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Radiative and Meson Decays of $Y(4230)$ in Flavor SU(3)

Luciano Maiani, Antonio Davide Polosa * and Verónica Riquer

Dipartimento di Fisica and INFN, Sapienza Università di Roma, Piazzale Aldo Moro 2, I-00185 Roma, Italy; luciano.maiani@roma1.infn.it (L.M.); veronica.riquer@cern.ch (V.R.)
* Correspondence: antonio.polosa@roma1.infn.it

Abstract: The charmonium-like exotic states $Y(4230)$ and the less known $Y(4320)$, produced in e^+e^- collisions, are sources of positive parity exotic hadrons in association with photons or pseudoscalar mesons. We analyze the radiative and pion decay channels in the compact tetraquark scheme, with a method that proves to work equally well in the most studied $D^* \to \gamma/\pi + D$ decays. The decay of the vector Y into a pion and a Z_c state requires a flip of charge conjugation and isospin that is described appropriately in the formalism used. Rates are found to depend on the fifth power of pion momentum, which would make the final states $\pi Z_c(4020)$ strongly suppressed with respect to $\pi Z_c(3900)$. The agreement with BES III data would be improved considering the $\pi Z_c(4020)$ events to be fed by the tail of the $Y(4320)$ resonance under the $Y(4230)$. These results should renovate the interest in further clarifying the emerging experimental picture in this mass region.

Keywords: exotic hadron spectroscopy; multiquark hadrons; quark model

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1. Introduction

The study of final states in e^+e^- high energy annihilation, with the pioneering contributions by BaBar, Belle and BES collaborations, has opened the way to the new spectroscopy of exotic hadrons.

The so-called Y states, unexpected charmonium-like states created by the initial lepton pair, are efficient sources of positive parity exotic hadrons produced in association with one photon, pion or K meson.

Decays of the lightest Y states, such as $Y(4230)$ into $\gamma/\pi/K + X/Z$, extensively studied by the BES III collaboration, have provided precious information on properties and quantum numbers of the lightest, $J^P = 1^+$ exotic states (see e.g., [1]), the latest result being the observation of the first, hidden charm, open strangeness $Z_{cs}(3985)$, produced in association with a charged K meson in [2] (exotic hadrons are extensively reviewed in [3–9]).

In this note, we adopt the compact tetraquark model for $X(3872)$, $Z_c(3900)$ and $Z_c(4020)$ as S-wave tetraquarks [10–12], and for Y states, as P-wave tetraquarks [11,13], to study radiative and pionic decays of $Y(4230)$

$$Y(4230) \to \gamma + X(3872) \tag{1}$$
$$Y(4230) \to \pi + Z_c(3900)/Z_c(4020), \tag{2}$$

observed by BES III in the reactions

$$e^+e^- \to Y(4230) \to \pi + Z_c(3900)/Z_c(4020) \text{ or } \gamma + X(3872) \tag{3}$$

For a Y resonance of valence composition $[cu][\bar{c}\bar{u}]$ or $[cd][\bar{c}\bar{d}]$, the photon in (1) is emitted from the light quark or antiquark. Decay (2) arises from the elementary transitions

$$u \to d\,\pi^+ \quad \text{or} \quad \bar{d} \to \bar{u}\,\pi^+ \tag{4}$$

and similar for π^- and π^0. The same transitions are operative in $D^* \to \gamma/\pi + D$ decays [14].

Our results for $Z_c(3900)$ and $X(3872)$ are in quantitative agreement with earlier studies of D^* decays. The agreement is, of course, welcome but not unexpected and it supports the picture of compact tetraquarks bound by QCD forces.

We find a strong dependence of decay rates from the pion momentum, $\Gamma \propto q^5$. As a consequence, the decay $Y(4230) \to \pi Z_c(4020)$ is strongly suppressed with respect to the decay into $\pi Z_c(3900)$, which does not seem to be supported by the cross sections reported by BES III. One possible explanation could be that the $Z_c(4020)$ events come from the second peak, $Y(4320)$. A clarification of the distribution of $\pi D^* \bar{D}^*$ events in the region as well as information on the decay modes of $Y(4320)$ would be very useful.

Production of exotic states in e^+e^- annihilation goes essentially via Y resonances. It is reasonable to assume that the open strangeness state $Z_{cs}(3985)$ seen in [2]

$$e^+e^- \to K^+ Z_{cs}^-(3985) \to K^+(D_s^{*-} D^0 + D_s^- D^{*0}) \tag{5}$$

also arises from a Y-like resonance with $[cu][\bar{c}\bar{u}]$ or $[cs][\bar{c}\bar{s}]$ valence quark composition that decays to the final $K^+ Z_{cs}$ state by the elementary processes

$$u \to s\,K^+ \quad \text{or} \quad \bar{s} \to \bar{u}\,K^+ \tag{6}$$

If the hypothesis is correct, our analysis of K meson transitions shows that strange members of the two nonets associated to $X(3872)$ and $Z_c(3900)$ should both appear in the final states of (5), i.e., the $D_s \bar{D}^{*0} + c.c.$ spectrum should include as well the $Z_{cs}(4003)$ recently observed by LHCb in B^+ decay [15] (the classification of the newly discovered Z_{cs} resonances is considered in [16]). This is a crucial feature that can be tested in higher luminosity experiments.

2. Production and Decay Modes of $Y(4230)$ in e^+e^- Annihilation

A $J^{PC} = 1^{--}$ resonance, $Y(4620)$, was first observed by BaBar and confirmed by Belle in e^+e^- annihilation with initial state radiation (ISR) [17,18]. BES III later studied the 4620 structure with higher resolution and demonstrated that it is resolved in two lines, now indicated as $Y(4230)$ and $Y(4320)$ (see [1]).

Y states as P-wave tetraquarks have been described in [11,13]. One expects four states $Y_1, \ldots, Y_4$, the two lightest ones with spin composition

$$Y_1 = |(0,0), L = 1\rangle_{J=1} \tag{7}$$

$$Y_2 = \frac{1}{\sqrt{2}}\big(|(1,0), L = 1\rangle + |(0,1), L = 1\rangle\big)_{J=1} \tag{8}$$

Valence quark composition is $[cq][\bar{c}\bar{q}]$, diquark and antidiquark spin are indicated in parenthesis and L is the orbital angular momentum.

