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On Automorphism Groups of Finite Chain Rings

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Abstract: A finite ring with an identity is a chain ring if its lattice of left ideals forms a unique chain. Let R be a finite chain ring with invaraints p, n, r, k, k', m. If $n = 1$, the automorphism group $Aut(R)$ of R is known. The main purpose of this article is to study the structure of $Aut(R)$ when $n > 1$. First, we prove that $Aut(R)$ is determined by the automorphism group of a certain commutative chain subring. Then we use this fact to find the automorphism group of R when $p \nmid k$. In addition, $Aut(R)$ is investigated under a more general condition; that is, R is very pure and p need not divide k. Based on the j-diagram introduced by Ayoub, we were able to give the automorphism group in terms of a particular group of matrices. The structure of the automorphism group of a finite chain ring depends essentially on its invaraints and the associated j-diagram.

Keywords: finite chain ring; automorphism group; Galois ring; j-diagram

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1. Introduction

All rings considered in this paper are finite and have an identity. Chain rings are rings whose left (right) ideals form a unique chain under inclusion. These rings have been used in geometry as coordinatizing rings of Klingenberg planes and Pappian Hjelmslev planes [1,2]. In recent years, chain rings have found significant applications in new places: in coding theory for creating more compact codes with higher capabilities of error correction [3,4]; in combinatorics for constructing bent functions, partial difference sets and relative difference sets [5–7]. In addition, such rings arise in the p-adic fields as factor-rings of rings of integers in suitable finite extensions of the field of p-adic numbers, $\mathbb{Q}_p$ [8]. The purpose of the present paper is to investigate the automorphism groups of chain rings. The results of this work give immediate corollaries for the applications mentioned above.

For a general background of finite chain rings, we refer to [8–13]. Let R denote a chain ring of characteristic p^n with non-zero (Jacobson) radical $J(R)$ of nilpotencey index m. The case when $J(R) = 0$, R is a field. The residue field $F = R/J(R)$ is a finite field of order p^r. R contains a subring (coefficient subring) R_0 of the form $R_0 = GR(p^n, r) \cong Z_{p^n}[a]$, where a is an element of R_0 of the multiplicative order $p^r - 1$. Moreover, there exist $\pi \in J(R) \setminus J^2(R)$ and $\sigma \in Aut(R_0)$ such that $J(R) = \pi R$ and $\pi u = \sigma(u)\pi$, for each $u \in R_0$. If k is the greatest integer $i \leq m$ such that $p \in J^i(R)$, R can be written as:

$$R = \oplus_{i=0}^{k-1} R_0 \pi^i \tag{1}$$

(as R_0-module). This implies $\pi^k = p \sum_{i=0}^{k-1} u_i \pi^i$, where $u_i \in R_0$ and u_0 is a unit, i.e., π is a root of an Eisenstein polynomial over R_0,

$$g(x) = x^k - p \sum_{i=0}^{k-1} u_i x^i. \tag{2}$$

If there exists $\pi \in R$ such that $\pi^k = p\beta$, where $\beta \in < a >$. Then, R is called a very pure chain ring. It is also known that σ is uniquely determined by R and R_0, and thus it is called

the associated automorphism of R with respect to R_0. If $n > 1$, then $\sigma^k = id$, and if k' is the order of σ, $k' \mid k$. The integers p, n, r, k, k', m are called the invariants of R. Furthermore, there is t, $1 \leq t \leq k$ such that $m = (n-1)k + t$.

When $n = 1$, the automorphism group $Aut(R)$ is determined in [14]. If $n > 1$, relatively little is known about $Aut(R)$. Indeed, a special class of automorphisms has been considered in [15]. Our main goal in this article is to investigate the structure of $Aut(R)$ when $n > 1$. We first show that $Aut(R)$ is given in terms of the automorphism group of a certain commutative chain subring; thus, it suffices to find $Aut(R)$, R is commutative ($k' = 1$). Next, we use this idea to determine $Aut(R)$ when R is a chain ring with $(p, k) = 1$. In addition, $Aut(R)$ is studied under a more general condition; R is very pure. If $p \nmid k$ or R is complete, we manage to give $Aut(R)$ in terms of a specific group of matrices.

2. Preliminaries and Notations

In this section, we mention some facts and introduce notations that will be used in the subsequent discussions.

In the sequel, R is a finite chain ring with invariants p, n, r, k, k', m. Let R_1 be the centralizer of R_0 in R, then

$$R_1 = \oplus_{i=0}^{k_1-1} R_0 \pi^{ik'}, \tag{3}$$

where $k_1 = \frac{k}{k'}$ and $m_1 = \lfloor \frac{m}{k'} \rfloor + 1$. From (3), the radical of R_1, $J(R_1) = \pi^{k'} R_1$. However, R_1 is a commutative chain ring with invariants p, n, r, k_1, m_1. Moreover, R_1 is the only maximal commutative subring of R containing R_0 and it is unique up to the inner automorphisms of R [9].

Let $S = R_0^{\sigma}$ be the fixed subring of R_0 by σ which is of the form $GR(p^n, r_0) = Z_{p^n}[b]$, b is an element in $< a >$ of the multiplicative order $p^{r_0} - 1$, where $r_0 = r/k'$. Denote $Z(R)$ the center of R, then,

$$Z(R) = \oplus_{i=0}^{k_1-1} S \pi^{k'i} + \Omega, \tag{4}$$

where $\Omega = J^{m_1-1}(R_1) = J^{m-1}(R)$ if $k' | (m-1)$ and $\Omega = 0$ otherwise. Let

$$R_2 = \oplus_{i=0}^{k_1-1} S \pi^{k'i}. \tag{5}$$

It is easy to check that $R_2 = Z(R)$ if $k' \nmid (m-1)$, and hence $Z(R)$ in such a case is a commutative chain ring with invariants p, n, r_0, k_1, m_1. When $k' \mid (m-1)$, then $Z(R) = R_2 + \Omega$, which is not a chain subring of R. However, $Z(R)/\Omega$ is a commutative chain ring with invariants p, n, r_0, k_1, $m_1 - 1$ [9]. Note that $\pi^k \in Z(R)$, i.e., π^k can be written as:

$$\pi^k = \sum_{i=0}^{k_1-1} u_i \pi^{k'i} + u_{m-1} \pi^{m-1} = p\beta_1 h_1 + u_{m-1} \pi^{m-1}, \tag{6}$$

where $\beta_1 \in < b >$, $h_1 = 1 + \sum_{i=1}^{k_1-1} u_i' \pi^{k'i}$, $pu_i' = \beta_1^{-1} u_i$, $u_i \in S$ for $0 \leq i \leq k_1 - 1$, and $u_{m-1} \in < a >$.

