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A New Representation of Semiopenness of L-fuzzy Sets in RL-fuzzy Bitopological Spaces

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Abstract: In this paper, we introduce a new representation of semiopenness of L-fuzzy sets in RL-fuzzy bitopological spaces based on the concept of pseudo-complement. The concepts of pairwise RL-fuzzy semicontinuous and pairwise RL-fuzzy irresolute functions are extended and discussed based on the (i,j)-RL-semiopen gradation. Further, pairwise RL-fuzzy semi-compactness of an L-fuzzy set in RL-fuzzy bitopological spaces are given and characterized. As RL-fuzzy bitopology is a generalization of L-bitopology, RL-bitopology, L-fuzzy bitopology, and RL-fuzzy topology, the results of our paper are more general.

Keywords: RL-fuzzy bitopology; (i,j)-RL-semiopen gradation; pairwise RL-fuzzy semicontinuous; pairwise RL-fuzzy irresolute; pairwise RL-fuzzy semi-compactness

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1. Introduction

In 1963, Levine [1] introduced the notion of semiopen set and its corresponding associated function in the realm of general topology. Afterwards, Azad [2] extended this notion and its related functions to the setting of L-topology. Thakur and Malviya [3] introduced and studied the concepts of (i,j)-semiopen and (i,j)-semiclosed L-fuzzy sets, pairwise fuzzy semicontinuous, and pairwise fuzzy semiopen functions in L-bitopology in the case of $L = [0,1]$. In [4], Shi introduced the notion of L-fuzzy semiopen and preopen gradations in L-fuzzy topological spaces. Furthermore, he introduced the notions of L-fuzzy semicontinuous functions, L-fuzzy precontinuous functions, L-fuzzy irresolute functions, and L-fuzzy pre-irresolute functions, and discussed some of their elementary properties. Shi's operators have been found very useful in defining other gradations and also in studying many topological characteristics. In 2011, Ghareeb [5] used L-fuzzy preopen operator to introduce the degree of pre-separatedness and the degree of preconnectedness in L-fuzzy topological spaces. Many characterizations of the degree of preconnectedness are discussed in L-fuzzy topological spaces. Later, Ghareeb [6] introduced the concept of L-fuzzy semi-preopen operator in L-fuzzy topological spaces and studied some of its properties. The concepts of L-fuzzy SP-compactness and L-fuzzy SP-connectedness in L-fuzzy pretopological spaces are introduced and studied [7]. Further, a new operator in L-fuzzy topology introduced in [8] to measure the **F**-openness of an L-fuzzy set in L-fuzzy topological spaces. Moreover, the new operator is used to introduce a new form of **F**-compactness. Recently, we used the new operators to generalize several kinds of functions between L-fuzzy topological spaces [9–12].

Recently, Li and Li [13] defined and studied the concept of RL-topology as an extension of L-topology. Moreover, RL-compactness by means of an inequality and RL-continuous mapping are introduced and discussed in detail. In [14], they presented RL-fuzzy topology on an L-fuzzy set as a generalization of RL-topology and L-fuzzy topology. Some relevant properties of RL-fuzzy compactness in RL-fuzzy topological spaces are further investigated. Later on, Zhang et al. [15] defined the degree of Lindelöf property and the degree of countable RL-fuzzy compactness of an L-fuzzy set, where L is a complete DeMorgan algebra. Since L-fuzzy topology in the sense of Kubiak and Šostak is a special case of RL-fuzzy topology, the degree of RL-fuzzy compactness and the degree of Lindelöf property are extensions of the corresponding degrees in L-fuzzy topology.

The purpose of this paper is to introduce the (i,j)-RL-semiopen gradation in RL-fuzzy bitopological spaces based on the concept of pseudo-complement of L-fuzzy sets. We also define and characterize pairwise RL-fuzzy semicontinuous, pairwise RL-fuzzy irresolute functions, and pairwise RL-fuzzy semi-compactness. Our results are more general than those of the corresponding notions in L-bitopology, RL-bitopology, RL-fuzzy topology, L-fuzzy topology, and L-fuzzy bitopology.

2. Preliminaries

In this section, we give some basic preliminaries required for this paper. By $(L, \bigvee, \bigwedge, ')$, we denote a complete DeMorgan algebra [16,17] (i.e., L is a completely distributive lattice with an order reversing involution $'$, where $\bigvee$ and $\bigwedge$ are join and meet operations, respectively), $X \neq \emptyset$ is a set, and L^X is the family of each L-fuzzy sets defined on X. The largest and the smallest members in L and L^X are denoted by $\top$, $\perp$, and $\top_X$, $\perp_X$, respectively. For each any two L-fuzzy sets $B \in L^X$, $C \in L^Y$, and any mapping $f : X \longrightarrow Y$, we define $f_L^{\rightarrow}(B)(y) = \bigvee\{B(x) : f(x) = y\}$ for all $y \in Y$ and $f_L^{\leftarrow}(C)(x) = \bigvee\{B(x) : f_L^{\rightarrow}(B) \leq C\} = C(f(x))$ for all $x \in X$. For each α, $\beta \in L$, $\alpha \prec \beta$ means that the element α is wedge below β in L [18], i.e., $\alpha \prec \beta$ if for every arbitrary subset $\mathcal{D} \subseteq L$, $\bigvee \mathcal{D} \geq \beta$ implies $\alpha \leq \gamma$ for some $\gamma \in \mathcal{D}$. An element $\alpha \in L$ is said to be co-prime if $\alpha \leq \beta \vee \gamma$ implies that $\alpha \leq \beta$ or $\alpha \leq \gamma$ and α is said to be prime if and only if α' is co-prime. The family of non-zero co-prime (resp. non-unit prime) members in L is denoted by $J(L)$ (resp. $P(L)$). By $\boldsymbol{\alpha}(\beta) = \bigvee\{\alpha \in L : \alpha \prec \beta\}$ and $\boldsymbol{\beta}(\beta) = \bigvee\{\alpha \in L : \alpha' \prec \beta'\}$, we denote the greatest minimal family and the greatest maximal family of β, respectively. $\boldsymbol{\alpha}^*(\alpha) = \boldsymbol{\alpha}(\alpha) \cap J(L)$ and $\boldsymbol{\beta}^*(\alpha) = \boldsymbol{\beta}(\alpha) \cap P(L)$ for all $\alpha \in L$.

An L-fuzzy set $A \in L^X$ is called *valuable* if $A \not\leq A'$. The collection of valuable L-fuzzy sets on X is denoted by $\mathscr{V}_X^L$. In other words, $\mathscr{V}_X^L = \{A \in L^X : A \not\leq A'\}$. For each $A \in \mathscr{V}_X^L$, we define the collection $\mathscr{F}_X^L(A)$ by $\mathscr{F}_X^L(A) = \{B \in L^X : B \leq A\}$. In fact, $\mathscr{F}_X^L(A)$ introduces the powerset of L-fuzzy set $A \in L^X$. Let $A \in \mathscr{V}_X^L$ and $B \in \mathscr{V}_Y^L$, the restriction of $f_L^{\rightarrow}$ on A, i.e., $f_L^{\rightarrow}|_A : \mathscr{F}_X^L(A) \longrightarrow L^Y$ provided that $D \in \mathscr{F}_X^L(A) \mapsto f_L^{\rightarrow}(D)$, is said to be the restriction of L-fuzzy function (RL-fuzzy function, in short) from A to B, given by $f_{L,A}^{\rightarrow} : A \longrightarrow B$ if $f_L^{\rightarrow}(A) \leq B$. The inverse of an L-fuzzy set $C \in \mathscr{F}_Y^L(B)$ under $f_{L,A}^{\rightarrow}$ is defined by $f_{L,A}^{\leftarrow}(C) = \bigvee\{D \in \mathscr{F}_X^L(A) : f_L^{\rightarrow}(D) \leq C\}$. It is clear that $f_{L,A}^{\leftarrow}(C) = A \wedge f_L^{\leftarrow}(C)$. The *pseudo-complement* of B relative to A [13,14], denoted by $\langle_L^A B$, is given by:

$$\langle_L^A B = \begin{cases} A \wedge B', & \text{if } B \neq A, \\ \perp_X, & \text{if } B = A. \end{cases}$$

where $A \in \mathscr{V}_X^L$ and $B \in \mathscr{F}_X^L(A)$. Some properties of pseudo-complement operation $\langle_L^A$ are listed in the following proposition:

Proposition 1. *[13,14] If $A \in \mathscr{V}_X^L$, $B, C \in \mathscr{F}_X^L(A)$, and $\{B_i\}_{i \in I} \subseteq \mathscr{F}_X^L(A)$, then:*

(1) *$\langle_L^A B = A$ if and only if $B \leq A'$.*
(2) *$B \leq C$ implies $\langle_L^A C \leq \langle_L^A B$.*
(3) *$\langle_L^A \bigwedge_{i \in I} B_i = \bigvee_{i \in I} \langle_L^A B_i$.*

(4) $\langle_L^A \bigvee_{i\in I} B_i \leq \bigwedge_{i\in I} \langle_L^A B_i$ *and* $\langle_L^A \bigvee_{i\in I} B_i = \bigwedge_{i\in I} \langle_L^A B_i$ *if* $\bigvee_{i\in I} B_i \neq A$.