It was noted in [13] that the mass difference of $Y_{1,2}$ arises from two contrasting contributions: the hyperfine interaction, which pushes Y_1 down, and the spin–orbit interaction, which pushes Y_2 down. We had chosen $M_1 < M_2$ on the basis of a preliminary indication that the $\gamma + X(3872)$ decay was associated with $Y(4320)$, since this decay may arise from the Y_2 structure in Equation (8) and not from Y_1, Equation (7).

Later information [19] indicates that the source of the $\gamma + X(3872)$ decay is instead $Y(4230)$. Consequently, we are led to change the assignment and propose $M_2 < M_1$, that is

$$Y_2 = Y(4230) \quad Y_1 = Y(4320) \text{ or higher} \tag{9}$$

Table 1 summarizes the cross sections of different final states produced in e^+e^- annihilation at the $Y(4230)$ peak. Cross sections are related to the width $\Gamma(Y(4230) \to f)$ by the formula

$$\sigma_{\rm peak}(e^+e^- \to Y(4230) \to f) = \frac{12\pi}{M_Y^2}\frac{\Gamma_e\Gamma_f}{\Gamma^2} \tag{10}$$

(Γ_e is the width to an electron pair). The total width of $Y(4230)$ is estimated in [1]

$$\Gamma(Y(4230)) = (56.0 \pm 3.6 \pm 6.9)\ \text{MeV} \tag{11}$$

Data from BES III indicate that $Y(4230)$ is isoscalar [20,21]. Thus, denoting by $Y_{u,d}$ the Y_2 states with $u\bar{u}$ and $d\bar{d}$ valence quarks, we take

$$Y(4230) = Y_2 = \frac{Y_u + Y_d}{\sqrt{2}} \tag{12}$$

Table 1. e^+e^- annihilation cross sections into exotic hadrons, determined by BES III in the $Y(4230)$ region.

Ref.	Z(Mass)	$\sqrt{s}$ (GeV)	$e+e \to Y(4230) \to \ldots$	Q	σ (pb)
[22]	$Z_c(3885)$	4.226	$\pi^0 Z_c^0 \to \pi^0(D\bar{D}^* + c.c.)^0$	197	77 ± 21
[23]	$Z_c(3885)$	4.23	$[\pi^+ Z_c^- + c.c.] \to [\pi^+(D\bar{D}^*)^- + c.c.]$	197	141 ± 14
[24]	$Z_c(4020)$	4.26	$[\pi^+ Z_c^- + c.c.] \to [\pi^+(D^*\bar{D}^*)^- + c.c.]$	65	$(0.65 \pm 0.11) \cdot (137 \pm 17) = 89 \pm 19$
[25]	$Z_c(4020)$	4.23	$\pi^0 Z_c^0 \to \pi^0(D^*\bar{D}^*)^0$	65	62 ± 12
[26]	$X(3872)$	4.226	$\gamma X \to \gamma\pi^+\pi^- J/\psi$	$--$	$0.27 \pm 0.09 \pm 0.12$
[27]			γX	354	$5.5^{+2.8}_{-3.6}$
[2]	$Z_{cs}(3982)$	4.681	$K^+ Z_{cs}^- \to K^+(D_s^{*-}D^0 + D_s^- D^{*0})$	199	4.4 ± 0.9

3. $Y(4230)$ Transitions to S-Wave Tetraquarks

We consider the decays

$$Y_2 \to \gamma + X \tag{13}$$
$$Y_2 \to \pi + Z \tag{14}$$
$$Y_2 \to \pi + Z' \tag{15}$$

where X, Z and Z' are the S-wave tetraquarks

$$X = \frac{1}{\sqrt{2}}\left(|(1,0),\ L=0\rangle + |(0,1),\ L=0\rangle\right)_{J=1} \tag{16}$$
$$Z = \frac{1}{\sqrt{2}}\left(|(1,0),\ L=0\rangle - |(0,1),\ L=0\rangle\right)_{J=1} \tag{17}$$
$$Z' = |(1,1),\ L=0\rangle_{J=1} \tag{18}$$

The decay (13) as a dipole transition $L=1 \to L=0$, $\Delta S = 0$ has been considered in [28]. Here we re-derive the result as an introduction to pionic transitions.

We work in the non-relativistic approximation and describe the states with wave functions in spin and coordinate space. In the rest frame of Y_2

$$|Y_2^a\rangle = N_Y\left[\epsilon_{abc}S^{b(+)}\frac{\xi^c}{r}R_{2P}(r)\right] \tag{19}$$

N_Y is a normalization constant and the spin wave function from (8) and (16) is

$$S^{a(+)} = \frac{(c\sigma_2\sigma^a u)_x(\bar{c}\sigma_2\bar{u})_y + (c\sigma_2 u)_x(\bar{c}\sigma_2\sigma^a\bar{u})_y}{2\sqrt{2}} \tag{20}$$

We indicate with a bar the charge–conjugate quark fields, $\boldsymbol{x}$, $\boldsymbol{y}$ are diquark and antidiquark coordinates, $\boldsymbol{\xi} = \boldsymbol{x} - \boldsymbol{y}$ the relative coordinate and $r = |\boldsymbol{\xi}|$ the relative radius. The plus sign in $S^{a(+)}$ reminds of the charge conjugation, as defined on the rhs of (20).

Considering the decay into X, we take

$$|X^a\rangle = N_X \left[S^{a(+)}\, R_{1S}(r)\right] \tag{21}$$

and normalize spin w.f. according to

$$S^{a(+)} \cdot S^{b(+)} = \delta_{ab} \tag{22}$$

Thus

$$\begin{aligned} \delta_{ab} &= \langle Y_2^a | Y_2^b\rangle = N_Y^2\, \delta_{ab} \frac{8\pi}{3} \int dr\, y_{2P}(r)^2 \\ \delta_{ab} &= \langle X^a | X^b\rangle = N_X^2\, \delta_{ab} 4\pi \int dr\, y_{1S}(r)^2 \end{aligned}$$

with $y(r) = rR(r)$, and $R(r)$ the radial wave function.

