Article

Surjective Identifications of Convex Noetherian Separations in Topological (C, R) Space

Susmit Bagchi

Department of Aerospace and Software Engineering (Informatics), Gyeongsang National University, Jinju 660701, Korea; profsbagchi@gmail.com

Abstract: The interplay of symmetry of algebraic structures in a space and the corresponding topological properties of the space provides interesting insights. This paper proposes the formation of a predicate evaluated P-separation of the subspace of a topological (C, R) space, where the P-separations form countable and finite number of connected components. The Noetherian P-separated subspaces within the respective components admit triangulated planar convexes. The vertices of triangulated planar convexes in the topological (C, R) space are not in the interior of the Noetherian P-separated open subspaces. However, the P-separation points are interior to the respective locally dense planar triangulated convexes. The Noetherian P-separated subspaces are surjectively identified in another topological (C, R) space maintaining the corresponding local homeomorphism. The surjective identification of two triangulated planar convexes generates a quasiloop–quasigroupoid hybrid algebraic variety. However, the prime order of the two surjectively identified triangulated convexes allows the formation of a cyclic group structure in a countable discrete set under bijection. The surjectively identified topological subspace containing the quasiloop–quasigroupoid hybrid admits linear translation operation, where the right-identity element of the quasiloop–quasigroupoid hybrid structure preserves the symmetry of distribution of other elements. Interestingly, the vertices of a triangulated planar convex form the oriented multiplicative group structures. The surjectively identified planar triangulated convexes in a locally homeomorphic subspace maintain path-connection, where the right-identity element of the quasiloop–quasigroupoid hybrid behaves as a point of separation. Surjectively identified topological subspaces admitting multiple triangulated planar convexes preserve an alternative form of topological chained intersection property.

Citation: Bagchi, S. Surjective Identifications of Convex Noetherian Separations in Topological (C, R) Space. *Symmetry* **2021**, *13*, 783. <https://doi.org/10.3390/sym13050783>

Academic Editor: Alexei Kanel-Belov

Received: 30 March 2021

Accepted: 29 April 2021

Published: 1 May 2021

Keywords: topological spaces; Noetherian space; separation; groupoid; predicate

MSC: 54E05; 54F65; 54H10; 54D65

Publisher's Note: MDPI stays neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Copyright: © 2021 by the author. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

1. Introduction

The notion of relative size of sets in a topological space provides an interesting insight to the inherent structure of the space. Let $X = \{A_i : i \in \mathbb{Z}^+\}$ be a family of sets (without considering any specific topological structure at this point). A proper subset $S \subset X$ is called *saturated* if every proper collection of subsets $V \subset X \setminus S$ is the family of disjoint and countable sets [1]. If X is a topological space and $Cl(X)$ is a family of closed subsets of X then $D \subset Cl(X)$ is called *discrimination in X* if $X \setminus \bigcup_i (A_i \in V)$ is uncountable for every countable family $V \subset D$ [1]. The main aim of preparing the sets of discrimination in a topological space is to identify the size of the elements in a topological space. The size of elements of a topological space determines the structure of the underlying space as well as the nature of convexity of subspaces and its Baire categorical meagerness in terms of measure, if any. For example, if U denotes a universe set then the *nano* topological space on a set $X \subseteq U$ is defined as $\tau_R(X) = \{\phi, U, L_R(X), U_R(X), B_R(X)\}$ where $L_R(X)$, $U_R(X)$ and $B_R(X)$ signify the lower approximation, upper approximation and the boundary region of set $X \subseteq U$ [2]. The nano topological space is a space containing the sets of highly reduced

size with at most five elements [2]. However, note that the proposed constructions in this paper are generalized in nature and as a result it does not consider any notion of size of a set in the topological (C, R) space.

Interestingly, a new topology can be constructed from a given topology by employing the irreducible sets. For example, the Scott topology on a poset is formulated from the Alexandroff topology, which is based on the notion of the Scott irreducible family of open sets [3]. Note that the Alexandroff topology $\tau(p)$ on a poset p is constructed by employing all the upper subsets of a set in the directed sets partially ordered by p . There is a relationship between the relative openness of a set in Scott topology and Alexandroff topology. A set $A \subset X$ is open in Scott topology if, and only if, it is also open in Alexandroff topology. It is known that if the topological space is a discrete space of Alexandroff variety then there exist Alexandroff topologies given by G_0 and G_1 admitting topological groupoids [4]. Moreover, in this case the topological groupoids are in order-preserving poset category and can be associated to the monoid $M = N / \sim$ [4].

This paper proposes the topological P-separation of subspaces in a topological (C, R) space based on the predicate evaluation. The separation is an upper Noetherian variety, and it forms a planar convex triangulation within a respective convex component. The algebraic as well as topological properties of the triangulated planar convex and the corresponding surjective identification through local homeomorphism are presented in this paper. A brief description of topological algebraic sets and manifolds are presented in Section 1.1 to enhance familiarity of the field to the wider audience of readers. The motivation of this work and the contributions made are presented in Section 1.2. In this paper the symbols C, R, Z and N represent sets of complex numbers, extended real numbers, integers and natural numbers, respectively.

1.1. Topological Algebraic Sets and Manifolds

The combinatorial topological properties of algebraic sets are important to understand the semi-algebraic triangulations of a bounded algebraic set. If $V \subset R^n$ is a bounded algebraic set, then there is a semi-algebraic triangulation of $V \subset R^n$ given by $\Phi: |K| \rightarrow V$ [5]. It is interesting to note that Alexandroff compactification can be admitted in the algebraic set $V \subset R^n$ because it is a bounded algebraic set. Moreover, the semi-algebraic triangulation of $V \subset R^n$ allows the possibility of stratification of $V \subset R^n$ by a family of polynomials refining the triangulation [5]. Suppose we consider an irreducible 3-manifold M_3 of the compact and orientable variety. The fundamental group of M_3 can be represented by $\pi_1(M_3)$. The nature of $\pi_1(M_3)$ is the type of $SL(2, C)$, which is a homeomorphism of $\pi_1(M_3)$ to $SL(2, C)$ [6]. It is important to note that the space of representation of fundamental group $\pi_1(M_3)$ in the $SL(2, C)$ structure is a complex algebraic set [6]. If $\bar{\rho}$ is a representation of the fundamental group $\pi_1(M_3)$ of a 3-manifold in $PSL(2, C)$ then $\bar{\rho}(\pi_1(M_3))$ becomes an Abelian group and any two non-trivial elements of $\bar{\rho}(\pi_1(M_3))$ have the same fixed-point in the corresponding sphere [6]. If we consider that M_3 is a Haken manifold, then the image $\bar{\rho}(\pi_1(M_3))$ is contained in a cyclic group structure. The result of Nash isotopy shows that if $M \subset R^n$ is a compact smooth manifold then there is a real algebraic set $V \subset R^n$ such that a sheet of $V \subset R^n$ is not a connected component of $V \subset R^n$ [7].

