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On (p, q) -Analogues of Laplace-Typed Integral Transforms and Applications

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Abstract: In this paper, we establish (p, q) -analogues of Laplace-type integral transforms by using the concept of (p, q) -calculus. Moreover, we study some properties of (p, q) -analogues of Laplace-type integral transforms and apply them to solve some (p, q) -differential equations.

Keywords: (p, q) -laplace-typed integral transforms; (p, q) -derivative; (p, q) -integral; (p, q) -calculus; (p, q) -difference equations; (p, q) -convolution theorem

MSC: 05A30; 44A10; 39A60



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1. Introduction

Integral transform techniques are very important for solving many problems in applied mathematics, physics, astronomy, economics and engineering. The integral transform techniques have contributed largely to a variety of theories and applications, such as Laplace, Sumudu, σ -Integral Laplace, Mohand, Sawi, Kamal and Pourreza transforms. In the sequence of such integral transforms, in 2017, H. Kim [1] introduced the Laplace-typed integral transform or ${}_aG$ -transform, which is defined by

$${}_aG(f(t); u) = u^\alpha \int_0^\infty e^{-t/u} f(t) dt,$$

where $\alpha \in \mathbb{R}$. The ${}_aG$ -transform can be applied directly to a suitable problem by choosing α appropriately. In Table 1, we list a few of them with their definitions and set u, α for converting ${}_aG$ -transform into appropriate transforms.

Table 1. Definitions of some integral transforms

Transform	Definition	${}_aG$ -Transform
Laplace [2]	$\int_0^\infty f(t) e^{-st} dt$	$u = 1/s$ and $\alpha = 0$
Sumudu [3,4]	$\frac{1}{s} \int_0^\infty f(t) e^{-t/s} dt$	$u = s$ and $\alpha = -1$
σ -Integral Laplace [5]	$\int_0^\infty f(t) e^{-t/(s)^\frac{1}{\sigma}} dt$	$u = 1/s^\frac{1}{\sigma}$ and $\alpha = 0$
Mohand [6]	$s^2 \int_0^\infty f(t) e^{-st} dt$	$u = 1/s$ and $\alpha = -2$
Sawi [7]	$\frac{1}{s^2} \int_0^\infty f(t) e^{-t/s} dt$	$u = s$ and $\alpha = -2$
Kamal [8]	$\int_0^\infty f(t) e^{-t/s} dt$	$u = s$ and $\alpha = 0$
Pourreza [9]	$s \int_0^\infty f(t) e^{s^2 t} dt$	$u = 1/s^2$ and $\alpha = -1/2$

In 2017, H. Kim [10] investigated the solution of Laguerre's equation by using ${}_aG$ -transform with $\alpha = -2$, and then, in the same year, H. Kim [11] used ${}_aG$ -transform to solve Volterra integral equation and semi-infinite string. In 2019, S. Sattaso et al. [12] studied the properties of Laplace-typed integral transforms for solving differential equations and presented some examples to illustrate the effectiveness of its applicability. In 2020, Y.H. Geum et al. [13] applied the matrix expression of convolution and its generalized continuous form with the ${}_aG$ -transform. Next, in 2021, S.R. Sararha et al. [14] introduced the fractional ${}_aG$ -transform by using modified Riemann–Liouville derivative to solve fractional nonlinear differential equations and applied to the spreading problem of a non-fatal disease within a population.

Quantum calculus, or q -calculus, is referred to as the study of calculus without limits and has also been applied to many areas of mathematics, applied mathematics and physics. It was first studied in the early eighteenth century by a mathematician Euler and developed by Gauss and Ramnaujan. In 1910, F.H. Jackson [15,16] introduced q -derivative and q -integral, which are known as Jackson derivative and Jackson integral. Many researchers have generalized and developed the q -calculus as found in [17–25] and the references cited therein. The book by V. Kac and P. Cheung [26] covers the basic theoretical concept of q -calculus.

The topic of q -integral transform has been scrutinized extensively by many researchers. In 2013, D. Albayrak et al. [27] investigated q -analogues of Sumudu transform and derived some properties. In 2014, W.S. Chung et al. [28] investigated the q -analogues of the Laplace transform and some properties of the q -Laplace transform. In 2020, S.K.Q. Al-Omari [29] proposed the q -analogues and properties of the Laplace-type integral operator in the quantum calculus; see [30–33] for more details.

Post-quantum calculus, or (p, q) -calculus, is a generalization of q -calculus. It was first studied in 1991 by R. Chakrabarti and R. Jagannathan [34]. In 2013, P.N. Sadjang [35] studied the concept of the (p, q) -derivative, the (p, q) -integration, (p, q) -Taylor formulas and the fundamental theorem of (p, q) -calculus. Many researchers studied and developed the (p, q) -calculus as found in [36–47] and the references cited therein.

Recently, there has been a good deal of extensive research about (p, q) -integral transforms. In 2017, P.N. Sadjang [48] studied the properties of (p, q) -analogues of the Laplace transform and applied them to solve some (p, q) -difference equations. In 2019, P.N. Sadjang [49] studied the (p, q) -analogues of the Sumudu transform and gave some properties to solve (p, q) -difference equations. In 2020, A. Tassaddiq [50] proposed (p, q) -Laplace and (p, q) -Sumudu transforms with (p, q) -Aleph-function. The results make a major contribution to the theory of integral transforms and special functions.

Inspired by the above mentioned-literature, we propose to study (p, q) -analogues of the Laplace-typed integral transform as well as giving some properties that encompass almost all existing (p, q) -integral transforms and to apply them to solve some (p, q) -differential equations.

The paper is organized as follows: in Section 2, we give some basic knowledge and notation that is used in the next sections; in Section 3, we present some properties of the (p, q) -analogues of the Laplace-typed integral transform; in Section 4, we apply the (p, q) -analogues of the Laplace-typed integral transform to some differential equations; and in the last section, we give the conclusion.

2. Preliminaries

In this section, we give basic knowledge that will be used in our work. Throughout this paper, let $0 < q < p \leq 1$ be constants.

Let us introduce (p, q) -analogue or (p, q) -number for $n \in \mathbb{N}$, which is defined by

$$[n]_{p,q} = \frac{p^n - q^n}{p - q}. \quad (1)$$

If $p = 1$ in (1), then (1) is q -analogue of n or q -number; see [26] for more details.

The (p, q) -factorial is defined by

$$[0]_{p,q}! = 1 \text{ and } [n]_{p,q}! = \prod_{j=1}^n [j]_{p,q}, \quad n \geq 1. \quad (2)$$

If $p = 1$ in (2), then (2) is q -factorial; see [26] for more details.

The (p, q) -binomial coefficients are defined by

$$\binom{n}{j}_{p,q} = \frac{[n]_{p,q}!}{[j]_{p,q}![n-j]_{p,q}!} = \binom{n}{n-j}_{p,q} \text{ for } 0 \leq j \leq n. \quad (3)$$

If $p = 1$ in (3), then (3) reduces to the q -binomial coefficients; see [26] for more details.

Definition 1 ([35]). If f is an arbitrary function, then

$$D_{p,q}f(x) = \frac{f(px) - f(qx)}{(p-q)x}, \quad x \neq 0, \quad (4)$$

is the (p, q) -derivative of the function f .

If $p = 1$ in (4), then $D_{p,q}f(x) = D_qf(x)$, which is the q -derivative of the function f ; in addition, if $q \rightarrow 1$ in (4), then we get the classical derivative.

Example 1. Define function $f : \mathbb{R} \rightarrow \mathbb{R}$ by $f(x) = x^2 + 2x + c$ and $x \neq 0$, where c is a constant; then, we have

$$\begin{aligned} D_{p,q}(x^2 + 2x + c) &= \frac{(p^2x^2 + 2px + c) - (q^2x^2 + 2qx + c)}{(p-q)x} \\ &= \frac{(p^2 - q^2)x^2 + 2(p-q)x}{(p-q)x} \\ &= (p+q)x + 2. \end{aligned}$$

Proposition 1. The (p, q) -derivatives of the product and quotient rules of functions f and g are as follows:

$$D_{p,q}(f(x)g(x)) = f(px)D_{p,q}g(x) + g(qx)D_{p,q}f(x), \quad (5)$$

$$D_{p,q}\left(\frac{f(x)}{g(x)}\right) = \frac{g(qx)D_{p,q}f(x) - f(qx)D_{p,q}g(x)}{g(px)g(qx)}, \quad g(x) \neq 0. \quad (6)$$

The proof of Proposition 1 is given in [35].

Definition 2. ([35]) If f is an arbitrary function, then the (p, q) -integral of f on $[0, \infty)$ is defined by

$$\int_0^\infty f(x) d_{p,q}x = (p-q) \sum_{j=-\infty}^\infty \frac{q^j}{p^{j+1}} f\left(\frac{q^j}{p^{j+1}}\right). \quad (7)$$

If $p = 1$ in (7), then (7) reduces to the q -integral of the function f ; also, if $q \rightarrow 1$ in (7), then we get the classical integral.