If $\mu \in Aut(R_0)$, $\lambda \in U(R_1)$ and $\xi \in U(R)$. Then, we define the following correspondences:

$$\phi_{\lambda}^{\mu}(\sum u_i \pi^i) = \sum \mu(u_i)(\lambda\pi)^i, \tag{7}$$

$$\psi_{\xi}(\sum u_i \pi^i) = \xi \sum u_i \pi^i \xi^{-1}, \tag{8}$$

$$\alpha_{\sigma}(\sum u_i \pi^i) = \sum \sigma(u_i) \pi^i. \tag{9}$$

The following statements are related to the commutative case [16,17]. Let $U(R)$ denote the unit group of R, then $U(R) = < a > \otimes H$, where a is an element of order $p^r - 1$ and $H = 1 + J(R)$ is the $p-$Sylow subgroup of $U(R)$. Let $H_s = 1 + J^s(R)$, $s \in P_m = \{1, 2, \ldots, m\}$. Consider

$$H = H_1 > H_2 > H_3 > \cdots > H_m = < 1 >, \tag{10}$$

joined with a function j defined by:

$$j(s) = \begin{cases} \min(ps, m), & s \leq u, \\ \min(s + k, m), & s > u, \end{cases} \tag{11}$$

where $u = \lfloor \frac{k}{p-1} \rfloor$. We refer to the series (10) when we mention the j-diagram. We call R an incomplete (complete) chain ring if H has an incomplete (complete) j-diagram in the sense that is given by Ayoub. For more details about j-diagrams, see [16,17].

Definition 1. *Let R be a chain ring, then we call R complete (incomplete) if its R_1 is complete (incomplete).*

Let e_{iq} denote the matrix with the identity of F in the (i,q) position and zeros elsewhere. The group M_d of all 1-triangular matrices:

$$1 + \sum \alpha_{iq} e_{iq},$$

where $\alpha_{iq} \in F$ is the $p-$Sylow subgroup of $GL_d(F)$, which is the general linear group over F [18]. Let E_d be the set of all matrices of the form $[a_{iq}]$:

$$a_{iq} = \sum_{e+f=i} a_{e,1} a_{f,q-1}.$$

Note that E_d is a subgroup of M_d of order p^{rd}.

All symbols shall retain their meanings throughout the article as stated above, in addition, for a given chain ring R, we denote T_R all pairs (R_0, π) which fulfill the aforementioned conditions.

3. The Automorphism Group Aut(R)

Throughout the section, R is a chain ring with invariants p, n, r, k, k', m. If $k = 1$, then $R = R_0$ and $Aut(R)$ is cyclic of order r generated by ρ, the Frobenius map. The case when $n = 1$ ($\pi^k = 0$), i.e., $m = k$, $Aut(R)$ is known [14]. Thus, we assume that $k > 1$ and $n > 1$ in our discussions.

Proposition 1. *Let R be a chain ring with p, n, r, k, k', m. Then,*

$$\pi^k = p\beta h, \tag{12}$$

$$\begin{cases} \beta \in < a >, h = 1, & \text{if } m - 1 = k, \\ \beta \in < b >, h = h_1 + \alpha_0 \pi^{m-k-1}, & \text{otherwise}, \end{cases}$$

where $h_1 \in R_2 \cap H(R_1)$ and $\alpha_0 \in R_0$.

Proof. First, if $m - 1 = k$, i.e., $n = 2$ and $t = 1$. Then, by (6), $p\beta_1 h_1 = p\beta_1$. This means, $\pi^k = p\beta_1 + u_{m-1}\pi^k$, i.e., $(1 - u_{m-1})\pi^k = p\beta_1$, and this implies $\pi^k = p\beta$, where $\beta = \beta_1(1 - u_{m-1})^{-1} \in < a >$. On the other hand, if $m - 1 > k$, we consider two cases. The case when $k' \nmid (m-1)$, it is easy to prove the result since $Z(R) = R_2$. Now, assume that $k' \mid (m-1)$, then $k' \mid (t-1)$ because $m - 1 = (n-1)k + t - 1$. Let $t - 1 = t_1 k'$ for some t_1 positive integer, then,

$$\begin{aligned} u_{m-1}\pi^{m-1} &= u_{m-1}\pi^{(n-1)k}\pi^{t_1k'} \\ &= u_{m-1}(p\beta_1h_1 + u_{m-1}\pi^{m-1})^{n-1}\pi^{t_1k'} \\ &= u_{m-1}(p^{n-1}\beta_1^{n-1}h_1^{n-1})\pi^{t_1k'} \\ &= u_{m-1}p^{n-1}\beta_1^{n-1}\pi^{t_1k'}, \end{aligned}$$

where $\beta_1 \in < b >$. Moreover, since $t_1 < k$, then Equation (6) yields,

$$\begin{aligned} \pi^k &= p\beta_1h_1 + u_{m-1}p^{n-1}\beta_1^{n-1}\pi^{t_1k'} \\ &= p\beta(h_1 + p^{n-2}u_{m-1}\beta^{n-1}\beta^{-1}\pi^{t_1k'}) \ \ (n > 2) \\ &= p\beta h, \end{aligned}$$

where $\beta = \beta_1$, $h = h_1 + \alpha_0\pi^{m-k-1}$ and $\alpha_0 = u_{m-1}\beta^{n-1}\beta^{-1}$. □

Remark 1. *By the proof of the previous proposition, we can write:*

$$Z(R) = R_2 + p^{n-1}R_0\pi^{t_1k'}. \tag{13}$$

Remark 2. *Note that if $k \neq m-1$ and $k' \mid (m-1)$, then by the proof of Proposition 1,*

$$h = \sum_{i=0}^{k_1-1} u_i\pi^{ik'}, \tag{14}$$

where $u_i \in S$ $(u_0 = 1)$ if $i \neq t_1$ and $u_{t_1} \in R_0$.

