Noether Invariants for Nonholonomic Systems

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Abstract: The aim of this paper is to construct Noether invariants for Lagrangian non-holonomic dynamics with affine or nonlinear constraints, considered to be adapted to a foliation on the base manifold. A set of illustrative examples is given, including linear and nonlinear Appell mechanical systems.

Keywords: nonholonomic constraints; Noether invariants

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1. Introduction

A general geometric setting to study nonlinear constraints for nonholonomic spaces is to involve foliations. This was motivated by [1], and it was used in [2]. The foliation setting is a basic one, and it is followed effectively in this paper. In [2], it can be seen that the foliated setting is simpler and more suitable to study nonlinear constraints, than using fibered manifolds. The simple foliations are just fibered manifolds, and most examples in the literature fit in this setting. The dynamics that follows use some constraints C and a Lagrangian $L : TM \rightarrow \mathbb{R}$, while the foliation $\mathcal{F}$ on the given manifold M seems to play a secondary role, by means of the space of leaves $M/\mathcal{F}$ and the normal bundle $\pi_{N\mathcal{F}} : N\mathcal{F} = TM/T\mathcal{F} \rightarrow M$. The constraints $C : N\mathcal{F} \rightarrow TM$ give rise to a submanifold of TM , as well as an induced Lagrangian $L_c = L \circ C$ on $N\mathcal{F}$. In the case when C is a linear or an affine map, the dynamics is a non-conservative Lagrangian system having the type (27), but the general case is that of a generalized nonconservative Lagrangian system having the type (35). Regularity conditions can be used in both cases, so that the dynamics are given by semi-spray-type equations. A brief presentation of these facts is given in the third section, but more technical details can be found in [2].

An important ingredient of the paper is given by using infinitesimal symmetries in a foliated setting. In order to obtain these infinitesimal symmetries, we follow local transformations (15) with an ε -parameter local group of transformations on the base manifold M that locally depends on the coordinates of the normal bundle $N\mathcal{F}$. The classical case is when M is considered as a trivial foliation by points, normal bundle TM . This case was studied in [3], or giving the same infinitesimal result, in [4] (see also [5] in the control setting). The Killing equations are obtained in the case when the symmetries depend on the Lagrangian, but not on the dynamics of the semi-spray that it comes from (see the classical case in [3] or the nonholonomic case in [6]). On the other hand, symmetries and Noether invariants of nonholonomic spaces considered in [6,7] come from symmetries on the ambient space, while in our approach, the symmetries come from the induced Lagrangian on the normal bundle, using the semi-spray that comes from the differential system that gives the dynamics of a nonholonomic space.

In the last two sections, we follow a classical way to construct Noether invariants of nonholonomic spaces in the general cases of affine and of nonlinear constraints, using infinitesimal symmetries. The case of non-conservative Lagrangian systems having the type

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(27) is given by the results we prove in Theorems 1 and 2; in this way, we cover the case of nonholonomic spaces with affine constraints. The general case of the dynamics given by a generalized nonconservative Lagrangian system of type (35) is given by the results we prove in Theorems 3 and 4; in this way, we cover the general case of nonholonomic spaces with nonlinear constraints. In both cases, we give some illustrative examples of infinitesimal symmetries and Noether invariants: linear and nonlinear Appell constraints, as well as the Appell–Hammel dynamic system in an elevator.

2. Foliations, Constraints, and Lagrangians: A Brief Mathematical Background

In this section, we want to show that our constructions and results involve certain global objects on manifolds. The mathematical background can mostly be skipped by the reader not really interested in a rigorous mathematical setting.

Involving the foliations in dynamics was motivated by [1], even though the term foliation was not explicitly used. In this section, we follow [2] to approach affine and nonlinear constraints, as well as Lagrangians and dynamics generated in a foliated setting.

A codimension n foliation $\mathcal{F}$ on a connected and orientable manifold M of dimension $(n + m)$ is defined by a foliated cocycle. This means there are local coordinates $(x^u, x^{\bar{u}})$, $u = 1, \dots, m$, $\bar{u} = 1, \dots, n$, where (x^u) are tangent to the leaves and $(x^{\bar{u}})$ are transverse, and they change according to the rules:

$$x^{u'} = x^{u'}(x^u, x^{\bar{u}}), \quad x^{\bar{u}'} = x^{\bar{u}'}(x^{\bar{u}}).$$

A particular case of a foliation is a simple foliation or a fibered manifold. In this case, the leaves are the fibers of a differentiable manifold surjective submersion $\pi : M \rightarrow M'$, and the coordinates $(x^{\bar{u}})$ come from the local coordinates on M' . In the sequel, we call $\pi : M \rightarrow M'$ a fibered manifold, where M and M' are the total space manifold and the base space manifold, respectively, while π is the canonical projection. For the sake of simplicity, we refer to the total space M as the fibered manifold, if no confusion arises. A particular case of a fibered manifold is a locally trivial fibration. Notice that a foliation is locally a simple one by the local submersion $M \supset U \rightarrow \bar{U}$, $(x^u, x^{\bar{u}}) \rightarrow (x^{\bar{u}})$, where $\bar{U}$ is called a transverse manifold.

Related to a foliation $\mathcal{F}$, $T\mathcal{F}$ is the tangent bundle, and $N\mathcal{F} = TM/T\mathcal{F}$ is its normal bundle. A vector field on M is transverse if it projects locally to a vector field on the local transverse manifold that has $(x^{\bar{u}})$ as local coordinates. It reads in local coordinates that the transverse components have transverse coefficients, depending only on transverse coordinates.

In order to consider affine constraints, we consider the following short exact sequence of vector bundle morphisms:

$$0 \rightarrow T\mathcal{F} \xrightarrow{I_0} TM \xrightarrow{\Pi_0} N\mathcal{F} \rightarrow 0. \quad (1)$$

In the simple foliation case $\pi : M \rightarrow M'$, then $T\mathcal{F} := VM = \ker \pi_*$ and $N\mathcal{F} := \pi^*TM'$.

The linear and affine constraints considered below are in accordance with those considered in [8]. A linear constraint adapted to a foliation $\mathcal{F}$ is a left splitting C of the inclusion $T\mathcal{F} \xrightarrow{I_0} TM$. The existence of C is equivalent to the existence of a right splitting D of the projection Π_0 or an inclusion of $N\mathcal{F}$ as $H\mathcal{F} = D(N\mathcal{F}) \subset TM$ by the injective morphism D . In this case, there is a Whitney sum decomposition:

$$TM = T\mathcal{F} \oplus H\mathcal{F},$$

and the linear constraint given above as a map can be uniquely defined by this decomposition, thus by the subbundle $H\mathcal{F} \subset TM$. In the nonlinear case, the above decomposition cannot exist; the submanifold $H\mathcal{F}$ of TM is defined by the map C , but the converse is not assured by the given data, i.e., C or D does not follow from $H\mathcal{F}$.

An affine constraint is an affine map $C : TM \rightarrow T\mathcal{F}$ that can be given as $C = C' + b$, where $C' : TM \rightarrow T\mathcal{F}$ is a linear constraint and $b \in \Gamma(T\mathcal{F})$ is a vector field on M tangent to the levels of $\mathcal{F}$. We have $C(X) = C'X + b$, $(\forall) X \in \mathcal{X}(M)$.

The nonlinear constraints compatible with foliations, considered in our paper, are in accordance with those defined in [2]. The foliated setting was motivated by the constructions and results in [1]. We used the notations from [2], where the mechanical principle described in the very beginning of the next section is also used, in order to obtain the dynamical equations. This is a generalized Lagrangian principle, concordant with the Chetaev one.

The canonical almost tangent structure $J \in \text{End}(TTM)$ (see [9]) induces by projection an endomorphism $\tilde{J} \in \text{End}(TN\mathcal{F})$ of fibers of $TN\mathcal{F}$. Let us denote by $VN\mathcal{F} \subset TN\mathcal{F}$ the vertical vector bundle of $N\mathcal{F} = TM/T\mathcal{F}$ and by $\Gamma_0 \in \Gamma(VN\mathcal{F})$ the transverse Liouville vector field. In local coordinates, we have:

$$\frac{\partial}{\partial x^u} \tilde{J} \rightarrow 0, \frac{\partial}{\partial x^{\bar{u}}} \tilde{J} \rightarrow \frac{\partial}{\partial y^{\bar{u}}}, \frac{\partial}{\partial y^{\bar{u}}} \tilde{J} \rightarrow 0, \Gamma_0 = y^{\bar{u}} \frac{\partial}{\partial y^{\bar{u}}}. \quad (2)$$

The local sections in $\Gamma(VN\mathcal{F})$ are generated by local bases $\left\{ \frac{\partial}{\partial y^{\bar{u}}} \right\}$. Notice that the above formulas are imposed by the local definition of the canonical almost tangent structure $J \in \text{End}(TTM)$ (see [9] for more details). In the foliated case, the normal vector bundle plays the role of a tangent bundle, transverse to leaves; thus, many geometrical objects, including the transverse Liouville vector field, or (non)linear connections can be considered. However, the normal bundle $N\mathcal{F}$ on the base M is a vector bundle, not a tangent bundle. Since $N\mathcal{F} = TM/T\mathcal{F}$, the geometrical objects of $N\mathcal{F}$ are considered as the projected ones from the vector bundle $TM \rightarrow M$ on the vector bundle $N\mathcal{F} \rightarrow M$.