Proposition 2. If f and g are arbitrary functions, then

$$\int_a^b f(px)(D_{p,q}g(x)) d_{p,q}x = f(b)g(b) - f(a)g(a) - \int_a^b g(qx)(D_{p,q}f(x)) d_{p,q}x, \quad (8)$$

is the (p, q) integration by parts. Note that $b = \infty$ is allowed.

The proofs of Proposition 2 are given in [35].

Definition 3 ([35]). If $z \in \mathbb{R}$, then the (p, q) -exponential functions are defined by

$$e_{p,q}(z) = \sum_{n=0}^{\infty} \frac{p^{\binom{n}{2}}}{[n]_{p,q}!} z^n, \quad (9)$$

$$E_{p,q}(z) = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}}{[n]_{p,q}!} z^n. \quad (10)$$

If $p = 1$ in (9) and (10), then we have the q -exponential function [26]; moreover, if $q \rightarrow 1$, then (9) and (10) reduce to the classical exponential function.

Proposition 3 ([35]). If $n \in \mathbb{R}$, then the following identities hold:

$$D_{p,q} e_{p,q}(nx) = n e_{p,q}(npx), \quad (11)$$

$$D_{p,q} E_{p,q}(nx) = n E_{p,q}(nqx). \quad (12)$$

The proofs of the following Propositions are given in [48].

Proposition 4. If $a, b \in \mathbb{R}$, then

$$D_{p,q}^n \left(\frac{1}{ax - b} \right) = \frac{(-a)^n [n]_{p,q}!}{\prod_{k=0}^n (ap^{n-k} q^k x - b)} = \frac{(-a)^n [n]_{p,q}!}{(ap^n x - b)(ap^{n-1} q x - b) \cdots (aq^n x - b)}. \quad (13)$$

Proposition 5. If $n \in \mathbb{N} \cup \{0\}$, then the (p, q) -cosine and the (p, q) -sine functions are as follows:

$$\cos_{p,q}(z) = \frac{e_{p,q}(iz) + e_{p,q}(-iz)}{2} = \sum_{n=0}^{\infty} \frac{(-1)^n p^{\binom{2n}{2}}}{[2n]_{p,q}!} z^{2n}, \quad (14)$$

$$\text{Cos}_{p,q}(z) = \frac{E_{p,q}(iz) + E_{p,q}(-iz)}{2} = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{2n}{2}}}{[2n]_{p,q}!} z^{2n}, \quad (15)$$

$$\sin_{p,q}(z) = \frac{e_{p,q}(iz) - e_{p,q}(-iz)}{2i} = \sum_{n=0}^{\infty} \frac{(-1)^n p^{\binom{2n+1}{2}}}{[2n+1]_{p,q}!} z^{2n+1}, \quad (16)$$

$$\text{Sin}_{p,q}(z) = \frac{E_{p,q}(iz) - E_{p,q}(-iz)}{2i} = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{2n+1}{2}}}{[2n+1]_{p,q}!} z^{2n+1}. \quad (17)$$

Proposition 6. If $n \in \mathbb{N} \cup \{0\}$, then the (p, q) -hyperbolic cosine and the (p, q) -hyperbolic sine functions are as follows:

$$\cosh_{p,q}(z) = \frac{e_{p,q}(z) + e_{p,q}(-z)}{2} = \sum_{n=0}^{\infty} \frac{p^{\binom{2n}{2}}}{[2n]_{p,q}!} z^{2n}, \quad (18)$$

$$\text{Cosh}_{p,q}(z) = \frac{E_{p,q}(z) + E_{p,q}(-z)}{2} = \sum_{n=0}^{\infty} \frac{q^{\binom{2n}{2}}}{[2n]_{p,q}!} z^{2n}, \quad (19)$$

$$\sinh_{p,q}(z) = \frac{e_{p,q}(z) - e_{p,q}(-z)}{2} = \sum_{n=0}^{\infty} \frac{p^{\binom{2n+1}{2}}}{[2n+1]_{p,q}!} z^{2n+1}, \quad (20)$$

$$\text{Sinh}_{p,q}(z) = \frac{E_{p,q}(z) - E_{p,q}(-z)}{2} = \sum_{n=0}^{\infty} \frac{q^{\binom{2n+1}{2}}}{[2n+1]_{p,q}!} z^{2n+1}. \quad (21)$$

Proposition 7. If $a \in \mathbb{R} \setminus \{0\}$, then we have

$$\int_0^\infty f(at) d_{p,q}t = \frac{1}{a} \int_0^\infty f(t) d_{p,q}t. \quad (22)$$

Example 2. If $f(t) = E_{p,q}(-t)$ in (22) and uses (12), we obtain

$$\int_0^\infty E_{p,q}(-2t) d_{p,q}t = -\frac{q}{2} \int_0^\infty D_{p,q} E_{p,q} \left(-\frac{2t}{q} \right) d_{p,q}t = -\frac{q}{2} \left[E_{p,q} \left(-\frac{2t}{q} \right) \right]_0^\infty = \frac{q}{2},$$

and

$$\frac{1}{2} \int_0^\infty E_{p,q}(-t) d_{p,q}t = -\frac{q}{2} \int_0^\infty D_{p,q} E_{p,q} \left(-\frac{t}{q} \right) d_{p,q}t = -\frac{q}{2} \left[E_{p,q} \left(-\frac{t}{q} \right) \right]_0^\infty = \frac{q}{2}.$$

Definition 4 ([37]). For $n \in \mathbb{N} \cup \{0\}$, the (p, q) -gamma function is defined by

$$\Gamma_{p,q}(n+1) = \frac{(p-q)_{p,q}^n}{(p-q)^n} = [n]_{p,q}!. \quad (23)$$

Definition 5 ([38]). For $s, t \in \mathbb{N}$, the (p, q) -beta function is defined by

$$B_{p,q}(s, t) = \int_0^1 x^{s-1} (1 \ominus qx)_{p,q}^{t-1} d_{p,q}x. \quad (24)$$

Theorem 1. For $s, t \in \mathbb{N}$, the relation between the (p, q) -gamma function and the (p, q) -beta function is

$$B_{p,q}(s, t) = p^{\frac{(t-1)(2s+t-2)}{2}} \frac{\Gamma_{p,q}(s) \Gamma_{p,q}(t)}{\Gamma_{p,q}(s+t)}. \quad (25)$$

The proof of this Theorem is given in [38].

3. Properties of (p, q) -Analogues of Laplace-Typed Integral Transform

In this section, we introduce (p, q) -analogues of the Laplace-typed integral transform in the form ${}_1G_{p,q}$ and ${}_2G_{p,q}$, which are called ${}_1G_{p,q}$ -transform of type one and type two, respectively.

Let

$$A = \left\{ f(t) \mid \exists M, j_1, j_2 > 0, |f(t)| < M E_{p,q} \left(\frac{|t|}{j_k} \right), t \in (-1)^k \times [0, \infty), k = 1, 2 \right\}, \quad (26)$$

and

$$B = \left\{ f(t) \mid \exists M, j_1, j_2 > 0, |f(t)| < M e_{p,q} \left(\frac{|t|}{j_k} \right), t \in (-1)^k \times [0, \infty), k = 1, 2 \right\}, \quad (27)$$

Now the definition of the ${}_1G_{p,q}$ -transform of type one and type two is given by:

Definition 6. The ${}_1G_{p,q}(f(t); u)$ over the set A in (26) and the ${}_2G_{p,q}(f(t); u)$ over the set B in (27) are defined as follows:

$${}_1G_{p,q}(f(t); u) = u^\alpha \int_0^\infty E_{p,q} \left(-\frac{qt}{u} \right) f(t) d_{p,q}t \quad \text{for } \alpha \in \mathbb{R}, \quad j_1 < u < j_2, \quad (28)$$

and

$${}_2G_{p,q}(f(t); u) = u^\alpha \int_0^\infty e_{p,q} \left(-\frac{pt}{u} \right) f(t) d_{p,q}t \quad \text{for } \alpha \in \mathbb{R}, \quad j_1 < u < j_2, \quad (29)$$

If $u = 1/s$, $p = 1$ and $\alpha = 0$, then (28) and (29) reduce to

$$L_q^1(f(t); s) = \int_0^\infty E_q(-qst)f(t)d_qt,$$

and

$$L_q^2(f(t); s) = \int_0^\infty e_q(-st)f(t)d_qt,$$

respectively, which appeared in [28]. If $u = s$ and $\alpha = -1$, then (28) and (29) reduce to

$$S_{p,q}^1(f(t); s) = \frac{1}{s} \int_0^\infty E_{p,q}\left(-\frac{qt}{s}\right)f(t)d_{p,q}t,$$

and

$$S_{p,q}^2(f(t); s) = \frac{1}{s} \int_0^\infty e_{p,q}\left(-\frac{pt}{s}\right)f(t)d_{p,q}t,$$

respectively, which appeared in [49].