1.2. Motivation and Contributions

The fundamental groupoids (i.e., groupoids in view of algebraic topology) are a general groupoid algebraic structure if the topological space is not a locally contractible space. For example, the fundamental groupoids of one-dimensional topological spaces are the extensions of corresponding fundamental groupoids of graphs through the reduced representations of path-homotopy classes by employing reparameterization [8]. It is important to note that the extension is infinitary in this case. Interestingly, if the fundamental group $\pi(X, x_0)$ in the topological space (X, τ_X) is Abelian then the homotopically Hausdorff property becomes equivalent to the transfinite π_1 -products in (X, τ_X) [8]. Note that in this case the topological space (X, τ_X) is considered to be a path-connected and metrizable space

admitting universal cover. It is mentioned earlier in this paper that a multidimensional bounded algebraic set admits Alexandroff compactification and it can have semi-algebraic triangulation as well as stratification by a family of polynomials [5].

The topological (C, R) space (X, τ_X) is a path-connected and multidimensional quasi-normed topological space [9]. Moreover, it is shown earlier that the topological (C, R) space (X, τ_X) admits homotopically Hausdorff property by employing the discrete-loop homotopy classes in the locally dense subspaces [10]. Note that a pseudocompact Baire set in any topological space is a zero-set whereas the realcompact as well as complete Baire set in the corresponding topological space is a realcompact as well as complete subspace [11]. In the case of topological (C, R) space, the locally dense subspace is a compactible Baire space (not a meager category). Moreover, the Nash isotopy theory provides an indication that the disconnected component of a real algebraic set can exist under a specific condition [7]. Hence, the motivating questions are: (1) is it possible to generate any planar and finite variety of convex triangulation of disconnected subspaces in a topological (C, R) space (X, τ_X) and, (2) what are the algebraic as well as topological properties of such triangulated planar subspaces in isolation on one topological space and under surjective identification in another topological space? Moreover, the question is: how to formulate a predicate evaluated separation in such topological structures? Furthermore, the restriction to maintain in this case is to preserve *local* homeomorphism during identification in the codomain. This paper addresses these questions in relative details.

The main contributions made in this paper can be summarized as follows. The concept of completely P-separated subspace of a topological (C, R) space is introduced generating multiple path-connected components. A set of Noetherian P-separated subspaces are formulated within the respective P-separated components and the corresponding P-separated triangulated planar convexes are formed. The vertices of planar convexes are not interior to the Noetherian subspace and these vertices form oriented multiplicative group algebraic structures. The P-separation points are interior to the locally dense triangulated convexes. The surjective identification of two triangulated planar convexes give rise the quasiloop–quasigroupoid algebraic variety. The algebraic as well as topological analysis of the properties of the algebraic structures and the locally homeomorphic subspaces under surjective identification are presented in this paper in detail. It is shown that the right-identity element of the quasiloop–quasigroupoid algebraic variety in a topological (C, R) space is a point of separation preserving the symmetry of distribution of elements and the path-connection in the identified topological subspace. Moreover, the topological property of surjectively identified subspace shows that it retains path-connectivity while the identified triangulated subspace becomes non-convex preserving an alternative form of topological finite intersection property.

The rest of the paper is organized as follows. The preliminary and existing concepts are presented Section 2 in brief for clarity and completeness. The definitions and descriptions of Noetherian P-separation and properties of a CR-quasigroupoid are presented in Section 3. The analyses of algebraic properties are presented in Section 4 in detail. Section 5 presents the algebraic as well as topological properties of surjective identification of triangulated Noetherian convexes. Finally, Section 6 concludes the paper.

2. Preliminary Concepts

Let (X, τ_X) be any arbitrary topological space and $A \subset X$ be a subset such that $A \neq \emptyset$ and the symbol $\oplus$ represents XOR operation. The subset $A \subset X$ is called as *irreducible* if $\exists \bar{E} \subset X, \exists \bar{F} \subset X$ such that $(A \subseteq \bar{E} \cup \bar{F}) \Rightarrow (A \subseteq \bar{E}) \oplus (A \subseteq \bar{F})$. If a topological space (X, τ_X) is T_0 then the partial order $p \equiv \leq$ is called a specialization order if $(x \leq y) \Leftrightarrow (x \in \overline{\{y\}})$. A poset p is called *upper Noetherian* if it satisfies the ascending chain condition $p \equiv \leq$ (or, equivalently it is called as *lower Noetherian* if it maintains the descending chain condition $p \equiv \geq$). The subset $A \subset X$ is called as τ_X – *irreducible* in (X, τ_X) if $A \subseteq \bar{E} \cup \bar{F}$ then $(A \subseteq \bar{E}) \oplus (A \subseteq \bar{F})$ condition is maintained by following the improper subset relation. The τ_X – *irreducible* sets of a topological space (X, τ_X) are denoted as

$\nabla(X)$. The τ_X – *irreducible* sets of a topological space maintain the following properties: (1) $[A \in \nabla(X)] \Leftrightarrow [\bar{A} \in \nabla(X)]$, (2) if $D = \{A_i : A_i \in \nabla(X), i \in \mathbb{Z}^+\}$ is a family of directed sets in τ_X – *irreducible* variety then $\bigcup_i A_i \in \nabla(X)$ and (3) if $f : (X, \tau_X) \rightarrow (Y, \tau_Y)$ is a continuous function then $[A \in \nabla(X)] \Rightarrow [f(A) \in \nabla(Y)]$. Moreover, if (X, τ_X) is a T_0 topological space then $D \subseteq X$ is an irreducible, where $D \subseteq X$ is a directed set under $p \equiv \leq$.

The groupoid is a generalization of a group algebraic structure where the group operation is a partial function. Often the groupoids are considered as an algebraic structure with many identities, which support topological monodromy and topological groupoids isomorphism [12]. A non-trivial groupoid algebraic structure can be equipped with binary-type morphing structure. For example, a qubit groupoid structure has the forward and reverse transformation operations given as α, α^{-1} between two binary states $\{+, -\}$ [13]. A generalized definition of a groupoid is represented by an algebraic structure given as a tuple $G = \langle G_0, G_1, m, d, r, u, (\cdot)^{-1} \rangle$ such that [14]:

- (I) G_0, G_1 are sets with mappings $d, r : G_1 \rightarrow G_0$ and $u : G_0 \rightarrow G_1$,
- (II) the function $m : (x, y) \rightarrow xy$ is associative, where $\{(x, y) : r(x) = d(y)\}$,
 $d(xy) = d(x), r(xy) = r(y)$,
- (III) $\forall x \in G_1, xu(r(x)) = x = u(d(x))x$ and
- (IV) the operator $(\cdot)^{-1} : G_1 \rightarrow G_1$ maintains $xx^{-1} = u(d(x)), x^{-1}x = u(r(x))$.