A nonlinear constraint is a map $C : N\mathcal{F} \rightarrow TM$, which can also be viewed as a section $C \in \Gamma(\pi_{N\mathcal{F}}^* TM)$, provided that $\tilde{J}(C) = \Gamma_0$. In local coordinates, we have,

$$(x^u, x^{\bar{u}}, y^{\bar{u}}) \xrightarrow{C} (x^u, x^{\bar{u}}, -C^u(x^{\bar{v}}, x^{\bar{v}}, y^{\bar{v}}), y^{\bar{u}}), C = C^u \frac{\partial}{\partial x^u} + y^{\bar{u}} \frac{\partial}{\partial x^{\bar{u}}}. \quad (3)$$

Let us note that $C : N\mathcal{F} \rightarrow TM$ gives rise to some local vector fields $C_V \in \mathcal{X}(V)$, $V = N\mathcal{F}_U$. Notice that C and C_V have the same formulas $C^u \frac{\partial}{\partial x^u} + y^{\bar{u}} \frac{\partial}{\partial x^{\bar{u}}}$, but they are different objects; $C : N\mathcal{F} \rightarrow TM$ is a map, but $C_V \in \mathcal{X}(V)$ is a local vector field.

In this general case, the exact sequence of vector bundle morphisms (1) is replaced by the following new one:

$$0 \rightarrow \pi_{NM}^* VM \xrightarrow{I''_0} \pi_{NM}^* TM \xrightarrow{\Pi''_0} \pi_{NM}^* NM \rightarrow 0. \quad (4)$$

Then C gives a left splitting C'' or, equivalently, a right splitting D'' of (4).

Using local coordinates, the map C'' has the local form:

$$X^u \frac{\partial}{\partial x^u} + X^{\bar{u}} \frac{\partial}{\partial x^{\bar{u}}} \xrightarrow{C''} \left(X^u - \frac{\partial C^u}{\partial y^{\bar{u}}} X^{\bar{u}} \right) \frac{\partial}{\partial x^u}. \quad (5)$$

There is an inclusion of $\pi_{N\mathcal{F}}^* N\mathcal{F}$, via the injective morphism D'' , as $N''\mathcal{F} = D''(\pi_{N\mathcal{F}}^* N\mathcal{F}) \subset \pi_{N\mathcal{F}}^* TM$, that gives the Whitney sum decomposition:

$$\pi_{N\mathcal{F}}^* TM = \pi_{N\mathcal{F}}^* T\mathcal{F} \oplus N''\mathcal{F}. \quad (6)$$

We consider now the simple foliation case given by a fibered manifold $\pi : M \rightarrow M'$. In this case, Γ_0 and $\tilde{J}$ are just the lifts of the Liouville vector field and of the almost tangent structure on M' , respectively, according to the vertical vector bundle $VNM = V(NM)$ of $NM = TM/VM = \pi^* TM'$. In this particular case, a nonlinear constraint is a fibered manifold map $C : VM \rightarrow TM$, viewed also as a section $C \in \Gamma(\pi_{NM}^* TM)$, provided that $\tilde{J}(C) = \Gamma_0$. Then, (4) has the same form.

We approach the time-dependent constraint case as follows. Instead of $N\mathcal{F}$, we consider $N^T\mathcal{F} = N\mathcal{F} \times \mathbb{R}$ or $N^T\mathcal{F} = N\mathcal{F} \times \mathbb{S}^1$, where $\mathbb{S}^1$ is the Euclidean circle. We consider on $N^T\mathcal{F}$ the foliation $\overline{\mathcal{F}}^T$ induced by a foliation $\overline{\mathcal{F}} = \mathcal{F}_{N\mathcal{F}}$ on $N\mathcal{F}$, provided that the canonical projection $N^T\mathcal{F} \rightarrow N\mathcal{F}$ is a diffeomorphism of leaves. It follows that the new parameter t is transverse.

We define a time-dependent nonlinear constraint on M as a map $C : N^T\mathcal{F} \rightarrow TM$ that can be viewed also as a section $C \in \Gamma(\pi_{N^T\mathcal{F}}^* TM)$, provided that $\tilde{J}(C) = \Gamma_0$. In local coordinates, where additionally t appears, we have:

$$(x^u, x^{\bar{u}}, y^{\bar{u}}, t) \xrightarrow{C} (C^u(x^v, x^{\bar{v}}, y^{\bar{v}}, t), y^{\bar{u}}), C = -C^u \frac{\partial}{\partial x^u} + y^{\bar{u}} \frac{\partial}{\partial x^{\bar{u}}}. \quad (7)$$

The short exact sequence (4) induces a new exact sequence:

$$0 \rightarrow \pi_{N^T\mathcal{F}}^* T\mathcal{F} \xrightarrow{I_0''} \pi_{N^T\mathcal{F}}^* TM \xrightarrow{\Pi_0''} \pi_{N^T\mathcal{F}}^* N\mathcal{F} \rightarrow 0. \quad (8)$$

Analogous to the time-independent case, left and right splittings, C'' and D'' , respectively, can be considered, as well as the corresponding Formulas (5) and (6).

Notice that linear and affine constraints are particular nonlinear constraints; this is the case when C is a linear or an affine map, respectively, on the fibers. We see below that using a similar approach to the linear and affine cases, we can obtain similar equations of motion in the general case of nonlinear constraints. These general equations of motion fit, in their turn, to the classical equations of motion in the linear and affine cases as those in [8].

The Mechanical Considerations and the Consequent Equations

For the sake of simplicity, we consider in our exposition below the case of a time-independent Lagrangian, but the constructions and results are valid also in the time-dependent case.

We consider a foliation $\mathcal{F}$ on a manifold M , a nonlinear constraint $C : N\mathcal{F} \rightarrow TM$, and a Lagrangian $L : TM \rightarrow \mathbb{R}$.

In this general case, as in the previous section (see [2] for more details), there is a left splitting $\mathcal{C}$ of I_0' or, equivalently, a right splitting $\mathcal{D}$ of the projection Π_0'' from (8).

We follow [2] in order to get the equations of motion governed by a Lagrangian and some nonlinear constraints. We proceed by adapting d'Alembert's principle from the linear or affine constraint case ([8], Section 5.2) to the nonlinear constraint case. In this regard, we impose the principle to apply the variation first and then project the Lagrange equations according to the constraint.

According to the decomposition (6), the Lagrange equations under the constraint effect have the form:

$$\begin{aligned} \left(\frac{d}{dt} \frac{\partial L}{\partial y^u} - \frac{\partial L}{\partial x^u} \right) \delta x^u + \left(\frac{d}{dt} \frac{\partial L}{\partial y^{\bar{u}}} - \frac{\partial L}{\partial x^{\bar{u}}} \right) \delta x^{\bar{u}} &= 0, \\ \delta x^u + C_{\bar{u}}^u \delta x^{\bar{u}} &= 0, \end{aligned}$$

where $C_{\bar{u}}^u = \frac{\partial C^u}{\partial y^{\bar{u}}}$ and δ is subject to $t = \text{const}$. Replacing $\delta x^{\bar{u}}$, obtained from the second equation above, in the first Lagrange equation, we find the induced constrained Lagrange equation:

$$-\delta L = \left(\frac{d}{dt} \frac{\partial L}{\partial y^u} - \frac{\partial L}{\partial x^u} \right) - C_{\bar{u}}^u \left(\frac{d}{dt} \frac{\partial L}{\partial y^u} - \frac{\partial L}{\partial x^u} \right) = 0. \quad (9)$$

The above equation is concordant with the Chetaev conditions.

Using the Lagrangian L and the constraint C , the composition $N\mathcal{F} \xrightarrow{C} TM \xrightarrow{L} \mathbb{R}$ gives the constrained Lagrangian $L_c = L \circ C$ on $N\mathcal{F}$, which has the local form:

$$L_c(x^u, x^{\bar{u}}, y^{\bar{u}}) = L(x^u, x^{\bar{u}}, -C^u, y^{\bar{u}}). \quad (10)$$

The variation of L_c is:

$$-\delta L_c = \left(\frac{d}{dt} \frac{\partial L_c}{\partial y^{\bar{u}}} - \frac{\partial L_c}{\partial x^{\bar{u}}} - C_{\bar{u}}^u \frac{\partial L_c}{\partial x^u} \right) dx^{\bar{u}}.$$

Using this, the constrained Lagrange Equation (9) has the local form:

$$-\delta L_c = \frac{\partial L}{\partial y^u} \left(\frac{\partial^2 C^u}{\partial y^{\bar{u}} \partial y^{\bar{v}}} \frac{dy^{\bar{v}}}{dt} - \left[C_V, \left[C_V, \frac{\partial}{\partial y^{\bar{u}}} \right] \right]^u \right) dx^{\bar{u}}, \quad (11)$$

where the Lie brackets are as follows:

$$-\left[C_V, \left[C_V, \frac{\partial}{\partial y^{\bar{u}}} \right] \right]^u = \frac{\partial C_{\bar{u}}^u}{\partial x^{\bar{v}}} y^{\bar{v}} + \frac{\partial C_{\bar{u}}^u}{\partial x^v} C^v - \frac{\partial C^u}{\partial x^{\bar{u}}} - C_{\bar{u}}^v \frac{\partial C^u}{\partial x^v}$$

$$\text{and } C_V = C^u \frac{\partial}{\partial x^u} + y^{\bar{u}} \frac{\partial}{\partial x^{\bar{u}}}.$$

The linear and affine cases have in common the relation $\frac{\partial^2 C^u}{\partial y^{\bar{u}} \partial y^{\bar{v}}} = 0$, along the constrained manifold. In general, if $\frac{\partial L}{\partial y^u} \frac{\partial^2 C^u}{\partial y^{\bar{u}} \partial y^{\bar{v}}} = 0$ along the constrained manifold, then the constrained Lagrange equations lead to the foliated Lagrangian dynamic system:

$$\begin{aligned} \frac{d}{dt} \frac{\partial L_c}{\partial y^{\bar{u}}} - \frac{\partial L_c}{\partial x^{\bar{u}}} &= Q_{\bar{u}}, \\ \frac{dx^u}{dt} &= -C^u, \end{aligned} \quad (12)$$

where:

$$Q_{\bar{u}}(x^u, x^{\bar{u}}, y^{\bar{u}}) = C_{\bar{u}}^u \frac{\partial L_c}{\partial x^u} - \frac{\partial L}{\partial y^u} \left[C_V, \left[C_V, \frac{\partial}{\partial y^{\bar{u}}} \right] \right]^u.$$

The regularity condition imposed on this dynamic system is expressed by the regularity condition on L_c , such that its transverse vertical Hessian is non-degenerated.