Theorem 2. (Linearity): If $f_1, g_1 \in A$ and $f_2, g_2 \in B$, then for constants c and d , we have

$${}_1G_{p,q}(cf_1(t) + dg_1(t); u) = c{}_1G_{p,q}(f_1(t); u) + d{}_1G_{p,q}(g_1(t); u), \quad (30)$$

$${}_2G_{p,q}(cf_2(t) + dg_2(t); u) = c{}_2G_{p,q}(f_2(t); u) + d{}_2G_{p,q}(g_2(t); u). \quad (31)$$

Proof. The theorem follows immediately from Definition 6. \square

Theorem 3. (Scaling): If $f_1 \in A$ and $g_1 \in B$, then the following formulas hold for non-zero constants β and γ :

$${}_1G_{p,q}(f_1(\beta t); u) = \frac{1}{\beta^{\alpha+1}} {}_1G_{p,q}(f_1(t); \beta u), \quad (32)$$

$${}_2G_{p,q}(g_1(\gamma t); u) = \frac{1}{\gamma^{\alpha+1}} {}_2G_{p,q}(g_1(t); \gamma u). \quad (33)$$

Proof. Using (28) and Proposition 7, we have

$$\begin{aligned} {}_1G_{p,q}(f(\beta t); u) &= u^\alpha \int_0^\infty E_{p,q}\left(-\frac{qt}{u}\right)f(\beta t)d_{p,q}t \\ &= \frac{\beta^\alpha u^\alpha}{\beta^{\alpha+1}} \int_0^\infty E_{p,q}\left(-\frac{qt}{\beta u}\right)f(t)d_{p,q}t \\ &= \frac{1}{\beta^{\alpha+1}} {}_1G_{p,q}(f(t); \beta u). \end{aligned}$$

The proof of (33) is similar to (32), and therefore the proof is completed. \square

Theorem 4. Let $\alpha \in \mathbb{R}$; then the following formulas hold:

$${}_1G_{p,q}(1; u) = u^{\alpha+1}, \quad (34)$$

$${}_2G_{p,q}(1; u) = u^{\alpha+1}. \quad (35)$$

Proof. Using (28) and (12) to prove (34), we get

$$\begin{aligned} {}^1_a G_{p,q}(1; u) &= u^\alpha \int_0^\infty E_{p,q} \left(-\frac{qt}{u} \right) d_{p,q} t \\ &= -u^{\alpha+1} \int_0^\infty D_{p,q} E_{p,q} \left(-\frac{t}{u} \right) d_{p,q} t \\ &= -u^{\alpha+1} \lim_{a \rightarrow \infty} \left[E_{p,q} \left(-\frac{t}{u} \right) \right]_0^a \\ &= u^{\alpha+1}. \end{aligned}$$

The proof of the part (35) utilizes a similar process as for (34). Therefore, the proof is completed. \square

Remark 1. If $u = 1/s$, $p = 1$ and $\alpha = 0$, then (34) reduces to

$$L_q(1; s) = \frac{1}{s},$$

which appeared in [28]. If $u = s$ and $\alpha = -1$, then (34) reduces to

$$S_{p,q}(1; s) = 1,$$

which appeared in [49].

Theorem 5. If $n \in \mathbb{N}$, then the following identities hold:

$$\begin{aligned} (i) \quad {}^1_a G_{p,q}(t; u) &= \frac{u^{\alpha+2}}{p}; \\ (ii) \quad {}^2_a G_{p,q}(t; u) &= \frac{u^{\alpha+2}}{q}; \\ (iii) \quad {}^1_a G_{p,q}(t^n; u) &= \frac{u^{\alpha+n+1} [n]_{p,q}!}{p^{\binom{n+1}{2}}}; \\ (iv) \quad {}^2_a G_{p,q}(t^n; u) &= \frac{u^{\alpha+n+1} [n]_{p,q}!}{q^{\binom{n+1}{2}}}. \end{aligned}$$

Proof. Using (8) and (28) to prove (i), we have

$$\begin{aligned} {}^1_a G_{p,q}(t; u) &= u^\alpha \int_0^\infty E_{p,q} \left(-\frac{qt}{u} \right) t d_{p,q} t \\ &= -u^{\alpha+1} \int_0^\infty t D_{p,q} E_{p,q} \left(-\frac{t}{u} \right) d_{p,q} t \\ &= -\frac{u^{\alpha+1}}{p} \int_0^\infty (pt) D_{p,q} E_{p,q} \left(-\frac{t}{u} \right) d_{p,q} t \\ &= -\frac{u^{\alpha+1}}{p} \left[\lim_{a \rightarrow \infty} \left[t E_{p,q} \left(-\frac{t}{u} \right) \right]_0^a - \frac{u^\alpha}{u^\alpha} \int_0^\infty E_{p,q} \left(-\frac{qt}{u} \right) d_{p,q} t \right] \\ &= \frac{u^{\alpha+1}}{p} \left[\frac{u^{\alpha+1}}{u^\alpha} \right] = \frac{u^{\alpha+2}}{p}. \end{aligned}$$

We prove (iii) by mathematical induction: obviously, (iii) is true for $n = 1$. Assuming that (iii) is true and using the (p, q) -integration by parts, we obtain

$$\begin{aligned}
 {}^1_{\alpha}G_{p,q}(t^{n+1}; u) &= u^{\alpha} \int_0^{\infty} E_{p,q}\left(-\frac{qt}{u}\right) t^{n+1} d_{p,q}t \\
 &= -\frac{u^{\alpha+1}}{p^{n+1}} \int_0^{\infty} (pt)^{n+1} D_{p,q}E_{p,q}\left(-\frac{t}{u}\right) d_{p,q}t \\
 &= -\frac{u^{\alpha+1}}{p^{n+1}} \left[\lim_{a \rightarrow \infty} \left[t^{n+1} E_{p,q}\left(-\frac{t}{u}\right) \right]_0^a - [n+1]_{p,q} \int_0^{\infty} t^n E_{p,q}\left(-\frac{qt}{u}\right) d_{p,q}t \right] \\
 &= \frac{u[n+1]_{p,q} G_{p,q}(t^n; u)}{p^{n+1}} \\
 &= \frac{u[n+1]_{p,q} u^{\alpha+n+1} [n]_{p,q}!}{p^{n+1} p^{\binom{n+1}{2}}} \\
 &= \frac{u^{\alpha+n+2} [n+1]_{p,q}!}{p^{\binom{n+2}{2}}}.
 \end{aligned}$$

The proofs of (ii) and (iv) use (8) and (29); then we follow a similar process for (i) and (iii), respectively. Therefore, the proof is completed. \square

Remark 2. If $u = 1/s$, $p = 1$ and $\alpha = 0$, then Theorem 5 (i) and (iii) reduce to

$$L_q(t; s) = \frac{1}{s^2} \text{ and } L_q(t^n; s) = \frac{[n]_q!}{s^{n+1}},$$

respectively, which appeared in [28]. If $u = s$ and $\alpha = -1$, then Theorem 5 (i) and (iii) reduce to

$$S_{p,q}(t; s) = \frac{s}{p} \text{ and } S_{p,q}(t^n; s) = \frac{s^n [n]_{p,q}!}{p^{\binom{n+1}{2}}},$$

respectively, which appeared in [49].

Theorem 6. If $a \in \mathbb{R} \setminus \{0\}$, then the following identities hold:

$$\begin{aligned}
 (i) \quad & {}^1_{\alpha}G_{p,q}(e_{p,q}(at); u) = \frac{u^{\alpha+1}p}{p-au}, \quad u < \left|\frac{p}{a}\right|; \\
 (ii) \quad & {}^2_{\alpha}G_{p,q}(e_{p,q}(at); u) = u^{\alpha+1} \sum_{n=0}^{\infty} \frac{p^{\binom{n}{2}}(au)^n}{q^{\binom{n+1}{2}}}; \\
 (iii) \quad & {}^1_{\alpha}G_{p,q}(E_{p,q}(at); u) = u^{\alpha+1} \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}(au)^n}{p^{\binom{n+1}{2}}}; \\
 (iv) \quad & {}^2_{\alpha}G_{p,q}(E_{p,q}(at); u) = \frac{u^{\alpha+1}q}{q-au}, \quad u < \left|\frac{q}{a}\right|.
 \end{aligned}$$