Radiative decay. We work in the radiation gauge, $A^0 = 0$ and $\boldsymbol{\nabla}\cdot\boldsymbol{A} = 0$. The photon couples to u and to other quarks with the basic Lagrangian

$$\mathcal{L}_{\text{e.m.}} = eQ_u\, \bar{u}(\boldsymbol{x})\, \boldsymbol{\gamma}\cdot\boldsymbol{A}(\boldsymbol{x})\, u(\boldsymbol{x}) + (u \to d) \tag{23}$$

The elementary transition amplitudes are

$$\mathcal{M}_u = eQ_u\, \chi^\dagger \left[\frac{\boldsymbol{p}_u\cdot\boldsymbol{\epsilon}}{m_u} + \frac{i\boldsymbol{q}\wedge\boldsymbol{\epsilon}\cdot\boldsymbol{\sigma}_u}{2m_u}\right]\chi \tag{24}$$

$$\mathcal{M}_{\bar{u}} = -eQ_u\, \chi^\dagger \left[\frac{\boldsymbol{p}_{\bar{u}}\cdot\boldsymbol{\epsilon}}{m_{\bar{u}}} + \frac{i\boldsymbol{q}\wedge\boldsymbol{\epsilon}\cdot\boldsymbol{\sigma}_{\bar{u}}}{2m_u}\right]\chi \tag{25}$$

The minus sign in $\mathcal{M}_{\bar{u}}$ arises from charge conjugation. In view of large mass denominators, we neglect radiation from the charm quarks.

The right-hand sides of these equations contain products of operators acting on the spin and space wave functions of the initial tetraquark multiplied by variables of the electromagnetic field. As usual, we identify

$$\boldsymbol{p}_u = -i\boldsymbol{\partial}_{\boldsymbol{x}} \qquad \boldsymbol{p}_{\bar{u}} = -i\boldsymbol{\partial}_{\boldsymbol{y}} \tag{26}$$

Acting on functions of $\boldsymbol{\xi} = \boldsymbol{x} - \boldsymbol{y}$

$$\boldsymbol{\partial}_{\boldsymbol{x}} = \boldsymbol{\partial}_{\boldsymbol{\xi}} = i\boldsymbol{p}_{\boldsymbol{\xi}} \qquad \boldsymbol{\partial}_{\boldsymbol{y}} = -\boldsymbol{\partial}_{\boldsymbol{\xi}} = -i\boldsymbol{p}_{\boldsymbol{\xi}} \tag{27}$$

Further, we set

$$\frac{\boldsymbol{p}_{\boldsymbol{\xi}}}{m_u} = \boldsymbol{v} = \frac{d\boldsymbol{\xi}}{dt} = i\omega\boldsymbol{\xi} \qquad \omega = |\boldsymbol{q}| \tag{28}$$

and the Hamiltonian acting on tetraquark wave functions is

$$H_I = eQ_u \left[\boldsymbol{\xi}\cdot\boldsymbol{E} + \frac{1}{2m_u}\left(\boldsymbol{\sigma}_u - \boldsymbol{\sigma}_{\bar{u}}\right)\cdot\boldsymbol{B}\right] \tag{29}$$

with $\boldsymbol{E}$ and $\boldsymbol{B}$ the electric and magnetic fields. The first term corresponds to the well known electric dipole transition that changes by one unit the orbital angular momentum, leaving the spin wave function unchanged [29]. One obtains

$$\begin{aligned}\mathcal{M}^{ab}(Y_2 \to \gamma + X) &= eQ_u \,\langle X^a|\xi^c|Y_2^b\rangle \, i\omega\epsilon^c = \epsilon_{abc}\,\epsilon^c \,\frac{\omega}{\sqrt{6}}\,\langle r\rangle_{2P\to 1S} \\ \langle r\rangle_{2P\to 1S} &= \frac{\int dr\,[y_{1S}(r)\, r\, y_{2P}(r)]}{\sqrt{\int dr\, y_{1S}(r)^2 \int dr\, y_{2P}(r)^2}}\end{aligned} \tag{30}$$

with $\omega = M_Y - M_X = \omega_X$ and

$$\Gamma(Y_2 \to \gamma + X) = \frac{4\alpha}{9}\, Q_u^2\, \omega_X^3 \langle r\rangle^2_{2P\to 1S} \tag{31}$$

For isoscalar $Y(4230)$ we use (12). Summing incoherently over the final states X_u and X_d, see [30], we get

$$Q_u^2 \to Q_{eff}^2 = \frac{1}{2}(Q_u^2 + Q_d^2) = \frac{5}{18}$$

and

$$\Gamma(Y(4320) \to \gamma + X) = 0.322\ \text{MeV}\ (\omega_X\ \langle r\rangle_{2P\to 1S})^2 \tag{32}$$

π^0 ***emission.*** We assume that quarks couple to pions via the isovector, axial vector current (to our knowledge, the quark–pion, axial vector interaction to describe pionic hadron decays has been first introduced in [31]):

$$\mathcal{L}_I^\pi = \frac{g}{f_\pi}\,\bar{q}\gamma^\mu\gamma_5(\partial_\mu \boldsymbol{\pi})q \tag{33}$$

$$\boldsymbol{\pi} = \frac{\tau^i \pi^i}{\sqrt{2}} = \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} & \pi^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} \end{pmatrix} \tag{34}$$

We follow [14] for the definition of the coupling g and

$$f_\pi = 132\ \text{MeV} \tag{35}$$

The Lagrangian contains the time derivative of the pion field. Applying the Legendre transformation, the interaction Hamiltonian is

$$\mathcal{H}_I^\pi = \frac{g}{f_\pi}\, \boldsymbol{A}\cdot\boldsymbol{\nabla}\boldsymbol{\pi} \tag{36}$$

The elementary quark transition is determined by

$$\boldsymbol{q}\cdot\boldsymbol{A}_u = \bar{u}(p+q)\,\boldsymbol{q}\cdot\boldsymbol{\gamma}\gamma_5\, u(p) = [\boldsymbol{q}\cdot\boldsymbol{\sigma} + \frac{q^2}{4m_u}\frac{\boldsymbol{p}}{m_u}\cdot\boldsymbol{\sigma} + \ldots] \tag{37}$$