Definition 2. *Let T_1 be a commutative chain ring, which is cyclic Galois over a commutative chain ring T_2. Let $G =< \chi >$ be the group of all T_2-automorphisms of T_1. Define $N_\chi(y) : U(T_1) \to U(T_2)$ as:*

$$N_\chi(y) = \prod_{\eta\in G} \eta(y), \tag{15}$$

where N_χ is called the norm function.

Lemma 1. *Let R be a chain ring. Then, the homomorphism ψ defined from $U(T_1)$ into $U(T_2)$ by: $\psi(\lambda) = N_\sigma(\lambda)$ is surjective, where $T_1 = R_1/J^{m_1-1}(R_1)$ and $T_2 = Z(R)/\Omega$.*

Proof. Note that every finite commutative chain ring is a quotient ring of the ring of integers of an appropriate extension of $\mathbb{Q}_p$ [8]. In addition, the extension T_1 over T_2 corresponds to an unramified extension over $\mathbb{Q}_p$, then it is a Galois extension with $Aut_{T_2}(T_1) =< \alpha_\sigma >$. Thus, by (Proposition 3, page 82 [19]), $N_\sigma(U(T_1)) = U(T_2)$. □

Proposition 2. *With the same assumptions of Lemma 1,*

$$ker\ N_\sigma = \{\alpha_\sigma(\lambda)\lambda^{-1} : \lambda \in U(T_1)\}. \tag{16}$$

Moreover, $ker\ N_\sigma \cong U(T_1)/U(T_2)$.

Proof. The first claim is trivial. Consider the map $\psi : U(T_1) \to ker\ N_\sigma$, defined by $\psi(\lambda) = \alpha_\sigma(\lambda)\lambda^{-1}$. Let $\lambda_1, \lambda_2 \in U(T_1)$, then by the definition of α_σ, $\alpha_\sigma(\lambda_1\lambda_2) = \alpha_\sigma(\lambda_1)\alpha_\sigma(\lambda_2)$. This implies that ψ is a group homomorphism. Moreover, by the first claim, ψ is surjective with $U(T_2)$ as its kernel. □

Remark 3. *If R_2 is a subring, we consider $T_1 = R_1$ and $T_2 = R_2$ in Lemma 1 and Proposition 2.*

Proposition 3. *Let R be a chain ring with invariants p, n, r, k, k', m. Then, R is very pure if and only if its R_1 is very pure.*

Proof. Let R_1 be very pure. If $k' = 1$, then $R = R_1$, and this ends the proof. Now, if $k' > 1$, let (R_0, π_1) be an element of T_{R_1} such that $\pi_1^{k_1} = p\beta$, where $\beta \in < a >$. Hence, $(\pi_1) = (\pi^{k'}) = J^{k'}(R)$. Moreover, $\pi_1 = \beta_1 \delta \pi^{k'}$, where $\beta_1 \in < b >$ and $\delta \in H(T_2)$ since $\pi^{k'} \in Z(R)$. By Lemma 1, there exist $\beta_2 \in < a >$ and $\zeta \in H(T_1)$ such that $\beta_1 = N_\sigma(\beta_2)$ and $\delta = N_\sigma(\zeta)$. Now, let $\theta = \beta_2 \zeta \pi$, then it is easy to verify that (R_0, θ) is an element of T_R and $\theta^k = p\beta$. Therefore, R is very pure. The converse is trivial. □

Corollary 1 ([9]). *A chain ring R with $p \nmid k$ is very pure.*

Corollary 2. *If R is a chain ring with $p \nmid k$ and $m > k + 1$, then R_2 is a subring.*

Remark 4. *If σ can be extended to an automorphism ψ of R fixing $\pi^{k'}$, then $\psi(\pi^k) = \pi^k$, and thus $\pi^k \in R_2$, i.e., R_2 is a subring. Conversely, if R_2 is a subring, then it is clear that α_σ is the required automorphism.*

Proposition 4. *Assume that R is a very pure chain ring with invariants p, n, r, k, k', m. Then, σ can be extended to an automorphism of R if and only if $\sigma(\beta)\beta^{-1} \in < b^e >$, where $e = (k_1, p^{r_0} - 1)$.*

Proof. Note that if $m > k+1$, then R_2 is a subring. Thus, the proof is obvious by Remark 4. Assume that $m = k + 1$ and ϕ is an extension of σ to R. Note that $\phi(R_1) = \xi R_1 \xi^{-1}$ for some $\xi \in U(R)$. Let $\psi = \phi\phi'$, where ϕ' is the conjugation by ξ^{-1}. Then, $\psi(R_1) = R_1$ and $\psi(\pi) = \alpha\zeta\pi$, where $\alpha \in < a >$ and $\zeta \in H(R_1)$. It follows that,

$$\begin{aligned} p\sigma(\beta) = \psi(\pi^k) &= \psi(\pi)^k \\ &= (\alpha\zeta\pi)^k \\ &= N_\sigma^{k_1}(\alpha) N_\sigma^{k_1}(\zeta) \pi^k \\ &= p N_\sigma^{k_1}(\alpha) \beta N_\sigma^{k_1}(\zeta). \end{aligned}$$

This implies $\sigma(\beta)\beta^{-1} \in < N_\sigma^{k_1}(\alpha) >$; consequently, $\sigma(\beta)\beta^{-1} \in < b^e >$, where $e = (k_1, p^{r_0} - 1)$. For the other direction, consider the correspondence ψ, defined as:

$$\psi(\sum_{i=0}^{k-1} u_i \pi^i) = \sum_{i=0}^{k-1} \sigma(u_i)(\alpha\pi)^i, \tag{17}$$

where α is an element of $< a >$ with $\sigma(\beta)\beta^{-1} \in < N_\sigma(\alpha)^{k_1} >$. Clearly, ψ is an automorphism which is an extension of σ to R. □