Proof. Using (8), (9), (10) and (28) to prove (i) and (iii), we have

$$\begin{aligned}
 {}^1_{\alpha}G_{p,q}(e_{p,q}(at); u) &= u^{\alpha} \int_0^{\infty} E_{p,q}\left(-\frac{qt}{u}\right) e_{p,q}(at) d_{p,q}t \\
 &= u^{\alpha} \int_0^{\infty} E_{p,q}\left(-\frac{qt}{u}\right) \sum_{n=0}^{\infty} \frac{p^{\binom{n}{2}}(at)^n}{[n]_{p,q}!} d_{p,q}t \\
 &= \sum_{n=0}^{\infty} \frac{p^{\binom{n}{2}}a^n u^{\alpha}}{[n]_{p,q}!} \int_0^{\infty} E_{p,q}\left(-\frac{qt}{u}\right) t^n d_{p,q}t \\
 &= \sum_{n=0}^{\infty} \frac{p^{\binom{n}{2}}a^n}{[n]_{p,q}!} \frac{u^{\alpha+n+1}[n]_{p,q}!}{p^{\binom{n+1}{2}}} \\
 &= u^{\alpha+1} \sum_{n=0}^{\infty} \frac{p^{\binom{n}{2}}(au)^n}{p^{\binom{n+1}{2}}} \\
 &= u^{\alpha+1} \sum_{n=0}^{\infty} \left(\frac{au}{p}\right)^n \\
 &= \frac{u^{\alpha+1}p}{p-au},
 \end{aligned}$$

and

$$\begin{aligned}
 {}^1_{\alpha}G_{p,q}(E_{p,q}(at); u) &= u^{\alpha} \int_0^{\infty} E_{p,q}\left(-\frac{qt}{u}\right) E_{p,q}(at) d_{p,q}t \\
 &= u^{\alpha} \int_0^{\infty} E_{p,q}\left(-\frac{qt}{u}\right) \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}(at)^n}{[n]_{p,q}!} d_{p,q}t \\
 &= \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}a^n u^{\alpha}}{[n]_{p,q}!} \int_0^{\infty} E_{p,q}\left(-\frac{qt}{u}\right) t^n d_{p,q}t \\
 &= \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}a^n}{[n]_{p,q}!} \frac{u^{\alpha+n+1}[n]_{p,q}!}{p^{\binom{n+1}{2}}} \\
 &= u^{\alpha+1} \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}(au)^n}{p^{\binom{n+1}{2}}}.
 \end{aligned}$$

The proofs of (ii) and (iv) adopt (8), (9), (10), and (29), and then follow the similar process for (i) and (iii), respectively. Therefore, the proof is completed. \square

Remark 3. If $u = 1/s$, $p = 1$ and $\alpha = 0$, then Theorem 6 (i) and (iii) reduce to

$$L_q(e_q(at); s) = \frac{1}{s-a},$$

and

$$L_q(E_q(at); s) = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}a^n}{s^{n+1}},$$

respectively, which appeared in [28]. Furthermore, if $u = s$ and $\alpha = -1$, then Theorem 6 (i) and (iii) reduce to

$$S_{p,q}(e_{p,q}(at); s) = \frac{p}{p-as},$$

and

$$S_{p,q}(E_{p,q}(at); s) = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}(au)^n}{p^{\binom{n+1}{2}}},$$

respectively, which appeared in [49].

Theorem 7. If $a \in \mathbb{R} \setminus \{0\}$, then the following identities hold:

- (i) ${}_1G_{p,q}(\cos_{p,q}(at); u) = \frac{u^{\alpha+1}p^2}{p^2 + a^2u^2}, \quad u < \left|\frac{p}{a}\right|;$
- (ii) ${}_2G_{p,q}(\cos_{p,q}(at); u) = u^{\alpha+1} \sum_{n=0}^{\infty} \frac{(-1)^n p^{\binom{2n}{2}} (au)^{2n}}{q^{\binom{2n+1}{2}}};$
- (iii) ${}_1G_{p,q}(\text{Cos}_{p,q}(at); u) = u^{\alpha+1} \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{2n}{2}} (au)^{2n}}{p^{\binom{n+1}{2}}};$
- (iv) ${}_2G_{p,q}(\text{Cos}_{p,q}(at); u) = \frac{u^{\alpha+1}q^2}{q^2 + a^2u^2}, \quad u < \left|\frac{q}{a}\right|.$

Proof. Using (8), (14), (15) and (28) to prove (i) and (iii), we have

$$\begin{aligned} {}_1G_{p,q}(\cos(at); u) &= u^{\alpha} \int_0^{\infty} E_{p,q}\left(-\frac{qt}{u}\right) \cos_{p,q}(at) d_{p,q}t \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n p^{\binom{2n}{2}} a^{2n} u^{\alpha}}{[2n]_{p,q}!} \int_0^{\infty} E_{p,q}\left(-\frac{qt}{u}\right) t^{2n} d_{p,q}t \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n p^{\binom{2n}{2}} a^{2n}}{[2n]_{p,q}!} \frac{u^{\alpha+2n+1} [2n]_{p,q}!}{p^{\binom{2n+1}{2}}} \\ &= u^{\alpha+1} \sum_{n=0}^{\infty} (-1)^n \left(\frac{a^2 u^2}{p^2}\right)^n \\ &= \frac{u^{\alpha+1} p^2}{p^2 + a^2 u^2}, \end{aligned}$$

and

$$\begin{aligned} {}_1G_{p,q}(\text{Cos}(at); u) &= u^{\alpha} \int_0^{\infty} E_{p,q}\left(-\frac{qt}{u}\right) \text{Cos}_{p,q}(at) d_{p,q}t \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{2n}{2}} a^{2n} u^{\alpha}}{[2n]_{p,q}!} \int_0^{\infty} E_{p,q}\left(-\frac{qt}{u}\right) t^{2n} d_{p,q}t \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{2n}{2}} a^{2n}}{[2n]_{p,q}!} \frac{u^{\alpha+2n+1} [2n]_{p,q}!}{p^{\binom{n+1}{2}}} \\ &= u^{\alpha+1} \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{2n}{2}} (au)^{2n}}{p^{\binom{n+1}{2}}}. \end{aligned}$$

The proofs of (ii) and (iv) use (8), (14), (15) and (29), then follow a similar process for (i) and (iii), respectively. Hence, the proof is completed. \square

Remark 4. If $u = 1/s$, $p = 1$ and $\alpha = 0$, then Theorem 7 (i) reduces to

$$L_q(\cos_q(at); s) = \frac{s}{s^2 + a^2},$$

which appeared in [28]. Furthermore, if $u = s$ and $\alpha = -1$, then Theorem 7 (i) reduces to

$$S_{p,q}(\cos_{p,q}(at); s) = \frac{p^2}{p^2 + a^2 s^2},$$

which appeared in [49].

Theorem 8. If $a \in \mathbb{R} \setminus \{0\}$, then the following identities hold:

$$\begin{aligned}
(i) \quad {}^1_\alpha G_{p,q}(\sin_{p,q}(at); u) &= \frac{pau^{\alpha+2}}{p^2 + a^2u^2}, \quad u < \left| \frac{p}{a} \right|; \\
(ii) \quad {}^2_\alpha G_{p,q}(\sin_{p,q}(at); u) &= au^{\alpha+2} \sum_{n=0}^{\infty} \frac{(-1)^n p^{\binom{2n+1}{2}} (au)^{2n}}{q^{\binom{2n+2}{2}}}; \\
(iii) \quad {}^1_\alpha G_{p,q}(\text{Sin}_{p,q}(at); u) &= au^{\alpha+2} \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{2n+1}{2}} (au)^{2n}}{p^{\binom{2n+2}{2}}}; \\
(iv) \quad {}^2_\alpha G_{p,q}(\text{Sin}_{p,q}(at); u) &= \frac{qau^{\alpha+2}}{q^2 + a^2u^2}, \quad u < \left| \frac{q}{a} \right|.
\end{aligned}$$

Proof. Using (8), (16), (17) and (28) to prove (i) and (iii), we obtain

$$\begin{aligned}
{}^1_\alpha G_{p,q}(\sin(at); u) &= u^\alpha \int_0^\infty E_{p,q} \left(-\frac{qt}{u} \right) \sin_{p,q}(at) d_{p,q}t \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n p^{\binom{2n+1}{2}} a^{2n+1} u^\alpha}{[2n+1]_{p,q}!} \int_0^\infty E_{p,q} \left(-\frac{qt}{u} \right) t^{2n+1} d_{p,q}t \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n p^{\binom{2n+1}{2}} a^{2n+1}}{[2n+1]_{p,q}!} \frac{u^{\alpha+2n+2} [2n+1]_{p,q}!}{p^{\binom{2n+2}{2}}} \\
&= \frac{u^{\alpha+2} a}{p} \sum_{n=0}^{\infty} (-1)^n \left(\frac{a^2 u^2}{p^2} \right)^n \\
&= \frac{pau^{\alpha+2}}{p^2 + a^2u^2},
\end{aligned}$$

and

$$\begin{aligned}
{}^1_\alpha G_{p,q}(\text{Sin}(at); u) &= u^\alpha \int_0^\infty E_{p,q} \left(-\frac{qt}{u} \right) \text{Sin}_{p,q}(at) d_{p,q}t \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{2n+1}{2}} a^{2n+1} u^\alpha}{[2n+1]_{p,q}!} \int_0^\infty E_{p,q} \left(-\frac{qt}{u} \right) t^{2n+1} d_{p,q}t \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{2n+1}{2}} a^{2n+1}}{[2n+1]_{p,q}!} \frac{u^{\alpha+2n+2} [2n+1]_{p,q}!}{p^{\binom{2n+2}{2}}} \\
&= au^{\alpha+2} \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{2n+1}{2}} (au)^{2n}}{p^{\binom{2n+2}{2}}}.
\end{aligned}$$

The proofs of (ii) and (iv) adopt (8), (16), (17) and (29), and then follow the similar process for (i) and (iii), respectively. The proof of the theorem 8 is finished. \square

Remark 5. If set $u = 1/s$, $p = 1$ and $\alpha = 0$, then Theorem 8 (i) reduces to

$$L_q(\sin_q(at); s) = \frac{a}{s^2 + a^2},$$

which appeared in [28]. Furthermore, if $u = s$ and $\alpha = -1$, then Theorem 8 (i) reduces to

$$S_{p,q}(\sin_{p,q}(at); s) = \frac{aps}{p^2 + a^2s^2},$$

which appeared in [49].