Lemma 1. *[13] Let* $A \in \mathscr{V}_X^L$, $B \in \mathscr{V}_Y^L$, $f_{L,A}^{\rightarrow} : A \longrightarrow B$ *be RL-fuzzy function, and* $D \in \mathscr{F}_X^L(A)$. *Then for any* $\mathcal{U} \subseteq \mathscr{F}_X^L(A)$, *we have*

$$\bigvee_{y\in Y}\Big(f_{L,A}^{\rightarrow}(D)(y) \wedge \bigwedge_{E\in\mathcal{U}} E(y)\Big) = \bigvee_{x\in X}\Big(D(x) \wedge \bigwedge_{E\in\mathcal{U}} f_{L,A}^{\leftarrow}(E)(x)\Big).$$

Equivalently [15],

$$\bigwedge_{y\in Y}\Big(\langle_L^A f_{L,A}^{\rightarrow}(D)(y) \vee \bigvee_{E\in\mathcal{P}} E(y)\Big) = \bigwedge_{x\in X}\Big(\langle_L^A D(x) \vee \bigvee_{E\in\mathcal{P}} f_{L,A}^{\leftarrow}(E)(x)\Big).$$

An L-topology [16,17,19] (L-t, for short) τ is a subfamily of L^X which contains $\bot_X$, $\top_X$ and is closed for any suprema and finite infima. Moreover, (X,τ) is called an L-topological space on X. Further, members of τ are called open L-fuzzy sets and their complements are called closed L-fuzzy sets. A mapping $f : (X,\tau_1) \longrightarrow (Y,\tau_2)$ is called L-continuous if and only if $f_L^{\leftarrow}(C) \in \tau_1$ for any $C \in \tau_2$. The notion of L-topology was generalized by Kubiak [20] and Šostak [21] independently as follows:

Definition 1. *[20–22] An L-fuzzy topology on the set X is the function* $\tau : L^X \longrightarrow L$, *which satisfies the following conditions:*

(O1) $\tau(\bot_X) = \tau(\top_X) = \top$.
(O2) $\tau(A \wedge B) \geq \tau(A) \wedge \tau(B)$, *for each* $A, B \in L^X$.
(O3) $\tau(\bigvee_{i\in I} A_i) \geq \bigwedge_{i\in I} \tau(A_i)$, *for each* $\{A_i\}_{i\in I} \subseteq L^X$.

The pair (X,τ) *is called an L-fuzzy topological space (L-fts, for short). The value* $\tau(A)$ *and* $\tau^*(A) = \tau(A')$ *represent the degree of openness and the degree of closeness of an L-fuzzy set A, respectively. A function* $f : (X,\tau_1) \longrightarrow (Y,\tau_2)$ *is called L-fuzzy continuous iff* $\tau_1(f_L^{\leftarrow}(C)) \geq \tau_2(C)$ *for any* $C \in L^Y$.

One of the attempts to generalize L-topological spaces was the definition of RL-topology $\varkappa$ on an L-fuzzy set A by Li and Li [13] as follows:

Definition 2. *[13] Let* $A \in \mathscr{V}_X^L$. *A relative L-topology (RL-t, for short)* $\varkappa$ *on an L-fuzzy set A, is a subfamily of* $\mathscr{F}_X^L(A)$, *that satisfies the following statements:*

(1) $A \in \varkappa$ *and* $B \in \varkappa$, *for each* $B \leq A'$.
(2) $B_1 \wedge B_2 \in \varkappa$, *for any* $B_1, B_2 \in \varkappa$.
(3) $\bigvee_{i\in I} B_i \in \varkappa$, *for any* $\{B_i\}_{i\in I} \subseteq \varkappa$.

The pair $(A,\varkappa)$ *is said to be a relative L-topological space on A (RL-ts, for short). The elements of* $\varkappa$ *are called relative open L-fuzzy sets (RL-open fuzzy set, for short) and an L-fuzzy set B is called relative L-closed fuzzy set (RL-closed fuzzy set, for short) if and only if* $\langle_L^A B \in \varkappa$. *The collection of all RL-closed fuzzy sets with respect to* $\varkappa$ *is denoted by* $\langle_L^A \varkappa$, *i.e.,* $\langle_L^A \varkappa = \{C : \langle_L^A C \in \varkappa\}$. *Let* $A \in \mathscr{V}_X^L$, $B \in \mathscr{V}_Y^L$, *and* $(A,\varkappa_1)$, $(B,\varkappa_2)$ *be two RL-ts's. The relative L-fuzzy function* $f_{L,A}^{\rightarrow} : A \longrightarrow B$ *is said to be an RL-continuous iff* $f_{L,A}^{\leftarrow}(C) \in \langle_L^A \varkappa_1$ *for any* $C \in \langle_L^A \varkappa_2$. *Equivalently,* $f_{L,A}^{\rightarrow} : A \longrightarrow B$ *is said to be an RL-continuous iff* $f_{L,A}^{\leftarrow}(C) \in \varkappa_1$ *for any* $C \in \varkappa_2$. *A triple* $(A,\varkappa_1,\varkappa_2)$ *consisting of an L-fuzzy set* $A \in \mathscr{V}_X^L$ *endowed with RL-topologies* $\varkappa_1$ *and* $\varkappa_2$ *on A is called an RL-bitopological space (RL-bts, for short). For any* $B \in \mathscr{F}_X^L(A)$, $\varkappa_i$*-RL-open (resp. closed) fuzzy set refers to the open (resp. closed) L-fuzzy set in* $(A,\varkappa_i)$, *for* $i = 1,2$. *It is clear that we get L-topology and L-bitopology as a special case if* $A = \top_X$.

The following two definitions extend the notions of (strong) $\boldsymbol{\beta}_\alpha$-cover, Q_α-cover, (strong) α-shading, (strong) α-remote collection [23] to the setting of RL-topological spaces:

Definition 3. *For any $A \in \mathscr{V}_X^L$, RL-topology $\varkappa$ on A, $B \in \mathscr{F}_X^L(A)$, and $\alpha \in L_\perp$, a collection $\mathcal{U} \subseteq \mathscr{F}_X^L(A)$ is called:*

(1) *$\boldsymbol{\beta}_\alpha$-cover of B if for any $x \in X$, it follows that $\alpha \in \boldsymbol{\beta}(\langle_L^A B(x) \vee \bigvee_{A \in \mathcal{U}} A(x))$ and $\mathcal{U}$ is called strong $\boldsymbol{\beta}_\alpha$-cover of B if $a \in \boldsymbol{\beta}(\bigwedge_{x \in X}(\langle_L^A B(x) \vee \bigvee_{A \in \mathcal{U}} A(x)))$.*

(2) *Q_α-cover of B if for any $x \in X$, it follows that $\langle_L^A B(x) \vee \bigvee_{A \in \mathcal{U}} A(x) \geq \alpha$.*

Definition 4. *For any $A \in \mathscr{V}_X^L$, RL-topology $\varkappa$ on A, $\alpha \in L_\top$ and $B \in \mathscr{F}_X^L(A)$, a collection $\mathcal{A} \subseteq \mathscr{F}_X^L(A)$ is called:*

(1) *α-shading of B if for any $x \in X$, $(\langle_L^A B(x) \vee \bigvee_{A \in \mathcal{A}} A(x)) \not\leq \alpha$.*

(2) *strong α-shading of B if $\bigwedge_{x \in X}(\langle_L^A B(x) \vee \bigvee_{A \in \mathcal{A}} A(x)) \not\leq \alpha$.*