The topological groupoids can be placed in a poset category preserving the order relation. In general, a groupoid in poset category is called an Alexandroff (topological) groupoid. Note that, there is a relationship between an Alexandroff groupoid and other varieties of topological groupoids. If G is an Alexandroff groupoid and the corresponding groupoids G_0 and G_1 are equipped with Alexandroff topology preserving the poset-order algebraic relation, then G is a topological groupoid. In this case, the partial ordering relation is an upper Noetherian type where the open sets are upward closed subsets within the respective topological space. Let the set $E = \{(a_k, s_k) : a_k \in Z, s_k \in S\}$ be denoting the set of ordered pairs where S is set of left-actions. The *etale* poset over topological groupoid G is given by $\pi : E \rightarrow G_0$ with right-action on G represented as $\alpha : E \times_{G_0} G_1 \rightarrow E$ preserving the partial order relation. Note that $\alpha : E \times_{G_0} G_1 \rightarrow E$ maintains the axioms of groupoid actions. If D_g is a discrete group and p is a poset then $G = (p, D_g)$ is an *action groupoid* with respect to right-action. In this case the Alexandroff groupoid is defined as $G_0 = p$ and $G_1 = p \times D_g$. A *locally star topological groupoid* is represented as a pair (G, X) , where G is a groupoid, X is a topological space and G_x is a local star of x [12]. It maintains the following set of properties:

- (I) all identities of G (given as I_G) are in X ,
- (II) if A_x is subspace then $x \in I_G, A_x = X \cap G_x$.

Note that a star topological groupoid is a groupoid such that each G_x has a topology preserving the homeomorphisms under right and left translation operations. Moreover, the transitive groupoids are also called as the connected groupoids, whereas the covering morphism between two groupoids is a bijective variety [15].

3. Noetherian P-Separation and CR-Quasigroupoid

It is well known that a topological space (X, τ_X) is a separated space if $\exists A \subset X, \exists B \subset X$ such that $A \cup B = X$ and $A \cap B = \emptyset$. First, we present the concept of complete separation of a topological subspace inline to the concept of topological separation, where the separation is *not* a discrete variety indicating that the separations are not formed by single-point closed subspaces. In this case, the complete separation of a subspace generates multiple components. Let (X, τ_X) be a topological (C, R) space and $Y \subset X$ be a corresponding subspace. If it is true that $\exists n \in \mathbb{Z}^+, 1 < n < +\infty$ such that $Y = \bigcup_{1 \leq i \leq n} (X_i \subset X)$ and $X_i \cap X_k = \emptyset$ if $i \neq k$ then the topological subspace $Y \subset X$ is called a finite and complete separation in (X, τ_X) . Once the finite and complete separation of a topological subspace is constructed, it is possible to establish the additional

constrains on it. The definitions related to complete P-separation, its Noetherian property, convexity and triangulation are presented in Sections 3.1–3.4. The formulation of resulting CR-Quasigroupoid algebraic structure is defined in Section 3.5.

3.1. Complete P-Separation

Let $\forall x_p \in X, P(x_p) \in \{0, 1\}$ be a predicate evaluation defined in the topological (C, R) space (X, τ_X) . If it is true that $\forall X_i \subset Y, \exists x_i \in X_i$ such that $P(x_1) \wedge P(x_2) \wedge \dots \wedge P(x_n) = 1$ then $Y \subset X$ is called a completely P-separated topological subspace. If we maintain that $n \in \mathbb{Z}^+, 1 < n < +\infty$ then the subspace $Y \subset X$ is a countably finite P-separation in (X, τ_X) .

Example 1. Suppose we consider a continuous function $f : [0, 1] \rightarrow X$ in the topological (C, R) space (X, τ_X) and the corresponding 2D planar real subspace within X is denoted as $Pl_0 \subset X$ such that $\forall x_i \in Pl_0, \text{Im}(\pi_C(x_i)) = 0$. Note that $\pi_C : X \rightarrow C$ is the complex projective subspace and $\text{Im}(z \in C) \in R$ according to the standard convention generating a non-compact Pl_0 planar subspace. Let the zero sets of $f : [0, 1] \rightarrow X$ be defined as $Z_e(f) \subset [0, 1]$ such that $f(r \in Z_e(f)) \in Pl_0$. If $a_i \in R, f_X(w \in R) = \sum_i a_i w^i$ is a polynomial over reals, then a zero of the corresponding polynomial $f_X(w \in R)$ is considered to be at $w_k \in R$ if, and only if, $(z_k = w_k, f_X(w_k)) = f(r \in Z_e(f))$. Hence, the corresponding P-separation with respect to polynomial $f_X(w \in R)$ is the irreducible $X_i \subset Pl_0$ such that $P(x_k) = 1$, where $x_k = (z_k = w_k, r_k = f_X(w_k)) \in X_i$. Moreover, the predicate maintains the condition that $\forall x_i \in X_i \setminus \{x_k\}, P(x_i) = 0$ within that P-separation component in $Pl_0 \subset X$.

3.2. Noetherian P-Separated Subspaces

Let $Y \subset X$ be a completely P-separated subspace in the topological (C, R) space (X, τ_X) . The subspace $X_i \subset Y$ is called Noetherian P-separation if there is a monotone class chain $A_1 \subset A_2 \subset A_3 \subset \dots \subset X_i$ such that $\exists m \in \mathbb{Z}^+, 1 < m < +\infty$ and $\forall k \geq m$ the P-separated subspace maintains that $A_{k+1} = A_k$ in $X_i \subset Y$. Note that in the corresponding Noetherian subspace $x_i \in A_k, P(x_i) = 1$ predicate evaluation is continued to be maintained. The complete P-separated subspace $Y \subset X$ is defined as Noetherian P-separated if the aforesaid condition is attained in each $X_i \subset Y$.

3.3. Noetherian Triangulated Convex

Let $X_i \subset Y$ be a Noetherian P-separation in $Y \subset X$. If the irreducible closed subset in the Noetherian P-separated subspace is $A_k \subset X_i$ then $\Delta_i^3 = \{x_a, x_b, x_c\}$ is a Noetherian triangulation in $A_k \subset X_i$ where $\Delta_i^3 \subset \partial A_k$ and $A_k = \overline{\Delta_i^3}$. Note that the Noetherian triangulation maintains the condition that $\exists x_i \in A_k, P(x_i) = 1$. As a result, the Noetherian P-separated triangulated planar convex is generated by vertices given in $\Delta_i^3 \subset \partial A_k$ in A_k which is defined as $B_\Delta^i \subset A_k$ such that it admits $x_i \in (B_\Delta^i)^0$ within the respective planar convex. It is important to note that the topological (C, R) space is dense and a corresponding Noetherian triangulated planar convex B_Δ^i is locally dense and not a meager category (i.e., it maintains the condition that $(B_\Delta^i)^0 \neq \emptyset$).

3.4. Identifications of Noetherian Convexes

Let B_Δ^i and B_Δ^k be two Noetherian P-separated planar convexes in respective topological subspaces generated by Δ_i^3 and Δ_k^3 in $X_i \subset Y$ and $X_k \subset Y$, respectively. If $f : X_i \cup X_k \rightarrow W$ is a locally homeomorphic embedding in (W, τ_W) then $f(\cdot)$ identifies Δ_i^3 and Δ_k^3 in (W, τ_W) if $f(\Delta_i^3) \cap f(\Delta_k^3) = \{w_c\} \subset W$.

Remark 1. Note that the identification function $f : X_i \cup X_k \rightarrow W$ is a surjection. Moreover, the local homeomorphism property of embedding ensures that the resulting subspace $f(X_i \cup X_k) \subset W$ is a convex subspace if, and only if, $X_i \subset Y, X_k \subset Y$ are also convex. Furthermore, it is important to note that $f(\Delta_i^3) \cup f(\Delta_k^3) \subset W$ is a planar but not a convex embedding in (W, τ_W) under surjection.