Corollary 1. If $a \in \mathbb{R} \setminus \{0\}$, then we have

$$(i) \quad {}^1_\alpha G_{p,q}(\cosh_{p,q}(at); u) = \frac{u^{\alpha+1} p^2}{p^2 - a^2u^2}, \quad u < \left| \frac{p}{a} \right|;$$

$$(ii) \quad {}^1_{\alpha}G_{p,q}(\sinh_{p,q}(at); u) = \frac{pau^{\alpha+2}}{p^2 - a^2u^2}, \quad u < \left| \frac{p}{a} \right|.$$

The proofs of Corollary 1 follow (18) and (20); therefore, the details of Theorems 7 and 8 are omitted.

Theorem 9. (Transforms of integrals): Let $f \in A$ and $\tilde{f} \in B$, then the following identities hold:

$$\begin{aligned} (i) \quad & {}^1_{\alpha}G_{p,q}\left(\int_0^t f(x)d_{p,q}x; pu\right) = (p^{\alpha+1}u)_{\alpha}^1G_{p,q}(f(t); u); \\ (ii) \quad & {}^1_{\alpha}G_{p,q}\left(\int_0^t \int_0^x f(\omega)d_{p,q}\omega d_{p,q}x; pu\right) = (p^{\alpha+2}u^2)_{\alpha}^1G_{p,q}(f(t); u); \\ (iii) \quad & {}^1_{\alpha}G_{p,q}\left(\underbrace{\int_0^t \left(\int_0^{x_n} \cdots \left(\int_0^{x_2} f(x_1)d_{p,q}x_1\right) \cdots d_{p,q}x_{n-1}\right) d_{p,q}x_n}_{n \text{ times}}; pu\right) = (p^{\alpha+n}u^n)_{\alpha}^1G_{p,q}(f(t); u); \\ (iv) \quad & {}^2_{\alpha}G_{p,q}\left(\int_0^t \tilde{f}(x)d_{p,q}x; pu\right) = (q^{\alpha}p^{\alpha}u^{\alpha})_{\alpha}^2G_{p,q}\left(\tilde{f}(t); \frac{pu}{q}\right); \\ (v) \quad & {}^2_{\alpha}G_{p,q}\left(\int_0^t \int_0^x \tilde{f}(\omega)d_{p,q}\omega d_{p,q}x; pu\right) = (q^{\alpha}p^{2\alpha}u^{2\alpha})_{\alpha}^2G_{p,q}\left(\tilde{f}(t); \frac{pu}{q^2}\right); \\ (vi) \quad & {}^2_{\alpha}G_{p,q}\left(\underbrace{\int_0^t \left(\int_0^{x_n} \cdots \left(\int_0^{x_2} \tilde{f}(x_1)d_{p,q}x_1\right) \cdots d_{p,q}x_{n-1}\right) d_{p,q}x_n}_{n \text{ times}}; pu\right) \\ & = (q^{\alpha}p^{n\alpha}u^{n\alpha})_{\alpha}^2G_{p,q}\left(\tilde{f}(t); \frac{pu}{q^n}\right). \end{aligned}$$

Proof. Using (8) and (28) to prove (i) – (iii), we have

$${}^1_{\alpha}G_{p,q}\left(\int_0^t f(x)d_{p,q}x; u\right) = u^{\alpha} \int_0^{\infty} E_{p,q}\left(-\frac{qt}{u}\right) \int_0^t f(x)d_{p,q}x d_{p,q}t.$$

We give $g(t) = E_{p,q}\left(-\frac{t}{u}\right)$, $h(t) = \int_0^t f(x)d_{p,q}x$ and apply the formula of (p, q) -integration by parts, we obtain

$$\int_0^{\infty} h(pt)D_{p,q}E_{p,q}\left(-\frac{t}{u}\right)d_{p,q}t = \lim_{a \rightarrow \infty} \left[h(t)E_{p,q}\left(-\frac{t}{u}\right) \right]_0^a - \int_0^{\infty} E_{p,q}\left(-\frac{qt}{u}\right)D_{p,q}h(t)d_{p,q}t.$$

Next, we get

$$-\frac{1}{u} \int_0^{\infty} h(pt)E_{p,q}\left(-\frac{t}{u}\right)d_{p,q}t = - \int_0^{\infty} E_{p,q}\left(-\frac{qt}{u}\right)f(t)d_{p,q}t.$$

Consequently,

$${}^1_{\alpha}G_{p,q}\left(\int_0^t f(x)d_{p,q}x; pu\right) = (p^{\alpha+1}u)_{\alpha}G_{p,q}(f(t); u).$$

Let $h(x) = \int_0^x f(\omega) d_{p,q}\omega$; then we get

$$\begin{aligned} {}^1_{\alpha}G_{p,q}\left(\int_0^t \int_0^x f(\omega) d_{p,q}\omega d_{p,q}x; pu\right) &= {}^1_{\alpha}G_{p,q}\left(\int_0^t h(x) d_{p,q}x; pu\right) \\ &= (p^{\alpha+1}u) {}^1_{\alpha}G_{p,q}(h(t); u) \\ &= (p^{\alpha+1}u) {}^1_{\alpha}G_{p,q}\left(\int_0^t f(\omega) d_{p,q}\omega; u\right) \\ &= (pu) {}^1_{\alpha}G_{p,q}\left(\int_0^t f(\omega) d_{p,q}\omega; pu\right) \\ &= (p^{\alpha+2}u^2) {}^1_{\alpha}G_{p,q}(f(t); u). \end{aligned}$$

After continuing this process, we obtain the sequence

$${}^1_{\alpha}G_{p,q}\left(\underbrace{\int_0^t \left(\int_0^{x_n} \cdots \left(\int_0^{x_2} f(x_1) d_{p,q}x_1\right) \cdots d_{p,q}x_{n-1}\right) d_{p,q}x_n}_{n \text{ times}}; pu\right) = (p^{\alpha+n}u^n) {}^1_{\alpha}G_{p,q}(f(t); u).$$

The proofs of (iv) – (vi) utilize (8) and (29), and then follow the similar process for (i) – (iii). The proof is completed. \square

Remark 6. If $p = 1$, then Theorem 9 (iii) reduces to

$${}^1_{\alpha}G_q\left(\underbrace{\int_0^t \left(\int_0^{x_n} \cdots \left(\int_0^{x_2} f(x_1) d_{p,q}x_1\right) \cdots d_{p,q}x_{n-1}\right) d_{p,q}x_n}_{n \text{ times}}; pu\right) = (u^n) {}^1_{\alpha}G_q(f(t); u). \quad (36)$$

Furthermore, if $q \rightarrow 1$, then (36) reduces to the ${}_{\alpha}G$ -transform of integrals, which appeared in [12].