(3) *α-remote collection of B if for any $x \in X$, $(B(x) \wedge \bigwedge_{D \in \mathcal{A}} D(x)) \not\geq \alpha$.*

(4) *strong α-remote collection of B if $\bigvee_{x \in X}(B(x) \wedge \bigwedge_{D \in \mathcal{A}} D(x)) \not\geq \alpha$.*

Theorem 1. *[13] For any RL-ts $(A, \varkappa)$, the following statements are true:*

(1) *$A \in \langle_L^A \varkappa$ and $B \in \langle_L^A \varkappa$ for all $B \leq A'$.*

(2) *$B_1 \vee B_2 \in \langle_L^A \varkappa$ for each $B_1, B_2 \in \langle_L^A \varkappa$,*

(3) *$\bigwedge_{i \in I} B_i \in \langle_L^A \varkappa$ for each $\{B_i : i \in I\} \subseteq \langle_L^A \varkappa$.*

Definition 5. *[14] Let $A \in \mathscr{V}_X^L$. An RL-fuzzy topology on A is a function $\varkappa : \mathscr{F}_X^L(A) \longrightarrow L$ such that $\varkappa$ satisfying the following conditions:*

($\mathcal{R}$1) *$\varkappa(A) = \top$, for each $B \leq A'$, $\varkappa(B) = \top$.*

($\mathcal{R}$2) *$\varkappa(B_1 \wedge B_2) \geq \varkappa(B_1) \wedge \varkappa(B_2)$, for each $B_1, B_2 \in \mathscr{F}_X^L(A)$.*

($\mathcal{R}$3) *$\varkappa(\bigvee_{i \in I} B_i) \geq \bigwedge_{i \in I} \varkappa(B_i)$, for each $\{B_i\}_{i \in I} \subseteq \mathscr{F}_X^L(A)$.*

The pair $(A, \varkappa)$ is said to be an RL-fuzzy topological space (RL-fts, for short) on A. For any $B \in \mathscr{F}_X^L(A)$, the gradation $\varkappa(B)$ (resp. $\varkappa(\langle_L^A B)$) can be viewed as the openness degree (resp. closeness degree) of B relative to $\varkappa$, respectively. Further, $\varkappa(B) = \top$ (resp. $\varkappa(\langle_L^A B) = \top$) confirms the RL-openness (resp. RL-closeness) of an L-fuzzy set B. Obviously if $A = \top_X$, then RL-fuzzy topology on A degenerates into Kubiak-Šostak's L-fuzzy topology, that is, RL-fuzzy topology on A is a generalization of L-fuzzy topology. If $(A, \varkappa)$ is an RL-topological space and $\chi_\varkappa : \mathscr{F}_X^L(A) \longrightarrow L$ is a function given by $\chi_\varkappa(B) = \top$ if $B \in \varkappa$, and $\chi_\varkappa(B) = \perp$ if $B \notin \varkappa$, then $(A, \chi_\varkappa)$ represents a special RL-fts, i.e., $(A, \varkappa)$ can also be seen as RL-fts.

Theorem 2. *[14] For each $A \in \mathscr{V}_X^L$ and RL-fts $(A, \varkappa)$ on A. The function $\langle_L^A \varkappa : \mathscr{F}_X^L(A) \longrightarrow L$ given by $\langle_L^A \varkappa(B) = \varkappa(\langle_L^A B)$ for any $B \in \mathscr{F}_X^L(A)$, satisfies the following conditions:*

(1) *$\langle_L^A \varkappa(A) = \top$, for each $B \leq A'$, $\langle_L^A \varkappa(B) = \top$.*

(2) *$\langle_L^A \varkappa(B_1 \vee B_2) \geq \langle_L^A \varkappa(B_1) \wedge \langle_L^A \varkappa(B_2)$, for each $B_1, B_2 \in \mathscr{F}_X^L(A)$.*

(3) *$\langle_L^A \varkappa(\bigwedge_{i \in I} B_i) \geq \bigwedge_{i \in I} \langle_L^A \varkappa(B_i)$, for each $\{B_i\}_{i \in I} \subseteq \mathscr{F}_X^L(A)$.*

$\langle_L^A \varkappa$ is said to be an RL-fuzzy cotopology (RL-cft, for short) on A and the pair $(A, \langle_L^A \varkappa)$ is said to be an RL-fuzzy cotopological space (RL-cfts, for short).

Definition 6. *[14] Let $A \in \mathscr{V}_X^L$, $B \in \mathscr{V}_Y^L$, and $(A, \varkappa_1)$, $(B, \varkappa_2)$ be two RL-fuzzy topological spaces on A and B, respectively. The relative L-fuzzy function $f_{L,A} : A \longrightarrow B$ is said to be an RL-fuzzy continuous iff*

$$\varkappa_1(f_{L,A}^{\leftarrow}(C)) \geq \varkappa_1(C),$$

equivalently,

$$\varkappa_1(\langle_L^A f_{L,A}^{\leftarrow}(C)) \geq \varkappa_1(\langle_L^B C),$$

for each $C \in \mathscr{F}_Y^L(B)$*. If* $(A, \langle_L^A \varkappa_1)$ *and* $(B, \langle_L^B \varkappa_2)$ *are the associated RL-fuzzy cotopological spaces of* $(A, \varkappa_1)$ *and* $(B, \varkappa_2)$ *respectively, then* $f_{L,A}^{\rightarrow}$ *is said to be an RL-fuzzy continuous iff*

$$\langle_L^A \varkappa_1(f_{L,A}^{\leftarrow}(C)) \geq \langle_L^B \varkappa_2(C),$$

for each $C \in \mathscr{F}_X^L(B)$*.*

Shi [24] introduced L-fuzzy closure operators in L-fuzzy topological spaces. In the following definition, we introduce its equivalent form in RL-fuzzy topological spaces.

Definition 7. *Let* $A \in \mathscr{V}_X^L$*, and* $(A, \varkappa)$ *be an RL-fts on A. The function* $Cl^{\varkappa} : \mathscr{F}_X^L(A) \rightarrow L^{J(\mathscr{F}_X^L(A))}$ *defined by*

$$Cl^{\varkappa}(B)(x_\lambda) = \bigwedge_{x_\lambda \not\leq D \geq B} \langle_L^A \Big(\varkappa(\langle_L^A D)\Big)$$

for each $x_\lambda \in J(\mathscr{F}_X^L(A))$ *and* $B \in \mathscr{F}_X^L(A)$ *is called an RL-fuzzy closure operator induced by* $\varkappa$*.*

Definition 8. *[14] For any* $A \in \mathscr{V}_X^L$ *and an RL-fts* $(A, \varkappa)$ *on A, an L-fuzzy set* $B \in \mathscr{F}_X^L(A)$ *is called an RL-fuzzy compact with respect to* $\varkappa$ *if for any* $\mathcal{P} \subseteq \mathscr{F}_X^L(A)$*, the following inequality holds:*

$$\bigvee_{D \in \mathcal{P}} \varkappa(\langle_L^A D) \vee \bigvee_{x \in X} \Big(B(x) \wedge \bigwedge_{D \in \mathcal{P}} D(x)\Big) \geq \bigwedge_{\mathcal{R} \in 2^{\mathcal{P}}} \bigvee_{x \in X} \Big(B(x) \wedge \bigwedge_{D \in \mathcal{R}} D(x)\Big).$$

Theorem 3. *[14] If* $A = \top_X$*, then following statements hold:*

- **(1)** $\langle_L^A B = B'$, $B \in \mathscr{F}_X^L(A) \Leftrightarrow B \in L^X$.
- **(2)** *RL-fuzzy compactness is reduced to L-fuzzy compactness.*
- **(3)** *B is RL-fuzzy compact if and only if B is L-fuzzy compact.*

Theorem 4. *[14] For any* $A \in \mathscr{V}_X^L$ *and an RL-ft* $\varkappa$ *on A, we have following conclusions:*

- **(1)** *If* $B_1, B_2 \in \mathscr{F}_X^L(A)$ *and* B_1, B_2 *are RL-fuzzy compact, then* $B_1 \vee B_2$ *is RL-fuzzy compact.*
- **(2)** *If* $B_1, B_2 \in \mathscr{F}_X^L(A)$ *such that* B_1 *is an RL-fuzzy compact and* B_2 *is an RL-closed fuzzy set, then* $B_1 \wedge B_2$ *is an RL-fuzzy compact.*

3. The Gradation of Semiopenness in RL-fuzzy Bitopological Spaces

A system $(A, \varkappa_1, \varkappa_2)$ consisting of an L-fuzzy set $A \in \mathscr{V}_X^L$ with two RL-fuzzy topologies $\varkappa_1$ and $\varkappa_2$ on A is called an RL-fuzzy bitopological space. Throughout this paper i, $j = 1, 2$ where $i \neq j$ and if P is any topological property then $\varkappa_i$-P refers to the property P with respect to the RL-fuzzy topology $\varkappa_i$. An L-fuzzy set $B \in \mathscr{F}_X^L(A)$ of an RL-bitopological space $(A, \varkappa_l, \varkappa_2)$ is called an (i,j)-RL-semiopen if there exists an L-fuzzy set $C \in \varkappa_i$ such that $C \leq B \leq Cl^{\varkappa_j}(C)$.

