_\varepsilon| |\partial_t \partial_x u_\varepsilon| dx &\leq 2\varepsilon^2 \|\partial_x^3 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \frac{1}{2} \|\partial_t \partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\
 &\leq C + \frac{1}{2} \|\partial_t \partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2.
 \end{aligned}$$

Consequently, by (38),

$$\frac{1}{2} \|\partial_t \partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \leq C,$$

which gives (35).

Finally, we prove (36). Thanks to (30) and (35) and the Hölder inequality,

$$\begin{aligned}
 (\partial_t u_\varepsilon(t, x))^2 &= 2 \int_{-\infty}^x \partial_t u_\varepsilon \partial_t \partial_x u_\varepsilon dy \leq 2 \int_{\mathbb{R}} |\partial_t u_\varepsilon| |\partial_t \partial_x u_\varepsilon| dx \\
 &\leq 2 \|\partial_t u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})} \|\partial_t \partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})} \leq C.
 \end{aligned}$$

Hence,

$$\|\partial_t u_\varepsilon\|_{L^\infty((0,T) \times \mathbb{R})}^2 \leq C,$$

which gives (36). $\square$

We are ready for the proof of Theorem 1.

Proof of Theorem 1. The convergence stated in (8) and the bounds proved in the previous lemmas gives the regularity on u claimed in (4).

We prove (5). Let u_1 and u_2 be two solutions of (1), which verify (4), that is

$$\begin{cases} \partial_t u_1 + \partial_x u_1 = 0, & 0 < t < T, x \in \mathbb{R}, \\ u_1(0, x) = u_{1,0}(x), & x \in \mathbb{R}, \\ \partial_t u_2 + u_2 \partial_x u_2 = 0, & 0 < t < T, x \in \mathbb{R}, \\ u_2(0, x) = u_{2,0}(x), & x \in \mathbb{R}. \end{cases}$$

Then, the function

$$\omega = u_1 - u_2 \tag{39}$$

is the solution of the following Cauchy problem:

$$\begin{cases} \partial_t \omega + u_1 \partial_x \omega + \partial_x u_2 \omega = 0, & 0 < t < T, x \in \mathbb{R}, \\ \omega(0, x) = u_{1,0}(x) - u_{2,0}(x), & x \in \mathbb{R}. \end{cases} \tag{40}$$

Multiplying (40) by 2ω , an integration on $\mathbb{R}$ gives

$$\begin{aligned} \frac{d}{dt} \|\partial_t \omega(t, \cdot)\|_{L^2(\mathbb{R})}^2 &= -2 \int_{\mathbb{R}} u_1 \omega \partial_x \omega dx - 2 \int_{\mathbb{R}} \partial_x u_2 \omega^2 \\ &= 2 \int_{\mathbb{R}} \partial_x u_1 \omega^2 dx - 2 \int_{\mathbb{R}} \partial_x u_2 \omega^2. \end{aligned} \quad (41)$$

Since $u_1, u_2 \in H^2(\mathbb{R})$ for every $0 \leq t \leq T$

$$\|\partial_x u_1\|_{L^\infty((0,T) \times \mathbb{R})}, \|\partial_x u_2\|_{L^\infty((0,T) \times \mathbb{R})} \leq C. \quad (42)$$

Therefore, by (41) and (42),

$$\frac{d}{dt} \|\omega(t, \cdot)\|_{L^2(\mathbb{R})}^2 \leq C \|\omega(t, \cdot)\|_{L^2(\mathbb{R})}^2.$$

The Gronwall Lemma and (39) give (5). $\square$

3. Discussion

The goal of this paper is to investigate the blow-up of the H^2 norm of the solution of the (inviscid) Burgers equation. The interest for this result is twofold. First of all, we prove an easy to verify relation between the H^2 norm of the initial datum and the blow-up time for the H^2 -norm of the solution. Moreover, we show a symmetry of the (inviscid) Burgers equation that consists in the invariance of the H^2 space under the action of that nonlinear equation. Finally, we show that as soon as the solution of the (inviscid) Burgers equation stays in H^2 is L^2 stable with respect to the initial datum. The arguments are based on energy and stability estimates on the the solution of the (viscous) Burgers equation.

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