Theorem 10. (Transforms of derivatives): If $f \in A$ and $D_{p,q}^n$ has the ${}^1_{\alpha}G(p, q)$ -transform of type one for each $n \in \mathbb{N}$, then the transforms of the first, second and n -th derivatives of f can be written in the following forms:

$$\begin{aligned} (i) \quad & {}^1_{\alpha}G_{p,q}(D_{p,q}f(t); u) = -u^{\alpha}f(0) + \frac{{}^1_{\alpha}G_{p,q}(f(t); up)}{up^{\alpha+1}}; \\ (ii) \quad & {}^1_{\alpha}G_{p,q}(D_{p,q}^{(2)}f(t); u) = -u^{\alpha}f'(0) - \frac{u^{\alpha-1}f(0)}{p} + \frac{{}^1_{\alpha}G_{p,q}(f(t); up^2)}{u^2p^{2\alpha+3}}; \\ (iii) \quad & \end{aligned}$$

$${}^1_{\alpha}G_{p,q}(D_{p,q}^{(n)}f(t); u) = \begin{cases} -u^{\alpha}f(0) + \frac{{}^1_{\alpha}G_{p,q}(f(t); up)}{up^{\alpha+1}}, & \text{for } n = 1 \\ \frac{{}^1_{\alpha}G_{p,q}(f(t); up^n)}{u^n p^{n\alpha} p^{\binom{n+1}{2}}} - \sum_{k=0}^{n-3} \frac{u^{\alpha-n+k+1} (D_{p,q}^k f)(0)}{p^{\frac{(n-k-2)(n-k+1)}{2} - \alpha}} \\ - \sum_{k=n-2}^{n-1} \frac{u^{\alpha-n+k+1} (D_{p,q}^k f)(0)}{p^{n-k-1}} & \text{for } n = 2, 3, \dots \end{cases}.$$

Proof. Using (8) and (28) to prove (i), we have

$$\begin{aligned}
 {}^1_0 G_{p,q}(D_{p,q}f(t); u) &= u^\alpha \int_0^\infty E_{p,q}\left(-\frac{qt}{u}\right) D_{p,q}f(t) d_{p,q}t \\
 &= u^\alpha \left[\lim_{a \rightarrow \infty} \left[f(t) E_{p,q}\left(-\frac{t}{u}\right) \right]_0^a - \int_0^\infty f(pt) D_{p,q}E_{p,q}\left(-\frac{t}{u}\right) d_{p,q}t \right] \\
 &= u^\alpha \left[-f(0) + \frac{1}{u} \int_0^\infty f(pt) E_{p,q}\left(-\frac{qt}{u}\right) d_{p,q}t \right] \\
 &= u^\alpha \left[-f(0) + \frac{1}{up} \int_0^\infty f(t) E_{p,q}\left(-\frac{qt}{up}\right) d_{p,q}t \right] \\
 &= -u^\alpha f(0) + \frac{(up)^\alpha}{up^{\alpha+1}} \int_0^\infty f(t) E_{p,q}\left(-\frac{qt}{up}\right) d_{p,q}t \\
 &= -u^\alpha f(0) + \frac{{}^1_0 G_{p,q}(f(t); up)}{up^{\alpha+1}}.
 \end{aligned}$$

Applying the equation above with $n = 2$ to prove (ii), we get

$$\begin{aligned}
 {}^1_0 G_{p,q}(D_{p,q}^{(2)}f(t); u) &= u^\alpha \int_0^\infty E_{p,q}\left(-\frac{qt}{u}\right) D_{p,q}^{(2)}f(t) d_{p,q}t \\
 &= -u^\alpha f'(0) + \frac{{}^1_0 G_{p,q}(f'(t); up)}{up^{\alpha+1}} \\
 &= -u^\alpha f'(0) + \frac{1}{up^{\alpha+1}} \left[-(up)^\alpha f(0) + \frac{{}^1_0 G_{p,q}(f(t); up^2)}{up^{\alpha+2}} \right] \\
 &= -u^\alpha f'(0) - \frac{u^{\alpha-1}f(0)}{p} + \frac{{}^1_0 G_{p,q}(f(t); up^2)}{u^2 p^{2\alpha+3}}.
 \end{aligned}$$

In (iii), if $n = 1$, it is not difficult to see that

$${}^1_0 G_{p,q}(D_{p,q}^{(n)}f(t); u) = \frac{{}^1_0 G_{p,q}(f(t); up^n)}{u^n p^{n\alpha} p^{\binom{n+1}{2}}} - \sum_{k=0}^{n-1} \frac{u^{\alpha-n+k+1} (D_{p,q}^k f)(0)}{p^{n-k-1}}.$$

If $n > 1$, we apply the results of $n = 1$ by changing p^{n-k-1} to $p^{\frac{(n-k-2)(n-k+1)}{2} - \alpha}$ and $n = 2$. We can write

$$\begin{aligned}
 {}^1_0 G_{p,q}(D_{p,q}^{(n)}f(t); u) &= -u^\alpha (D_{p,q}^{(n-1)}f)(0) - \frac{u^{\alpha-1} (D_{p,q}^{(n-2)}f)(0)}{p} + \frac{{}^1_0 G_{p,q}(D_{p,q}^{(n-2)}f(t); up^2)}{u^2 p^{2\alpha+3}} \\
 &= -u^\alpha (D_{p,q}^{(n-1)}f)(0) - \frac{u^{\alpha-1} (D_{p,q}^{(n-2)}f)(0)}{p} + \frac{1}{u^2 p^{2\alpha+3}} \left[\frac{{}^1_0 G_{p,q}(f(t); up^n)}{u^{n-2} p^{2n-4+n\alpha-2\alpha} p^{\binom{n-1}{2}}} - \sum_{k=0}^{n-3} \frac{(up^2)^{\alpha-n+k+3} (D_{p,q}^{(k)}f)(0)}{p^{\frac{(n-k-4)(n-k-1)}{2} - \alpha}} \right] \\
 &= -u^\alpha (D_{p,q}^{(n-1)}f)(0) - \frac{u^{\alpha-1} (D_{p,q}^{(n-2)}f)(0)}{p} + \frac{{}^1_0 G_{p,q}(f(t); up^n)}{u^n p^{n\alpha} p^{\binom{n+1}{2}}} - \frac{1}{u^2 p^{2\alpha+3}} \sum_{k=0}^{n-3} \frac{(up^2)^{\alpha-n+k+3} (D_{p,q}^{(k)}f)(0)}{p^{\frac{(n-k-4)(n-k-1)}{2} - \alpha}} \\
 &= -u^\alpha (D_{p,q}^{(n-1)}f)(0) - \frac{u^{\alpha-1} (D_{p,q}^{(n-2)}f)(0)}{p} + \frac{{}^1_0 G_{p,q}(f(t); up^n)}{u^n p^{n\alpha} p^{\binom{n+1}{2}}} - \sum_{k=0}^{n-3} \frac{u^{\alpha-n+k+1} (D_{p,q}^k f)(0)}{p^{\frac{(n-k-2)(n-k+1)}{2} - \alpha}} \\
 &= \frac{{}^1_0 G_{p,q}(f(t); up^n)}{u^n p^{n\alpha} p^{\binom{n+1}{2}}} - \sum_{k=0}^{n-3} \frac{u^{\alpha-n+k+1} (D_{p,q}^k f)(0)}{p^{\frac{(n-k-2)(n-k+1)}{2} - \alpha}} - \sum_{k=n-2}^{n-1} \frac{u^{\alpha-n+k+1} (D_{p,q}^k f)(0)}{p^{n-k-1}}.
 \end{aligned}$$

Therefore, the proof is completed. \square

The proof of ${}_a G_{p,q}$ -transform of the type two in (29) is similar to the one for Theorem 10, and is therefore omitted.

Remark 7. If $u = 1/s$, $p = 1$ and $\alpha = 0$, then Theorem 10 (iii) reduces to

$$L_q(D_q^{(n)} f(t); s) = s^n L_q(f(t); s) - \sum_{k=0}^{n-1} s^{n-k-1} (D_q^k f)(0),$$

which appeared in [28].

Theorem 11. (Derivative of transforms): For $n \in \mathbb{N}$, the following formulas hold:

$$\begin{aligned} (i) \quad & {}_a^1 G_{p,q}(tf(t); u) = -\frac{u^\alpha}{q^\alpha} \frac{D_{p,q}}{D_{p,q}\left(\frac{1}{u}\right)} \left(\frac{{}_a^1 G_{p,q}(f(t); qu)}{u^\alpha} \right); \\ (ii) \quad & {}_a^1 G_{p,q}(t^2 f(t); u) = \frac{u^\alpha}{q^{2\alpha-1}} \frac{D_{p,q}}{D_{p,q}\left(\frac{1}{u}\right)} \left(\frac{D_{p,q}}{D_{p,q}\left(\frac{1}{u}\right)} \left(\frac{{}_a^1 G_{p,q}(f(t); q^2 u)}{u^\alpha} \right) \right); \\ (iii) \quad & {}_a^1 G_{p,q}(t^n f(t); u) = \frac{(-1)^n u^\alpha}{q^{n\alpha - \binom{n}{2}}} \left(\frac{D_{p,q}}{D_{p,q}\left(\frac{1}{u}\right)} \right)^n \left(\frac{{}_a^1 G_{p,q}(f(t); q^n u)}{u^\alpha} \right). \end{aligned}$$

Proof. Using (28) to prove (i) – (iii), we have

$${}_a^1 G_{p,q}(f(t); qu) = q^\alpha u^\alpha \int_0^\infty f(t) E_{p,q} \left(-\frac{t}{u} \right) d_{p,q} t. \quad (37)$$

Taking (p, q) -derivative on both sides with respect to $1/u$, we get

$${}_a^1 G_{p,q}(tf(t); u) = -\frac{u^\alpha}{q^\alpha} \frac{D_{p,q}}{D_{p,q}\left(\frac{1}{u}\right)} \left(\frac{{}_a^1 G_{p,q}(f(t); qu)}{u^\alpha} \right).$$

From (37), taking the second (p, q) -derivative on both sides with respect to $1/u$ to prove (ii), we have

$${}_a^1 G_{p,q}(t^2 f(t); u) = \frac{u^\alpha}{q^{2\alpha-1}} \frac{D_{p,q}}{D_{p,q}\left(\frac{1}{u}\right)} \left(\frac{D_{p,q}}{D_{p,q}\left(\frac{1}{u}\right)} \left(\frac{{}_a^1 G_{p,q}(f(t); q^2 u)}{u^\alpha} \right) \right).$$

Following the same process, we can prove (iii). Therefore, the proof is completed. \square

The proof of ${}_a G_{p,q}$ -transform of the type two in (29) is similar to the one for Theorem 11 and therefore is omitted.