The topological (C, R) space is suitable to establish various abstract algebraic structures. Earlier it is reported that topological group structures can be formed within the topological (C, R) space under certain conditions. However, the Noetherian P-separations and associated planar convexes under surjective identification enable the formation of a quasiloop–quasigroupoid structure in the surjectively identified triangulated planar convexes. First, we present the algebraic construction of a quasigroupoid in a subset $A \subset X$ of a topological (C, R) space. Note that the subset $A \subset X$ is considered to be an arbitrary subspace to establish the algebraic structure without enforcing any subspace topology on it at this point.

3.5. Quasigroupoid in (C, R) Space

Let $A \subset X$ be a countable set in a topological (C, R) space (X, τ_X) and the binary variety of an abstract algebraic operation is denoted in such space as $*_{\Delta} : A^2 \rightarrow A$. An algebraic CR-quasigroupoid structure of order $|A| = n$ in the corresponding topological (C, R) space is represented as $G_{crq}(X, n)$, where the structure $G_{crq}(X, n)$ must satisfy the following axioms.

$$\begin{aligned} &\forall x_a, x_b, e_A \in A, x_a \neq x_b \neq e_A, \\ &\forall x_a \exists x_b : x_a *_{\Delta} x_b \in A, \\ &\forall x_a \forall x_b, x_a *_{\Delta} x_b \neq x_b *_{\Delta} x_a, \\ &\forall x_a \in A, e_A *_{\Delta} e_A = e_A, \\ &x_a *_{\Delta} e_A = x_a, e_A *_{\Delta} x_a \neq x_a, \\ &\forall x_a \exists (x_a)^{-1} : x_a *_{\Delta} (x_a)^{-1} = (x_a)^{-1} *_{\Delta} x_a = e_A. \end{aligned} \quad (1)$$

It is important to note that the partial function $*_{\Delta} : A^2 \rightarrow A$ is closed but not total in $A \subset X$. Moreover, the $G_{crq}(X, n)$ structure does not admit commutativity including the identity element e_A . The identity element is a right-identity variety and as a result it is sensitive to the order of operation because $x_a *_{\Delta} e_A \neq e_A *_{\Delta} x_a$ condition is maintained within $G_{crq}(X, n)$. However, the identity element commutes with itself as a stationary point. Moreover, the algebraic CR-quasigroupoid structure includes inverse elements, where the algebraic operation between an element and its inverse commutes.

Remark 2. The algebraic $G_{crq}(X, n)$ structure is distinct as compared to an n – order algebraic groupoid structure $G_o(n)$. The reason is that a $G_o(n)$ not necessarily always admits binary operation $*_{\Delta}$ and the algebraic operation $*_{\Delta}$ preserves associativity in $G_o(n)$. On the other hand, $G_{crq}(X, n)$ does not preserve associativity property and the algebraic operation is strictly a binary variety. The similarity between $G_{crq}(X, n)$ and $G_o(n)$ is that both structures maintain partial function which is not total. If $L_o(n)$ is an algebraic loop, then $G_{crq}(X, n)$ is also a quasiloop structure. The comparison of properties of $L_o(n)$, $G_o(n)$, semigroup and $G_{crq}(X, n)$ is presented in Table 1.

Table 1. Comparison of CR-quasigroupoid algebraic structure with other varieties.

Algebraic Structures	Total (T)/Partial (P) Function	Associativity, Commutativity	Identity	Commutative Invertibility
Loop	T	No, No	Yes, commutative	Yes
Groupoid	P	Yes, No	Yes, commutative	Yes
Semigroup	T	Yes, No	No	No
CR-quasigroupoid	P	No, No	Right-identity, self-commuting	Yes

Before proceeding further into detailed algebraic as well as topological analysis, we first show that the structure $G_{crq}(X, n)$ is indeed a quasiloop–quasigroupoid variety as presented in the following proposition.

Proposition 1. *If $L_o(n)$ is an algebraic loop and $G_o(n)$ is an algebraic groupoid then $G_{crq}(X, n)$ is a quasiloop–quasigroupoid hybrid variety.*

Proof. Let $L_o(n)$ be an algebraic loop and $G_o(n)$ be an algebraic groupoid of order n in $A \subset X$. The partial function $*_{\Delta} : A^2 \rightarrow A$ is total only in $L_o(n)$ by definition. Additionally, the structure $G_o(n)$ preserves associativity of $*_{\Delta} : A^2 \rightarrow A$ but $L_o(n)$ does not. However, the $L_o(n)$ and $G_o(n)$ algebraic structures always admit identity and inverse, where both commute. The algebraic CR-quasigroupoid structure $G_{crq}(X, n)$ preserves properties of right-identity and inverse but does not admit complete commutativity of identity element with respect to partial function $*_{\Delta} : A^2 \rightarrow A$. The identity element is only self-commuting as a stationary point and the algebraic operation is non-commutative for the rest of the elements in $G_{crq}(X, n)$. Moreover, the structure $G_{crq}(X, n)$ does not support associativity and the partial function is not total in $G_{crq}(X, n)$. Hence, the CR-quasigroupoid $G_{crq}(X, n)$ in a topological (C, R) space is a hybrid variety of quasiloop and quasigroupoid generated from $L_o(n)$ and $G_o(n)$, respectively. $\square$

4. Algebraic Properties of CR-Quasigroupoid

The algebraic and topological analyses presented in this paper consider that the P-separation is a complete and countable finite variety. In the algebraic analysis presented in Sections 4.1 and 4.2, the topological space (X, τ_X) is identified under surjective $f : V_i \cup V_k \rightarrow X$ from another Noetherian P-separated topological (C, R) space (V, τ_V) .

4.1. Algebraic Analysis of CR-Quasigroupoid

In this section, the algebraic analysis of a $G_{crq}(X, n)$ variety is presented within an identified topological (C, R) space (X, τ_X) where $n = 5$ due to surjective identification of two Noetherian P-separated triangulated convex planar subspaces. Let a finite countable set be given as $\{x_i : i = a, b, c, d\} \cup \{e_A\} \subset A \subset X$ such that $\{x_i\} \in \tau_X$ is closed in the Hausdorff topological space. According to the definition of a $G_{crq}(X, n)$, we can derive the following equations.

$$\begin{aligned} f(\Delta_i^3) \cap f(\Delta_k^3) &= \{e_A\} \subset A, \\ (x_a *_{\Delta} x_c) &= (x_c *_{\Delta} x_a) = (x_b *_{\Delta} x_d) = (x_d *_{\Delta} x_b) = e_A, \\ e_A *_{\Delta} e_A &= e_A, \\ x_d *_{\Delta} x_c &= x_a, x_c *_{\Delta} x_d = x_b, \\ x_b *_{\Delta} x_a &= x_c, x_a *_{\Delta} x_b = x_d. \end{aligned} \quad (2)$$

