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Dense and σ -Porous Subsets in Some Families of Darboux Functions

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Abstract: G. Ivanova and E. Wagner-Bojakowska shown that the set of Darboux quasi-continuous functions with nowhere dense set of discontinuity points is dense in the metric space of Darboux quasi-continuous functions with the supremum metric. We prove that this set also is σ -strongly porous in such space. We obtain the symmetrical result for the family of strong Świątkowski functions, i.e., that the family of strong Świątkowski functions with nowhere dense set of discontinuity points is dense (thus, “large”) and σ -strongly porous (thus, asymmetrically, “small”) in the family of strong Świątkowski functions.

Keywords: quasi-continuity; strong Świątkowski property; cliquish functions; porosity; Darboux property

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1. Introduction

The studies concerning relationships between subsets in a metric space, also in the case if it is a space of functions, have been investigated by many mathematicians. The classical result of this type is the Banach theorem. It is said that the set of nowhere differentiable functions is residual in the space of continuous functions. The problem of the comparison of subsets in the space of functions was considered by Grande, Maliszewski, Rosen and others (see [1–3]).

In this paper, we compare subsets of the family of functions with the Darboux property in porosity terms. For the first time, the notion of porosity was introduced by Denjoy [4] and Khinchine [5]. The porosity in spaces of functions was considered in [3,6–8]. Interesting examples of sets, which are porous, but not strongly porous in function spaces, were presented by the authors of [6,8].

The family of σ -porous sets on the real line forms a σ -ideal, which is between the σ -ideal of countable subsets and the σ -ideal of meager nullsets.

Examples of subsets simultaneously dense and σ -porous in some function spaces were given by G. Ivanova and E. Wagner-Bojakowska Ivanova [9]. In a similar way, we construct examples of sets which are dense and σ -porous in some subspaces of the space of Darboux functions.

In this paper, we consider only real-valued functions defined on the real line. Symbols $C(f)$ and $D(f)$ mean sets of continuity and discontinuity points of the function f , respectively. To simplify our notations, we write:

$$\langle a, b \rangle = (\min\{a, b\}, \max\{a, b\})$$

for some $a, b \in \mathbb{R}$.

We say that:

1. f is *quasi-continuous at a point* x ([10]) if for every neighborhood U of x and for every neighborhood V of $f(x)$ there exists a non-empty open subset G of U such that

- $f(G) \subset V$. A function f is *quasi-continuous* (briefly, $f \in \mathcal{Q}$) if it is quasi-continuous at each point.
2. f is *internally quasi-continuous* ([11]) (briefly, $f \in \mathcal{Q}_i$) if it is quasi-continuous and $D(f)$ is nowhere dense.
 3. f is *cliquish at a point* x if for every neighborhood U of x and for each $\varepsilon > 0$ there exists a non-empty open subset G of U such that $|f(y) - f(z)| < \varepsilon$ for each $y, z \in G$. A function f is *cliquish* (briefly, $f \in \mathcal{C}_q$) if it is cliquish at each point.
 4. f has the strong Świątkowski property or f is strong Świątkowski function ([12]) (briefly, $f \in \mathcal{D}_s$) if, for each interval $(a, b) \subset \mathbb{R}$ and for each $\lambda \in \langle f(a), f(b) \rangle$, there exists a point $x_0 \in (a, b)$ such that $f(x_0) = \lambda$ and f is continuous at x_0 .
 5. f has the internally strong Świątkowski property (compare [11]) (briefly, $f \in \mathcal{D}_{si}$) if it has the strong Świątkowski property and $D(f)$ is nowhere dense.
 6. f has the Darboux property (briefly, $f \in \mathcal{D}$) if the image of any interval under f is also an interval.

Clearly, $\mathcal{Q} \subset \mathcal{C}_q$. We denote by $\mathcal{DQ}$ a family of Darboux quasi-continuous functions. It is known that $\mathcal{D}_s \subsetneq \mathcal{DQ}$ (see, e.g., [2]). Obviously, $\mathcal{D}_{si} \subset \mathcal{D}_s$, $\mathcal{DQ}_i \subset \mathcal{DQ}$. We show that the second inclusion is proper. Unfortunately, we do not know if there exists a function $f \in \mathcal{D}_s \setminus \mathcal{D}_{si}$.

For further considerations, the following technical lemma is useful. The proof of this lemma follows from the proof of Theorem 14 in [8].