Specifically, when C are affine functions, having a local form $C_V = (C_{\bar{u}}^u(x^u, x^{\bar{u}}) y^{\bar{u}} - b^u(x^u, x^{\bar{u}})) \frac{\partial}{\partial x^u} + y^{\bar{u}} \frac{\partial}{\partial x^{\bar{u}}}$, then:

$$-\left[\left[\frac{\partial}{\partial y^{\bar{u}}}, C_V \right], C_V \right] = (\gamma_{\bar{u}}^u + B_{\bar{u}\bar{v}}^u y^{\bar{v}}) \frac{\partial}{\partial x^u},$$

where $B_{\bar{u}\bar{v}}^u = \frac{\partial C_{\bar{u}}^u}{\partial x^{\bar{v}}} - \frac{\partial C_{\bar{v}}^u}{\partial x^{\bar{u}}} + C_{\bar{v}}^v \frac{\partial C_{\bar{u}}^u}{\partial x^v} - C_{\bar{u}}^v \frac{\partial C_{\bar{v}}^u}{\partial x^v}$ and $\gamma_{\bar{u}}^u = \frac{\partial b^u}{\partial x^{\bar{u}}} - C_{\bar{u}}^v \frac{\partial b^u}{\partial x^v} + b^v \frac{\partial C_{\bar{u}}^u}{\partial x^v}$.

In the time-dependent case, there is an analogous Formula (12), fitting as well in the non-conservative Lagrangian system case, expressed by Formula (27). More exactly, in this case:

$$Q_{\bar{u}}(x^u, x^{\bar{u}}, y^{\bar{u}}) = C_{\bar{u}}^u \frac{\partial L_c}{\partial x^u} - \frac{\partial L}{\partial y^u} \left[C_V, \left[C_V, \frac{\partial}{\partial y^{\bar{u}}} \right] \right]^u + \frac{\partial^2 C^u}{\partial t \partial y^{\bar{u}}} \frac{\partial L}{\partial y^u}.$$

In the general case, when $\frac{\partial L}{\partial y^u} \frac{\partial^2 C^u}{\partial y^{\bar{u}} \partial y^{\bar{v}}}$ does not vanish along the constrained manifold, then the regularity condition shows that the local matrices $(h_{\bar{u}\bar{v}})$ are invertible, where:

$$h_{\bar{u}\bar{v}} = \frac{\partial^2 L_c}{\partial y^{\bar{u}} \partial y^{\bar{v}}} - \frac{\partial L}{\partial y^u} \frac{\partial^2 C^u}{\partial y^{\bar{u}} \partial y^{\bar{v}}}.$$

Notice that the above local matrices correspond to restrictions of the vertical Hessian matrices of Lagrangian L to the constrained manifold (see [2] for more details).

In the time independent case, the constrained Lagrange equations have the following form:

$$\begin{aligned} \frac{d}{dt} \frac{\partial L_c}{\partial y^{\bar{u}}} - \frac{\partial L_c}{\partial x^{\bar{u}}} &= Q_{\bar{u}} + \frac{\partial L}{\partial y^{\bar{u}}} \frac{\partial^2 C^u}{\partial y^{\bar{u}} \partial y^{\bar{v}}} \frac{dy^{\bar{v}}}{dt}, \\ \frac{dx^u}{dt} &= -C^u, \end{aligned} \quad (13)$$

where:

$$Q_{\bar{u}}(x^u, x^{\bar{u}}, y^{\bar{u}}) = C_{\bar{u}}^u \frac{\partial L_c}{\partial x^u} - \frac{\partial L}{\partial y^{\bar{u}}} \left[C_V, \left[C_V, \frac{\partial}{\partial y^{\bar{u}}} \right] \right]^u.$$

The regularity condition on this dynamic system is expressed by the regularity condition on matrix h , i.e., h is invertible. A generalized nonconservative Lagrangian system follows, expressed by Formula (35).

In the time-dependent case, there is an analogous form of constrained Lagrange equations, depicted also by Formula (13), but here, it is as follows:

$$Q_{\bar{u}}(t, x^u, x^{\bar{u}}, y^{\bar{u}}) = C_{\bar{u}}^u \frac{\partial L_c}{\partial x^u} - \frac{\partial L}{\partial y^{\bar{u}}} \left[C_V, \left[C_V, \frac{\partial}{\partial y^{\bar{u}}} \right] \right]^u + \frac{\partial^2 C^u}{\partial t \partial y^{\bar{u}}} \frac{\partial L}{\partial y^u}.$$

These formulas come from the fact that Equation (11) is in this case replaced by:

$$-\delta L_c = \left(\frac{\partial^2 C^u}{\partial y^{\bar{u}} \partial y^{\bar{v}}} \frac{dy^{\bar{v}}}{dt} \frac{\partial L}{\partial y^u} + \frac{\partial^2 C^u}{\partial t \partial y^{\bar{u}}} \frac{\partial L}{\partial y^u} - \frac{\partial L}{\partial y^{\bar{u}}} \left[C_V, \left[C_V, \frac{\partial}{\partial y^{\bar{u}}} \right] \right]^u \right) dx^{\bar{u}}. \quad (14)$$

3. Symmetries and Invariants of Lagrangians in a Foliated Constrained Setting

Consider a foliation $\mathcal{F}$ on the manifold M and local foliated coordinates $(x^u, x^{\bar{u}})$ on M . We can consider a foliation $\mathcal{F}^{\mathbb{R}}$ on $\mathbb{R} \times M$, where the real parameter is added to the transverse part. If the foliation, $\mathcal{F}$ is simple, and $f : M \rightarrow \bar{M}$ is the corresponding submersion on a manifold $\bar{M}$ with coordinates $(x^{\bar{u}})$, then (x^u) are the coordinates on the leaf $f^{-1}(x^{\bar{u}})$, giving together the coordinates $(x^u, x^{\bar{u}})$ on M . Thus, the parameter $t \in \mathbb{R}$ in coordinates $(t, x^u, x^{\bar{u}})$ on $\mathbb{R} \times M$ is transverse, such as $(x^{\bar{u}})$.

We follow some ideas used in [3], giving the same infinitesimal result as in [4], adapted here for a regular Lagrangian and a nonconservative system case (see also [10], Section 7.1). Using foliations, we give here a global form for the invariant objects. The classical setting is as in [10–22].

Let $L : N\mathcal{F}^{\mathbb{R}} \rightarrow \mathbb{R}$ be a (transverse) Lagrangian, where a real parameter can (but non-necessarily) be involved. This Lagrangian can be obtained as $L = L' \circ C$, using a Lagrangian $L' : \mathbb{R} \times TM \rightarrow \mathbb{R}$ and a nonlinear constraint $C : \mathbb{R} \times N\mathcal{F} \rightarrow \mathbb{R} \times TM$.

We say that a Lagrangian L action is invariant under a set of an ε -parameter local group of foliated transformations:

$$\begin{cases} \bar{t}(t) = t + \varepsilon \tau(t, x^u(t), x^{\bar{u}}(t), y^{\bar{u}}(t)) + o(\varepsilon^2), \\ \bar{x}^u(t) = x^u(t) + \varepsilon \zeta^u(t, x^u(t), x^{\bar{u}}(t), y^{\bar{u}}(t)) + o(\varepsilon^2), \\ \bar{x}^{\bar{u}}(t) = x^{\bar{u}}(t) + \varepsilon \bar{\zeta}^{\bar{u}}(t, x^u(t), x^{\bar{u}}(t), y^{\bar{u}}(t)) + o(\varepsilon^2), \end{cases} \quad (15)$$

if there is another Lagrangian $\Lambda : N\mathcal{F}^{\mathbb{R}} \rightarrow \mathbb{R}$ such that:

$$\begin{aligned} &L\left(\bar{t}, \bar{x}^u(\bar{t}), \bar{x}^{\bar{u}}(\bar{t}), \frac{d\bar{x}^{\bar{u}}}{d\bar{t}}(\bar{t})\right) \frac{d\bar{t}}{dt} \\ &= L\left(t, x^u(t), x^{\bar{u}}(t), \frac{dx^{\bar{u}}}{dt}(t)\right) + \varepsilon \frac{d}{dt} \Lambda\left(t, x^u(t), x^{\bar{u}}(t), \frac{dx^{\bar{u}}}{dt}(t)\right) + o(\varepsilon^2). \end{aligned} \quad (16)$$

We say that the local group is infinitesimally exact if the vector field:

$$X_0 = \tau(t, x^u, x^{\bar{u}}, y^{\bar{u}}) \frac{\partial}{\partial t} + \zeta^u(t, x^u, x^{\bar{u}}, y^{\bar{u}}) \frac{\partial}{\partial x^u} + \bar{\zeta}^{\bar{u}}(t, x^u, x^{\bar{u}}, y^{\bar{u}}) \frac{\partial}{\partial x^{\bar{u}}} = \tau \frac{\partial}{\partial t} + X_0^{(t)}, \quad (17)$$

called an infinitesimal action, is a global foliated vector field on $\pi_{N\mathcal{F}\mathbb{R}}^*T(M \times \mathbb{R})$.