Remark 5. *In the light of Lemma 1 and Proposition 2,*

$$\begin{aligned} N_\sigma(< a >) &=< b >, \\ N_\sigma(H(T_1)) &= H(T_2). \end{aligned}$$

Let $\alpha_1, \ldots, \alpha_{r_0}, \alpha_{r_0+1} \ldots, \alpha_r$ be a representative system in T_1 for a basis of $F = GF(p^r)$ over $\mathbb{Z}_p$ such that $\alpha_1, \ldots, \alpha_{r_0}$ is a basis of $K = GF(p^{r_0})$ over $\mathbb{Z}_p$. Then, $\{1 + \alpha_i \pi^{sk'}\}_{1 \le i \le r}$ are generators of $U_s(T_1)$, and $\{1 + \alpha_i \pi^{sk'}\}_{1 \le i \le r_0}$ are generators of $U_s(T_2)$, where $1 \le s \le m_1$ [16]. In addition,

$$H(T_1) = H(T_2) \otimes ker\ N_\sigma. \tag{18}$$

Hence, we can consider $1 + \alpha_i \pi^{sk'}$, $r_0 + 1 \leq i \leq r$ *as generators of*

$$U_s(ker\ N_\sigma) = U_s(T_1)/U_s(T_2). \tag{19}$$

Lemma 2. *For* $1 \leq s \leq m_1 - 1$, *let* $N_{\sigma s}$ *be the restriction of* N_σ *on* $H_s(T_1)$. *Then,*

(i) $N_\sigma(H_s(T_1)) = H_s(T_2)$.

(ii) $N_{\sigma s}$ *is a surjective homomorphism, defined from* $H_s(T_1)$ *into* $H_s(T_2)$. *Moreover,*

$$ker\ N_{\sigma s} = ker\ N_\sigma \cap H_s(T_1) = H_s(ker\ N_\sigma). \tag{20}$$

Proof. (i) It is clear that $N_\sigma(H_s(T_1)) \subseteq H_s(T_2)$, and thus by Proposition 2 and Remark 5, we have

$$N_\sigma^{-1}(H_s(T_2)) = L \otimes ker\ N_\sigma,$$

where L is a subgroup of $H_s(T_1)$. Moreover, if we take the restriction χ of N_σ on $L \otimes ker\ N_\sigma$, then it follows that $L \otimes ker\ N_\sigma / ker\ \chi \cong Im\ \chi$. As $ker\ \chi = ker\ N_\sigma$, then $Im\ \chi = L$. Furthermore, since χ is surjective, $L = H_s(T_2)$. Thus, (i) is proved. For (ii), the result follows from (i) and Remark 5. □

Proposition 5. *Let* R *be a chain ring with invariants* p, n, r, k, k', m. *Then,* ϕ *is an automorphism of* R *if and only if*

$$\phi(\sum_{i=0}^{k-1} u_i \pi^i) = \zeta \sum_{i=0}^{k-1} \mu(u_i)(\alpha w \pi)^i \zeta^{-1}, \tag{21}$$

where μ *is an automorphism of* R_0, $\zeta \in U(R)$, $\alpha \in < a >$ *and* $w \in H(R_1)$ *such that*

$$N_\sigma(\alpha)^{k_1} = \mu(\beta)\beta^{-1}, \tag{22}$$

$$pN_\sigma(w)^{k_1} = p\phi(h)h^{-1}. \tag{23}$$

Proof. Let $\phi \in Aut(R)$, then $\phi(R_1)$ is the centralizer of $\phi(R_0)$ in R. Hence, there exists $\zeta \in U(R)$ such that $\phi(R_1) = \zeta R_1 \zeta^{-1}$. Let ψ be the composition of the conjugation by ζ^{-1} and ϕ. It follows that $\psi(R_1) = R_1$. Then, $\psi(\pi) = \lambda\pi$, and this means that $\phi(\pi) = \zeta\lambda\pi\zeta^{-1}$, where $\lambda \in U(R_1)$. Note that $\lambda = \alpha w$, where $\alpha \in < a >$ and $w \in H(R_1)$. If $\pi^k = p\beta h$, where $\beta \in < a >$ and $h \in H(R_1)$, then as in the proof of Proposition 4,

$$\begin{aligned} p\mu(\beta)\phi(h) = \phi(\pi^k) = \phi(\pi)^k &= (\alpha w \pi)^k \\ &= N_\sigma(\alpha)^{k_1} N_\sigma(w)^{k_1} \pi^k \\ &= p\beta N_\sigma(\alpha)^{k_1} h N_\sigma(w)^{k_1}, \end{aligned}$$

where μ is the restriction of ϕ on R_0. Thus, $N_\sigma(\alpha)^{k_1} = \mu(\beta)\beta^{-1}$ and $pN_\sigma(w)^{k_1} = p\phi(h)h^{-1}$. Moreover, $\phi = \psi_\zeta \phi_\lambda^\mu$. Conversely, if ϕ is defined as in (21), then, it is clear that ϕ is an automorphism of R if and only if $\phi(\pi^k) = \phi(\pi)^k$. However, the conditions in Equation (22) and Equation (23) guarantee $\phi(\pi^k) = \phi(\pi)^k$. Thus, ϕ is an automorphism of R. □

Corollary 3. *If* R *is a chain ring. Then,* $(R_0, \theta) \in T_R$ *if and only if there is* $\phi_\lambda^\mu \in Aut(R)$ *such that* $\phi_\lambda^\mu(\pi) = \theta$.