Remark 8. If $u = 1/s$, $p = 1$ and $\alpha = 0$, then Theorem 11 (iii) reduces to

$$L_q(t^n f(t); s) = \frac{(-1)^n}{q^{-\binom{n}{2}}} \left(\frac{D_{p,q}}{D_{p,q} s} \right)^n L_q(f(t); \frac{s}{q^n}). \quad (38)$$

Furthermore, if $q \rightarrow 1$, then (38) reduces to the derivative of Laplace transform, which appeared in [28].

Corollary 2. If $n \in \mathbb{N}$ and $f \in A$, then

$${}_a^1 G_{p,q}(t^n e_{p,q}(at); u) = \frac{u^\alpha q^{-\binom{n+1}{2}} [n]_{p,q}!}{(1/u - a/p)(pq^{-1}/u - a/p) \cdots (p^n q^{-n}/u - a/p)}, \quad (39)$$

$${}_a^1 G_{p,q}(e_{p,q}(at)f(t); u) = \sum_{n=0}^{\infty} \frac{(-a)^n}{[n]_{p,q}!} u^\alpha q^{\binom{n}{2} - na} \left(\frac{D_{p,q}}{D_{p,q}\left(\frac{1}{u}\right)} \right)^n \left(\frac{{}_a^1 G_{p,q}(f(t); q^n u)}{u^\alpha} \right). \quad (40)$$

Proof. Using Theorem 11 (iii) and (13) to prove (39), we have

$$\begin{aligned} {}^1_{\alpha}G_{p,q}(t^n e_{p,q}(at); u) &= (-1)^n u^{\alpha} q^{-n\alpha + \binom{n}{2}} \left(\frac{D_{p,q}}{D_{p,q}\left(\frac{1}{u}\right)} \right)^n \left(\frac{{}^1_{\alpha}G_{p,q}(f(t); q^n u)}{u^{\alpha}} \right) \\ &= (-1)^n u^{\alpha} q^{-n\alpha + \binom{n}{2}} \left(\frac{D_{p,q}}{D_{p,q}\left(\frac{1}{u}\right)} \right)^n \left(\frac{1}{1/uq^n - a/p} \right) \\ &= \frac{u^{\alpha} q^{-\binom{n+1}{2}} [n]_{p,q}!}{(1/u - a/p)(pq^{-1}/u - a/p) \cdots (p^n q^{-n}/u - a/p)}. \end{aligned}$$

Using (28) and Theorem 11, we can prove (40). The proof is completed. \square

The proof of ${}_{\alpha}G_{p,q}$ -transform of the type two in (29) is similar to the one for Corollary 2, but changes $e_{p,q}(at)$ to $E_{p,q}(at)$, and therefore is omitted.

Remark 9. If $u = 1/s$, $p = 1$ and $\alpha = 0$, then Theorem 11 (i) and (ii) reduce to

$$L_q(t^n e_q(at); s) = \frac{q^{-(\frac{n+1}{2})} [n]_q!}{(s-a)(q^{-1}s-a) \cdots (q^{-n}s-a)},$$

and

$$L_q(e_q(at)f(t); s) = \sum_{k=0}^{\infty} \frac{(-a)^k}{[k]_q!} q^{\binom{k}{2}} \left(\frac{D_{p,q}}{D_{p,q}s} \right)^k L_q(f(t); q^{-k}s),$$

respectively, which appeared in [28].

Theorem 12. (Transforms of the Heaviside function): For $a \geq 0$, let

$$H(t-a) = \begin{cases} 1, & \text{for } t \geq a; \\ 0, & \text{for } 0 \leq t \leq a. \end{cases}$$

Then, we have

$${}^1_{\alpha}G_{p,q}(H(t-a); u) = u^{\alpha+1} E_{p,q}\left(-\frac{a}{u}\right), \quad (41)$$

$${}^2_{\alpha}G_{p,q}(H(t-a); u) = u^{\alpha+1} e_{p,q}\left(-\frac{a}{u}\right). \quad (42)$$

Proof. Using (28) to prove (41), we obtain

$$\begin{aligned} {}^1_{\alpha}G_{p,q}(H(t-a); u) &= u^{\alpha} \int_0^{\infty} E_{p,q}\left(-\frac{qt}{u}\right) H(t-a) d_{p,q}t \\ &= u^{\alpha} \int_a^{\infty} E_{p,q}\left(-\frac{qt}{u}\right) d_{p,q}t \\ &= -u^{\alpha+1} \int_0^{\infty} D_{p,q} E_{p,q}\left(-\frac{t}{u}\right) d_{p,q}t \\ &= -u^{\alpha+1} \lim_{a \rightarrow \infty} \left[E_{p,q}\left(-\frac{t}{u}\right) \right]_0^a \\ &= u^{\alpha+1} E_{p,q}\left(-\frac{a}{u}\right). \end{aligned}$$

The proof of (42) uses (29), and then follows the similar process in (41). Therefore, the proof is completed. \square

Remark 10. If $p = 1$, then (41) reduces to

$${}_1G_q(H(t-a); u) = u^{\alpha+1} E_q\left(-\frac{a}{u}\right). \quad (43)$$

Furthermore, if $q \rightarrow 1$, then (43) reduces to the ${}_aG$ -transform of Heaviside function, which appeared in [1].

Theorem 13. (Transforms of the Dirac delta function): For $a \geq 0$, let

$$f_k(t-a) = \begin{cases} \frac{1}{k}, & \text{for } a \leq t \leq a+k; \\ 0, & \text{otherwise.} \end{cases}$$

If $\delta(t-a)$ denotes the limit of f_k as $k \rightarrow 0$, then we have

$${}_1G_{p,q}(\delta(t-a); u) = u^\alpha E_{p,q}\left(-\frac{aq}{u}\right), \quad (44)$$

$${}_2G_{p,q}(\delta(t-a); u) = u^\alpha e_{p,q}\left(-\frac{ap}{u}\right), \quad (45)$$

where δ is the Dirac delta function.

Proof. Using (28) to prove (44), we obtain

$$\begin{aligned} {}_1G_{p,q}(f_k(t-a); u) &= u^\alpha \int_0^\infty E_{p,q}\left(-\frac{qt}{u}\right) f_k(t-a) d_{p,q}t \\ &= u^\alpha \int_a^{a+k} E_{p,q}\left(-\frac{qt}{u}\right) d_{p,q}t \\ &= -\frac{u^{\alpha+1}}{k} \int_a^{a+k} D_{p,q} E_{p,q}\left(-\frac{t}{u}\right) d_{p,q}t \\ &= -\frac{u^{\alpha+1}}{k} \left[E_{p,q}\left(-\frac{(a+k)}{u}\right) - E_{p,q}\left(-\frac{a}{u}\right) \right]. \end{aligned}$$

If we take the limit of f_k as $k \rightarrow 0$, then

$${}_1G_{p,q}(\delta(t-a); u) = \lim_{k \rightarrow 0} {}_1G_{p,q}(f_k(t-a); u) = u^\alpha E_{p,q}\left(-\frac{aq}{u}\right).$$

The proof of (45) uses (29), and then follows the similar process for (44). Therefore, the proof is completed. \square

Remark 11. If $p = 1$, then (44) reduces to

$${}_1G_q(\delta(t-a); u) = u^{\alpha+1} E_q\left(-\frac{aq}{u}\right). \quad (46)$$

Furthermore, if $q \rightarrow 1$, then (46) reduces to the ${}_aG$ -transform of Dirac delta function, which appeared in [1].