Notice that a foliated section $\bar{\xi} \in \pi_{N\mathcal{F}\mathbb{R}}^*(N\mathcal{F})$ having the local form:

$$\bar{\xi} = \bar{\xi}^{\bar{u}}(t, x^u, x^{\bar{u}}, y^{\bar{u}}) \overline{\frac{\partial}{\partial x^{\bar{u}}}}$$

can be lifted, via the constraints C , to a global foliated vector field $\xi \in \pi_{N\mathcal{F}\mathbb{R}}^*T(M \times \mathbb{R})$, using the formula:

$$\xi = C^u(t, x^u, x^{\bar{u}}, \bar{\xi}^{\bar{u}}) \frac{\partial}{\partial x^u} + \bar{\xi}^{\bar{u}}(t, x^u, x^{\bar{u}}, y^{\bar{u}}) \frac{\partial}{\partial x^{\bar{u}}}. \quad (18)$$

For example, the Liouville-type section $\bar{\Gamma}_0 \in \pi_{N\mathcal{F}\mathbb{R}}^*(\mathbb{R} \times N\mathcal{F})$, $\bar{\xi} = y^{\bar{u}} \overline{\frac{\partial}{\partial x^{\bar{u}}}}$ lifts to a C-Liouville-type section:

$$\bar{\Gamma}'_0 = \frac{\partial}{\partial t} + C^u(t, x^u, x^{\bar{u}}, y^{\bar{u}}) \frac{\partial}{\partial x^u} + y^{\bar{u}} \frac{\partial}{\partial x^{\bar{u}}}. \quad (19)$$

In the case when the vector field (17) has the component $X_0^{(t)}$ of the form (18), we say that the infinitesimal action is compatible with the constraints.

The energy of L is the Lagrangian $\mathcal{E}(L) = D_v(L) - L$, where D_v is the vertical derivation $D_v(L) = y^{\bar{v}} \frac{\partial L}{\partial y^{\bar{v}}}$. Let us consider the differential form $\delta(L) \in \mathcal{X}^*(N\mathcal{F})$, given by $\delta L = dL + \mathcal{E}(L)d\tau$ and called the Cartan form of L . Using local coordinates, we have:

$$\begin{aligned} \mathcal{E}(L) &= \frac{\partial L}{\partial y^{\bar{v}}}(t, x^u, x^{\bar{u}}, y^{\bar{u}}) y^{\bar{v}} - L(t, x^u, x^{\bar{u}}, y^{\bar{u}}), \\ \delta L &= \frac{\partial L}{\partial t} dt + \frac{\partial L}{\partial x^u} dx^u + \frac{\partial L}{\partial x^{\bar{u}}} dx^{\bar{u}} + \frac{\partial L}{\partial y^{\bar{u}}} dy^{\bar{u}} + \left(y^{\bar{v}} \frac{\partial L}{\partial y^{\bar{v}}} - L \right) dt. \end{aligned}$$

A vector field $S \in \mathbb{R} \times N\mathcal{F}$ that projects on the C-Liouville-type section is an almost transverse semi-spray. Using local coordinates, S has the form:

$$S = \frac{\partial}{\partial t} + C^u(x^u, x^{\bar{u}}, y^{\bar{u}}) \frac{\partial}{\partial x^u} + y^{\bar{u}} \frac{\partial}{\partial x^{\bar{u}}} + S^{\bar{u}}(t, x^u, x^{\bar{u}}, y^{\bar{u}}) \frac{\partial}{\partial y^{\bar{u}}}. \quad (20)$$

The vector field S has the solutions of the differential equation system as the integral curves:

$$\begin{aligned} \frac{dx^u}{dt} &= -C^u(x^u, x^{\bar{u}}, y^{\bar{u}}), \\ \frac{dx^{\bar{u}}}{dt} &= y^{\bar{u}}, \\ \frac{dx^{\bar{u}}}{dt} &= S^{\bar{u}}(t, x^u, x^{\bar{u}}, y^{\bar{u}}) \end{aligned}$$

We denote below by $\frac{d}{dt}$ the action of S on real functions on $T(N\mathcal{F}\mathbb{R})$.

We say that the Lagrangian L is invariant up to a gauge term, if for every almost transverse semi-spray S , corresponding to a certain non-linear constraint C , there is an S -related vector field $X \in \mathcal{X}(N\mathcal{F}\mathbb{R})$, called an infinitesimal symmetry, and a Lagrangian $\Lambda : N\mathcal{F}\mathbb{R}$, called the gauge term, such that:

$$\delta L(X) = S(\Lambda). \quad (21)$$

We obtain the next proposition.

Proposition 1. *If the Lagrangian action of a regular Lagrangian on $N\mathcal{F}^{\mathbb{R}}$ is invariant under (15) and the local group is infinitesimally exact, then L is invariant up to a gauge term Λ , having an infinitesimal symmetry given by an infinitesimal action.*

Proof. If we differentiate the equality (16) with respect to ε and then put $\varepsilon = 0$, we obtain:

$$\begin{aligned} & \tau \frac{\partial L}{\partial t} + \zeta^u \frac{\partial L}{\partial x^u} + \zeta^{\bar{u}} \frac{\partial L}{\partial x^{\bar{u}}} + \left(\frac{d}{d\varepsilon} \frac{d\bar{x}^{\bar{u}}}{d\bar{t}}(\bar{t}) \right) \Big|_{\varepsilon=0} \frac{\partial L}{\partial y^{\bar{u}}} + \frac{d\tau}{dt} L \\ &= \frac{d}{dt} \Lambda \left(t, x^u(t), x^{\bar{u}}(t), \frac{dx^{\bar{u}}}{dt}(t) \right). \end{aligned} \quad (22)$$

However,

$$\begin{aligned} & \left(\frac{d}{d\varepsilon} \frac{d\bar{x}^{\bar{u}}}{d\bar{t}}(\bar{t}) \right) \Big|_{\varepsilon=0} = \left(\frac{d}{d\varepsilon} \left(\frac{1}{\frac{d\bar{t}}{dt}} \frac{d\bar{x}^{\bar{u}}}{d\bar{t}} \right) \right) \Big|_{\varepsilon=0} \\ &= \left(\frac{d}{d\varepsilon} \left(\frac{1}{\frac{d\bar{t}}{dt}} \right) \right) \Big|_{\varepsilon=0} \frac{dx^{\bar{u}}}{dt} + \left(\frac{d}{d\varepsilon} \left(\frac{dx^{\bar{u}}}{d\bar{t}} \right) \right) \Big|_{\varepsilon=0} = -\frac{d\tau}{dt} \frac{dx^{\bar{u}}}{dt} + \frac{d\zeta^{\bar{u}}}{dt}. \end{aligned}$$

Thus, Relation (22) becomes:

$$\begin{aligned} & \tau \frac{\partial L}{\partial t} + \zeta^u \frac{\partial L}{\partial x^u} + \zeta^{\bar{u}} \frac{\partial L}{\partial x^{\bar{u}}} + \frac{d\zeta^{\bar{u}}}{dt} \frac{\partial L}{\partial y^{\bar{u}}} + \frac{d\tau}{dt} \left(L - \frac{dx^{\bar{u}}}{dt} \frac{\partial L}{\partial y^{\bar{u}}} \right) \\ &= \frac{d}{dt} \Lambda \left(t, x^u(t), x^{\bar{u}}(t), \frac{dx^{\bar{u}}}{dt}(t) \right). \end{aligned} \quad (23)$$

Notice that the action of the operator $\frac{d}{dt} \in \mathcal{X}(\mathbb{R} \times N\mathcal{F})$ has the form:

$$\begin{aligned} \frac{d}{dt} &= \frac{\partial}{\partial t} + C^u(x^u, x^{\bar{u}}, \frac{dx^u}{dt}) \frac{\partial}{\partial x^u} + \frac{dx^{\bar{u}}}{dt} \frac{\partial}{\partial x^{\bar{u}}} + \frac{d^2 x^{\bar{u}}}{dt^2} \frac{\partial}{\partial y^{\bar{u}}} \\ &= \frac{\partial}{\partial t} + C^u \frac{\partial}{\partial x^u} + y^{\bar{u}} \frac{\partial}{\partial x^{\bar{u}}} + \frac{dy^{\bar{u}}}{dt} \frac{\partial}{\partial y^{\bar{u}}}. \end{aligned} \quad (24)$$

If S is an almost transverse semi-spray, then its action on real functions from $\mathcal{F}(N\mathcal{F}^{\mathbb{R}})$ is the same as the action of the operator $\frac{d}{dt}$ in (24), but along the integral curves of S . Thus, considering τ , ζ , and S , one can construct X . Since the relation (23) can be written as (21), the conclusion follows. $\square$

In particular, if $\bar{\Gamma}'_0$ is the C-Liouville-type section given by Formula (19), then $X = \frac{\partial}{\partial t} + \bar{\Gamma}'_0$ is an infinitesimal symmetry, having L as the gauge term; it can be easily checked using Formula (23). Thus, the set of infinitesimal transformations that satisfies Equation (22) is non-void.