For $f : \mathbb{R} \rightarrow \mathbb{R}$ and $A \in \mathbb{R}$, let

$$\text{osc}(f, A) = \sup_{x \in A} \{f(x)\} - \inf_{x \in A} \{f(x)\}.$$

Lemma 1. Let $f \in \mathcal{C}_q$ and $\varepsilon > 0$. Then, there exist sequences of disjoint intervals $\{(a_n, b_n)\}_{n \in \mathbb{N}}$ and $\{(y_k^n, y_{k+1}^n)\}_{k \in \mathbb{Z}, n \in \mathbb{N}}$, such that:

(i) for each $n \in \mathbb{N}$

$$(a_n, b_n) = \bigcup_{k \in \mathbb{Z}} [y_k^n, y_{k+1}^n];$$

(ii) $\text{osc}\left(f, [y_k^n, y_{k+1}^n]\right) < \varepsilon$ for each $n \in \mathbb{N}, k \in \mathbb{Z}$;

(iii) for each $n \in \mathbb{N}$ we have $\lim_{k \rightarrow \infty} y_k^n = b_n$ and $\lim_{k \rightarrow -\infty} y_k^n = a_n$; and

(iv) the set $\mathbb{R} \setminus \bigcup_{n \in \mathbb{N}} (a_n, b_n)$ is contained in $D(f)$ and is nowhere dense.

To compare families in porosity terms, we need some definitions.

Let X be an arbitrary metric space. By $B(x, r)$ we denote the open ball with center x and of radius $r > 0$. Additionally, $B(x, 0) = \emptyset$. Fix $M \subset X, x \in X$ and $r > 0$. Here and subsequently, let

$$\gamma(x, r, M) = \sup\{t \geq 0 : \exists z \in X : B(z, t) \subset B(x, r) \setminus M\}.$$

Define the porosity of M at x as

$$p(M, x) = 2 \limsup_{r \rightarrow 0^+} \frac{\gamma(x, r, M)}{r}.$$

Definition 1 ([13]). The set $M \subset X$ is

1. porous if $p(M, x) > 0$ for each $x \in M$;
2. strongly porous if $p(M, x) = 1$ for each $x \in M$;
3. σ -porous if it can be represented as a countable union of porous sets; and
4. σ -strongly porous if it can be represented as a countable union of strongly porous sets.

In the proofs, we use the following kinds of function:

Let I, J be arbitrary intervals. We say that f is an (I, J) -left side surjective if $f((\inf I, t)) = J$ for all $t \in I, t > \inf I$. Analogously, the function f is an (I, J) -right side surjective if, for each $t \in I, t < \sup I$, we obtain $f((t, \sup I)) = J$. The function f is an (I, J) -bi-surjective function if it is both left and right side surjective (compare [14]).

Lemma 2. For each of the intervals I and J , there exists a function $f : I \rightarrow \mathbb{R}$ which is (I, J) -bi-surjective and continuous on the interior of I .

Proof. Let I and J be arbitrary intervals. Fix sequences $\{a_n^I\}_{n \in \mathbb{N}}, \{b_n^I\}_{n \in \mathbb{N}}, \{a_n^J\}_{n \in \mathbb{N}}, \{b_n^J\}_{n \in \mathbb{N}}$ such that $a_n^I \searrow \inf I, b_n^I \nearrow \sup I$ (i.e. $\{a_n^I\}$ decreasing and tending to $\inf I$ and $\{b_n^I\}$ increasing and tending to $\sup I$) $a_1^J < b_1^J, a_n^J \searrow \inf J, b_n^J \nearrow \sup J, a_1^J < b_1^J$. If $\inf J \in J$, then for each $n \in \mathbb{N}$ put $a_n = \inf J$. In the opposite case for each $n \in \mathbb{N}$, put $a_n = a_n^J$. Analogously, if $\sup J \in J$, then for each $n \in \mathbb{N}$ put $b_n = \sup J$. In the opposite case for each $n \in \mathbb{N}$, put $b_n = b_n^J$.