Notation 1. *Denote* $T_1 = R_1/J^{m_1-1}(R_1)$ *and* $T_2 = Z(R)/\Omega$. *Let*

$$G_0(R) = \{\phi_\alpha^\mu \in Aut(R) : \alpha \in < a >, \mu \in Aut(R_0)\}, \tag{24}$$

$$G_1(R) = \{\phi_w \in Aut(R) : w \in H(T_1)\}, \tag{25}$$

$$G_2(R) = \{\phi_w \in Aut(R) : w \in H(T_2)\}. \tag{26}$$

Theorem 1. *Let R be a chain ring with invariants p, n, r, k, k', m. Then,*

$$Aut(R) = \begin{cases} Inn(R) \rtimes Aut(R_2), & \text{if } R_2 \text{ is a subring,} \\ (Inn(R) \rtimes Aut(R_1))/Aut_{R_0}(R), & \text{otherwise.} \end{cases} \tag{27}$$

Proof. (i) If R_2 is a subring, then $\pi^k = p\beta h \in R_2$ (Proposition 1). As $\phi_{w_1} = \phi_{w_2}$ if and only if $w_1 = w_2 \bmod H_{m_1-1}(R_1)$. Then, we can only consider $\phi_w \in G_1(R)$. Moreover, since $H(T_1) = ker\ N_\sigma \times H(T_2)$ (Lemma 2), $w = w_1 w_2$, where $w_1 \in ker\ N_\sigma$ and $w_2 \in H(T_2)$. This follows by Proposition 2, $w_1 = \alpha_\sigma(\xi)\xi^{-1}$, for some $\xi \in H(T_1)$. Note that $\phi_w(\pi) = \phi_{w_1 w_2}(\pi) = \psi_{\xi^{-1}}\phi_{w_2}(\pi)$, i.e., $\phi_w = \psi_{\xi^{-1}}\phi_{w_2}$, where $\psi_{\xi^{-1}} \in Inn(R)$. Thus by Proposition 5, the set of all automorphisms of the form ϕ_w, $w \in H(T_2)$, is $G_2(R_2)$. Similarly, if $\alpha \in < a > \cap\ ker\ N_\sigma$ and $\phi_\alpha^\mu \in Aut(R)$, then $\phi_\alpha^\mu = \phi_\alpha^{id}\phi_1^\mu$, where $\phi_\alpha^{id} \in Inn(R)$. Hence, $G_0(R) = G_\sigma \times G_0(R_2)$, where G_σ is the subgroup of $Inn(R)$ contains all automorphisms ϕ_α^{id}, $\alpha \in ker\ N_\sigma$. Since $Aut(R_2) = G_2(R_2) \times G_0(R_2)$, then by Proposition 5,

$$Aut(R) = Inn(R) \times Aut(R_2). \tag{28}$$

Furthermore, note that if $\phi \in Inn(R) \cap Aut(R_2)$, then $\phi = \psi_\xi$, where $\xi \in U(R)$. Moreover, $\phi \in Aut(R_2)$, then $\phi = \phi_\lambda^\mu$, where $\mu \in Aut(S)$ and $\lambda \in U(R_2)$. Thus, $\psi_\xi = \phi_\lambda^\mu$, and hence $\mu = id$ and $\lambda = \xi\alpha_\sigma(\xi^{-1}) = \sigma(\alpha)\alpha^{-1}\alpha_\sigma(w)w^{-1}$ is an element of $ker\ N_\sigma$, $\alpha \in < a >$, $w \in H(R_1)$ satisfying Equations (22) and (23), respectively. Since $ker\ N_\sigma \cap H(T_2) = < 1 >$ (Remark 5), then $\lambda = 1$. This yields $Inn(R) \cap Aut(R_2) = < id >$, and this ends the proof.
(ii) Assume that R_2 is not a subring. Note that, in this case, $\pi^k \notin R_2$. By a similar argument, one can check that $G_1(R) = G_1(R_1)$. Furthermore, $G_0(R) = G_0(R_1)$. Hence, $Aut(R) = Inn(R) \times Aut(R_1)$. Moreover, $Inn(R) \cap Aut(R_1) = Aut_{R_0}(R)$. Therefore, the result follows. □

Remark 6. *By Theorem 1, it is enough to determine $Aut(R)$ when R is a commutative chain ring, i.e., $k' = 1$.*

In what follows, unless otherwise mentioned, R is a very pure chain ring with $\pi^k = p\beta$ and $\beta = a^s$.

Proposition 6. *If R is a commutative chain ring with invariants p, n, r, k, m. Then, $\phi \in Aut(R)$ if and only if*

$$\phi(\sum_{i=0}^{k-1} u_i \pi^i) = \sum_{i=0}^{k-1} \mu(u_i)(\alpha w \pi)^i, \tag{29}$$

where $\alpha \in < a >, w \in H$ satisfying:

$$\alpha^k = \sigma(\beta)\beta^{-1}, \tag{30}$$

$$w^k = 1\ mod\ \pi^{m-k}, \tag{31}$$

for some $\mu \in Aut(R_0)$.

Proof. The proof is direct from that of Proposition 5 with $k' = 1$ and $h = 1$. □

Remark 7. *From Proposition 6, it is obvious that if $\phi \in Aut(R)$, then $\phi = \phi_w \phi_\alpha^\mu$.*

Remark 8. *If we consider a commutative chain ring R' with invariants $p, n-1, r, k, m$, i.e., $R' = R/J^{m-k}(R)$. Thus, we can write Equation (31) as $w^k = 1$.*

Next, we focus on automorphisms $\phi_\alpha^\mu \in G_0(R)$ which satisfy Equation (30), where $G_0(R)$ is defined in (24). Assume that $\alpha = a^z$, we denote such ϕ_α^μ by ϕ_z^μ, and we write G_0 instead of $G_0(R)$. Let

$$\begin{cases} d = (p^r - 1, s),\ c = \frac{p^r-1}{d}, \\ q = (d, k),\ b = \frac{d}{q}, \\ e = (\frac{p^r-1}{b}, k). \end{cases} \tag{32}$$

Lemma 3. *Let R be a commutative chain ring with invariants p, n, r, k, m.*
(i) Let ϕ_z^μ be an automorphism of R. Then, $a^z = a^{bi}$ for some i.
(ii) Let i be any integer. There exists an automorphism ϕ_{bi}^μ of R if and only if there exists a non-negative integer f such that $(p^r - 1) \mid s(p^f - 1) - bik$.