Definition 9. *Let* $A \in \mathscr{V}_X^L$ *and* $(A, \varkappa_1, \varkappa_2)$ *be an RL-fuzzy bitopological space on A. For any* $B \in \mathscr{F}_X^L(A)$*, define a function* (i,j)-$\mathcal{S} : \mathscr{F}_X^L(A) \rightarrow L$ *by*

$$(i,j)\text{-}\mathcal{S}(B) = \bigvee_{C \leq B} \left\{ \varkappa_i(C) \wedge \bigwedge_{x_\lambda \prec B} \bigwedge_{x_\lambda \not\leq D \geq C} \langle_L^A \Big(\varkappa_j(\langle_L^A D)\Big) \right\}.$$

Then (i,j)-$\mathcal{S}(B)$ *is called an* (i,j)*-RL-semiopenness gradation of B induced by* $\varkappa_i$ *and* $\varkappa_j$ *such that* $i \neq j$*, where* (i,j)-$\mathcal{S}(B)$ *represents the degree to which B is* (i,j)*-RL-semiopen and* (i,j)-$\mathcal{S}^*(B) = (i,j)$-$\mathcal{S}(\langle_L^A B)$ *represents the degree to which B is* (i,j)*-RL-semiclosed.*

Based on the above definition and Definition 7, we can state the following corollary:

Corollary 1. *Let $A \in \mathscr{V}_X^L$ and $(A, \varkappa_1, \varkappa_2)$ be an RL-fuzzy bitopological space on A. Then for each $B \in \mathscr{F}_X^L(A)$, we have*

$$(i,j)\text{-}\mathcal{S}(B) = \bigvee_{C \leq B} \left\{ \varkappa_i(C) \wedge \bigwedge_{x_\lambda \prec B} Cl^{\varkappa_j}(C)(x_\lambda) \right\}.$$

Theorem 5. *Let $A \in \mathscr{V}_X^L$, $\varkappa_1$, $\varkappa_2 : \mathscr{F}_X^L(A) \to \{\perp, \top\}$ be RL-topologies on A, and (i,j)-$\mathcal{S} : \mathscr{F}_X^L(A) \to \{\perp, \top\}$ be the gradation of (i,j)-RL-semiopenness induced by $\varkappa_i$ and $\varkappa_j$ such that $i \neq j$. Then (i,j)-$\mathcal{S}(B) = \top$ iff B is an (i,j)-RL-semiopen.*

Proof. The proof can be obtained simply from the following inequality:

$$\begin{aligned}
(i,j)\text{-}\mathcal{S}(B) = \top \quad & \text{iff } \bigvee_{C \leq B} \left\{ \varkappa_i(C) \wedge \bigwedge_{x_\lambda \prec B} Cl^{\varkappa_j}(C)(x_\lambda) \right\} = \top \\
& \text{iff } \exists C \leq B \text{ such that } \varkappa_i(C) = \top \text{ and } \bigwedge_{x_\lambda \prec B} Cl^{\varkappa_j}(C)(x_\lambda) = \top \\
& \text{iff } \exists C \leq B \text{ such that } \varkappa_i(C) = \top \text{ and for each } x_\lambda \prec B,\ Cl^{\varkappa_j}(C)(x_\lambda) = \top \\
& \text{iff } \exists C \in \varkappa_i \text{ such that } C \leq B \leq Cl^{\varkappa_j}(C) \\
& \text{iff } B \text{ is } (i.j)\text{-}RL\text{-semiopen}.
\end{aligned}$$

□

Theorem 6. *Let $A \in \mathscr{V}_X^L$, $(A, \varkappa_1, \varkappa_2)$ be an RL-fuzzy bitopological space on A, and (i,j)-$\mathcal{S}$ be the gradation of (i,j)-RL-semiopenness induced by $\varkappa_i$ and $\varkappa_j$ such that $i \neq j$. Then for each $B \in \mathscr{F}_X^L(A)$, we have $\varkappa_i(B) \leq (i,j)$-$\mathcal{S}(B)$.*

Proof. The proof can be obtained simply from the following inequality:

$$\begin{aligned}
(i,j)\text{-}\mathcal{S}(B) \quad &= \bigvee_{C \leq B} \left\{ \varkappa_i(C) \wedge \bigwedge_{x_\lambda \prec B} Cl^{\varkappa_j}(C)(x_\lambda) \right\} \geq \varkappa_i(B) \wedge \bigwedge_{x_\lambda \prec B} Cl^{\varkappa_j}(B)(x_\lambda) \\
&= \varkappa_i(B) \wedge \top = \varkappa_i(B).
\end{aligned}$$

□

Corollary 2. *Let $A \in \mathscr{V}_X^L$, $(A, \varkappa_1, \varkappa_2)$ be an RL-fuzzy bitopological space on A, and (i,j)-$\mathcal{S}$ be the gradation of (i,j)-RL-semiopenness induced by $\varkappa_i$ and $\varkappa_j$ such that $i \neq j$. Then for each $B \in \mathscr{F}_X^L(A)$, we have $\langle_L^A \varkappa_i(B) \leq (i,j)$-$\mathcal{S}^*(B)$.*

Theorem 7. *If $A \in \mathscr{V}_X^L$, $(A, \varkappa_1, \varkappa_2)$ be an RL-fuzzy bitopological space on A, and (i,j)-$\mathcal{S}$ be the gradation of (i,j)-RL-semiopenness induced by $\varkappa_i$ and $\varkappa_j$ such that $i \neq j$, then (i,j)-$\mathcal{S}\left(\bigvee_{i \in I} B_i\right)$ $\geq \bigwedge_{i \in I} (i,j)$-$\mathcal{S}(B_i)$ for each $\{B_i\}_{i \in I} \subseteq \mathscr{F}_X^L(A)$.*

Proof. Let $\alpha \in L$ and $\alpha \prec \bigwedge_{i \in I} (i,j)$-$\mathcal{S}(B_i)$, then there exists $C_i \leq B_i$ such that $\alpha \prec \varkappa_i(C_i)$ and $\alpha \prec \bigwedge_{x_\lambda \prec B_i} \bigwedge_{x_\lambda \not\leq D \geq C_i} \langle_L^A \left(\varkappa_j(\langle_L^A D) \right)$ for any $i \in I$. Hence $\alpha \leq \bigwedge_{i \in I} \varkappa_i(C_i) \leq \varkappa_i\left(\bigvee_{i \in I} C_i\right)$ and $\alpha \leq \bigwedge_{i \in I} \bigwedge_{x_\lambda \prec B_i} \bigwedge_{x_\lambda \not\leq D \geq C_i} \langle_L^A \left(\varkappa_j(\langle_L^A D) \right)$. Since $\{x_\lambda : x_\lambda \prec \bigvee_{i \in I} B_i\} = \bigcup_{i \in I} \{x_\lambda : x_\lambda \prec B_i\}$, we have

$$(i,j)\text{-}\mathcal{S}\left(\bigvee_{i \in I} B_i\right) \quad = \quad \bigvee_{C \leq \bigvee_{i \in I} B_i} \left\{ \varkappa_i(C) \wedge \bigwedge_{x_\lambda \prec \bigvee_{i \in I} B_i} \bigwedge_{x_\lambda \not\leq D \geq C} \langle_L^A \left(\varkappa_j(\langle_L^A D) \right) \right\}$$

$$\begin{aligned} &\geq \varkappa_i\left(\bigvee_{i\in I} C_i\right) \wedge \bigwedge_{i\in I}\bigwedge_{x_\lambda\prec B_i}\bigwedge_{x_\lambda\not\leq D\geq\bigvee_{i\in I}C_i} \langle^A_L\left(\varkappa_j(\langle^A_L D)\right) \\ &\geq \varkappa_i\left(\bigvee_{i\in I} C_i\right) \wedge \bigwedge_{i\in I}\bigwedge_{x_\lambda\prec B_i}\bigwedge_{x_\lambda\not\leq D\geq C_i} \langle^A_L\left(\varkappa_j(\langle^A_L D)\right) \\ &\geq \alpha. \end{aligned}$$