The regularity condition on Lagrangians and Lagrangian actions we consider is to verify the hypothesis of Proposition 1. Thus, an allowed Lagrangian action corresponds to a regular Lagrangian on $N\mathcal{F}^{\mathbb{R}}$; the Lagrangian action is invariant under (15); and the local group is infinitesimally exact.

The existence of Λ in Formula (21) raises the problem of whether it depends on S or not. We have to notice that if Λ comes from a local group action, as in the hypothesis of Proposition 1, then Λ does not depend on S . In this case, taking into account Formula (20), the equality (23) implies:

$$\begin{aligned} & \tau \frac{\partial L}{\partial t} + \zeta^u \frac{\partial L}{\partial x^u} + \zeta^{\bar{u}} \frac{\partial L}{\partial x^{\bar{u}}} + \left(\frac{\partial \zeta^{\bar{u}}}{\partial t} + C^u \frac{\partial \zeta^{\bar{u}}}{\partial x^u} + y^{\bar{v}} \frac{\partial \zeta^{\bar{u}}}{\partial x^{\bar{v}}} \right) \frac{\partial L}{\partial y^{\bar{u}}} + \\ & \left(\frac{\partial \tau}{\partial t} + C^u \frac{\partial \tau}{\partial x^u} + y^{\bar{v}} \frac{\partial \tau}{\partial x^{\bar{v}}} \right) \left(L - \frac{dx^{\bar{u}}}{dt} \frac{\partial L}{\partial y^{\bar{u}}} \right) \\ &= \frac{\partial \Lambda}{\partial t} + C^u \frac{\partial \Lambda}{\partial x^u} + y^{\bar{v}} \frac{\partial \Lambda}{\partial x^{\bar{v}}}. \end{aligned} \quad (25)$$

and:

$$\frac{\partial \xi^{\bar{v}}}{\partial y^{\bar{u}}} \frac{\partial L}{\partial y^{\bar{v}}} + \frac{\partial \tau}{\partial y^{\bar{u}}} \left(L - \frac{dx^{\bar{u}}}{dt} \frac{\partial L}{\partial y^{\bar{u}}} \right) = \frac{\partial \Lambda}{\partial y^{\bar{u}}}. \quad (26)$$

The two relations (25) and (26) are called Killing equations in the classical case [3], as well as in the nonholonomic case [6].

4. The Case of Nonconservative Lagrangian Systems

Consider a foliation $\mathcal{F}$ on the manifold M and a Lagrangian $L : TM \rightarrow \mathbb{R}$.

A nonconservative Lagrangian system has the form:

$$\begin{aligned} \frac{d}{dt} \frac{\partial L}{\partial y^{\bar{u}}} &= \frac{\partial L}{\partial x^{\bar{u}}} + Q_{\bar{u}}(t, T, x^u, x^{\bar{u}}, y^{\bar{u}}), \\ \frac{dx^u}{dt} &= -C^u(t, T, x^u, x^{\bar{u}}, y^{\bar{u}}), \\ \frac{dT}{dt} &= \tau. \end{aligned} \quad (27)$$

This is the case when there is a linear or an affine constraint $C : N^T\mathcal{F} \rightarrow TM$. The constrained Lagrange equations, depicted above in (12), have the form (27). They were studied, for example, in [2,8].

Using second-order derivatives, we extend in the next section the above definition.

Two global forms associated with a Lagrangian L are $d_v L \in \Gamma(\pi_{N\mathcal{F}\mathbb{R}}^*(N^*\mathcal{F}))$ (the vertical differential) and $H_v L \in \Gamma(\pi_{N\mathcal{F}\mathbb{R}}^*(N^*\mathcal{F} \otimes N^*\mathcal{F}))$ (the vertical Hessian), given by $d_v L(Z) = Z(L)$ and $H_v L(Z_1, Z_2)$, respectively, for vertical lifts Z, Z_1 , and Z_2 . In local coordinates:

$$d_v L = \frac{\partial L}{\partial y^{\bar{u}}} dx^{\bar{u}}, \quad H_v L = \frac{\partial^2 L}{\partial y^{\bar{u}} \partial y^{\bar{v}}} dx^{\bar{u}} \otimes dx^{\bar{v}}.$$

If the vertical Hessian of L is non-degenerate, then L is regular, and the nonconservative system (27) is said to be also regular. We prove the following Proposition.

Proposition 2. *If the Lagrangian L is regular, then the curves on M that are solutions of a nonconservative Lagrangian system (27) are exactly the integral curves of an almost transverse semi-spray S (called the canonical semi-spray associated with the nonconservative Lagrangian system).*

Proof. Equation (27) reads:

$$\frac{\partial^2 L}{\partial t \partial y^{\bar{u}}} + y^u \frac{\partial^2 L}{\partial x^u \partial y^{\bar{u}}} + y^{\bar{v}} \frac{\partial^2 L}{\partial x^{\bar{v}} \partial y^{\bar{u}}} + \frac{dy^{\bar{v}}}{dt} \frac{\partial^2 L}{\partial y^{\bar{v}} \partial y^{\bar{u}}} = \frac{\partial L}{\partial x^{\bar{u}}} + Q_{\bar{u}}$$

or, denoting $g_{\bar{u}\bar{v}} = \frac{\partial^2 L}{\partial y^{\bar{u}} \partial y^{\bar{v}}}$ and $(g^{\bar{u}\bar{v}}) = (g_{\bar{u}\bar{v}})^{-1}$, we have:

$$\frac{dy^{\bar{u}}}{dt} = g^{\bar{u}\bar{v}} \left(\frac{\partial L}{\partial x^{\bar{v}}} - y^u \frac{\partial^2 L}{\partial x^u \partial y^{\bar{u}}} - y^{\bar{v}} \frac{\partial^2 L}{\partial x^{\bar{v}} \partial y^{\bar{u}}} - \frac{dy^{\bar{v}}}{dt} \frac{\partial^2 L}{\partial y^{\bar{v}} \partial y^{\bar{u}}} + Q_{\bar{v}} \right) \equiv S^{\bar{u}}. \quad (28)$$

We can directly check that Formula (20) gives a global almost transverse semi-spray. $\square$

When L is time-independent (i.e., $\frac{\partial L}{\partial t} = 0$), the term $\frac{\partial}{\partial t}$ does not appear in Formula (20) of an almost transverse semi-spray.

Theorem 1. *Consider an allowed Lagrangian action that corresponds to a regular Lagrangian L on $N\mathcal{F}^{\mathbb{R}}$, and let S be the canonical semi-spray of a nonconservative Lagrangian system (27). Then:*

$$\frac{d}{dt} (\tau \mathcal{E}(L) - d_v L(\xi) + \Lambda) + Q_{\bar{u}}(\xi^{\bar{u}} - \tau y^{\bar{u}}) - (\xi^u - \tau C^u) \frac{\partial L}{\partial x^u} = 0. \quad (29)$$

Proof. We have, along the integral curves of S :

$$\frac{d}{dt} = \frac{\partial}{\partial t} + C^u \frac{\partial}{\partial x^u} + y^{\bar{u}} \frac{\partial}{\partial x^{\bar{u}}} + S^{\bar{u}}(t, x^u, x^{\bar{u}}, y^{\bar{u}}) \frac{\partial}{\partial y^{\bar{u}}}. \quad (30)$$

Using Formulas (27) and (28), we obtain:

$$\begin{aligned} \frac{d}{dt} \mathcal{E}(L) &= \frac{d}{dt} \left(y^{\bar{u}} \frac{\partial L}{\partial y^{\bar{u}}} - L \right) = S^{\bar{u}} \frac{\partial L}{\partial y^{\bar{u}}} + y^{\bar{u}} \frac{d}{dt} \frac{\partial L}{\partial y^{\bar{u}}} - \frac{d}{dt} L \\ &= S^{\bar{u}} \frac{\partial L}{\partial y^{\bar{u}}} + y^{\bar{u}} \left(\frac{\partial L}{\partial x^{\bar{u}}} + Q_{\bar{u}} \right) - \frac{\partial L}{\partial t} - C^u \frac{\partial L}{\partial x^u} - \frac{dx^{\bar{u}}}{dt} \frac{\partial L}{\partial x^{\bar{u}}} - S^{\bar{u}} \frac{\partial L}{\partial y^{\bar{u}}} \\ &= y^{\bar{u}} Q_{\bar{u}} - C^u \frac{\partial L}{\partial x^u} - \frac{\partial L}{\partial t}. \end{aligned}$$