Let us define $f : \text{Int}(I) \rightarrow J$, where $\text{Int}(I)$ is interior of I , in the following way:

$$f(x) = \begin{cases} a_n & \text{for } x \in \bigcup_{n \in \mathbb{N}} \{a_{2n-1}^I, b_{2n-1}^I\}, \\ b_n & \text{for } x \in \bigcup_{n \in \mathbb{N}} \{a_{2n}^I, b_{2n}^I\}, \\ \text{linear} & \text{on } [a_1, b_1], [a_{n+1}^I, a_n^I], [b_n^I, b_{n+1}^I], n \in \mathbb{N}. \end{cases}$$

It is easy to see that f is $(\text{Int}(I), J)$ -bi-surjective and continuous. To complete the proof, for $x \in \{\inf I, \sup I\} \cap I$ put $f(x) = a_1$. $\square$

2. Main Results

Lemma 3. There exists a function $f \in \mathcal{DQ} \setminus \mathcal{DQ}_i$.

Proof. Let $A = \{a_1, a_2, a_3, \dots, a_n, \dots\}$ be countable and dense subset of $\mathbb{R}$. Fix sequence $\{p_n\}_{n \in \mathbb{N}}$ of positive real numbers such that $\sum_{n=1}^{\infty} p_n < \infty$.

Now we proceed by induction and construct a sequence of functions $\{f_n\}_{n \in \mathbb{N}}$.

Let $n = 1$. We can choose sequences $a_k^1 \searrow a_1$ and $b_k^1 \nearrow a_1$ such that for each $k \in \mathbb{N}$ it holds

$$b_k^1 < b_{k+1}^1 < a_1 < a_{k+1}^1 < a_k^1.$$

Then, the set

$$A_1 = \{a_1\} \cup \bigcup_{i=1}^{\infty} \{a_k^1, b_k^1\}$$

is closed, nowhere dense and bounded ($\sup A_1 = a_1^1, \inf A_1 = b_1^1$), so the set $(b_1^1, a_1^1) \setminus A_1$ is open and dense in (b_1^1, a_1^1) . The set of all middle points of component interval of $(b_1^1, a_1^1) \setminus A_1$ we denote by B_1 .

Let us define a function f_1 as follows

$$f_1(x) = \begin{cases} 0 & \text{for } x \in \{a_1\} \cup (-\infty, b_1^1) \cup (a_1^1, \infty) \\ \frac{\text{dist}(x, A_1)}{\text{dist}(x, A_1) + \text{dist}(x, B_1)} & \text{for } x \in [b_1^1, a_1^1] \setminus \{a_1\}, \end{cases}$$

where, for each $x \in \mathbb{R}$ and $A \subset \mathbb{R}$, we have $\text{dist}(x, A) = \inf\{|y - x|, y \in A\}$.

Fix $l \in \mathbb{N}$. Assume that we define sets $A_1, \dots, A_{l-1}$ and $B_1, \dots, B_{l-1}$ and functions $f_1, \dots, f_{l-1}$. Let us take sequences $a_k^l \searrow a_l$ and $b_k^l \nearrow a_l$ such that for every $k, i \in \mathbb{N}$ and $j \in \{1, 2, \dots, l-1\}$:

1. $b_k^l < b_{k+1}^l < a_l < a_{k+1}^l < a_k^l$;
2. $a_k^l \neq a_i^l, a_k^l \neq a_j$ and $a_k^l \neq b_i^l$; and
3. $b_k^l \neq b_i^l, b_k^l \neq a_j$ and $b_k^l \neq a_i^l$.

Then, the set

$$A_l = \bigcup_{j=1}^l \left(\{a_j\} \cup \bigcup_{i=1}^{\infty} \{a_i^j, b_i^j\} \right)$$

is closed, nowhere dense and bounded, so the set $(\inf A_l, \sup A_l) \setminus A_l$ is open and dense in $(\inf A_l, \sup A_l)$. The set of all middle points of component interval of $(\inf A_l, \sup A_l) \setminus A_l$ is denoted by B_l .

Let us define a function f_l as follows

$$f_l(x) = \begin{cases} 0 & \text{for } x \in \bigcup_{i=1}^l \{a_i\} \cup (-\infty, b_1^1) \cup (a_1^1, \infty) \\ \frac{\text{dist}(x, A_l)}{\text{dist}(x, A_l) + \text{dist}(x, B_l)} & \text{for } x \in [\inf A_l, \sup A_l] \setminus \bigcup_{i=1}^l \{a_i\}. \end{cases}$$

Let us observe that this function is continuous on $\mathbb{R} \setminus \bigcup_{i=1}^l \{a_i\}$.