This shows that $(i,j)\text{-}\mathcal{S}\left(\bigvee_{i\in I} B_i\right) \geq \bigwedge_{i\in I}(i,j)\text{-}\mathcal{S}(B_i)$. □

Corollary 3. *Let $A \in \mathscr{V}_X^L$, $(A, \varkappa_1, \varkappa_2)$ be an RL-fuzzy bitopological space on A, and (i,j)-$\mathcal{S}$ be the gradation of (i,j)-RL-semiopenness induced by $\varkappa_i$ and $\varkappa_j$ such that $i \neq j$. Then $(i,j)\text{-}\mathcal{S}^*\left(\bigwedge_{i\in I} B_i\right) \geq \bigwedge_{i\in I}(i,j)\text{-}\mathcal{S}^*(B_i)$ for any $\{B_i\}_{i\in I} \subseteq \mathscr{F}_X^L(A)$.*

4. Pairwise Fuzzy Semicontinuous Functions Between *RL*-fuzzy Bitopological Spaces

Let $A \in \mathscr{V}_X^L$, $B \in \mathscr{V}_Y^L$, and $(A, \varkappa_1, \varkappa_2)$, $(B, \varkappa_1^*, \varkappa_2^*)$ be *RL*-fbts's on A and B, respectively. An *RL*-fuzzy function $f_{L,A} : A \longrightarrow B$ is said to be pairwise *RL*-fuzzy continuous (resp. open) iff $f_{L,A} : (A, \varkappa_1) \longrightarrow (B, \varkappa_1^*)$ and $f_{L,A} : (A, \varkappa_2) \longrightarrow (B, \varkappa_2^*)$ are *RL*-fuzzy continuous (resp. open).

Definition 10. *Let $A \in \mathscr{V}_X^L$, $B \in \mathscr{V}_Y^L$, $(A, \varkappa_1, \varkappa_2)$ and $(B, \varkappa_1^*, \varkappa_2^*)$ be RL-fbts's on A and B, respectively, and (i,j)-$\mathcal{S}_1$, (i,j)-$\mathcal{S}_2$ their corresponding gradations of (i,j)-RL-semiopenness. An RL-fuzzy function $f_{L,A} : A \longrightarrow B$ is called:*

- **(1)** *pairwise RL-fuzzy semicontinuous iff $\varkappa_i^*(C) \leq (i,j)\text{-}\mathcal{S}_1(f_{L,A}^{\leftarrow}(C))$ holds for each $C \in \mathscr{F}_X^L(B)$.*
- **(2)** *pairwise RL-fuzzy irresolute iff $(i,j)\text{-}\mathcal{S}_2(C) \leq (i,j)\text{-}\mathcal{S}_1(f_{L,A}^{\leftarrow}(C))$ holds for each $C \in \mathscr{F}_X^L(B)$.*

Corollary 4. *Let $A \in \mathscr{V}_X^L$, $B \in \mathscr{V}_Y^L$, $(A, \varkappa_1, \varkappa_2)$ and $(B, \varkappa_1^*, \varkappa_2^*)$ be RL-fbts's on A and B, respectively, and (i,j)-$\mathcal{S}_1$, (i,j)-$\mathcal{S}_2$ their corresponding gradations of (i,j)-RL-semiopenness. Then:*

- **(1)** *$f_{L,A}$ is pairwise RL-fuzzy semicontinuous iff $\langle^B_L \varkappa_i^*(C) \leq (i,j)\text{-}\mathcal{S}_1^*(f_{L,A}^{\leftarrow}(C))$ for each $C \in \mathscr{F}_X^L(B)$.*
- **(2)** *$f_{L,A}$ is pairwise RL-fuzzy irresolute iff $(i,j)\text{-}\mathcal{S}_2^*(C) \leq (i,j)\text{-}\mathcal{S}_1^*(f_{L,A}^{\leftarrow}(C))$ for each $C \in \mathscr{F}_X^L(B)$.*

Theorem 8. *Let $A \in \mathscr{V}_X^L$, $B \in \mathscr{V}_Y^L$, $(A, \varkappa_1, \varkappa_2)$ and $(B, \varkappa_1^*, \varkappa_2^*)$ be RL-fbts's on A and B, respectively, and (i,j)-$\mathcal{S}_1$, (i,j)-$\mathcal{S}_2$ their corresponding gradations of (i,j)-RL-semiopenness. Then:*

- **(1)** *$f_{L,A} : (A, \varkappa_1, \varkappa_2) \to (B, \varkappa_1^*, \varkappa_2^*)$ is pairwise RL-fuzzy semicontinuous iff $f_{L,A} : (A, \varkappa_{1[\alpha]}, \varkappa_{2[\alpha]}) \to (B, \varkappa^*_{1[\alpha]}, \varkappa^*_{2[\alpha]})$ is pairwise RL-semicontinuous for each $\alpha \in J(L)$.*
- **(2)** *$f_{L,A} : (A, \varkappa_1, \varkappa_2) \to (B, \varkappa_1^*, \varkappa_2^*)$ is pairwise RL-fuzzy irresolute iff $f_{L,A} : (A, \varkappa_{1[\alpha]}, \varkappa_{2[\alpha]}) \to (B, \varkappa^*_{1[\alpha]}, \varkappa^*_{2[\alpha]})$ is pairwise RL-irresolute for each $\alpha \in J(L)$.*

Proof.

- **(1)** Let $C \in \varkappa^*_{i\,[\alpha]}$ for each $C \in \mathscr{F}_X^L(B)$ and $\alpha \in J(L)$, then $\varkappa_i^*(C) \geq \alpha$. Since $f_{L,A} : (A, \varkappa_1, \varkappa_2) \to (B, \varkappa_1^*, \varkappa_2^*)$ is pairwise *RL*-fuzzy semicontinuous, then $(i,j)\text{-}\mathcal{S}_1(f_{L,A}^{\leftarrow}(C)) \geq \varkappa_i^*(C) \geq \alpha$, i.e., $(i,j)\text{-}\mathcal{S}_1(f_{L,A}^{\leftarrow}(C)) \geq \alpha$. Therefore $f_{L,A}^{\leftarrow}(C)$ is (i,j)-*RL*-semiopen *L*-fuzzy set in $(A, \varkappa_{1[\alpha]}, \varkappa_{2[\alpha]})$. Hence $f_{L,A} : (A, \varkappa_{1[\alpha]}, \varkappa_{2[\alpha]}) \to (B, \varkappa^*_{1[\alpha]}, \varkappa^*_{2[\alpha]})$ is pairwise *RL*-semicontinuous function.

Conversely, let $\varkappa_i^*(C) \geq \alpha$ for each $C \in \mathscr{F}_X^L(B)$ and $\alpha \in J(L)$, then $C \in \varkappa^*_{i\,[\alpha]}$. By the pairwise semicontinuity of $f_{L,A} : (A, \varkappa_{1[\alpha]}, \varkappa_{2[\alpha]}) \rightarrow (B, \varkappa^*_{1\,[\alpha]}, \varkappa^*_{2\,[\alpha]})$, we have $f^{\leftarrow}_{L,A}(C)$ is (i,j)-RL-semiopen with respect to $(A, \varkappa_{1[\alpha]}, \varkappa_{2[\alpha]})$. Accordingly, (i,j)-$\mathcal{S}_1(f^{\leftarrow}_{L,A}(C)) \geq \alpha$ for each $\alpha \in J(L) \cap J(\varkappa_i^*(C))$, where $J(\varkappa_i^*(C)) = \{\alpha \in J(L) | \alpha \leq \varkappa_i^*(C)\}$. It follows that (i,j)-$\mathcal{S}_1(f^{\leftarrow}_{L,A}(C)) \geq \bigvee J(\varkappa_i^*(C)) = \varkappa_i^*(C)$.