Note that the partial function $*_{\Delta} : A^2 \rightarrow A$ is not total because the algebraic operations $x_a *_{\Delta} x_d, x_d *_{\Delta} x_a$ and $x_b *_{\Delta} x_c, x_c *_{\Delta} x_b$ are not defined in $G_{crq}(X, 5)$. It is known that in a fibered topological (C, R) space a composite algebraic operation $(+T)$ can be admitted, where the function $T : X \rightarrow X$ is a linear and finite translation operation. Let us consider that $G_{crq}(X, 5)$ can be constructed by employing $*_{\Delta} \equiv (+T)$ in (X, τ_X) . This results in the following set of equations.

$$\begin{aligned} \forall x_i \in X, x_i &= (z_i \in C, r_i \in R), \\ \forall x_i, T(x_i) &= (z_i^T \in C, r_i^T \in R), \\ x_a(+T)x_b &= x_d = (z_a + z_b^T, r_a + r_b^T), \\ x_b(+T)x_a &= x_c = (z_b + z_a^T, r_b + r_a^T), \\ x_c(+T)x_d &= x_b = (z_c + z_d^T, r_c + r_d^T), \\ x_d(+T)x_c &= x_a = (z_d + z_c^T, r_d + r_c^T). \end{aligned} \quad (3)$$

Note that if $T(e_A) = e_A$ signifies a fixed and unique right-identity element in $G_{crq}(X, 5)$ then it can be concluded that $e_A(+T)e_A = e_A$. Thus it is evident from the set of equations that $G_{crq}(X, 5)$ symmetrically generates the elements with respect to the stationary identity element e_A , where the surjectively identified triangulation $f(\Delta_i^3) \cup f(\Delta_k^3)$ is not a convex in X .

4.2. Analysis of Identity at Origin

The analysis of behavior of identity element at the origin of a topological (C, R) space and the resulting structural properties of $G_{crq}(X, 5)$ are presented in this section. Recall that $x_0 = (z_0, 0)$ is the origin of a topological (C, R) space, where $z_0 \in C$ is the Gauss origin. If we consider that $x_0 \equiv e_A$ in $G_{crq}(X, 5)$ is the right-identity as well as a stationary point, then it results in the following set of conditions to be maintained in $G_{crq}(X, 5)$.

$$\begin{aligned} A &= \{x_i : i = a, b, c, d\}, \\ \forall x_i \in A, z_i &\neq z_0, r_i \neq 0, \\ (e_A)^{-1} &= e_A. \end{aligned} \quad (4)$$

Remark 3. It is important to observe that if we consider that identity is located at $x_0 = (z_0, 0)$ in $G_{crq}(X, 5)$ then $(e_A)^{-1} \neq ((z_0)^{-1}, (r_a)^{-1})$ and the inverse of identity element is to be specifically defined as $(e_A)^{-1} = e_A$. Note that if we define $(e_A)^{-1} = e_A$ then the properties of a stationary point delegated to the right-identity at origin is not violated because $T(x_0) = x_0$ preserves the translation invariance of origin of a topological (C, R) space. Moreover, the identity element located at origin $x_0 = (z_0, 0)$ with $(e_A)^{-1} = e_A$ and the $T(x_0) = x_0$ translation invariance condition successfully preserves the properties of right-identity element at origin as presented in the following derivation.

$$\begin{aligned} x_0 + x_0 &= (z_0 + z_0, 0 + 0) = x_0, \\ e_A(+T)(e_A)^{-1} &= (e_A)^{-1}(+T)e_A = e_A(+T)e_A, \\ e_A(+T)e_A &= x_0(+T)x_0 = (x_0 \equiv e_A). \end{aligned} \quad (5)$$

The further derivations by following the presented definitions and the associated conditions lead to the following observations.

$$\begin{aligned} (z_a, r_a)^{-1} &\equiv (z_a^{-1} \in C, r_a^{-1} \in R) = (z_c, r_c), \\ \Rightarrow (z_a, r_a)^{-1}(+T)(z_c, r_c) &= (z_c, r_c)^{-1}(+T)(z_a, r_a) = (z_0, 0), \\ \Rightarrow (z_a^{-1} + z_c^T, r_a^{-1} + r_c^T) &= (z_c^{-1} + z_a^T, r_c^{-1} + r_a^T) = (z_0, 0). \end{aligned} \quad (6)$$

The rearrangement of terms results in the following conditions, where $(-1).x_0 = x_0$ and $x_0 + x_0 = x_0$ properties are maintained preserving the multiplicative as well as self-additive invariance of stationary right-identity at origin within the topological (C, R) space.

$$\begin{aligned} z_a^{-1} - z_a^T &= z_0, r_a^{-1} = r_a^T, \\ z_c^{-1} - z_c^T &= z_0, r_c^{-1} = r_c^T. \end{aligned} \quad (7)$$

The observations drawn from the above conditions can be further generalized as $z_i^{-1} - z_i^T = z_0, r_i^{-1} = r_i^T$ for $i = a, b, c, d$ in $G_{crq}(X, 5)$.

5. Analysis of Noetherian and Surjective Identification

The algebraic as well as topological properties presented in this section consider that $f : X \rightarrow W$ is a surjective identification from a Noetherian P-separated topological (C, R) space (X, τ_X) to the dense topological (C, R) space (W, τ_W) . The algebraic analysis is presented in Section 5.1 and the topological analysis is presented in Section 5.2.

5.1. Analysis of Algebraic Properties

First, we show that the surjective identification of two Noetherian P -separated planar triangulated convexes forms the $G_{crq}(W, 5)$ algebraic structure. Note that the formulation $(\Delta_f^5, *_\Delta)$ explicitly represents a CR-quasigroupoid, where Δ_f^5 is a surjectively identified triangulation within a topological subspace and $*_\Delta$ denotes an algebraic operation in Δ_f^5 .

Theorem 1. *If $f : X \rightarrow W$ is a surjective identification of two Noetherian P -separated triangulated planar convexes between topological (C, R) spaces then $(\Delta_f^5, *_\Delta)$ is a $G_{crq}(W, 5)$ where $\Delta_f^5 = f(\Delta_i^3) \cup f(\Delta_k^3)$ and $*_\Delta \equiv (+T)$ in (W, τ_W) .*

Proof. Let $A_i \subset X_i$ and $A_k \subset X_k$ be two P -separated Noetherian subspaces in the topological (C, R) space (X, τ_X) such that $x_i \in B_\Delta^i$ and $x_k \in B_\Delta^k$ preserve the $P(x_i) \wedge P(x_k) = 1$ condition. Suppose (W, τ_W) is a topological (C, R) space with origin located at $w_0 \in W$. If we consider a surjective identification $f : (X_i \cup X_k \subset X) \rightarrow W$ within the topological (C, R) space (W, τ_W) such that $f(\Delta_i^3 \subset \partial A_i) \cap f(\Delta_k^3 \subset \partial A_k) = \{e_w \equiv w_0\}$ then $\Delta_f^5 = f(\Delta_i^3) \cup f(\Delta_k^3)$ is a triangulated path-connected component in (W, τ_W) . Thus if we consider a composite algebraic operation in (W, τ_W) represented as $*_\Delta \equiv (+T)$, where $T : W \rightarrow W$ is a linear finite translation, then $(+T) : \Delta_f^5 \times \Delta_f^5 \rightarrow \Delta_f^5$ is a $G_{crq}(W, 5)$ by definition and the algebraic construction under the surjective identification. Recall that as $w_0 \in W$ is the origin in (W, τ_W) , so $T(w_0) = w_0$ and $w_0 + w_0 = w_0$ preserve the invariance of origin as a stationary point. Hence, the algebraic structure $(\Delta_f^5, *_\Delta)$ is a $G_{crq}(W, 5)$ in (W, τ_W) . $\square$