The first term corresponds to $\Delta L = 0$, operative in $D^* \to D\pi$ [14], the second to $\Delta L = 1$, for Y and D_1 pionic decay, dots indicate terms with $\Delta L > 1$. Using charge conjugation symmetry, restricting to the $\Delta L = 1$ term and specializing to the π^0 case, we obtain the Hamiltonian

$$H_I^\pi = \frac{g}{f_\pi}\frac{q^2}{4m_u}\, i\omega\, \chi^\dagger[\boldsymbol{\xi}\cdot(\boldsymbol{\sigma}_u - \boldsymbol{\sigma}_{\bar{u}}) - (u\to d)]\chi\,\frac{\pi^0}{\sqrt{2}} \tag{38}$$

We note the results of applying the spin operators to the components of the spin wave function

$$\begin{aligned}\sigma_u^b(c\sigma_2 u) &= (c\sigma_2\sigma^b u) \\ \sigma_u^b(c\sigma_2\sigma^a u) &= (c\sigma_2\sigma^a\sigma^b u) = \left[\delta_{ab}(\bar{c}\sigma_2\bar{u}) + i\epsilon_{abc}(\bar{c}\sigma_2\sigma^c\bar{u})\right]\end{aligned}$$

$$\begin{aligned} -\sigma^b_{\bar{u}}(\bar{c}\sigma_2\bar{u}) &= -(\bar{c}\sigma_2\sigma^b\bar{u}) \\ -\sigma^b_{\bar{u}}(\bar{c}\sigma_2\sigma^a\bar{u}) &= -(\bar{c}\sigma_2\sigma^a\sigma^b\bar{u}) = -\left[\delta_{ab}(\bar{c}\sigma_2\bar{u}) + i\epsilon_{abc}(\bar{c}\sigma_2\sigma^c\bar{u})\right] \end{aligned} \tag{39}$$

Defining

$$S^{a(-)}_u = \frac{(c\sigma_2\sigma^a u)(\bar{a}\sigma_2\bar{u}) - (c\sigma_2 u)(\bar{c}\sigma_2\sigma^a\bar{u})}{2\sqrt{2}} = Z \tag{40}$$

$$S^a_u = i\epsilon_{abc}\frac{(c\sigma_2\sigma^b u)(\bar{c}\sigma_2\sigma^c\bar{u})}{2\sqrt{2}} = Z' \tag{41}$$

we obtain

$$(\sigma^b_u - \sigma^b_{\bar{u}})S^{a(+)}_u = i\epsilon_{abc}\left[S^{c(-)}_u + S^c_u\right] \tag{42}$$

Note that, going from Y to Z or Z', the minus sign between σ_u and $\sigma_{\bar{u}}$ changes the charge conjugation sign of the spin w.f.. Similarly, the minus sign between the u and d term in (38) changes the S_u and S_d combination from $I = 0$ (in Y) to $I = 1$ (in Z and Z').

In conclusion, we find

$$\mathcal{M}^{ab} = \langle Z^a|H^\pi_I|Y^b_2\rangle = \delta_{ab}\sqrt{\frac{2}{3}}\frac{g}{\sqrt{2}}\left(\frac{q^2}{4f_\pi m_u}\right)(\omega_Z\langle r\rangle_{2P\to 1S}) \tag{43}$$

and

$$\begin{aligned} \Gamma(Y(4230)\to Z^0\pi^0) &= |\mathcal{M}^{11}|^2\frac{1}{(2\pi)^3}\,4\pi\int\frac{q\omega d\omega}{2\omega}\,(2\pi)\delta(\Delta M-\omega) = \frac{q}{2\pi}\,|\mathcal{M}^{11}|^2 = \\ &= \frac{q}{6\pi}g^2\left(\frac{q^2}{4f_\pi m_u}\right)^2\left(\frac{\omega_Z}{\omega_X}\right)^2(\omega_X\langle r\rangle_{2P\to 1S})^2 \\ &= 4.36\text{ MeV}\;g^2(\omega_X\langle r\rangle_{2P\to 1S})^2 \end{aligned} \tag{44}$$

q is the decay momentum, $\omega_Z = M_Y - M_Z$, we have chosen to normalize the radius with ω_X, for comparison with Equation (32) and $m_u = 308$ MeV, from the constituent quark model spectrum of mesons (see e.g., [3,10]).

4. Charge Conjugation in Y and Other Tetraquark Nonets

A charge conjugation quantum number can be given to each self conjugate SU(3)$_f$ multiplet according to

$$\mathcal{C}T\mathcal{C} = \eta_T\tilde{T} \tag{45}$$

where $\mathcal{C}$ denotes the operator of charge conjugation, T the matrix representing the multiplet in SU(3) space and $\tilde{T}$ the transpose matrix. η_T is the sign taken by neutral members, but it can be attributed to all members of the multiplet. In the exact SU(3)$_f$ limit, η is conserved in strong and electromagnetic decays. $\eta = -1$ is given to the electromagnetic current J^μ and to Y^μ while $\eta_{K,\pi} = +1$.

We extend Y_2 to a full nonet that we write as (omitting the overall normalization for brevity)

$$\begin{aligned} Y_2(a,\bar{b})^\gamma &= \epsilon_{\gamma\alpha\beta}\left[(D^\alpha_a\bar{D}_{\bar{b}}) + (D_a\bar{D}^\alpha_{\bar{b}})\right]\frac{\xi^\beta}{r}\,F(\xi) \\ D^\alpha_a &= (c\sigma_2\sigma^\alpha q_a);\ \bar{D}_{\bar{b}} = (\bar{c}\sigma_2\bar{q}_b),\ \text{etc.} \end{aligned} \tag{46}$$

$F(\xi)$ is the wave function in the relative coordinate, even under $\xi\to-\xi$.