(2) Suppose that C is (i,j)-RL-semiopen L-fuzzy set in $(B, \varkappa^*_{1\,[\alpha]}, \varkappa^*_{2\,[\alpha]})$, then (i,j)-$\mathcal{S}_2(C) \geq \alpha$. Since $f_{L,A} : (A, \varkappa_1, \varkappa_2) \rightarrow (B, \varkappa_1^*, \varkappa_2^*)$ is pairwise RL-fuzzy irresolute, then (i,j)-$\mathcal{S}_1(f^{\leftarrow}_{L,A}(C)) \geq (i,j)$-$\mathcal{S}_2(C) \geq \alpha$, so (i,j)-$\mathcal{S}_1(f^{\leftarrow}_{L,A}(C)) \geq \alpha$, therefore $f^{\leftarrow}_{L,A}(C)$ is $(i.j)$-RL-semiopen L-fuzzy set in $(A, \varkappa_{1[\alpha]}, \varkappa_{2[\alpha]})$. So that $f_{L,A} : (A, \varkappa_{1[\alpha]}, \varkappa_{2[\alpha]}) \rightarrow (B, \varkappa^*_{1\,[\alpha]}, \varkappa^*_{2\,[\alpha]})$ is pairwise RL-irresolute.

Conversely, let (i,j)-$\mathcal{S}_2(C) \geq \alpha$ for each $\alpha \in J(L)$, then C is an $(i.j)$-RL-semiopen in $(B, \varkappa^*_{1\,[\alpha]}, \varkappa^*_{2\,[\alpha]})$. Since $f_{L,A} : (A, \varkappa_{1[\alpha]}, \varkappa_{2[\alpha]}) \rightarrow (B, \varkappa^*_{1\,[\alpha]}, \varkappa^*_{2\,[\alpha]})$ is pairwise RL-irresolute, then $f^{\leftarrow}_{L,A}(C)$ is (i,j)-RL-semiopen in $(A, \varkappa_{1[\alpha]}, \varkappa_{2[\alpha]})$. Accordingly, (i,j)-$\mathcal{S}_1(f^{\leftarrow}_{L,A}(C)) \geq \alpha$ for any $\alpha \in J(L) \cap J((i,j)\text{-}\mathcal{S}_2(C))$, where $J((i,j)\text{-}\mathcal{S}_2(C)) = \{\alpha \in J(L) | \alpha \leq (i,j)\text{-}\mathcal{S}_2(C)\}$. It follows that (i,j)-$\mathcal{S}_1(f^{\leftarrow}_{L,A}(C)) \geq \bigvee J((i,j)\text{-}\mathcal{S}_2(C)) = (i,j)$-$\mathcal{S}_2(C)$.

□

Theorem 9. *Let $A \in \mathscr{V}_X^L$, $B \in \mathscr{V}_Y^L$, and $(A, \varkappa_1, \varkappa_2)$, $(B, \varkappa_1^*, \varkappa_2^*)$ be RL-fbts's on A and B, respectively. If an RL-fuzzy function $f_{L,A} : A \longrightarrow B$ is pairwise RL-fuzzy continuous, then $f_{L,A}$ is also pairwise RL-fuzzy semicontinuous.*

Proof. Let $f_{L,A} : A \longrightarrow B$ be pairwise RL-fuzzy continuous, then $\varkappa_i^*(C) \leq \varkappa_i(f^{\leftarrow}_{L,A}(C))$ for each $C \in \mathscr{F}_X^L(B)$ and $i = 1,2$. By Theorem 6, we have

$$\varkappa_i^*(C) \leq \varkappa_i(f^{\leftarrow}_{L,A}(C)) \leq (i,j)\text{-}\mathcal{S}_1(f^{\leftarrow}_{L,A}(C)),$$

for each $C \in \mathscr{F}_X^L(B)$. Therefore $f_{L,A}$ is pairwise RL-fuzzy semicontinuous. □

Theorem 10. *Let $A \in \mathscr{V}_X^L$, $B \in \mathscr{V}_Y^L$, and $(A, \varkappa_1, \varkappa_2)$, $(B, \varkappa_1^*, \varkappa_2^*)$ be two RL-fbts's on A and B, respectively. If $f_{L,A} : (A, \varkappa_1, \varkappa_2) \longrightarrow (A, \varkappa_1^*, \varkappa_2^*)$ is pairwise RL-fuzzy irresolute, then $f_{L,A}$ is pairwise RL-fuzzy semicontinuous.*

Proof. Let $f_{L,A} : (A, \varkappa_1, \varkappa_2) \longrightarrow (B, \varkappa_1^*, \varkappa_2^*)$ be pairwise RL-fuzzy irresolute, then (i,j)-$\mathcal{S}_2(C) \leq (i,j)$-$\mathcal{S}_1(f^{\leftarrow}_{L,A}(C))$ for each $C \in \mathscr{F}_X^L(B)$. By Theorem 6, we have $\varkappa_i(C) \leq (i,j)$-$\mathcal{S}_2(C) \leq (i,j)$-$\mathcal{S}_1(f^{\leftarrow}_{L,A}(C))$. Therefore $f_{L,A}$ is pairwise RL-fuzzy semicontinuous. □

Theorem 11. *Let $A \in \mathscr{V}_X^L$, $B \in \mathscr{V}_Y^L$, $C \in \mathscr{V}_Z^L$, and $(A, \varkappa_1, \varkappa_2)$, $(B, \varkappa_1^*, \varkappa_2^*)$, $(C, \varkappa_1^{**}, \varkappa_2^{**})$ be RL-fbts's on A, B, and C, respectively. If $f_{L,A} : (A, \varkappa_1, \varkappa_2) \longrightarrow (B, \varkappa_1^*, \varkappa_2^*)$ is pairwise RL-fuzzy semicontinuous and $g_{L,B} : (B, \varkappa_1^*, \varkappa_2^*) \longrightarrow (C, \varkappa_1^{**}, \varkappa_2^{**})$ is pairwise RL-fuzzy continuous, then $(g \circ f)_{L,A} : (A, \varkappa_1, \varkappa_2) \longrightarrow (C, \varkappa_1^{**}, \varkappa_2^{**})$ is pairwise RL-fuzzy semicontinuous.*

Proof. Straightforward. □

5. Pairwise Fuzzy Semi-Compactness in RL-fuzzy Bitopological Spaces

Definition 11. *For any $A \in \mathscr{V}_X^L$ and RL-fbt $(\varkappa_1, \varkappa_2)$ on A, an L-fuzzy set $B \in \mathscr{F}_X^L(A)$ is said to be a pairwise RL-fuzzy semi-compact with respect to $(\varkappa_1, \varkappa_2)$ if for each $\mathcal{R} \subseteq \mathscr{F}_X^L(A)$, the following inequality holds:*

$$\bigwedge_{D \in \mathcal{R}} (i,j)\text{-}\mathcal{S}(D) \wedge \bigwedge_{x \in X} \left(\langle_L^A B(x) \vee \bigvee_{D \in \mathcal{R}} D(x) \right) \leq \bigvee_{\mathcal{Q} \in 2^{(\mathcal{R})}} \bigwedge_{x \in X} \left(\langle_L^A B(x) \vee \bigvee_{D \in \mathcal{Q}} D(x) \right),$$

where $2^{(\mathcal{R})}$ refers to the collection of all finite subcollection of $\mathcal{R}$.

Theorem 12. *Let $A \in \mathscr{V}_X^L$ and RL-fbt $(\varkappa_1, \varkappa_2)$ on A. An L-fuzzy set $B \in \mathscr{F}_X^L(A)$ is said to be a pairwise RL-fuzzy semi-compact with respect to $(\varkappa_1, \varkappa_2)$ if for each $\mathcal{W} \subseteq \mathscr{F}_X^L(A)$, it follows that*

$$\bigvee_{D \in \mathcal{W}} (i,j)\text{-}\mathcal{S}(\langle_L^A D) \vee \bigvee_{x \in X} \left(B(x) \wedge \bigwedge_{D \in \mathcal{W}} D(x) \right) \geq \bigwedge_{\mathcal{H} \in 2^{(\mathcal{W})}} \bigvee_{x \in X} \left(B(x) \wedge \bigwedge_{D \in \mathcal{H}} D(x) \right).$$

Proof. Straightforward. □

Theorem 13. *If $A \in \mathscr{V}_X^L$, $(\varkappa_1, \varkappa_2)$ be an RL-fbt on A, and $B \in \mathscr{F}_X^L(A)$, then the next statements are equivalent:*