Theorem 14. (Convolution theorem): If $f(t) = t^\gamma$ and $g(t) = t^{\beta-1}$ for $\gamma \geq 0, \beta \geq 1$, then we have

$${}_1G_{p,q}((f * g)_{p,q}; u) = \frac{p^{(\beta^2-3\beta-2\gamma)/2}}{u^\alpha} {}_1G_{p,q}(t^\gamma; u) {}_1G_{p,q}(t^{\beta-1}; u), \quad (47)$$

where

$$(f * g)_{p,q}(t) = \int_0^t f(\eta) g(t-q\eta) d_{p,q}\eta. \quad (48)$$

Proof. Using (48), we get

$$(f * g)_{p,q}(t) = \int_0^t \eta^\gamma (t - q\eta)^{\beta-1} d_{p,q}\eta.$$

We then change the variables in the equation above by $\eta = rt$ and use (24), which results in the following form:

$$\begin{aligned}(f * g)_{p,q}(t) &= t \int_0^1 r^\gamma t^\gamma (z - qrt)^{\beta-1} d_{p,q}r \\ &= t^{\gamma+\beta} \int_0^1 r^\gamma (1 - qr)^{\beta-1} d_{p,q}r \\ &= t^{\gamma+\beta} B_{p,q}(\gamma + 1, \beta).\end{aligned}$$

Thus, using (23) and (25) in the equation above, we get

$$\begin{aligned}\frac{u^{\alpha-1} {}_{\alpha}G_{p,q}((f * g)_{p,q}; u)}{p^{(\beta^2-3\beta-2\gamma)/2}} &= \frac{u^{2\alpha} B_{p,q}(\gamma + 1, \beta) \int_0^\infty E_{p,q}\left(-\frac{qt}{u}\right) t^{\gamma+\beta} d_{p,q}t}{p^{(\beta^2-3\beta-2\gamma)/2}} \\ &= \frac{p^{(\beta-1)(2\gamma+\beta)/2} \Gamma_{p,q}(\gamma + 1) \Gamma_{p,q}(\beta) u^{2\alpha+\gamma+\beta+1} [\gamma + \beta]_{p,q}!}{p^{(\beta^2-3\beta-2\gamma)/2} p^{(\gamma+\beta+1)} \Gamma_{p,q}(\gamma + \beta + 1)} \\ &= \frac{[\gamma]_{p,q}! [\beta - 1]_{p,q}! u^{2\alpha+\gamma+\beta+1}}{p^{(\gamma+1)} p^{(\beta)}} \\ &= {}_{\alpha}G_{p,q}(t^\gamma; u) {}_{\alpha}G_{p,q}(t^{\beta-1}; u).\end{aligned}$$

Hence, we obtain

$${}_{\alpha}G_{p,q}((f * g)_{p,q}; u) = \frac{p^{(\beta^2-3\beta-2\alpha)/2}}{u^\alpha} {}_{\alpha}G_{p,q}(f(t); u) {}_{\alpha}G_{p,q}(g(t); u).$$

Therefore, the proof is completed. \square

The proof of ${}_{\alpha}G_{p,q}$ -transform of the type two in (29) is similar to one for Theorem 14 and therefore is omitted.

Remark 12. If $u = 1/s$, $p = 1$ and $\alpha = 0$, then (47) reduces to

$$L_q((f * g)_q; s) = L_q(f(t); s) L_q(g(t); s),$$

which appeared in [28].

4. Examples

In this section, we solve the (p, q) -differential equations using the definition and properties of ${}_{\alpha}G_{p,q}$ -transform of type one. We consider the (p, q) -Cauchy problem and two second-order (p, q) -differential equations.

Example 3. The (p, q) -Cauchy problem is in the following form:

$$D_{p,q}f(t) + cf(pt) = 0, \quad f(0) = 1,$$

where c is a constant.

Applying ${}_{\alpha}G_{p,q}$ -transform of the type one, we get

$$-u^\alpha f(0) + \frac{{}_{\alpha}G_{p,q}(f(t); up)}{up^{\alpha+1}} + \frac{c {}_{\alpha}G_{p,q}(f(t); pu)}{p^{\alpha+1}} = 0.$$

Using the initial conditions $f(0) = 1$, we obtain

$$-u^\alpha + \left(\frac{1}{up^{\alpha+1}} + \frac{c}{p^{\alpha+1}} \right) {}^1_0G_{p,q}(f(t); up) = 0.$$

Hence

$${}_0^1G_{p,q}(f(t); up) = \frac{u^{\alpha+1}p^{\alpha+1}}{1+cu},$$

and so

$${}_0^1G_{p,q}(f(t); u) = \frac{u^{\alpha+1}p}{p+cu},$$

we obtain the solution

$$f(t) = e_{p,q}(-ct). \quad (49)$$

In addition, if $p = 1$ and $q \rightarrow 1$, then (49) reduces to

$$f(t) = \exp(-ct).$$

Example 4. The second order (p, q) -differential equation is in the following form:

$$D_{p,q}^{(2)}f(t) - f(p^2t) = t, f(0) = 1, f'(0) = 1.$$

Taking ${}_0^1G_{p,q}$ -transform of the type one on both sides, we have

$$-u^\alpha f'(0) - \frac{u^{\alpha-1}f(0)}{p} + \frac{{}_0^1G_{p,q}(f(t); up^2)}{u^2p^{2\alpha+3}} - \frac{{}_0^1G_{p,q}(f(t); up^2)}{p^{2\alpha+2}} = \frac{u^{\alpha+2}}{q}.$$

After simplifying the above equation, we get

$$\left(\frac{1}{u^2p^{2\alpha+3}} - \frac{1}{p^{2\alpha+2}} \right) {}_0^1G_{p,q}(f(t); up^2) = \frac{u^{\alpha+2}}{q} + u^\alpha + \frac{u^{\alpha-1}}{p}.$$

Hence

$${}_0^1G_{p,q}(f(t); up^2) = \frac{u^{\alpha+4}p^{4\alpha+5}}{qp^{2\alpha+2} - u^2qp^{2\alpha+3}} + \frac{u^{\alpha+2}p^{4\alpha+5}}{p^{2\alpha+2} + u^2p^{2\alpha+3}} + \frac{u^{\alpha+1}p^{4\alpha+5}}{p^{2\alpha+3} - u^2p^{2\alpha+4}},$$

and so

$${}_0^1G_{p,q}(f(t); u) = -\frac{1}{pq} \left(\frac{u^{\alpha+2}}{p} \right) + \left(1 + \frac{1}{pq} \right) \sqrt{p} \left(\frac{p^{\frac{u^{\alpha+2}}{\sqrt{p}}}}{p^2 - \left(\frac{u}{\sqrt{p}} \right)^2} \right) + \frac{u^{\alpha+1}p^2}{p^2 - \left(\frac{u}{\sqrt{p}} \right)^2}.$$

The solution is as follows:

$$f(t) = -\frac{t}{pq} + \left(1 + \frac{1}{pq} \right) \sqrt{p} \sinh_{p,q} \left(\frac{t}{\sqrt{p}} \right) + \cosh_{p,q} \left(\frac{t}{\sqrt{p}} \right). \quad (50)$$

In addition, if $p = 1$ and $q \rightarrow 1$, then (50) reduces to

$$f(t) = -t + 2 \sinh(t) + \cosh(t),$$

which appeared in [1].

Example 5. Find a solution of

$$D_{p,q}^{(2)}f(t) + f(p^2t) = 6 \sin_{p,q}(2t), f(0) = 3, f'(0) = 1.$$

Taking ${}_1G_{p,q}$ -transform with initial conditions, we have

$${}_1G_{p,q}(f(t); up^2) = \frac{12u^{\alpha+4}p^{4\alpha+6}}{(p^2 + 4u^2)(p^{2\alpha+2} + u^2p^{2\alpha+3})} + \frac{u^{\alpha+2}p^{2\alpha+5}}{p^2 + u^2p^3} + \frac{3u^{\alpha+1}p^{2\alpha+4}}{p^2 + u^2p^3}.$$

Hence,

$${}_1G_{p,q}(f(t); u) = \frac{3u^{\alpha+1}p^2}{p^2 + \left(\frac{u}{\sqrt{p}}\right)^2} + \sqrt{p} \left(\frac{12p}{4 - p^3} + 1 \right) \left(p \frac{\frac{u^{\alpha+2}}{\sqrt{p}}}{p^2 + \left(\frac{u}{\sqrt{p}}\right)^2} \right) - \left(\frac{6p^3}{4 - p^3} \right) \left(\frac{p \frac{2}{p^2} u^{\alpha+2}}{p^2 + \left(\frac{2u}{p^2}\right)^2} \right).$$

We have the solution:

$$f(t) = 3 \cos_{p,q} \left(\frac{t}{\sqrt{p}} \right) + \sqrt{p} \left(\frac{12p}{4 - p^3} + 1 \right) \sin_{p,q} \left(\frac{t}{\sqrt{p}} \right) - \left(\frac{6p^3}{4 - p^3} \right) \sin_{p,q} \left(\frac{2t}{p^2} \right). \quad (51)$$

In addition, if $p = 1$ and $q \rightarrow 1$, then (51) reduces to

$$f(t) = 3 \cos(t) + 5 \sin(t) - 2 \sin(t).$$

5. Conclusions

In this work, the properties of the ${}_aG_{p,q}$ -transform of type one and type two are introduced and proven. After that, we apply the properties of ${}_aG_{p,q}$ -transform of type one to some (p, q) -differential equations. The properties proposed and the results of the applications are compared with other papers.

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