The Lagrangian (33) generalizes to

$$\mathcal{L}_I^\pi = \frac{g}{f_\pi}\,\bar{q}\gamma^\mu\gamma_5(\partial_\mu \boldsymbol{M})q \tag{47}$$

where

$$\boldsymbol{M} = \frac{\lambda^i M^i}{\sqrt{2}} = \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -2\frac{\eta_8}{\sqrt{6}} \end{pmatrix} \tag{48}$$

(λ^i are the Gell–Mann matrices). Correspondingly, the action on each diquark of the Hamiltonian derived from (47) is (for brevity, we omit two-dimensional spinors $\chi^\dagger$ and χ, which should bracket all the expressions below)

$$\begin{aligned} \mathcal{H}_I D_a^\alpha &\propto M_{aa'}(\boldsymbol{\sigma}\cdot\boldsymbol{\xi})^{\alpha\beta} D_{a'}^\beta = (MD_{\sigma\xi})_a^\alpha \\ \mathcal{H}_I \bar{D}_{\bar{b}} &\propto -(\boldsymbol{\sigma}\cdot\boldsymbol{\xi})(M_{\bar{b}'\bar{b}}\bar{D})_{\bar{b}'} = -(\boldsymbol{\sigma}\cdot\boldsymbol{\xi})\bar{D}_{\bar{b}'}M_{\bar{b}'\bar{b}} = -(\bar{D}_{\sigma\xi}M)_{\bar{b}}, \text{ etc.} \end{aligned}$$

and

$$\mathcal{H}_I(D_a^\alpha \bar{D}_{\bar{b}}) \propto (MD_{\sigma\xi})_a^\alpha \bar{D}_{\bar{b}} + D_a^\alpha(\bar{D}_{\sigma\xi}M)_{\bar{b}} \tag{49}$$

where

$$(D_{\sigma\xi})_a^\alpha = (c\sigma_2(\sigma\cdot\xi)q_a) \qquad (\bar{D}_{\sigma\xi})_{\bar{b}} = (\bar{c}\sigma_2(\sigma\cdot\xi)\bar{q}_{\bar{b}})$$

Explicitly,

$$\begin{aligned} &\frac{\xi^\beta}{r}\left[(D_{\sigma\xi})_a^\alpha \bar{D}_{\bar{b}} + (D_{\sigma\xi})_a \bar{D}_{\bar{b}}^\alpha\right] = \\ &= \frac{\xi^\beta}{r}\xi^\rho\left[(c\sigma_2\sigma^\alpha\sigma^\rho q_a)(\bar{c}\sigma_2\bar{q}_{\bar{b}}) + (c\sigma_2\sigma^\rho q_a)(\bar{c}\sigma_2\sigma^\alpha q_{\bar{b}})\right] = \\ &= \frac{\xi^\beta}{r}\xi^\rho\left[\delta^{\alpha\rho}\,(c\sigma_2 q_a)(\bar{c}\sigma_2\bar{q}_{\bar{b}}) + i\epsilon^{\alpha\rho\nu}(c\sigma_2\sigma^\nu q_a)(\bar{c}\sigma_2\bar{q}_{\bar{b}}) + (c\sigma_2\sigma^\rho q_a)(\bar{c}\sigma_2\sigma^\alpha q_{\bar{b}})\right] \end{aligned} \tag{50}$$

Applying similar arguments to the second line of (49), we obtain

$$\begin{aligned} &\frac{\xi^\beta}{r}\left[(D)_a^\alpha(\bar{D}_{\sigma\xi})_{\bar{b}} + (D)_a(\bar{D}_{\sigma\xi})_{\bar{b}}^\alpha\right] = \\ &= \frac{\xi^\beta}{r}\xi^\rho\left[(c\sigma_2\sigma^\alpha q_a)(\bar{c}\sigma_2\sigma^\rho\bar{q}_{\bar{b}}) + (c\sigma_2 q_a)(\bar{c}\sigma_2\sigma^\alpha\sigma^\rho q_{\bar{b}})\right] = \\ &= \frac{\xi^\beta}{r}\xi^\rho\left[(c\sigma_2\sigma^\alpha q_a)(\bar{c}\sigma_2\sigma^\rho\bar{q}_{\bar{b}}) + \delta^{\alpha\rho}\,(c\sigma_2 q_a)(\bar{c}\sigma_2 q_{\bar{b}}) + i\epsilon_{\alpha\rho\nu}(c\sigma_2 q_a)(\bar{c}\sigma_2\sigma^\nu q_{\bar{b}})\right] \end{aligned} \tag{51}$$

The expressions in (50) and (51) are to be integrated with functions symmetric under $\xi^a \to -\xi^a$, so we can replace $\xi^\beta\xi^\rho \to \delta^{\beta\rho}\,r^2/3$. In addition, in the square brackets we can add and subtract terms that reconstruct the spin wave functions of tetraquarks of charge conjugation $+1$, spin $0, 1, 0, 2$, namely $X,\ X_0,\ X_0',\ X_2$ and of charge conjugation -1, spin 1, i.e., $Z,\ Z'$. Indicating for brevity only $X,\ Z,\ Z'$, Equations (16)–(18), we obtain

$$\mathcal{H}_I\, Y_2 \propto \frac{r}{3}\, i\epsilon_{\alpha\beta\nu}\,\frac{1}{2}\Big[(MX^\nu - X^\nu M) + (MZ^\nu + Z^\nu M) + (MZ'^\nu + Z'^\nu M) + ...\Big] \tag{52}$$

Multiplying by the $SU(3)$ matrix representing Y and taking the trace we obtain the exact $\mathrm{SU}(3)_f$ rules for the couplings of a $C = -1$ vector nonet to M plus an S-wave tetraquark of charge conjugation η_T:

$$\begin{aligned} \mathcal{H}_I &\propto \mathrm{Tr}[Y[M,X]] \quad (\eta_X = +1) \\ \mathcal{H}_I &\propto \mathrm{Tr}[Y\{M,Z\}] \quad (\eta_Z = -1). \end{aligned} \tag{53}$$

In particular, for $Y(4230)$: $Y = \mathrm{diag}(1/\sqrt{2}, 1/\sqrt{2}, 0)$, we obtain vanishing coupling $Y \to \pi X$ and Equation (43) for $Y_2 \to \pi^0 Z^0$.