Proof. (i) If $u \in R_0$, then $\phi_z^\mu(u) = \mu(u) = u^{p^f}$ for some $0 \le f \le r-1$. Then, $\phi_z^\mu(\pi^k) = \pi^k a^{zk}$ implies,

$$pa^{sp^f} = \phi_z^\mu(pa^s) = \phi_z^\mu(\pi^k) = pa^s a^{zk}.$$

Moreover, $a^{zk} \in < a^s > = < a^d >$. As $(k, d) = q$, we obtain $a^{az} \in < a^d >$, and as a result $b \mid z$.
(ii) Assume that such ϕ_{bi}^μ exists. Now, $\phi_{bi}^\mu(u) = u^{p^f}$ for some $f \ge 0$, where $u \in R_0$. Then, $\phi_{bi}^\mu(\pi^k) = \pi^k a^{bik}$ leads to

$$pa^{sp^f} = pa^{s+bik}.$$

This means, $a^{s(p^f-1)-bik} = 1$, i.e., $(p^r - 1) \mid s(p^f - 1) - bik$. Conversely, let such f exist. Then, we have an automorphism μ of R_0 such that $\mu(u) = u^{p^f}$, where $u \in R_0$. Consider $\pi_1 = \pi a^{bi}$, then $\pi_1^k = p\mu(a^s)$. This gives an automorphism ϕ_{bi}^μ that extends μ and for which $\phi_{bi}^\mu(\pi) = \pi_1$. □

Lemma 4. *Let R be a commutative chain ring, then G_0 is a subgroup of $Aut(R)$ of order $\frac{re}{\tau(c)}$, where $\tau(c)$ is the multiplicative order of p modulo c.*

Proof. Let $\phi_{z_1}^\mu$ and $\phi_{z_2}^\eta$, where $\mu, \eta \in Aut(R_0)$. Then,

$$\phi_{z_1}^\mu \phi_{z_2}^\eta(\pi) = \phi_{z_1}^\mu(a^{z_2}\pi) = a^{z_1}\mu(a^{z_2})\pi = a^{z_1 + p^i z_2}\pi = \phi_{z_1 + z_2 p^i}^{\mu\eta}(\pi),$$

where $\mu = \rho^i$. In other words, $\phi_{z_1}^\mu \phi_{z_2}^\eta = \phi_{z_1+z_2p^i}^{\mu\eta}$. Similarly, one can easily find that,

$$(\phi_z^\mu)^{-1} = \phi_{-zp^i}^\eta,$$

where $\eta = \mu^{-1} = \rho^i$, for some i positive integer $1 \le i \le r$. Thus, G_0 is a subgroup of $Aut(R)$. Next, we compute the order of G_0. Note that from Lemma 3, $b \mid z$. Fix σ, if $\phi_{z_2}^\mu \in G_0$, by (30), $\phi_{z_2}^\mu \in G_0$ if and only if $a^{z_1 - z_2} \in A$, where A is the set of all zeros of $x^e - 1$ in the group generated by $a^{(p^r-1)/b}$. Since there are exactly e different zeros of $x^e - 1$, hence there are e of automorphisms of the type ϕ_z^μ. On the other hand, let z be fixed. Equation (30) gives $\mu(\beta)\beta^{-1} = 1 \bmod < a^d >$, i.e., $\mu(\beta) = \beta$. Thus, it is enough to find the stabilizers of β modulo $< a^d >$ when $Aut(R_0)$ acts on $< a >$ in the usual way. It is well known that

$$\mid stab(\beta) \mid\mid orb(\beta) \mid = \mid Aut(R_0) \mid .$$

Moreover, since $\mathrm{orb}(\beta) = \{\rho^i(\beta) : 1 \le i \le r\}$; thus, $orb(\beta) = \{\rho^i(\beta) : 1 \le i \le \tau(c)\}$, where $\tau(c)$ is the multiplicative order of p modulo c. Therefore,

$$\mid stab(\beta) \mid = \frac{r}{\tau(c)},$$

which follows that $|\ G_0\ | = \frac{re}{\tau(c)}$. □

Proposition 7. *Let R be a commutative chain ring with invariants p, n, r, k, m. Then, G_0 is a solvable group. Moreover, if $d = 1$, then G_0 is cyclic.*

Proof. Let $G_{11} = < \phi_{z_0}^{id} >$, where $z_0 = \frac{p^r - 1}{e}$. Then, G_{11} is clearly a normal subgroup of G_0. Moreover, G_0/G_{11} is abelian, and hence G_0 is solvable. Now, if $d = 1$, then $c = p^r - 1$. Note that $a^{z_1 + z_2 p^{\tau(c)}} = a^{z_1 + z_2 + z_2 ci} = a^{z_1 + z_2}$ consequently,

$$\phi_{z_1}^{\mu} \phi_{z_2}^{\eta} = \phi_{z_1 + z_2}^{\mu\eta}.$$

Moreover, since $(a^{p^r - 1/e})^k = \rho^{\tau(c)}(\beta)\beta^{-1} = 1$, then $\phi_{z_0}^{\mu}$ lies in G_0, where $\mu = \rho^{\tau(c)}$. It is clear that the cyclic subgroup of G_0 generated by $\chi = \phi_{z_0}^{\mu}$ is of order $\frac{re}{\tau(c)}$, and then by Lemma 4, G_0 is cyclic generated by χ. However, in this case, $\tau(c) = r$, which means $\chi = \phi_{z_0}^{id}$ and the order of $G_0 = G_{11}$ is e. □

For every $s \in P_m$, we define $\nu(s)$ the least positive integer satisfying $j^{\nu(s)}(s) = m$. Also we write $k = (p-1)u + v, 0 \leq v < p - 1$.