Lemma 1. *If H is a countable as well as discretely (i.e., completely) disconnected space and $g : \Delta_f^5 \rightarrow H$ is a bijection then $G = (g(\Delta_f^5), \cdot)$ is a cyclic group in H .*

Proof. Let H be a countable and discrete set (i.e., topologically completely disconnected space). As the order of $(\Delta_f^5, *_\Delta)$ is a prime and $g : \Delta_f^5 \rightarrow H$ is a bijection hence there is a generator $\langle a \rangle \in H$ generating a cyclic group in H given by $G = (g(\Delta_f^5), \cdot)$ where $a \in g(\Delta_f^5)$. $\square$

Interestingly, if we consider a commutative total partial function with order $n = 3$ then the resulting algebraic structure $G_i = (\Delta_i^3, *_\Delta)$ successfully generates a group. This observation is presented in the following theorem.

Theorem 2. *In every Noetherian P -separated $X_i \subset X$ the $G_i = (\Delta_i^3, *_\Delta)$ is a group of order 3 if and only if $G_i = (\Delta_i^3, *_\Delta)$ is not a $G_{crq}(X, 3)$ and $*_\Delta$ is a commutative total.*

Proof. Let $A_i \subset X_i \subset X$ be a P -separated Noetherian subspace in the topological (C, R) space (X, τ_X) . Suppose we consider a triangulation Δ_i^3 in A_i and an abstract algebraic operation $*_\Delta : \Delta_i^3 \times \Delta_i^3 \rightarrow \Delta_i^3$. If we consider that $\Delta_i^3 = \{x_a, x_b\} \cup \{x_e\}$ such that each $\forall x_i \in \Delta_i^3, \{x_i\} \in \tau_X$ is closed in Hausdorff (X, τ_X) then it is possible to algebraically construct a Cayley structure as given below considering that $*_\Delta : \Delta_i^3 \times \Delta_i^3 \rightarrow \Delta_i^3$ is a commutative total in Δ_i^3 .

$$\begin{aligned} x_e *_\Delta x_e &= x_e, \\ x_a *_\Delta x_b &= x_b *_\Delta x_a = x_e, \\ i = a, b : x_i *_\Delta x_e &= x_e *_\Delta x_i = x_i. \end{aligned} \tag{8}$$

It follows that $\{x_e\} \in \tau_X$ is an identity and a stationary point in $G_i = (\Delta_i^3, *_\Delta)$. Hence, the algebraic structure $G_i = (\Delta_i^3, *_\Delta)$ is a group of order 3 under commutative total $*_\Delta : \Delta_i^3 \times \Delta_i^3 \rightarrow \Delta_i^3$ and as a result $G_i = (\Delta_i^3, *_\Delta)$ is not a $G_{crq}(X, 3)$. $\square$

Remark 4. It is relatively straightforward to observe that if $x_e \equiv x_0 \in X$ then $G_i = (\Delta_i^3, +)$ is an additive group where $x_a = (z_a, r_a)$, $x_b = (-z_a, -r_a)$ and $+(x_i, x_k) = (z_i + z_k, r_i + r_k)$. Accordingly, the structure $G_i = (\Delta_i^3, \cdot)$ is a multiplicative group if, and only if, $x_e \equiv ((1, 0)_i, 1) \in X$ and $x_b = (x_a)^{-1} = (z_a^{-1}, r_a^{-1})$, where $\cdot(x_i, x_k) = (z_i \cdot z_k, r_i \cdot r_k)$. It is important to note that $x_0 \notin (\Delta_i^3, \cdot)$, for obvious reasons, if we consider that $G_i = (\Delta_i^3, \cdot)$ is a finite variety. Moreover, the group structure $G_i = (\Delta_i^3, \cdot)$ will be either completely negatively oriented (i.e., $\{x_a, x_b\} \subset C^- \times R^-$) or completely positively oriented (i.e., $\{x_a, x_b\} \subset C^+ \times R^+$). Note that the notation follows the implications given as: $z_a \in C^- \Rightarrow (Re(z_a) < 0) \wedge (Im(z_a) < 0)$, $z_a \in C^+ \Rightarrow (Re(z_a) > 0) \wedge (Im(z_a) > 0)$ and $R^- \equiv R \setminus (R^+ \cup \{0\})$.

Example 2. Two different numerical examples are presented here revealing few interesting observations. First, if we consider $\Delta_i^3 = \{(z_0, r), (z_0, -r)\} \cup \{(z_0, 0)\}$ then $(\Delta_i^3, +)$ is an additive group on the fiber $\{z_0\} \times R$ and as a result $(\Delta_i^3, +)$ is not a proper triangulated subspace within the fibered (X, τ_X) . However, two planar varieties of oriented multiplicative groups can be generated, which are properly triangulated on a plane. For example, the P-separated Noetherian triangulated $\Delta_i^3 = \{(z_a, 1), (z_a^{-1}, 1)\} \cup \{((1, 0)_i, 1)\}$ is an example of positively oriented multiplicative group $(\Delta_i^3, \cdot)_+$ and alternatively $\Delta_i^3 = \{(-z_a, 1), (-z_a^{-1}, 1)\} \cup \{((1, 0)_i, 1)\}$ is an example of negatively oriented multiplicative group $(\Delta_i^3, \cdot)_-$ with respect to the corresponding plane.

5.2. Analysis of Topological Properties

The analyses of topological properties are presented in this section considering the existence of a surjective identification function between two topological (C, R) spaces maintaining the local homeomorphism. First, we show that there is a fiber in the surjectively identified and fibered topological (C, R) space which preserves the symmetry of $G_{crq}(W, 5)$ with respect to that corresponding fiber.

Theorem 3. If $f : (X, \tau_X) \rightarrow (W, \tau_W)$ is a local homeomorphism generating $(\Delta_f^5, *_\Delta)$, then there is a fiber in (W, τ_W) maintaining symmetry of $G_{crq}(W, 5) \equiv (\Delta_f^5, *_\Delta)$ with respect to the fiber.