Summarizing, we obtain the selection rules:

1. $Y_2(I = 0)$ does not decay into $\pi^{\pm,0} X^{\mp,0}$
2. $Y_2(I = 1)$ decays into $\pi^+ X^- - \pi^- X^+$
3. $Y_2(I = 0)$ decays into $\pi^+ Z^- + \pi^- Z^+ + \pi^0 Z^0$, same for Z'
4. $Y_2(I = 0, \text{or } 1)$ or $Y[cs\bar{c}\bar{s}]$ all decay into $(K^+ X_{cs}^- \ - \ c.c.)$ and $(K^+ Z_{cs}^- \ + \ c.c.)$
5. The decay $Z_{cs} \to J/\psi\, K$ is allowed in the exact $\mathrm{SU}(3)_f$ limit with

$$\mathcal{H}_I = \lambda\mu\ \psi\ (\mathrm{Tr}\{Z, M\}),\ [\mu] = \text{mass} \tag{54}$$

6. The decay $X_{cs} \to J/\psi\, K$ may occur to first order in $\mathrm{SU}(3)_f$ symmetry breaking with

$$\mathcal{H}_I = \lambda\ i\psi\ \mathrm{Tr}([\epsilon_8[X, M]) \sim \lambda\ (m_s - m_u)\ i\psi(X_{cs}^+ K^- \ - \ c.c.) \tag{55}$$

5. Radiative and Pionic Decays: D^* and D_1 Mesons

The decay $\boldsymbol{D^* \to \gamma + D}$. In its spin dependent part, the Hamiltonian (29) describes the radiative decay of D^*, $\Delta S = 1$ and no change in orbital angular momentum. Setting the charm quark in the origin, D^* and D are represented by

$$D^{(*a)} = \boldsymbol{V}^{(a)} \cdot \left(c^\dagger(0) \frac{\boldsymbol{\sigma}}{\sqrt{2}} u(\boldsymbol{x})\right) R(|\boldsymbol{x}|) \qquad D = \left(c^\dagger(0) \frac{1}{\sqrt{2}} u(\boldsymbol{x})\right) R(|\boldsymbol{x}|)$$

$$\mathcal{M}(D^{*0} \to \gamma + D^0) = \frac{eQ_u}{2m_u} \chi^\dagger\, \boldsymbol{V}^{(a)} \cdot \boldsymbol{q} \wedge \boldsymbol{\epsilon}^{(b)}\, \chi$$

and we obtain

$$\Gamma(D^{*0} \to \gamma + D^0) = \frac{\alpha}{3} \left(\frac{Q_u}{m_u}\right)^2 q^3 \tag{56}$$

Reference [14] can be consulted for a discussion of the D^* decay rate and the strong interaction corrections to (56).

The decay $\boldsymbol{D^{*+} \to \pi^0 D^+}$. From the Hamiltonian (36) and Equation (37), the relevant term in the Hamiltonian is

$$\mathcal{H}_I^\pi = \frac{g}{\sqrt{2} f_\pi}\, \boldsymbol{q} \cdot \boldsymbol{\sigma_u}$$

so that

$$\mathcal{M}(D^{*+} \to \pi^0 D^+) = \frac{g}{\sqrt{2} f_\pi}\, \boldsymbol{V} \cdot \boldsymbol{q}$$

and

$$\Gamma(D^{*+} \to \pi^0 D^+) = g^2\, \frac{(p^0)^3}{12\pi f_\pi^2}$$

with p^0 the decay momentum. Further,

$$\Gamma(D^{*+} \to \pi^+ D^0) = g^2 \frac{(p^+)^3}{6\pi f_\pi^2}$$

(decay momentum p^+). We reproduce the results of [14]. We assume D^{*+} decay to be dominated by πD final states and use the D^{*+} total width [32] to estimate the value of g, obtaining

$$g \sim 0.56. \tag{57}$$

$\boldsymbol{D_1^0 \to \pi^0\, D^{*0}}$ transition. $D_1(2420)$ is a well identified P-wave, positive parity charmed meson with total spin and angular momentum $S = J = 1$. We can use its decay into $D^* \pi$

to calibrate the $\Delta L = 1$ Hamiltonian (38). In analogy with (19), we write the D_1 wave function as:

$$\begin{aligned} |D_1^a\rangle &= N_1\left[\epsilon_{abc}\frac{\xi^b}{r}\,(\bar{c}\sigma_2\frac{\sigma^c}{\sqrt{2}}u)\,R_{2P,D}(r)\right] \\ \delta^{ab} &= \langle D_1^a|D_1^b\rangle = N_1^2\,\frac{8\pi}{3}\int dr\, y_{2P,D}(r)^2 \end{aligned} \tag{58}$$

where the subscript D indicates that the QCD couplings of the $\bar{c}u$ system are used.

The decay is induced by the ξ dependent part of the Hamiltonian, restricted to the u term. Proceeding as before, we find

$$\mathcal{M}^{ab} = \langle D^{*a}|H_I^\pi|D_1^b\rangle = \delta_{ab}\,\frac{g}{\sqrt{2}}\left(\frac{q^2}{4f_\pi m_u}\right)\sqrt{\frac{2}{3}}\,(\omega\langle r\rangle_{2P,D\to 1S,D})$$

and

$$\Gamma(D_1^0 \to \pi^0 + D^{*0}) = \frac{q_1}{6\pi}\,g^2\left(\frac{q^2}{4f_\pi m_u}\right)^2(\omega_1\langle r\rangle_{2P,D\to 1S,D})^2 \tag{59}$$

$$\Gamma(D_1^0 \to \pi + D^{*}) = 3\,\Gamma(D_1^0 \to \pi^0 + D^{*0}) \tag{60}$$

q_1 is the decay momentum, $\omega_1 = M_{D_1} - M_{D^*}$ and we have assumed that the πD^* modes saturate the total width. The transition radius in Equation (59) is computed in the next Section, see Table 2. Using the experimental width [32] we find:

$$g = 0.63 \pm 0.08 \tag{61}$$

the error is estimated from the D_1^0 and $D_1^\pm$ width errors and variations in the estimated radius.

Table 2. Values of the transition radius $2P \to 1S$, GeV^{-1}, for P-wave tetraquark and D_1.