Let

$$f(x) = \sum_{n=1}^{\infty} p_n f_n(x).$$

As $\mathbb{R} \setminus A \subset C(f)$ and A is countable, f is Baire 1 (i.e., it is the pointwise limit of some sequence of continuous functions).

We prove that f has the Young property, i.e. for each x_0 there exist sequences $y_k \nearrow x_0$ and $x_k \searrow x_0$ such that $f(x_0) = \lim_{k \rightarrow \infty} f(x_k) = \lim_{k \rightarrow \infty} f(y_k)$ (see [15]). For this purpose, fix $x_0 \in \mathbb{R}$. If x_0 does not belong to A , then f is continuous at this point. It implies the Young property at x_0 .

Assume now that $x_0 \in A$, then there exists $n \in \mathbb{N}$ with $x_0 = a_n$ and $f(x_0) = \sum_{j=1}^{n-1} p_j f_j(x_0)$, where f_j is continuous at x_0 for each $j \in \{1, \dots, n-1\}$. Fix $k \in \mathbb{N}$. We can find $\delta_k > 0$, $\delta_k < \frac{1}{k}$, such that, for each $x \in (x_0 - \delta_k, x_0 + \delta_k)$ and $j \in \{1, \dots, n-1\}$, we have $|p_j f_j(x_0) - p_j f_j(x)| < \frac{1}{(n-1)k}$. As $a_k^n \searrow a_n$ and $b_k^n \nearrow a_n$, there exists $m \in \mathbb{N}$ with $a_m^n, b_m^n \in (x_0 - \delta_k, x_0 + \delta_k)$. Put $x_k = a_m^n$ and $y_k = b_m^n$. Then,

$$\begin{aligned} |f(x_0) - f(x_k)| &= \left| \sum_{j=1}^{\infty} p_j f_j(x_0) - \sum_{j=1}^{\infty} p_j f_j(x_k) \right| = \left| \sum_{j=1}^{n-1} p_j f_j(x_0) - \sum_{j=1}^{n-1} p_j f_j(x_k) \right| \\ &\leq \sum_{j=1}^{n-1} |p_j f_j(x_0) - p_j f_j(x_k)| < \sum_{j=1}^{n-1} \frac{1}{(n-1)k} = \frac{1}{k}, \end{aligned}$$

analogously we can show that $|f(x_0) - f(y_k)| < \frac{1}{k}$, so $y_k \nearrow x_0$ and $x_k \searrow x_0$ and $f(x_0) = \lim_{k \rightarrow \infty} f(x_k) = \lim_{k \rightarrow \infty} f(y_k)$. Consequently, f has the Young property. Moreover, f is Baire 1, so f has the Darboux property (see [15]).

Functions f_k are quasi-continuous for each $k \in \mathbb{N}$, the sequence $\{f_k : k \in \mathbb{N}\}$ uniformly converges to f , so f is also quasi-continuous and, finally, $f \in \mathcal{DQ} \setminus \mathcal{DQ}_i$. $\square$

In the space of functions, we introduce a metric ρ defined as follows

$$\rho(f, g) = \min\{1, \sup\{|f(t) - g(t)| : t \in \mathbb{R}\}\}.$$

Theorem 1 ([16]). *The set of internally Darboux quasi-continuous functions is dense in $(\mathcal{DQ}, \rho)$.*

We prove that this set not only dense, but it is σ -strongly porous in $(\mathcal{DQ}, \rho)$.

Theorem 2. *The set of internally Darboux quasi-continuous functions is σ -strongly porous in $(\mathcal{DQ}, \rho)$.*

Proof. Let $\{(r_j, q_j)\}_{j \in \mathbb{N}}$ be a sequence of all intervals with rational endpoints. Put

$$\mathcal{F}_j = \{f \in \mathcal{DQ}_i : (r_j, q_j) \subset C(f)\}.$$

Of course, $\bigcup_{j \in \mathbb{N}} \mathcal{F}_j = \mathcal{DQ}_i$. Let us prove that for each $j \in \mathbb{N}$ the family $\mathcal{F}_j$ is strongly porous in $\mathcal{DQ}$. For this purpose, fix $j \in \mathbb{N}$ and $f \in \mathcal{F}_j$. We show that for each $r \in (0, 1)$ and $\varepsilon \in (0, r/4)$ there exists a function $g \in \mathcal{DQ}$ such that $B(g, r/2 - \varepsilon) \subset B(f, r) \setminus \mathcal{F}_j$.