Lemma 5. *Let R be a commutative chain ring, and let s_0 be the least positive integer such that $\nu(s_0) \leq l$, where $k = k_0 p^l$ and $(k_0, p) = 1$. Then,*

$$s_0 = \begin{cases} (n - l - 1)k + t, & \text{if } l < n - 1, \\ max\{t, u\}, & \text{if } l = n - 1, \\ \begin{cases} q_0, & \text{if } t \geq u, \\ \lceil \frac{u}{p^{l-n-1}} \rceil, & \text{if } t < u, \end{cases} & (l > n - 1) \end{cases} \tag{33}$$

where,

$$q_0 = \begin{cases} s_0', & \text{if } s_0' p^{l-n-1} \geq t, \\ s_0' p, & \text{if } s_0' p^{l-n-1} < t, \end{cases} \quad s_0' = \begin{cases} \lceil \frac{u}{p^{l-n-1}} \rceil, & \text{if } p^{l-n-1} \nmid u, \\ \frac{u}{p^{l-n-1}} + 1, & \text{otherwise.} \end{cases}$$

Proof. First if $l < n - 1$, then $j^l((n - l - 1)k + t) = m$ which means that $s_0 = (n - l - 1)k + t$. The case when $l = n - 1$, note that if $t \geq u$, then $j^l(t) = m$, and thus $s_0 = t$. Now, when $t < u$, $j^l(u) = m$ take $s_0 = u$ since $p(u - 1) < k + t$, i.e., $\nu(u - 1) > l$. Next, let $l > n - 1$. It suffices to find $s_0' \in P_m$ such that $j^{l-n-2}(s_0') \leq u < j^{l-n-1}(s_0')$, i.e., $s_0' p^{l-n-2} \leq u < s_0' p^{l-n-1}$. This implies $\frac{u}{p^{l-n-1}} < s_0' \leq \frac{u}{p^{l-n-2}}$. As s_0 is the least, then $s_0' = \lceil \frac{u}{p^{l-n-1}} \rceil$ if $p^{l-n-1} \nmid u$ and $s_0' = \frac{u}{p^{l-n-1}} + 1$ otherwise. Consider two cases. If $t < u$, then obviously $s_0 = \lceil \frac{u}{p^{l-n-1}} \rceil$ is the required number. For the second case $t \geq u$, observe that if $j^{l-n-1}(s_0') \geq t$, then clearly $s_0 = s_0'$, and if $j^{l-n-1}(s_0') < t$, take $s_0 = j(s_0') = p s_0'$. □

Remark 9. *Let R be a commutative chain ring. Denote $L(R) = \{w \in H(R) : w^k = 1 \ mod\ \pi^{m-k}\}$. Note that if $p \nmid k$, i.e., $l = 0$ or $m = k + 1$, then by Lemma 5 applied to $R/(\pi^{m-k})$, $s_0 = m - k$ and this means, $L(R) = H_{m-k}$. The structure of the subgroup of H, H_{s_0}, can be obtained from [16] via the j-diagram:*

$$H_{s_0} > H_{s_0 + 1} > \cdots > H_m = 1. \tag{34}$$

In addition, if $p \nmid k$ or R is a complete chain ring, then $L(R) = H_{s_0}(R)$. The case when $m = k + 1$, R is complete [16].

Proposition 8. *If R is a commutative chain ring with invariants p, n, r, k, m. Let $G_1 = \{\phi_w : w \in L(R)\}$. Then, G_1 is a normal subgroup of $Aut(R)$. In particular, if $p \nmid k$ or R is complete, $G_1 \cong E_{m - s_0 - 1}$.*

Proof. It is clear that G_1 is subgroup of $Aut(R)$. Every element ϕ of $Aut(R)$ can be written as $\phi = \phi_w \phi_z^\mu$, where $\phi_z^\mu \in G_0$ and $\phi_w \in G_1$, i.e.,

$$Aut(R) = G_1 \times G_0.$$

Moreover, if ϕ_λ in G_1 and $\phi = \phi_z^\mu$, then,

$$\phi^{-1}\phi_\lambda\phi = (\phi_z^\mu)^{-1}\phi_\lambda\phi_z^\mu = \phi_{\psi(\lambda)},$$

where $\psi = (\phi_z^\mu)^{-1}$. Consequently, G_1 is a normal subgroup of $Aut(R)$. Also observe that if $w_1, w_2 \in L(R)$, then $\phi_{w_1} = \phi_{w_2}$ if and only if $\pi w_1 = \pi w_2$ if and only if $1 - w_1 w_2 \in (\pi^{m-1})$ if and only if $w_1 H_{m-1} = w_2 H_{m-1}$. Hence, the order of G_1 is $\frac{|L(R)|}{p^r}$. Moreover, if $p \nmid k$ or R is complete, then by Lemma 5 and Remark 9, $L(R) = H_{s_0}$. This implies that,

$$\mid G_1 \mid = p^{r(m-s_0-1)}.$$

Furthermore, if we define a map g from G_1 into E_{m-s_0-1} by $g(\phi_w) = [a_{iq}]$, where $a_{iq} = \sum_{e+f=i} a_{e,1} a_{f,q-1}$, $w = 1 + \sum_{i=1}^{m-s_0-1} a_{i+1,1}\pi^i$ and $a_{i+1,1} \in F$. It can be verified that g is a group isomorphism. □

Corollary 4. *Assume that R is as in Proposition 8. Then, $(p,k) = 1$ or $m - 1 = k$ if and only if $G_1 = \{\phi_w : w \in H_{m-k}\}$. Moreover, $G_1 \cong E_{k-1}$ and, in this case,*

$$\mid G_1 \mid = p^{r(k-1)}. \tag{35}$$

Theorem 2. *Let R be a commutative chain ring, which is very pure with invariants p, n, r, k, m. Then,*

$$Aut(R) = G_1 \rtimes G_0. \tag{36}$$

In particular, if R is complete or $(p,k) = 1$,

$$Aut(R) \cong E_{m-s_0-1} \rtimes G_0. \tag{37}$$

Proof. First, note that from Proposition 8, $Aut(R) = G_1 \times G_0$. Now, suppose that $\phi \in G_1 \cap G_0$, then as $\phi \in G_1$, $\phi(u) = u$, where $u \in R_0$. Thus, $\phi = \phi_z^{id}$, i.e., $\phi(\pi) = \pi a^z = \pi^{bi}$ for some i. Moreover, $\phi \in G_1$ yields that $\phi(\pi) = \pi w$, $w \in H$. This implies that $\phi = id$, and thus $G_1 \cap G_0 =< id >$. The last assertion follows from Proposition 8. □

Corollary 5. *If R is as in Theorem 2. Then, $Aut_{R_0}(R) = G_1 \rtimes < \chi >$, where $\chi = \phi_{z_0}^{id}$.*

For the following results, denote $G_{qi} = G_q(R_i)$, $q = 0, 1$ and $i = 1, 2$.