Using once more (30), we have:

$$\begin{aligned} &\frac{d}{dt} (\tau \mathcal{E}(L) - d_v L(\xi) + \Lambda) \\ &= \frac{d\tau}{dt} \mathcal{E}(L) + \tau \frac{d\mathcal{E}(L)}{dt} - \frac{d\xi^{\bar{u}}}{dt} \frac{\partial L}{\partial y^{\bar{u}}} - \xi^{\bar{u}} \frac{d}{dt} \frac{\partial L}{\partial y^{\bar{u}}} \\ &= \frac{d\tau}{dt} \mathcal{E}(L) + \tau \left(y^{\bar{u}} Q_{\bar{u}} - C^u \frac{\partial L}{\partial x^u} - \frac{\partial L}{\partial t} \right) - \frac{d\xi^{\bar{u}}}{dt} \frac{\partial L}{\partial y^{\bar{u}}} - \\ &\quad \xi^{\bar{u}} \left(\frac{\partial L}{\partial x^{\bar{u}}} + Q_{\bar{u}} \right) \\ &= -Q_{\bar{u}} \left(\xi^{\bar{u}} - \tau \frac{dx^{\bar{u}}}{dt} \right) + \frac{\partial L}{\partial x^u} (\xi^u - \tau C^u); \end{aligned}$$

thus, the conclusion follows. $\square$

Corollary 1. Consider an allowed Lagrangian action that corresponds to a regular Lagrangian L on $N\mathcal{F}^{\mathbb{R}}$, and let S be the canonical corresponding semi-spray. Then:

$$\frac{d}{dt} (\tau \mathcal{E}(L) - d_v L(\xi) + \Lambda) + (\tau C^u - \xi^u) \frac{\partial L}{\partial x^u} = 0. \quad (31)$$

In the case $Q = 0$, we recover the case of a constrained Lagrangian system. The time-independent case is when $L : N\mathcal{F} \rightarrow \mathbb{R}$ or $L : \widehat{N\mathcal{F}} \rightarrow \mathbb{R}$. We study some special situations below.

If S is an almost transverse semi-spray, we say that an $h \in \mathcal{F}(N\mathcal{F}^{\mathbb{R}})$ is S -invariant if $S(h) = 0$.

We now obtain a special case of Theorem 1.

Theorem 2. Consider an allowed Lagrangian action that corresponds to a regular Lagrangian L on $N\mathcal{F}^{\mathbb{R}}$. Let S be the canonical semi-spray of a nonconservative Lagrangian system (27), and supplementary to this, suppose that there is an $f \in \mathcal{F}(N\mathcal{F}^{\mathbb{R}})$ such that:

$$\frac{df}{dt} = Q_{\bar{u}} (\xi^{\bar{u}} - \tau y^{\bar{u}}) - (\xi^u - \tau C^u) \frac{\partial L}{\partial x^u}, \quad (32)$$

where Γ is the Liouville vector field. Then, the function:

$$h = \tau \mathcal{E}(L) - d_v L(\xi) + \Lambda + f$$

is S -invariant.

The existence of a function f is not always possible, even locally, as can be seen below, in Example 1. In this case, we can use Theorem 1.

When the existence of a function f as in Theorem 2 is assured, Corollary 1 becomes:

Corollary 2. Consider an allowed Lagrangian action that corresponds to a regular Lagrangian on $N\mathcal{F}^{\mathbb{R}}$. Let S be the canonical semi-spray of the Lagrangian system, and suppose that there is an $f \in \mathcal{F}(N\mathcal{F}^{\mathbb{R}})$ such that:

$$\frac{df}{dt} = (\tau C^u - \xi^u) \frac{\partial L}{\partial x^u}. \quad (33)$$

Then, the function:

$$h = \tau \mathcal{E}(L) - d_v L(\xi) + \Lambda + f$$

is S -invariant.

Proof. This is the case of Theorem 2, where $Q_{\bar{u}} = 0$. $\square$

Corollary 3. Consider an allowed Lagrangian action that corresponds to a regular Lagrangian on $N\mathcal{F}^{\mathbb{R}}$, and let S be the canonical semi-spray corresponding to the Lagrangian system. Suppose that the infinitesimal symmetry is compatible with the constraints and also $\tau = 1$. Then, the function:

$$h = \mathcal{E}(L) - d_v L(\xi) + \Lambda$$

is S -invariant.

Proof. This is the case of Corollary 2, where $\xi^u = C^u$ and $\tau = 1$; thus, $f = 0$. $\square$

Example 1. We use the linear Appell constraints as considered in [2] (Example 5.1) (see also [11]). We consider the manifold $M = \mathbb{R}^3 \times \mathbb{T}^2$, where $\mathbb{T}^2$ is the Euclidean torus. The canonical projection is $\mathbb{R}^3 \times \mathbb{T}^2 \rightarrow \mathbb{T}^2$, and we consider the corresponding simple foliation. We denote by (x^1, x^2, x^3) and $(x^{\bar{1}}, x^{\bar{2}})$ the coordinates on $\mathbb{R}^3$ and $\mathbb{T}^2$, respectively. The formulas:

$$C^1 = Ry^{\bar{1}} \cos x^2, C^2 = Ry^{\bar{1}} \sin x^2, C^3 = ry^{\bar{1}} \quad (34)$$

define the linear Appell constraints.

The Lagrangian and the induced Lagrangian have the forms:

$$L = \frac{1}{2}\alpha \left((y^1)^2 + (y^2)^2 \right) + \frac{1}{2}\beta (y^3)^2 + \frac{1}{2}I_1 (y^{\bar{1}})^2 + \frac{1}{2}I_2 (y^{\bar{2}})^2 + \gamma x^3$$

and:

$$\begin{aligned} L_c(x^1, x^2, x^3, x^{\bar{1}}, x^{\bar{2}}, y^{\bar{1}}, y^{\bar{2}}) &= \frac{1}{2}(I_1 + \alpha R^2 + \beta r^2) (y^{\bar{1}})^2 + \frac{1}{2}I_2 (y^{\bar{2}})^2 + \gamma x^3 \\ &= \frac{1}{2}\alpha'' (y^{\bar{1}})^2 + \frac{1}{2}I_2 (y^{\bar{2}})^2 + \gamma x^3 \end{aligned}$$

respectively.

An infinitesimal symmetry is given by $\xi^{\bar{u}} = y^{\bar{u}}$, $\xi^3 = 0$, $\tau = \tau_0 = \text{const.}$ and $\Lambda = \frac{1}{2}\alpha'' (y^{\bar{1}})^2 + \frac{1}{2}I_2 (y^{\bar{2}})^2$, since it verifies the Killing relations (25) and (26), together with (23). We have $Q_{\bar{u}} = \gamma r \delta_{\bar{u}\bar{1}}$; thus, Equation (32) has the form:

$$\frac{df}{dt} = \gamma r \delta_{\bar{u}\bar{1}} y^{\bar{u}} (1 - \tau) + \gamma r \tau y^{\bar{1}} = r \gamma y^{\bar{1}}.$$

However, $y^{\bar{1}} = \frac{dx^{\bar{1}}}{dt}$; thus, we can take $f = r \gamma x^{\bar{1}} + c$, and we can use Theorem 2. It follows that $h = -(1 + \tau_0)\Lambda + \tau_0 x^3 + r \gamma x^{\bar{1}} + c$ is an invariant along the integral curves of the linear Appell constraints system, where c is a real constant.

Another infinitesimal symmetry is given by $\xi^{\bar{u}} = \frac{1}{2}x^{\bar{u}}$, $\xi^3 = -x^3$, $\tau = t$, and $\Lambda = 0$. Since Relation (23) holds, Relations (25) and (26) also hold; thus, this infinitesimal symmetry is a Killing one. In this case, Equation (32) has the form:

$$\frac{df}{dt} = \frac{r\gamma}{2}x^{\bar{1}} + \gamma x^3.$$

Since $x^{\bar{1}}$ and x^3 cannot be in the form $\frac{dg}{dt}$, we cannot find a global function f to satisfy the above equation, as in the case of the previous symmetry. In this case, we do not use Theorem 2, but Theorem 1. Since $\tau\mathcal{E}(L) - d_v L(\xi) + \Delta = (t-1)\Delta - t\gamma x^3$, where $\Delta = \frac{1}{2}\alpha''(y^{\bar{1}})^2 + \frac{1}{2}I_2(y^{\bar{2}})^2$, it follows that along the integral curves of the linear Appell constraints system, we obtain:

$$-\frac{d}{dt}(t\gamma x^3) + \frac{r\gamma}{2}x^{\bar{1}} + \gamma x^3 = 0.$$

Considering for x^3 and $x^{\bar{1}}$ corresponding polynomial expressions of degree at most one in t , we can find the solutions of the above differential equation along an integral curve. They are concordant with the general solution (see, for example, [2] for an explicit form for general solutions).

5. The Case of Generalized Nonconservative Lagrangian Systems

Consider a foliation $\mathcal{F}$ on the manifold M and a Lagrangian $L : TM \rightarrow \mathbb{R}$.

A generalized nonconservative Lagrangian system is a dynamic system of the form:

$$\begin{aligned} \frac{d}{dt} \frac{\partial L}{\partial y^{\bar{u}}} &= \frac{\partial L}{\partial x^{\bar{u}}} + b_{\bar{u}\bar{v}}(t, T, x^u, x^{\bar{u}}, y^{\bar{u}}) \frac{dy^{\bar{v}}}{dt} + Q_{\bar{u}}(t, T, x^u, x^{\bar{u}}, y^{\bar{u}}), \\ \frac{dx^u}{dt} &= C^u(t, T, x^u, x^{\bar{u}}, y^{\bar{u}}), \\ \frac{dT}{dt} &= \tau. \end{aligned} \quad (35)$$

This is the case when a nonlinear constraint $C : N^T\mathcal{F} \rightarrow TM$ is given. The constrained Lagrange equations, depicted above in (13), have the form (35).