- **(1)** *B is a pairwise RL-fuzzy semi-compact.*
- **(2)** *For all $\alpha \in J(L)$, every strong α-remote collection $\mathcal{R}$ of B such that $\bigwedge_{D \in \mathcal{R}} (i,j)\text{-}\mathcal{S}^*(D) \not\leq \alpha'$ has a finite subcollection $\mathcal{H}$ which is a (strong) α-remote collection of B.*
- **(3)** *For all $\alpha \in J(L)$, every strong α-remote collection $\mathcal{R}$ of B such that $\bigwedge_{D \in \mathcal{R}} (i,j)\text{-}\mathcal{S}^*(D) \not\leq \alpha'$, there exists a finite subcollection $\mathcal{H}$ of $\mathcal{R}$ and $\beta \in \boldsymbol{\beta}^*(\alpha)$ such that $\mathcal{H}$ is a (strong) β-remote collection of B.*
- **(4)** *For all $\alpha \in P(L)$, every strong α-shading $\mathcal{U}$ of B such that $\bigwedge_{D \in \mathcal{U}} (i,j)\text{-}\mathcal{S}(D) \not\leq \alpha$ has a finite subcollection $\mathcal{V}$ which is a (strong) α-shading of B.*
- **(5)** *For all $\alpha \in P(L)$, each strong α-shading $\mathcal{U}$ of B such that $\bigwedge_{D \in \mathcal{U}} (i,j)\text{-}\mathcal{S}(D) \not\leq \alpha$, there exists a finite collection $\mathcal{V}$ of $\mathcal{U}$ and $\beta \in \boldsymbol{\beta}^*(\alpha)$ such that $\mathcal{V}$ is a (strong) β-shading of B.*
- **(6)** *For all $\alpha \in J(L)$ and $\beta \in \boldsymbol{\beta}^*(\alpha)$, each Q_α-cover $\mathcal{U}$ of B such that $(i,j)\text{-}\mathcal{S}(D) \geq \alpha$ (for each $D \in \mathcal{U}$) has a finite subcollection $\mathcal{V}$ which is a Q_β-cover of B.*
- **(7)** *For all $\alpha \in J(L)$ and any $\beta \in \boldsymbol{\beta}^*(\alpha)$, Q_α-cover $\mathcal{U}$ of B such that $(i,j)\text{-}\mathcal{S}(D) \geq \alpha$ (for each $D \in \mathcal{U}$) has a finite subcollection $\mathcal{V}$ which is a (strong) $\boldsymbol{\beta}_\alpha$-cover of B.*

Proof. Straightforward. □

Theorem 14. *Let $A \in \mathscr{V}_X^L$, $(\varkappa_1, \varkappa_2)$ be an RL-fbt on A, $B \in \mathscr{F}_X^L(A)$, and $\boldsymbol{\beta}(\alpha \wedge \beta) = \boldsymbol{\beta}(\alpha) \wedge \boldsymbol{\beta}(\beta)$ for all α, $\beta \in L$, then the next statements are equivalent:*

- **(1)** *B is pairwise RL-fuzzy semi-compact.*
- **(2)** *For all $\alpha \in J(L)$, every strong $\boldsymbol{\beta}_\alpha$-cover $\mathcal{U}$ of B such that $\alpha \in \boldsymbol{\beta}(\bigwedge_{D \in \mathcal{U}} (i,j)\text{-}\mathcal{S}(D))$ has a finite subcollection $\mathcal{V}$ which is a (strong) $\boldsymbol{\beta}_\alpha$-cover of B.*
- **(3)** *For all $\alpha \in J(L)$, every strong $\boldsymbol{\beta}_\alpha$-cover $\mathcal{U}$ of B such that $\alpha \in \boldsymbol{\beta}(\bigwedge_{D \in \mathcal{U}} (i,j)\text{-}\mathcal{S}(D))$, there exists a finite subcollection $\mathcal{V}$ of $\mathcal{U}$ and $\beta \in J(L)$ with $\alpha \in \boldsymbol{\beta}^*(\beta)$ such that $\mathcal{V}$ is a (strongly) $\boldsymbol{\beta}_\beta$-cover of B.*

Proof. Straightforward. □

Definition 12. *Let $A \in \mathscr{V}_X^L$, $(A, \varkappa_1, \varkappa_2)$ be an RL-bitopological space, $\alpha \in J(L)$, and $B \in \mathscr{F}_X^L(A)$. An L-fuzzy set B is called an α-pairwise RL-fuzzy semi-compact iff for any $\beta \in \boldsymbol{\beta}(\alpha)$, Q_α-(i,j)-RL-semiopen cover $\mathcal{U}$ of B has a finite subcollection $\mathcal{V}$ which is a Q_β-(i,j)-RL-semiopen cover of B.*

Theorem 15. *Let $A \in \mathscr{V}_X^L$, and $(A, \varkappa_1, \varkappa_2)$ be an RL-bitopological space. An L-fuzzy set $B \in \mathscr{F}_X^L(A)$ is pairwise RL-fuzzy semi-compact iff B is α-pairwise fuzzy semi-compact for any $\alpha \in J(L)$.*

Proof. Let B be a pairwise RL-fuzzy semi-compact, then for any $\alpha \in L_\top$, $\beta \in \boldsymbol{\beta}(\alpha)$ and $\mathcal{U}$ be any Q_α-(i,j)-RL-semiopen cover of B, we have

$$\bigwedge_{x \in X} \left(\langle_L^A B(x) \vee \bigvee_{D \in \mathcal{U}} D(x) \right) \leq \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} \bigwedge_{x \in X} \left(\langle_L^A B(x) \vee \bigvee_{D \in \mathcal{V}} D(x) \right),$$

and $\alpha \leq \bigwedge_{x\in X}\left(\langle_L^A B(x) \vee \bigvee_{D\in\mathcal{U}} D(x)\right)$, so that

$$\alpha \leq \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} \bigwedge_{x\in X}\left(\langle_L^A B(x) \vee \bigvee_{D\in\mathcal{V}} D(x)\right).$$

By $\beta \in \boldsymbol{\beta}(\alpha)$, we have

$$\beta \leq \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} \bigwedge_{x\in X}\left(\langle_L^A B(x) \vee \bigvee_{D\in\mathcal{V}} D(x)\right).$$

Then there is $\mathcal{V} \in 2^{(\mathcal{U})}$ with $\beta \leq \bigwedge_{x\in X}\left(\langle_L^A B(x) \vee \bigvee_{D\in\mathcal{V}} D(x)\right)$. This proves that $\mathcal{V}$ is Q_β-(i,j)-RL-semiopen cover of B.

Conversely, suppose that each Q_α-(i,j)-RL-semiopen cover $\mathcal{U}$ of B has a finite subcollction $\mathcal{V}$ which is a Q_β-(i,j)-RL-semiopen cover of B for all $\beta \in \boldsymbol{\beta}(\alpha)$. Hence, $\alpha \leq \bigwedge_{x\in X}\left(\langle_L^A B(x) \vee \bigvee_{D\in\mathcal{U}} D(x)\right)$ yields to $\beta \leq \bigwedge_{x\in X}\left(\langle_L^A B(x) \vee \bigvee_{D\in\mathcal{U}} D(x)\right)$. Therefore $\alpha \leq \bigwedge_{x\in X}\left(\langle_L^A B(x) \vee \bigvee_{D\in\mathcal{U}} D(x)\right)$ implies that $\beta \leq \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} \bigwedge_{x\in X}\left(\langle_L^A B(x) \vee \bigvee_{D\in\mathcal{U}} D(x)\right)$. So $\alpha \leq \bigwedge_{x\in X}\left(\langle_L^A B(x) \vee \bigvee_{D\in\mathcal{U}} D(x)\right)$ implies that

$$\bigvee_{\beta\in\boldsymbol{\beta}(\alpha)} \beta \leq \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} \bigwedge_{x\in X}\left(\langle_L^A B(x) \vee \bigvee_{D\in\mathcal{U}} D(x)\right),$$

i.e,

$$\alpha \leq \bigwedge_{x\in X}\left(\langle_L^A B(x) \vee \bigvee_{D\in\mathcal{U}} D(x)\right),$$

implies that

$$\alpha \leq \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} \bigwedge_{x\in X}\left(\langle_L^A B(x) \vee \bigvee_{D\in\mathcal{U}} D(x)\right).$$

Hence

$$\bigwedge_{x\in X}\left(\langle_L^A B(x) \vee \bigvee_{D\in\mathcal{U}} D(x)\right) \leq \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} \bigwedge_{x\in X}\left(\langle_L^A B(x) \vee \bigvee_{D\in\mathcal{V}} D(x)\right).$$