Corollary 6. *Let R be a very pure chain ring with invariants p, n, r, k, k', m. Then,*

$$Aut(R) = \begin{cases} Inn(R) \rtimes G_{12} \rtimes G_{02}, & \text{if } R_2 \text{ is a subring,} \\ (Inn(R) \rtimes G_{11} \rtimes G_{01})/Aut_{R_0}(R), & \text{otherwise.} \end{cases} \tag{38}$$

In particular, if $(p,k) = 1$ or R is complete,

$$Aut(R) \cong \begin{cases} Inn(R) \rtimes E_{m_1-s_0-1} \rtimes G_{02}, & \text{if } R_2 \text{ is a subring,} \\ (Inn(R) \rtimes E_{m_1-s_0-1} \rtimes G_{01})/Aut_{R_0}(R), & \text{otherwise.} \end{cases} \tag{39}$$

Proof. The proof is just a direct application of Theorems 1 and 2. □

Corollary 7. *Let R be a chain ring with invariants p, n, r, k, k', m such that $(p, k) = 1$ and $m > k + 1$. If $d = 1$, then*

$$Aut(R) \cong Inn(R) \rtimes E_{k_1-1} \rtimes C_e, \tag{40}$$

where C_e is a cyclic group of order e.

Remark 10. *Note that by Remark 9, Corollarys 6 and 7, the structure of $Aut(R)$ is strongly dependent on the invariants p, n, r, k, k', m and on the associated j-diagram.*

Next, we introduce some results on the subgroup $Aut_{R_0}(R)$.

Proposition 9. *Let R be a commutative chain ring with invariants p, n, r, k, m. Then, $\mid Aut_{R_0}(R) \mid = p^{(k-1)r}$ if and only if $(k, p) = 1$ and $(k, p^r - 1) = 1$ or $m - 1 = k$ and $(k, p^r - 1) = 1$.*

Proof. From Corollary 5, $Aut_{R_0}(R) = G_1 \rtimes < \phi_{z_0}^{id} >$ and $\mid G_1 \mid = \frac{|L(R)|}{p^r} \mid < \phi_{z_0}^{id} > \mid = e$. Thus by Corollary 4, $\mid Aut_{R_0}(R) \mid = ep^{(k-1)r}$ if and only if $G_1 = \{\phi_w : w \in H_{m-k}\}$ if and only if $(p, k) = 1$ or $m - 1 = k$. Moreover, $\mid Aut_{R_0}(R) \mid = p^{(k-1)r}$ if and only if $e = 1$ if and only if $(k, p^r - 1) = 1$ Therefore, the proof follows. □

Remark 11. *Consider a complete chain ring R with $k' > 1$, i.e., R is non-commutative. Now, if $w = 1 + \sum_{i=1}^{m-s_0} a_{i+1,1}\pi^i$ and $a_{i+1,1} \in F$. Let D_1 be the subgroup of E_{m-s_0-1} contains all matrices of the form $[b_{iq}]$, where $b_{iq} = \sigma(a_{iq})a_{iq}^{-1}$. Define the mapping g as $g(\phi_\lambda^{id}) = [b_{iq}]\delta$, where $\lambda = \sigma(\alpha)\alpha^{-1}\eta_\sigma(w)w^{-1} \in ker\ N_\sigma$ and $\delta = \sigma(\alpha)\alpha^{-1}$. Then, it is not hard to show that g is an isomorphism from $Aut_{R_0}(R)$ into $D_1 \rtimes < \phi_{z_1}^{id} >$, where $\phi_{z_1}^{id} \in G_0$ of order $(e, p^r - 1/p^{r_0} - 1)$.*

Example 1. *Let R be a commutative chain ring with invariants p, n, r, k, m and with $\pi^{p-1} = -p$ (Example 2, [16]). Note that $-1 = a^{(p^r-1)/2}$, i.e., $s = (p^r - 1)/2$. Moreover, since $(p, k) = 1$, then,*

$$\mid G_1 \mid = p^{r(p-2)}.$$

Also $c = 2$, and then, $\tau(c) = 1$. Hence, $\mid G_0 \mid = re$. Now, if $(p^r - 1, k) = 1$, then $e = 1$, which implies,

$$Aut(R) \cong E_{k-1} \rtimes Aut(R_0). \tag{41}$$

In addition, $Aut_{R_0}(R) \cong E_{k-1}$.

Remark 12. *In the case when $n = 1$ ([14]), then obviously $\phi_z^\mu = \phi_z\alpha_\mu$, for any $1 \leq z \leq p^r - 1$ and $\mu \in Aut(F)$ ($R_0 = F$). Since $\pi^k = 0$, then ϕ_z and α_μ are automorphisms of R. This means,*

$$G_0 = F^* \rtimes Aut(F). \tag{42}$$

In addition, $G_1 \cong E_{k_1-1}$ in general. Since R_2 is a subring, Corollary 6 concludes that,

$$Aut(R) \cong Inn(R) \rtimes E_{k_1-1} \rtimes F^* \rtimes Aut(F). \tag{43}$$

Example 2. *Consider a finite chain ring R with invariants p, n, r, k, k', m and associated with $\pi^k = p$. Then, it is clear that ϕ_1^μ is an automorphism for every $\mu \in Aut(R_0)$. This means, $G_1 = < \chi > \rtimes Aut(R_0)$, where $\chi = \phi_{z_0}^{id}$. If $(p, k) = 1$ and $m > k + 1$, then,*

$$Aut(R) \cong Inn(R) \rtimes E_{k_1-1} \rtimes < \chi > \rtimes Aut(R_0). \tag{44}$$

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