The Lagrangian L is quasi-regular for the generalized nonconservative Lagrangian system (35) if the matrix $\left(\frac{\partial^2 L}{\partial y^{\bar{u}} \partial y^{\bar{v}}} - b_{\bar{u}\bar{v}} \right)$ is non-singular in every point.

The definition of a generalized nonconservative Lagrangian system can be extended by replacing, in the first relation in (35), $b_{\bar{u}\bar{v}}(t, T, x^u, x^{\bar{u}}, y^{\bar{u}}) \frac{dy^{\bar{v}}}{dt} + Q_{\bar{u}}(t, T, x^u, x^{\bar{u}}, y^{\bar{u}})$ by $Q_{\bar{u}}(t, T, x^u, x^{\bar{u}}, y^{\bar{u}}, \frac{dy^{\bar{u}}}{dt})$, then the non-singularity of the matrix $\left(\frac{\partial^2 L}{\partial y^{\bar{u}} \partial y^{\bar{v}}} - b_{\bar{u}\bar{v}} \right)$ by the non-singularity of the matrix $\left(\frac{\partial^2 L}{\partial y^{\bar{u}} \partial y^{\bar{v}}} - \frac{\partial Q_{\bar{u}}}{\partial y^{\bar{v}}} \right)$.

However, we do not need this general case in this paper, since the nonlinear constraint case fits in well with the case we study here.

We obtain the next result, which is analogous to Proposition 2.

Proposition 3. *If we assume that L is a quasi-regular Lagrangian, then the curves on M , which are solutions of a generalized nonconservative Lagrangian system (35), are exactly the integral curves of an almost transverse semi-spray S (called the canonical semi-spray of the generalized nonconservative Lagrangian system).*

We extend Theorem 1 in the following way, obtaining a similar conclusion.

Theorem 3. Consider an allowed Lagrangian action that corresponds to a regular Lagrangian L on $N\mathcal{F}^{\mathbb{R}}$, and let S be the canonical semi-spray of a generalized nonconservative Lagrangian system (35). Then, along the integral curves of S , we have:

$$\frac{d}{dt}(\tau\mathcal{E}(L) - d_v L(\xi) + \Lambda) + (Q_{\bar{u}} + b_{\bar{u}\bar{v}} S^{\bar{v}})(\xi^{\bar{u}} - \tau y^{\bar{u}}) - \frac{\partial L}{\partial x^u}(\xi^u - \tau C^u) = 0. \quad (36)$$

We also extend Theorem 2 as follows.

Theorem 4. Consider an allowed Lagrangian action that corresponds to a regular Lagrangian L on $N\mathcal{F}^{\mathbb{R}}$. Let S be the canonical semi-spray of a generalized nonconservative Lagrangian system (35), and supplementary to this, let us suppose that there is an $f \in \mathcal{F}(N\mathcal{F}^{\mathbb{R}})$ such that:

$$\frac{df}{dt} = (Q_{\bar{u}} + b_{\bar{u}\bar{v}} S^{\bar{v}})(\xi^{\bar{u}} - \tau y^{\bar{u}}) - \frac{\partial L}{\partial x^u}(\xi^u - \tau C^u). \quad (37)$$

Then, the function:

$$h = \tau\mathcal{E}(L) - d_v L(\xi) + \Lambda + f$$

is an S -invariant.

Proof. This is analogous to the proof of Theorem 2, but using Relations (35) and (37) instead of (27) and (32), respectively. $\square$

Example 2. In this example, we consider the nonlinear Appell constraints as in [2] (Example 5.2) (see also [11]). Let $M = \mathbb{R}^3$ be the manifold and the projection $\mathbb{R}^3 \rightarrow \mathbb{R}^2$, $(x^1, x^2, x^1) \rightarrow (x^1, x^2)$ be the fibered manifold projection. The nonlinear Appell constraint is given by the formula:

$$C^1 = \alpha \sqrt{(y^1)^2 + (y^2)^2}. \quad (38)$$

The Lagrangian and the induced Lagrangian have the forms:

$$L = \frac{\beta}{2} \left((y^1)^2 + (y^2)^2 \right) + \frac{\gamma}{2} (y^1)^2 + \delta x^1 \quad (39)$$

and:

$$L_c(x^1, x^2, x^3, x^1, x^2, y^1, y^2) = \frac{\alpha'}{2} \left((y^1)^2 + (y^2)^2 \right) + \delta x^1,$$

respectively, where $\alpha' = \frac{\beta + \alpha^2 \gamma}{2}$.

An infinitesimal symmetry is given by:

$$\xi^{\bar{u}} = y^{\bar{u}}, \xi^1 = 0, \tau = \tau_0 (= \text{const.}) \text{ and } \Lambda = \frac{\alpha'}{2} \left((y^1)^2 + (y^2)^2 \right).$$

We have $Q_{\bar{u}} = \frac{\partial C^1}{\partial y^{\bar{u}}} \frac{\partial L}{\partial x^1} = \frac{\delta \alpha y^{\bar{u}}}{\sqrt{\Delta}}$. Equation (37) has the form:

$$\frac{df}{dt} = \delta C^1.$$

Now, we can take $f = \delta x^1 + c$, where c is a real constant, since $\frac{dx^1}{dt} = C^1$. Thus, $h = \tau_0(\Lambda - \delta x^1) - 2\Lambda + \Lambda + \delta x^1 + c = (1 - \tau_0)(\delta x^1 - \Lambda) + c$ or:

$$h = (1 - \tau_0)(\delta x^1 - \Lambda) + c. \quad (40)$$

For $\tau_0 \neq 1$, we obtain a non-trivial invariant.

Example 3. The Appell–Hammel dynamic system in an elevator (see [2,12]) is an example of a time-dependent nonlinear constraint. The time-dependent constraint is given by:

$$\alpha^2 \left((y^1)^2 + (y^2)^2 \right) - (y^1 - v^0(t))^2 = 0. \quad (41)$$

The particular case when $v^0(t) = 0$ fits into the above nonlinear Appell constraint considered in Example 2.

We take $C^1(t, y^1, y^2) = v^0(t) + \alpha \sqrt{(y^1)^2 + (y^2)^2}$ and the Lagrangian (39), as in the case of a nonlinear Appell system (a particular case of this example, when $v^0(t) = 0$). In this case, the induced Lagrangian is:

$$\begin{aligned} L_c(x^1, x^1, x^2, y^1, y^2) \\ = \frac{\beta + \alpha^2 \gamma}{2} \left((y^1)^2 + (y^2)^2 \right) + \gamma v^0 \sqrt{(y^1)^2 + (y^2)^2} + \delta x^1 + \frac{1}{2} (v^0)^2 \end{aligned}$$

and the pseudo-curvature is $R_V = \frac{\partial L}{\partial y^u} \left[C_V, \left[C_V, \frac{\partial}{\partial y^u} \right] \right]^u = 0$.

Let us denote $\Delta = (y^1)^2 + (y^2)^2$ and $\alpha'' = \beta + \alpha^2 \gamma$; thus, the induced Lagrangian has the form:

$$L_c = \frac{\alpha''}{2} \Delta + \gamma v^0 \sqrt{\Delta} + \delta x^1 + \frac{1}{2} (v^0)^2.$$

An infinitesimal symmetry is given by $\xi^{\bar{u}} = \tau_0 y^{\bar{u}}$, $\xi^1 = \left(u - \frac{\tau_0}{\delta} \right) v^0 \dot{v}^0$, $\tau = \tau_0 = \text{const.}$, $u = \text{const.}$ and $\Lambda = \tau_0 \left(\frac{\alpha'' \Delta}{2} + \gamma v^0 \sqrt{\Delta} \right) + u \frac{(v^0)^2}{2}$. We have $Q_{\bar{u}} = \frac{\partial C^1}{\partial y^{\bar{u}}} \frac{\partial L_c}{\partial x^1} + \frac{\partial^2 C^u}{\partial t \partial y^{\bar{u}}} \frac{\partial L}{\partial y^u} = \frac{\alpha \delta y^{\bar{u}}}{\sqrt{\Delta}}$, $S^{\bar{u}} = \frac{(a\delta + \gamma \dot{v}^0) y^{\bar{u}}}{\sqrt{\Delta}}$, $b_{\bar{u}\bar{v}} = \frac{\partial L}{\partial y^1} \frac{\partial^2 C^1}{\partial y^{\bar{u}} \partial y^{\bar{v}}}$, $b_{\bar{u}\bar{v}} S^{\bar{u}} = 0$.