Transition $2P \to 1S$	Lattice, Equation (63)	Cornell, Equation (64)	Pure Confinement, Equation (65)
$Y(4230) \to X(3872)/Z_c(3900)$	2.17	1.84	2.15
$D_1(2420) \to D^*$	3.74	3.34	3.36

6. Transition Radius

The transition radius for a diquarkonium was estimated in [28], from the radial wave functions of a diquark–antidiquark system in a confining, QCD potential. We solve numerically the two body, radial Schrödinger Equation [33] with potential and diquark mass

$$V(r) = -\frac{\alpha_s}{r} + kr \qquad M_{[cq]} = 1.97\ \text{GeV} \tag{62}$$

Couplings are taken from lattice calculation of charmonium spectrum [34]

$$\alpha_s = 0.3 \qquad k = 0.15\ \text{GeV}^2 \tag{63}$$

Alternatively, Reference [28] uses the parameters of the Cornell potential [35] or a pure confinement case:

$$\alpha_s = 0.47 \qquad k = 0.19\ \text{GeV}^2\ (\text{Cornell}) \tag{64}$$

$$\alpha_s = 0 \qquad k = 0.25\ \text{GeV}^2\ (\text{confinement only}) \tag{65}$$

For the $D_1 \to D^*$ transition, we use the same potentials and $M_c = 1.7, m_u = 0.308$ GeV. Results are reported in Table 2

7. Summary

The value of g. The ratio of (44) to (32) depends on g^2 only:

$$R_\Gamma = \frac{\Gamma(Y4230) \to \pi^0 Z_c(3900))}{\Gamma(Y(4230) \to \gamma X(3872))} = 13.5\, g^2 \tag{66}$$

Assuming that $\Gamma(Z_c(3900)^0 \to (D^* \bar{D} + c.c)^0)$ saturates the $Z_c(3900)$ width, we obtain g by comparison to the ratio of the corresponding cross sections, Table 1:

$$R_\sigma = \frac{\sigma(e^+e^- \to \pi^0 Z_c(3900)^0 \to \pi^0 (D\bar{D}^* + c.c.)^0)}{\sigma(e^+e^- \to \gamma X(3872))} = 14 \tag{67}$$

at $\sqrt{s} = 4.226$ GeV. From $R_\Gamma = R_\sigma$, taking into account the errors of the cross sections, we find

$$g = 1.0^{+0.6}_{-0.3} \tag{68}$$

that compares well with with $g = 0.6 - 0.7$ obtained in (57) and (61).

The $Z_c(4020)$ ***puzzle.*** The axial $\Delta L = 1$ transition amplitude has a strong dependence from the pion momentum, which reflects in a steep dependence of the rate: $\Gamma \propto q^5$, see Equation (44). The pion momentum of $Y(4230) \to \pi Z_c(4020)$ implies a suppression factor $\sim$30 with respect to $Y(4230) \to \pi Z_c(3900)$, which does not seem to be supported by the cross sections in Table 1.

Would it be possible that the $Z_c(4020)$ events come from the second peak of the structure, $Y(4320)$? A clarification of the source of $D^*\bar{D}^*$ events in the region and of the decay modes of $Y(4320)$ would be very useful.

The total $Z_c(3900)$ ***width.*** Following (68) and the transition radius in Table 2 (lattice value), we estimate the total rate

$$\Gamma(Y(4230) \to \pi Z_c(3900)) = 8^{+10}_{-4}\ \text{MeV} \tag{69}$$

corresponding to a fraction (5–36)% of the total $Y(4230)$ rate, Equation (11).

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References

1. Yuan, C.Z. A recent review of BESIII results. *a*rXiv **2021**, arXiv:2102.12044.
2. Ablikim, M.; Achasov, M.N.; Adlarson, P.; Ahmed, S.; Albrecht, M.; Aliberti, R.; Amoroso, A.; An, Q.; Anita, X.; Bai, H.; et al. Observation of a near-threshold structure in the K^+ recoil-mass spectra in $e^+e^- \to K^+(D_s^- D^{*0} + D_s^{*-} D^0)$, *a*rXiv **2020**, arXiv:2011.07855.
3. Ali, A.; Maiani, L.; Polosa, A.D. *Multiquark Hadrons*; Cambridge University Press: Cambridge, UK, 2019.
4. Chen, H.X.; Chen, W.; Liu, X.; Zhu, S.L. The hidden-charm pentaquark and tetraquark states. *Phys. Rept.* **2016**, *639*, 1. [CrossRef]
5. Esposito, A.; Pilloni, A.; Polosa, A. Multiquark Resonances. *Phys. Rept.* **2017**, *668*, 1–97. [CrossRef]
6. Ali, A.; Lange, J.S.; Stone, S. Exotics: Heavy Pentaquarks and Tetraquarks. *Prog. Part. Nucl. Phys.* **2017**, *97*, 123. [CrossRef]
7. Guo, F.K.; Hanhart, C.; Meissner, U.G.; Wang, Q.; Zhao, Q.; Zou, B.S. Hadronic molecules. *arXiv* **2017**, arXiv:1705.00141.
8. Lebed, R.F.; Mitchell, R.E.; Swanson, E.S. Heavy-Quark QCD Exotica. *Prog. Part. Nucl. Phys.* **2017**, *93*, 143. [CrossRef]
9. Olsen, S.L.; Skwarnicki, T.; Zieminska, D. Non-Standard Heavy Mesons and Baryons, an Experimental Review. *arXiv* **2017**, arXiv:1708.04012.
10. Maiani, L.; Piccinini, F.; Polosa, A.D.; Riquer, V. Diquark-antidiquarks with hidden or open charm and the nature of X(3872). *Phys. Rev. D* **2005**, *71*, 014028. [CrossRef]
11. Maiani, L.; Piccinini, F.; Polosa, A.D.; Riquer, V. The Z(4430) and a New Paradigm for Spin Interactions in Tetraquarks. *Phys. Rev. D* **2014**, *89*, 114010. [CrossRef]