Indeed, let $r \in (0, 1)$ and $\varepsilon \in (0, r/4)$. An interval $[a, b] \subset (r_j, q_j)$ can be found that satisfies the inequality $\text{diam}(f([a, b])) < 2\varepsilon$. We define a number A

$$A = \frac{\inf\{f(x) : x \in [a, b]\} + \sup\{f(x) : x \in [a, b]\}}{2},$$

and intervals I, J :

$$I = \left(a, \frac{a+b}{2}\right), J = \left[A - \frac{r}{2}, A + \frac{r}{2}\right].$$

Let g_1 be (I, J) -bi-surjective function continuous on I .

We denote $\frac{a+b}{2}$ by x_0 . Put

$$g(x) = \begin{cases} g_1(x) & \text{for } x \in I \\ f(x) & \text{for } x \notin (a, b) \setminus (I \cup \{x_0\}) \\ A + \frac{r}{2} & \text{for } x = x_0. \end{cases}$$

Clearly, $g \in \mathcal{DQ}_i$. Let us show that $B(g, r/2 - \varepsilon) \cap \mathcal{F}_j = \emptyset$. For this purpose fix a function $h \in B(g, r/2 - \varepsilon)$. We consider a number $\delta > 0$ such that $(x_0 - \delta, x_0 + \delta) \subset (a, b)$.

Now, we prove that h is discontinuous at x_0 . Indeed, g is (I, J) -bi-surjective, so there exists x_1 in the set $I \cap (x_0 - \delta, x_0 + \delta)$ with $g(x_1) = A - \frac{r}{2}$. Therefore,

$$h(x_0) - h(x_1) \geq A + \frac{r}{2} - \rho(h, g) - \left(A - \frac{r}{2} + \rho(h, g)\right) = r - 2\rho(h, g) > 2\varepsilon.$$

Hence, $h((x_0 - \delta, x_0 + \delta))$ is not contained in $(f(x_0) - \varepsilon, f(x_0) + \varepsilon)$. Therefore, h is discontinuous at x_0 and $h \notin \mathcal{F}_j$. It means that

$$B(g, r/2 - \varepsilon) \cap \mathcal{F}_j = \emptyset.$$

Hence,

$$\gamma(f, r, \mathcal{F}_j) = \frac{r}{2}$$

so

$$p(\mathcal{F}_j, f) = 2 \limsup_{r \rightarrow 0^+} \frac{\gamma(f, r, \mathcal{F}_j)}{r} = 1$$

and $\mathcal{F}_j$ is strongly porous in $\mathcal{DQ}$. Finally, $\mathcal{DQ}_i$ is σ -strongly porous in $\mathcal{DQ}$. $\square$

Theorem 3. The set $\mathcal{D}_{si}$ is dense in $(\mathcal{D}_s, \rho)$.

Proof. Fix $f \in \mathcal{D}_s$ and $\varepsilon > 0$. By Lemma 1, there exist sequences of disjoint intervals $\{(a_n, b_n)\}_{n \in \mathbb{N}}$ and $\{(y_k^n, y_{k+1}^n)\}_{k \in \mathbb{Z}, n \in \mathbb{N}}$, having properties (i)–(iv).

Put

$$I_k^n = (y_k^n, y_{k+1}^n), J_k^n = [\inf\{f(x) : x \in \overline{I_k^n}\}, \sup\{f(x) : x \in \overline{I_k^n}\}],$$

where $\overline{I_k^n}$ means the closure of I_k^n . For each $n \in \mathbb{N}, k \in \mathbb{Z}$, let us choose (I_k^n, J_k^n) -bi-surjective functions g_k^n continuous on I_k^n . Define the function g as follows

$$g(x) = \begin{cases} f(x) & \text{for } x \notin \bigcup_{n \in \mathbb{N}} \bigcup_{k \in \mathbb{Z}} I_k^n, \\ g_k^n(x) & \text{for } x \in \bigcup_{n \in \mathbb{N}} \bigcup_{k \in \mathbb{Z}} I_k^n. \end{cases}$$

By Lemma 1(ii), we have $\rho(f, g) \leq \varepsilon$. Now, we prove that g has the strong Świątkowski property. Let us take an interval $(a, b) \subset \mathbb{R}$ such that $g(a) \neq g(b)$. Without loss of generality, assume that $g(a) < g(b)$. Let $\lambda \in (g(a), g(b))$. We consider the following cases:

1. There exist $n \in \mathbb{N}$ and $k \in \mathbb{Z}$ such that $a, b \in I_k^n$.
The function g is continuous on (a, b) , so the restriction $g|_{[a,b]}$ has the Darboux property. Hence, there exists $x \in (a, b)$ with $g(x) = \lambda$. Of course, g is continuous at x .
2. There exist $n \in \mathbb{N}$ and $k \in \mathbb{Z}$ such that $a \in I_k^n, b \notin I_k^n$ and $\lambda \in J_k^n$ (analogously, there exist $n \in \mathbb{N}$ and $k \in \mathbb{Z}$ such that $b \in I_k^n, a \notin I_k^n$ and $\lambda \in J_k^n$). Then, as g is (I_k^n, J_k^n) -bi-surjective, we can find a point $x \in (a, y_k^n)$ with $g(x) = \lambda$. Clearly, g is continuous at x .
3. There does not exist $n \in \mathbb{N}$ and $k \in \mathbb{Z}$ such that $\lambda \in J_k^n$ and $\{a, b\} \cap I_k^n = \emptyset$.
If $f(a) = g(a)$, then put $a' = a$.
If $f(a) \neq g(a)$, then we can find $n \in \mathbb{N}$ and $k \in \mathbb{Z}$ such that $a \in I_k^n$ and $b \notin I_k^n$. Put $a' = y_{k+1}^n$. Then, $a' \in (a, b)$ and, as $\lambda \notin J_k^n, f(a') = g(a') < \lambda$.
If $f(b) = g(b)$, put $b' = b$.
If $f(b) \neq g(b)$, there exist $n \in \mathbb{N}$ and $k \in \mathbb{Z}$ such that $a \notin I_k^n$ and $b \in I_k^n$. Put $b' = y_k^n$. Then, $b' \in (a, b)$ and, as $\lambda \notin J_k^n, f(b') = g(b') > \lambda$.
As f has the strong Świątkowski property and $f(a') \neq f(b')$, with $x' \in (a', b')$ such that $x' \in C(f)$ and $f(x') = \lambda$. Thus, we can find $n \in \mathbb{N}$ and $k \in \mathbb{Z}$ with $x' \in \overline{I_k^n}$ and $y_k^n \in (a', b')$ ($y_{k+1}^n \in (a', b')$). The function g is (I_k^n, J_k^n) -bi-surjective on I_k^n . Hence, there exists $x \in (a, y_k^n)$ such that $g(x) = g(x') = \lambda$ (respectively, there exists $x \in (y_{k+1}^n, b)$ such that $g(x) = g(x') = \lambda$).

Consequently, we obtain $g \in \mathcal{D}_{si}$. As $\rho(f, g) \leq \varepsilon$ and $g \in B(f, 2\varepsilon) \cap \mathcal{D}_{si}$, the set $\mathcal{D}_{si}$ is dense in $(\mathcal{D}_s, \rho)$. $\square$

Theorem 4. The set $\mathcal{D}_{si}$ is σ -strongly porous in $(\mathcal{D}_s, \rho)$.

Proof. Analogously, as above, let $\{(r_j, q_j)\}_{j \in \mathbb{N}}$ be a sequence of all intervals with rational endpoints. Assume that

$$\mathcal{F}_j = \{f \in \mathcal{D}_{si} : (r_j, q_j) \subset C(f)\}.$$

Obviously, $\bigcup_{j \in \mathbb{N}} \mathcal{F}_j = \mathcal{D}_{si}$. We show that for each $j \in \mathbb{N}$ the family $\mathcal{F}_j$ is strongly porous in $\mathcal{D}_s$.

For this purpose, fix $j \in \mathbb{N}$ and let $f \in \mathcal{F}_j$. We prove that for each $r \in (0, 1)$ and $\varepsilon \in (0, r/4)$ there exists a function $g \in \mathcal{D}_s$ such that $B(g, r/2 - \varepsilon) \subset B(f, r) \setminus \mathcal{F}_j$.