Proof. Let (X, τ_X) be a topological (C, R) space and $Y \subset X$ be a P-separated subspace. Suppose $X_i \subset Y$ and $X_k \subset Y$ are two Noetherian P-separated subspaces with respective triangulated planar convexes B_Δ^i and B_Δ^k . Let us consider (W, τ_W) to be a topological (C, R) space such that $f : (X, \tau_X) \rightarrow (W, \tau_W)$ is a local homeomorphism maintaining $\text{hom}(X_i, f(X_i))$ and $\text{hom}(X_k, f(X_k))$ conditions. If $f : (X, \tau_X) \rightarrow (W, \tau_W)$ is a surjective identification in (W, τ_W) , then $\exists w_e \in W$ such that $f(\Delta_i^3) \cap f(\Delta_k^3) = \{w_e\}$. It indicates that there is a non-compact fiber $\{z_e\} \times R$ in the fibered (W, τ_W) such that $f(\Delta_i^3 \setminus \{x_i\}) \subset f(X_i)$ and $f(\Delta_k^3 \setminus \{x_k\}) \subset f(X_k)$ where $f(\{x_i\} \cup \{x_k\}) = \{w_e\}$. Moreover, the Noetherian P-separated subspaces are disjoint as $X_i \cap X_k = \emptyset$ so $\exists W_i \subset f(X_i)$ and $\exists W_k \subset f(X_k)$ such that $f(\Delta_i^3 \setminus \{x_i\}) \subset W_i$ and $f(\Delta_k^3 \setminus \{x_k\}) \subset W_k$, respectively. Note that the surjective identification maintains $W_i \cap W_k = \emptyset$. Furthermore, the local homeomorphism preserves the condition that $\exists E_i \subset W, \exists E_k \subset W$ such that $W_i \subset E_i, \exists W_k \subset E_k$ and $E_i \cap E_k = \{w_e\}$. Hence, the fiber $\{z_e\} \times R$ maintains symmetry of $G_{crq}(W, 5) \equiv (\Delta_f^5, *_\Delta)$ in the topological space (W, τ_W) , where $\{w_e\} \subset \Delta_f^5$. $\square$

Corollary 1. It is relatively straightforward to conclude that the fiber $\{z_e\} \times R$ symmetrically separates $E_i \times R$ and $E_k \times R$ in (W, τ_W) such that $((E_i \setminus \{w_e\}) \times R) \cap (E_k \times R) = \emptyset$ and $(E_i \times R) \cap ((E_k \setminus \{w_e\}) \times R) = \emptyset$.

Recall that the surjectively identified two triangulated planar convexes fail to retain convexity within the identified space. However, the path-connection property is retained successfully under the surjective identification. This topological property is presented in the following theorem.

Theorem 4. If X_i, X_k are locally dense Noetherian P -separated subspaces in X , then $f(B_\Delta^i \cup B_\Delta^k)$ is path-connected under the surjective identification $f : (X, \tau_X) \rightarrow (W, \tau_W)$ in dense (W, τ_W) .

Proof. Let X_i and X_k be two Noetherian P -separated subspaces in (X, τ_X) such that $X_i \cap X_k = \emptyset$. Suppose $X_i \subset X$ and $X_k \subset X$ are locally dense subspaces in X , where $A_i \subset X_i$ and $A_k \subset X_k$ are Noetherian containing triangulated planar convexes B_Δ^i and B_Δ^k , respectively. Recall that $A_i \subset X_i$ and $A_k \subset X_k$ are closed dense subspaces such that $\Delta_i^3 \subset \partial A_i$ and $\Delta_k^3 \subset \partial A_k$. Thus, if we consider a surjective identification $f : (X, \tau_X) \rightarrow (W, \tau_W)$ in dense (W, τ_W) then $f(A_i) \cup f(A_k)$ is also locally dense in $f(X_i \cup X_k)$. As a result, the subspace $f(B_\Delta^i \cup B_\Delta^k) \subset f(A_i) \cup f(A_k)$ is not a meager category in (W, τ_W) according to Baire categorization and the algebraic structure $G_{crq}(W, 5)$ is established within $f(B_\Delta^i \cup B_\Delta^k)$ successfully. Note that it maintains the conditions presented as $\{w_e\} \subset f(B_\Delta^i \cup B_\Delta^k)$ and $f(\{x_i\} \cup \{x_k\}) \subset f(B_\Delta^i \cup B_\Delta^k)$, where $P(x_i) \wedge P(x_k) = 1$ in (X, τ_X) . Hence, it can be concluded that $\forall w_i \in f(B_\Delta^i), \forall w_k \in f(B_\Delta^k)$ there is a continuous function $p : [0, 1] \rightarrow f(B_\Delta^i \cup B_\Delta^k)$ such that $w_e \in p([0, 1])$ and as a result $f(B_\Delta^i \cup B_\Delta^k)$ is path-connected in dense (W, τ_W) . $\square$

Remark 5. The aforesaid observation can be further generalized by following the properties of dense subspaces under surjective identification saying that $f(X_i \cup X_k)$ is at least connected in dense (W, τ_W) . Moreover, if we take a continuous function $p : [0, 1] \rightarrow f(B_\Delta^i \cup B_\Delta^k)$ such that $p(0) = f(x_i)$ and $p(1) = f(x_k)$ then $p([0, 1])$ can be considered as a P -join of $f(B_\Delta^i)$ and $f(B_\Delta^k)$ in (W, τ_W) under $p(\cdot)$.

Interestingly, the retention of path-connection property under surjective identification can be further extended by employing the chain of a P -joined convexes maintaining finite intersection property. The following lemma presents this observation.

Lemma 2. If $\Lambda_X = \{X_i \subset Y : i \in \mathbb{Z}^+\}$ is a set of Noetherian P -separated subspaces, then the surjective identification $f : \Lambda_X \rightarrow W$ preserves path-connection if, and only if, $\Lambda_X = \{X_i \subset Y : i \in \mathbb{Z}^+\}$ maintains a chained finite intersection property given as $\forall X_i \in \Lambda_X, \exists X_k \in \Lambda_X : X_i \cap X_k \neq \emptyset$.

Proof. Let $\Lambda_X = \{X_i \subset Y : i \in \mathbb{Z}^+\}$ be a Noetherian P -separated subspace $Y \subset X$ and the separations are finite as well as countable such that $i \in (1, +\infty)$ maintaining the overall complete Noetherian separation $\bigcap_{m \in [1, i]} X_m = \emptyset$. Suppose we consider a relaxed variety of finite intersection property (indicating as a finite intersection property under chained P -join) such that $\forall X_i \in \Lambda_X, \exists X_k \in \Lambda_X : X_i \cap X_k = \{x_{ik}\} \neq \emptyset$ in $Y \subset X$. Note that in this case it is true that $\forall X_i \in \Lambda_X, \exists x_i \in X_i : P(x_i) = 1$ and as a result it can be further concluded that $P(x_1) \wedge P(x_2) \wedge P(x_3) \wedge \dots \wedge P(x_i) = 1$ in $\bigcup_{m \in [1, i]} X_m$. If we take a surjective identification $f : \Lambda_X \rightarrow W$ in a topological (C, R) space (W, τ_W) , then there is a continuous function given as $p : [0, 1] \rightarrow \bigcup_{m \in [1, i]} f(X_m)$, where $\bigcup_{m \in [1, i]} \{f(x_m)\} \subset p([0, 1])$ and $\bigcup_{\forall i, k} \{f(x_{ik})\} \subset p([0, 1])$. Hence, the surjectively identified subspace is path connected by $p(\cdot)$ in (W, τ_W) due to the formation of a chained P -join. $\square$

Recall that a topological (C, R) space is dense and as a result a finite linear fiber translation function can be admitted within the non-compactly fibered space. The finite linear fiber translation function in the surjectively identified space successfully recovers the P -separation property in the domain through the pre-image. This interesting observation is presented in the following theorem.