12. Maiani, L.; Polosa, A.D.; Riquer, V. Interpretation of Axial Resonances in J/psi-phi at LHCb. *Phys. Rev. D* **2016**, *94*, 054026. [CrossRef]
13. Ali, A.; Maiani, L.; Borisov, A.V.; Ahmed, I.; Aslam, M.J.; Parkhomenko, A.Y.; Polosa, A.D.; Rehman, A. A new look at the Y tetraquarks and Ω_c baryons in the diquark model. *Eur. Phys. J. C* **2018**, *78*, 29. [CrossRef]
14. Casalbuoni, R.; Deandrea, A.; Bartolomeo, N.D.; Gatto, R.; Feruglio, F.; Nardulli, G. Phenomenology of heavy meson chiral Lagrangians. *Phys. Rept.* **1997**, *281*, 145–238. [CrossRef]
15. Aaij, R.; Abellán Beteta, C.; Ackernley, T.; Adeva, B.; Adinolfi, M.; Afsharnia, H.; Aidala, C.A.; Aiola, S.; Ajaltouni, Z.; Akar, S. Observation of new resonances decaying to $J/\psi K^+$ and $J/\psi\phi$. *arXiv* **2021**, arXiv:2103.01803.
16. Maiani, L.; Polosa, A.D.; Riquer, V. The new resonances Z_cs(3985) and Z_cs(4003) (almost) fill two tetraquark nonets of broken SU(3)_f. *arXiv* **2021**, arXiv:2103.08331.
17. Aubert, B. Observation of a broad structure in the $\pi^+\pi^- J/\psi$ mass spectrum around 4.26-GeV/c^2. *Phys. Rev. Lett.* **2005**, *95*, 142001. [CrossRef]
18. Wang, X.L. Observation of Two Resonant Structures in e+e- to pi+ pi- psi(2S) via Initial State Radiation at Belle. *Phys. Rev. Lett.* **2007**, *99*, 142002. [CrossRef] [PubMed]
19. Ablikim, M.; Achasov, M.N.; Adlarson, P.; Ahmed, S.; Albrecht, M.; Alekseev, M.; Amoroso, A.; An, F.F.; An, Q.; Bai, Y. Study of $e^+e^- \to \gamma\omega J/\psi$ and Observation of $X(3872) \to \omega J/\psi$. *Phys. Rev. Lett.* **2019**, *122*, 232002. [CrossRef] [PubMed]
20. Ablikim, M. Study of the process $e^+e^- \to \pi^0\pi^0 J/\psi$ and neutral charmonium-like state $Z_c(3900)^0$. *Phys. Rev. D* **2020**, *102*, 012009. [CrossRef]
21. Ablikim, M.; Achasov, M.N.; Ahmed, S.; Ai, X.C.; Albayrak, O.; Albrecht, M.; Ambrose, D.J.; Amoroso, A.; An, F.F.; An, Q.; et al. Determination of the Spin and Parity of the $Z_c(3900)$. *Phys. Rev. Lett.* **2017**, *119*, 072001. [CrossRef]
22. Ablikim, M.; Achasov, M.N.; Ai, X.C.; Albayrak, O.; Albrecht, M.; Ambrose, D.J.; Amoroso, A.; An, F.F.; An, Q.; Bai, J.Z. Observation of a Neutral Structure near the $D\bar{D}^*$ Mass Threshold in $e^+e^- \to (D\bar{D}^*)^0\pi^0$ at $\sqrt{s}$ = 4.226 and 4.257 GeV. *Phys. Rev. Lett.* **2015**, *115*, 222002. [CrossRef]
23. Ablikim, M.; Achasov, M.N.; Ai, X.C.; Albayrak, O.; Albrecht, M.; Ambrose, D.J.; Amoroso, A.; An, F.F.; An, Q.; Bai, J.Z. Confirmation of a charged charmoniumlike state $Z_c(3885)^\mp$ in $e^+e^- \to \pi^\pm(D\bar{D}^*)^\mp$ with double D tag. *Phys. Rev. D* **2015**, *92*, 092006. [CrossRef]
24. Ablikim, M.; Achasov, M.N.; Albayrak, O.; Ambrose, D.J.; An, F.F.; An, Q.; Bai, J.Z.; Ferroli, R.B.; Ban, Y.; Becker, J. Observation of a charged charmoniumlike structure in $e^+e^- \to (D^*\bar{D}^*)^\pm\pi^\mp$ at $\sqrt{s} = 4.26$GeV. *Phys. Rev. Lett.* **2014**, *112*, 132001. [CrossRef]
25. Ablikim, M.; Achasov, M.N.; Ai, X.C.; Albayrak, O.; Albrecht, M.; Ambrose, D.J.; Amoroso, A.; An, F.F.; An, Q.; Bai, J.Z. Observation of a neutral charmoniumlike state $Z_c(4020)^0$ in $e^+e^- \to (D^*\bar{D}^*)^0\pi^0$. *Phys. Rev. Lett.* **2015**, *115*, 182002. [CrossRef]
26. Ablikim, M.; Achasov, M.N.; Ai, X.C.; Albayrak, O.; Ambrose, D.J.; An, F.F.; An, Q.; Bai, J.Z.; Ferroli, R.B.; Ban, Y. Observation of $e^+e^- \to \gamma X(3872)$ at BESIII. *Phys. Rev. Lett.* **2014**, *112*, 092001. [CrossRef]
27. Li, C.; Yuan, C.Z. Determination of the absolute branching fractions of X(3872) decays. *Phys. Rev. D* **2019**, *100*, 094003. [CrossRef]
28. Chen, H.X.; Maiani, L.; Polosa, A.D.; Riquer, V. $Y(4260) \to \gamma + X(3872)$ in the diquarkonium picture. *Eur. Phys. J. C* **2015**, *75*, 550. [CrossRef]
29. Lifshitz, E.M.; Berestetski, V.B.; *Pitaevskii, L.P. Quantum Electrodynamics (Course of Theoretical Physics, Volume 4)*; Pergamon Press: Oxford, UK, 1980.
30. Maiani, L.; Polosa, A.D.; Riquer, V. A Theory of X and Z Multiquark Resonances. *Phys. Lett. B* **2018**, *778*, 247. [CrossRef]
31. Van Royen, R.; Weisskopf, V.F. Hadron decay processes and the quark model. *Il Nuovo Cimento A (1971–1996)* **1967**, *50*, 617. [CrossRef]
32. Zyla, P.A. Review of Particle Physics. *Prog. Theor. Exp. Phys.* **2020**, *2020*, 083C01. [CrossRef]
33. Lucha, W.; Schoberl, F.F. Solving the Schrodinger equation for bound states with Mathematica 3.0. *Int. J. Mod. Phys. C* **1999**, *10*, 607. [CrossRef]
34. Kawanai, T.; Sasaki, S. Charmonium potential from full lattice QCD. *Phys. Rev. D* **2012**, *85*, 091503. [CrossRef]
35. Ikhdair, S.M.; Sever, R. Spectroscopy of B_c meson in a semi-relativistic quark model using the shifted large-N expansion method. *Int. J. Mod. Phys.* **2004**, *A 19*, 1771. [CrossRef]