Equation (37) has the form $\frac{df}{dt} = -\delta \left(\left(u - \frac{\tau_0}{\delta} \right) v^0 \dot{v}^0 - \tau_0 C^1 \right) + (\tau_0^2 - \tau_0) \left(\gamma \dot{v}^0 \sqrt{\Delta} + v^0 \dot{v}^0 \right) = v^0 \dot{v}^0 (\tau_0^2 - \delta u) + \tau_0 \delta C^1 = \frac{d}{dt} \left(\frac{(v^0)^2}{2} (\tau_0^2 - \delta u) + \tau_0 \delta x^1 \right)$. Thus, we can take:

$$f = \frac{(v^0)^2}{2} (\tau_0^2 - \delta u) + \tau_0 \delta x^1 + c,$$

where c is a real constant. We have $\mathcal{E}(L) = y^{\bar{u}} \frac{\partial L}{\partial y^{\bar{u}}} - L = y^{\bar{u}} \left(\alpha'' y^{\bar{u}} + \gamma v^0 \frac{y^{\bar{u}}}{\sqrt{\Delta}} \right) - \frac{\alpha'' \Delta}{2} - \gamma v^0 \sqrt{\Delta} - \delta x^1 - \frac{(v^0)^2}{2} = \frac{\alpha'' \Delta}{2} - \delta x^1 - \frac{(v^0)^2}{2}$.

Thus, $h = \frac{\alpha'' \Delta}{2} - \delta x^1 - \frac{(v^0)^2}{2} - y^{\bar{u}} \left(\alpha'' y^{\bar{u}} + \gamma v^0 \frac{y^{\bar{u}}}{\sqrt{\Delta}} \right) + \frac{\alpha'' \Delta}{2} + \gamma v^0 \sqrt{\Delta} + \frac{(v^0)^2}{2} (\tau_0^2 - \delta u) + \tau_0 \delta x^1 + c = \frac{(v^0)^2}{2} (\tau_0^2 - 1 - \delta u) + (\tau_0 - 1) \delta x^1 + c$.

We now consider the case when $v^0(t) = v_0^0 t + c_0$, with v_0^0 and c_0 real constants. An infinitesimal symmetry is given by $\xi^{\bar{u}} = y^{\bar{u}}$, $\xi^1 = \frac{\gamma}{\delta} v_0^0 \sqrt{\Delta}$, $\tau = 0$ and $\Lambda = \frac{\alpha'' \Delta}{2} + \gamma v^0 \sqrt{\Delta}$, where $\Delta = (y^1)^2 + (y^2)^2$. We have $Q_{\bar{u}} = \frac{\alpha \delta y^{\bar{u}}}{\sqrt{\Delta}}$, and Equation (37) has the form:

$$\frac{df}{dt} = Q_{\bar{u}} \xi^{\bar{u}} - \xi^u \frac{\partial L_c}{\partial x^u} = \frac{\alpha \delta y^{\bar{u}}}{\sqrt{\Delta}} y^{\bar{u}} - \delta \frac{\gamma}{\delta} v_0^0 \sqrt{\Delta} = (\alpha \delta - \gamma v_0^0) \sqrt{\Delta}.$$

Considering all the above,

$$\frac{df}{dt} = \frac{\alpha\delta - \gamma v_0^0}{\alpha} (v^0 + \alpha\sqrt{\Delta}) - \frac{(\alpha\delta - \gamma)}{\alpha} (v_0^0 t + c_0),$$

meaning:

$$\frac{df}{dt} = \frac{d}{dt} \left(\frac{\alpha\delta - \gamma v_0^0}{\alpha} \left(x^1 - v_0^0 \frac{t^2}{2} - c_0 t \right) + c \right).$$

Take $f = \delta' \left(x^1 - v_0^0 \frac{t^2}{2} - c_0 t \right) + c_1$, where c is a real constant and $\delta' = \frac{\delta\alpha - \gamma v_0^0}{\alpha}$.

$$\text{Thus, } h = -\alpha''\sqrt{\Delta} - \gamma v^0\sqrt{\Delta} + \frac{\alpha''\Delta}{2} + \gamma v^0\sqrt{\Delta} + \delta' \left(x^1 - v_0^0 \frac{t^2}{2} - c_0 t \right) + c_1 = -\frac{\alpha''\Delta}{2} + \delta' x^1 - \delta' \left(v_0^0 \frac{t^2}{2} + c_0 t \right) + c_1.$$

For $v_0^0 = c_0 = 0$, we obtain a non-trivial invariant, similar to the invariant (40) of a nonlinear Appell system, obtained previously.

6. Conclusions

The use of constraints is imposed by mechanical reasons relating them to Lagrangians. In our paper, the symmetries of the equation of motion of such a system were related to the classical ones of Noether's. The novelty is that the symmetries of constrained mechanical systems were studied in the presence of nonlinear constraints related to a foliation on a manifold. In our opinion, the foliated setting is more suitable and gives a simpler form for the constructed mathematical objects than the bundled or fibered setting. The Killing's conditions for symmetries were explained in our paper by their relations with the dynamics given by semi-spray equations. We obtained the Noether invariants in a foliated setting for linear and affine constraints, but also for the general case of nonlinear constraints. Using a more general setting, we formulated and obtained them also for conservative and generalized conservative Lagrangian systems, defined in the paper. A mathematical background was included in a distinct section, in order to explain that the objects we considered on the manifold were global ones, giving a new opportunity to mathematicians to use Lagrangians in geometry.

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References

1. Borisov, A.V.; Mamaev, I.S. Symmetries and reduction in nonholonomic mechanics. *Regul. Chaotic Dyn.* **2015**, *20*, 553–604. [CrossRef]
2. Popescu, P.; Ida, C. Nonlinear constraints in nonholonomic mechanics. *J. Geom. Mech.* **2014**, *6*, 527–547. [CrossRef]
3. Sarlet, W.; Cantrijn, F. Generalizations of Noether's theorem in classical mechanics. *Siam Rev.* **1981**, *23*, 467–494. [CrossRef]
4. Djukic, D.S. Noether's theorem for optimum control systems. *Int. J. Control* **1973**, *18*, 667–672. [CrossRef]
5. Federico, G.S.F.; Torres, D.F.M. Nonconservative Noether's theorem in optimal control. In Proceedings of the 13th IFAC Workshop on Control Applications of Optimisation, Paris, France, 26–28 April 2006.

6. Luo, S.-K.; Jia, L.-Q.; Cai, J.-L. Noether symmetry can lead to non-Noether conserved quantity of holonomic nonconservative systems in general Lie transformations. *Commun. Theor. Phys.* **2005**, *43*, 193–196.
7. Li, Z.; Jiang, W.; Luo, S. Lie symmetries, symmetrical perturbation and a new adiabatic invariant for disturbed nonholonomic systems. *Nonlinear Dyn.* **2012**, *67*, 445–455. [[CrossRef](#)]
8. Bloch, A.M. *Nonholonomic Mechanics and Control*; Springer: Berlin/Heidelberg, Germany, 2003; Volume 24.
9. Grifone, J. Structure presque tangente et connexions I. *Ann. L'Institut Fourier* **1972**, *22*, 287–334. [[CrossRef](#)]
10. Logan, J.D. *Applied Mathematics—A Contemporary Approach*; Wiley-Interscience Publication, John Wiley & Sons, Inc.: New York, NY, USA, 1987.
11. Marle, C.M. Various approaches to conservative and nonconservative nonholonomic systems. *Rep. Math. Phys.* **1998**, *42*, 211–229. [[CrossRef](#)]
12. Li, S.-M.; Berakdar, J. A generalization of the Chetaev condition for nonlinear nonholonomic constraints: The velocity-determined virtual displacement approach. *Rep. Math. Phys.* **2009**, *63*, 179–189. [[CrossRef](#)]
13. Bates, L. Nonholonomic reduction. *Rep. Math. Phys.* **1993**, *32*, 99–115. [[CrossRef](#)]
14. Bloch, A.M.; Krishnaprasad, P.S.; Marsden, J.E.; Murray, R.M. Nonholonomic mechanical systems with symmetry. *Arch. Ration. Mech. Anal.* **1996**, *136*, 21–99. [[CrossRef](#)]
15. Bucataru, I.; Miron, R. *Finsler-Lagrange Geometry: Applications to Dynamical Systems*; Editura Academiei Romane: Bucharest, Romania, 2007.
16. Cortés, J.; de León, M.; Marrero, J.C.; Martínez, E. Nonholonomic Lagrangian systems on Lie algebroids. *Discret. Contin. Dyn. Syst. A* **2009**, *24*, 213–271. [[CrossRef](#)]
17. Crampin, M. Constants of the motion in Lagrangian mechanics. *Int. J. Theor. Phys.* **1977**, *16*, 741–754. [[CrossRef](#)]
18. Crampin, M.; Mestdag, T. The Cartan form for constrained Lagrangian systems and the nonholonomic Noether theorem. *Int. J. Geom. Methods Mod. Phys.* **2011**, *8*, 897–923. [[CrossRef](#)]
19. Garaev, K.G. On the problem of modified theory of invariant variation problems construction. In *Geometry and Topology of Submanifolds IX*; World Scientific: Singapore, 1999; pp. 139–147.
20. Halder, A.K.; Paliathanasis, A.; Leach, P.G. Noether's Theorem and Symmetry. *Symmetry* **2018**, *10*, 744. [[CrossRef](#)]
21. Krupková, O. Geometric mechanics on nonholonomic submanifolds. *Commun. Math.* **2010**, *18*, 51–77.
22. Sun, X.; Yang, B.; Zhang, Y.; Xue, X.; Jia, L. Accurate conserved quantity and approximate conserved quantity deduced from Noether symmetry for a weakly Chetaev nonholonomic system. *Nonlinear Dyn.* **2015**, *81*, 1563–1568. [[CrossRef](#)]