Theorem 5. If the function $f : (X, \tau_X) \rightarrow (W, \tau_W)$ is a surjective identification and $T : W \rightarrow W$ is a finite linear fiber translation then $P(f^{-1}(w_i)) = P(f^{-1}(w_k))$ if $T^n(w_i) = w_k$, where $n \in \mathbb{Z}^+, n < +\infty, T^n \neq T$ and $f^{-1}(\cdot)$ denotes pre-image.

Proof. Let (X, τ_X) be a topological (C, R) space containing a finitely countable Noetherian P -separated subspaces and (W, τ_W) be a topological space with the corresponding surjective identification $f : (X, \tau_X) \rightarrow (W, \tau_W)$. Suppose the subset $\{x_i, x_k\} \subset X_i \cup X_k$ represents respective P -separation points in the Noetherian separation $X_i \cap X_k = \phi$ in $Y \subset X$ such that $P(x_i) \wedge P(x_k) = 1$. The surjective identification $f(X_i \cup X_k)$ maintains that $f(x_i) \neq f(x_k), |f(x_i)| = |f(x_k)| = 1$ in (W, τ_W) if $x_i \in B_\Delta^i$ and $x_k \in B_\Delta^k$. Let us consider that the surjection $f : (X, \tau_X) \rightarrow (W, \tau_W)$ identifies $f(x_i) = w_i \in f(X_i \cup X_k)$ and $f(x_k) = w_k \in f(X_i \cup X_k)$ in (W, τ_W) . Note that (W, τ_W) is Hausdorff and as a result $\{w_i\}, \{w_k\}$ are closed and separable in (W, τ_W) because $\exists N_i \subset W, w_i \in N_i$ and $\exists N_k \subset W, w_k \in N_k$ such that $N_i \cap N_k = \phi$, where N_i, N_k are respective open neighborhoods. Recall that in the fibered topological (C, R) space (W, τ_W) , the $\{z_i\} \times R$ and $\{z_k\} \times R$ are two non-compact fibers at w_i and w_k , respectively. If we consider a linear and finite fiber translation $T : W \rightarrow W$ such that $\forall w_a \in W, T((z_a, R)) = (T(z_a), R)$ then $T^n(w_i) = w_k$ for some $n \in \mathbb{Z}^+, 1 < n < +\infty$ where $T^n \neq T$ and $T(R) = R$. As a result, we can conclude that $f^{-1}(T^n(w_i)) = x_k \in B_\Delta^k$ in (X, τ_X) . Hence, it proves that $P(f^{-1}(w_i)) = P(f^{-1}(w_k))$ under the non-compact fiber translation. $\square$

Corollary 2. *It is relatively straightforward to observe that if $B_\Delta^i \subset A_i$ then B_Δ^i is locally dense in A_i and as a result $\{B_\Delta^i : i \in \mathbb{Z}^+\}$ is a family of countable locally dense convex sets which are P -separable in $Y \subset X$.*

6. Conclusions

The Noetherian P -separated subspaces in a topological (C, R) space admit respective triangulated planar convexes supporting the groups of order three. The surjective identification of two triangulated planar convexes generates a quasiloop–quasigroupoid hybrid algebraic structure. The algebraic structure admits right-identity element and the identified subspaces maintain local homeomorphism. There exists a fiber at right-identity element in the surjectively identified topological space maintaining the symmetry of the distribution of elements of quasiloop–quasigroupoid generated by two Noetherian P -separated triangulated planar convexes. Moreover, the bijection from the prime ordered quasiloop–quasigroupoid structure to a countable as well as completely separated set forms a cyclic group in the codomain. The P -separation points are interior to the locally dense triangulated planar convexes. However, the vertices of the triangulated planar convexes are not interior to the respective Noetherian open subspaces. The Noetherian P -separated convexes form multiple connected components within the topological (C, R) space. A surjectively identified topological subspace admitting multiple triangulated planar convexes generates an alternative form of topological chained intersection property. The finite linear translation operation in an identified subspace containing the triangulated convexes allows the recovery of P -separation points in the corresponding Noetherian completely separated topological subspace through pre-image. The locally dense Noetherian P -separated spaces maintain path connection under surjective identification if the space in codomain is also dense and the identification maintains local homeomorphism. A continuous path between the surjectively identified triangulated planar convexes introduces the concept of P -join within the identified topological subspace under a predefined predicate evaluation.

Funding: The research is funded by Gyeongsang National University, Jinju, Korea.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: Author likes to thank anonymous reviewers and editors for their valuable comments and suggestions.

Conflicts of Interest: The author declares no conflict of interest.

References

1. Pfeffer, W.F.; Prikry, K. A topological concept of smallness. *Proc. Natl. Acad. Sci. USA* **1987**, *84*, 3957–3958. [[CrossRef](#)] [[PubMed](#)]
2. El-Atik, A.E.F.A.; Hassan, H.Z. Some nano topological structures via ideals and graphs. *J. Egypt. Math. Soc.* **2020**, *28*, 1–21. [[CrossRef](#)]
3. Dongsheng, Z.; Ho, W.K. On topologies defined by irreducible sets. *J. Log. Algebraic Methods Program.* **2015**, *84*, 185–195. [[CrossRef](#)]
4. Hemelaer, J. A topological groupoid representing the topos of presheaves on a monoid. *Appl. Categ. Struct.* **2020**, *28*, 2749–2772. [[CrossRef](#)]
5. Bochnak, J.; Coste, M.; Roy, M.F. Topology of real algebraic varieties. In *Real Algebraic Geometry*; Springer: Berlin/Heidelberg, Germany, 1998; Volume 36, pp. 263–296.
6. Motegi, K. Haken manifolds and representations of their fundamental groups in $SL(2, \mathbb{C})$. *Topol. its Appl.* **1988**, *29*, 207–212. [[CrossRef](#)]
7. King, H. Survey of the topology of real algebraic sets. *Rocky Mt. J. Math.* **1984**, *14*, 821–830. [[CrossRef](#)]
8. Brazas, J. Transfinite product reduction in fundamental groupoids. *Eur. J. Math.* **2021**, *7*, 28–47. [[CrossRef](#)]
9. Bagchi, S. On the Topological Structure and Properties of Multidimensional (C, R) Space. *Symmetry* **2020**, *12*, 1542. [[CrossRef](#)]
10. Bagchi, S. Connected Fundamental Groups and Homotopy Contacts in Fibered Topological (C, R) Space. *Symmetry* **2021**, *13*, 500. [[CrossRef](#)]
11. Negrepontis, S. Baire sets in topological spaces. *Arch. Math.* **1967**, *18*, 603–608. [[CrossRef](#)]
12. Mucuk, O. Monodromy Principle and Universal Coverings. *Math. Comput. Appl.* **2000**, *5*, 119–127. [[CrossRef](#)]
13. Ibort, A.; Rodríguez, M. On the Structure of Finite Groupoids and Their Representations. *Symmetry* **2019**, *11*, 414. [[CrossRef](#)]
14. Palmigiano, A.; Re, R. Topological Groupoid Quantales. *Stud. Logica* **2010**, *95*, 125–137. [[CrossRef](#)]
15. Brown, R.; Danesh-Naruie, G.; Hardy, J.P.L. Topological groupoids: II. Covering morphisms and G -spaces. *Math. Nachrichten* **2009**, *74*, 143–156. [[CrossRef](#)]