Fix $r \in (0, 1)$ and $\varepsilon \in (0, r/4)$. There exists an interval $(a, b) \subset (r_j, q_j)$ satisfying $\text{diam}(f([a, b])) < 2\varepsilon$. Put

$$I = (a, b), J = \left(A - \frac{r}{2}, A + \frac{r}{2}\right),$$

where

$$A = \frac{\inf\{f(x) : x \in [a, b]\} + \sup\{f(x) : x \in [a, b]\}}{2}.$$

Let g_1 be (I, J) -bi-surjective function continuous on I . Put

$$g(x) = \begin{cases} f(x) & \text{for } x \notin (a, b), \\ g_1(x) & \text{for } x \in (a, b). \end{cases}$$

Then, $\rho(f, g) < \frac{r}{2} + \varepsilon$. Now, let us show that $f \in \mathcal{D}_s$. For this purpose, fix c, d such that $f(c) \neq f(d)$. Without loss of generality, we can assume that $g(c) < g(d)$. Let $\lambda \in (g(c), g(d))$.

Now, we consider the following cases:

1. $(c, d) \subset (a, b)$.
The function g is continuous on $[c, d]$, so $g|_{[c, d]}$ has the Darboux property. Hence, there exists $x \in (c, d)$ such that $g(x) = \lambda$. Obviously, g is continuous at x .
2. $(a, b) \subset (c, d)$.
Then, $f(a) = g(a)$, $f(b) = g(b)$ and, consequently, $\lambda \in (f(a), f(b))$. As f has the strong Świątkowski property, there exists $x \in C(f) \cap (c, d)$ such that $f(x) = \lambda$. If $x \notin [a, b]$, then $g(x) = f(x)$ and g is continuous at x . In the case, when $x \in (a, b]$, there exists $x' \in (a, x)$ with $g(x') = f(x) = \lambda$ and g is continuous at x' . If $x = a$, then there exists $x' \in (x, b)$ such that $g(x') = f(x) = \lambda$. Obviously, g is continuous at x .
3. $c \in (a, b)$, $d \notin (a, b)$. If $\lambda \in J$, then, by virtue of assumption, that g is (I, J) -bi-surjection, there exists $x \in (c, b) \cap C(g)$ such that $g(x) = \lambda$. If $\lambda \notin J$, then $\lambda > \sup J$, so $\lambda \in (f(b), f(d))$ and there exists $x \in (b, d) \cap C(g)$ with $g(x) = f(x) = \lambda$. Analogously, we prove for $c \notin (a, b)$, $d \in (a, b)$.

As a result, we show that g belongs to $\mathcal{D}_s$. Fix now $h \in B(g, \frac{r}{2} - \varepsilon)$. Let us take $\delta > 0$ with $(a, a + \delta) \subset (a, b)$. We prove that h is not continuous at a . Consider ε_h satisfying inequalities $0 < \varepsilon_h < \varepsilon$ and $x_1, x_2 \in (a, a + \delta)$ such that $\sup J - g(x_1) < \varepsilon_h$ and $g(x_2) - \inf J < \varepsilon_h$. Observe that

$$\begin{aligned} h(x_1) - h(x_2) &\geq g(x_1) - \rho(h, g) - (g(x_2) + \rho(h, g)) = \\ g(x_1) - g(x_2) - 2\rho(h, g) &\geq \sup J - \varepsilon_h - (\inf J + \varepsilon_h) - 2\rho(h, g) = \\ r - 2\varepsilon_h - 2\rho(h, g) &> 2\varepsilon - 2\varepsilon_h = \alpha > 0. \end{aligned}$$

We proved that for every $\delta > 0$ there exist $x_1, x_2 \in (a, a + \delta)$ such that

$$h(x_1) - h(x_2) > \alpha.$$

It means that h is not continuous at a . Therefore, h is discontinuous on the interval (r_j, q_j) and, consequently, $h \notin \mathcal{F}_j$. Then, we have

$$B(g, r/2 - \varepsilon) \cap \mathcal{F}_j = \emptyset.$$

Hence,

$$\gamma(f, r, \mathcal{F}_j) = \frac{r}{2}$$

and it implies

$$p(\mathcal{F}_j, f) = 2 \limsup_{r \rightarrow 0^+} \frac{\gamma(f, r, \mathcal{F}_j)}{r} = 1.$$

The set $\mathcal{F}_j$ is strongly porous in $\mathcal{D}_s$. As a result, $\mathcal{D}_{si}$ is σ -strongly porous in $(\mathcal{D}_s, \rho)$. $\square$

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