□

Theorem 16. *Let $A \in \mathscr{V}_X^L$, and $(A, \varkappa_1, \varkappa_2)$ be an RL-fuzzy bitopological space. An L-fuzzy set $B \in \mathscr{F}_X^L(A)$ is a pairwise RL-fuzzy semi-compact in $(A, \varkappa_1, \varkappa_2)$ if and only if B is an α-pairwise RL-fuzzy semi-compact in $(A, \varkappa_{1_{[\alpha]}}, \varkappa_{2_{[\alpha]}})$ for all $\alpha \in J(L)$.*

Proof. Let $B \in \mathscr{F}_X^L(A)$ be a pairwise RL-fuzzy semi-compact in $(A, \varkappa_1, \varkappa_2)$, then for each collection $\mathcal{U} \subseteq \mathscr{F}_X^L(A)$, we have

$$\bigwedge_{D\in\mathcal{U}} (i,j)\text{-}\mathcal{S}(D) \wedge \bigwedge_{x\in X}\left(\langle_L^A B(x) \vee \bigvee_{D\in\mathcal{U}} D(x)\right) \leq \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} \bigwedge_{x\in X}\left(\langle_L^A B(x) \vee \bigvee_{D\in\mathcal{V}} D(x)\right).$$

Then for all $\alpha \in J(L)$ and $\mathcal{U} \subseteq ((i,j)\text{-}\mathcal{S})_{[\alpha]}$, we have that

$$\alpha \leq \bigwedge_{x\in X}\left(\langle_L^A B(x) \vee \bigvee_{D\in\mathcal{U}} D(x)\right) \Rightarrow \alpha \leq \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} \bigwedge_{x\in X}\left(\langle_L^A B(x) \vee \bigvee_{D\in\mathcal{V}} D(x)\right).$$

Hence, for every $\beta \in \boldsymbol{\beta}(\alpha)$, there is $\mathcal{V} \in 2^{(\mathcal{U})}$ with $\beta \leq \bigwedge_{x\in X}\left(\langle_L^A B(x) \vee \bigvee_{D\in\mathcal{V}} D(x)\right)$. i.e., for all $\alpha \in J(L)$ and $\beta \in \boldsymbol{\beta}(\alpha)$, every Q_α-(i,j)-RL-semiopen cover $\mathcal{U}$ of B in $(A, \varkappa_{1_{[\alpha]}}, \varkappa_{2_{[\alpha]}})$

has a finite subcollection $\mathcal{V}$ which is a Q_α-$(i.j)$-RL-semiopen cover. Then for every $\alpha \in J(L)$, B is α-pairwise RL-fuzzy semi-compact in $(A, \varkappa_{1[\alpha]}, \varkappa_{2[\alpha]})$.

Conversely, suppose that for every $\alpha \in J(L)$, B is α-pairwise RL-fuzzy semi-compact in $(A, \varkappa_{1[\alpha]}, \varkappa_{2[\alpha]})$ and let $\alpha \leq \bigwedge_{D\in\mathcal{U}}(i,j)\text{-}\mathcal{S}(D) \wedge \bigwedge_{x\in X}\big(\langle^A_L B(x) \vee \bigvee_{D\in\mathcal{U}} D(x)\big)$ for every $\mathcal{U} \subseteq \mathscr{F}^L_X(A)$, then $\alpha \leq \bigwedge_{D\in\mathcal{U}}(i,j)\text{-}\mathcal{S}(D)$ and $\alpha \leq \bigwedge_{x\in X}\big(\langle^A_L B(x) \vee \bigvee_{D\in\mathcal{U}} D(x)\big)$, i.e, $\mathcal{U} \subseteq ((i,j)\text{-}\mathcal{S})_{[\alpha]}$ and $\alpha \leq \bigwedge_{x\in X}\big(\langle^A_L B(x) \vee \bigvee_{D\in\mathcal{U}} D(x)\big)$. Hence for all $\beta \in \boldsymbol{\beta}(\alpha)$, there is $\mathcal{V} \in 2^{(\mathcal{U})}$ with

$$\beta \leq \bigwedge_{x\in X}\left(\langle^A_L B(x) \vee \bigvee_{D\in\mathcal{V}} D(x)\right).$$

So that

$$\alpha \leq \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} \bigwedge_{x\in X}\left(\langle^A_L B(x) \vee \bigvee_{D\in\mathcal{V}} D(x)\right).$$

Then B is a pairwise RL-fuzzy semi-compact in $(A, \varkappa_1, \varkappa_2)$. □

Lemma 2. *Let $A \in \mathscr{V}^L_X$, and $(A, \varkappa_1, \varkappa_2)$ be an RL-bitopological space, $\alpha \in J(L)$, and $B, C \in \mathscr{F}^L_X(A)$. If B is α-pairwise RL-fuzzy semi-compact and C is (i,j)-RL-semiclosed, then $B \wedge C$ is α-pairwise RL-fuzzy semi-compact.*

The next theorem is an immediate consequence from Lemma 2:

Theorem 17. *Let $A \in \mathscr{V}^L_X$, and $(A, \varkappa_1, \varkappa_2)$ be an RL-fuzzy bitopological space, and $B, C \in \mathscr{F}^L_X(A)$. If B is a pairwise RL-fuzzy semi-compact and (i,j)-$\mathcal{S}^*(C) = \top$, then $B \wedge C$ is a pairwise RL-fuzzy semi-compact.*

Lemma 3. *Let $A \in \mathscr{V}^L_X$, and $(A, \varkappa_1, \varkappa_2)$ be an RL-bitopological space, $\alpha \in J(L)$, and $B, C \in \mathscr{F}^L_X(A)$. If B, C are α-pairwise RL-fuzzy semi-compact, then $B \vee C$ is α-pairwise RL-fuzzy semi-compact.*

Theorem 18. *Let $A \in \mathscr{V}^L_X$, and $(A, \varkappa_1, \varkappa_2)$ be an RL-fuzzy bitopological space, and $B, C \in \mathscr{F}^L_X(A)$. If B, C are pairwise RL-fuzzy semi-compact, then $B \vee C$ is pairwise RL-fuzzy semi-compact.*

Proof. Straightforward. □

Lemma 4. *Let $A \in \mathscr{V}^L_X$, $B \in \mathscr{V}^L_Y$, and $(A, \varkappa_1, \varkappa_2)$, $(B, \varkappa^*_1, \varkappa^*_2)$ be RL-bts's on A and B, respectively, $\alpha \in J(L)$, $D \in \mathscr{F}^L_X(A)$, and $f_{L,A} : A \longrightarrow B$ be a pairwise RL-irresolute mapping. If D is α-pairwise fuzzy semi-compact in $(A, \varkappa_1, \varkappa_2)$, then $f^{\rightarrow}_{L,A}(D)$ is α-pairwise fuzzy semi-compact in $(B, \varkappa^*_1, \varkappa^*_2)$.*

Theorem 19. *Let $A \in \mathscr{V}^L_X$, $B \in \mathscr{V}^L_Y$, and $(A, \varkappa_1, \varkappa_2)$, $(B, \varkappa^*_1, \varkappa^*_2)$ be two RL-fbts's on A and B, respectively, $D \in \mathscr{F}^L_X(A)$, and $f_{L,A} : A \longrightarrow B$ be a pairwise RL-fuzzy irresolute mapping. If D is a pairwise RL-fuzzy semi-compact in $(A, \varkappa_1, \varkappa_2)$, then $f^{\rightarrow}_{L,A}(D)$ is a pairwise RL-fuzzy semi-compact in $(B, \varkappa^*_1, \varkappa^*_2)$.*

Proof. Let D be a pairwise RL-fuzzy semi -compact in $(A, \varkappa_1, \varkappa_2)$. Based on Theorem 16, we have D is α-pairwise fuzzy semi-compact in $(A, \varkappa_{1[\alpha]}, \varkappa_{2[\alpha]})$ for all $\alpha \in J(L)$. By Theorem 16, $f_{L,A} : (A, \varkappa_{1[\alpha]}, \varkappa_{2[\alpha]}) \rightarrow (B, \varkappa^*_{1[\alpha]}, \varkappa^*_{2[\alpha]})$ is pairwise RL-irresolute. Therefore by using Lemma 4, $f^{\rightarrow}_{L,A}(D)$ is α-pairwise RL-fuzzy semi-compact in $(B, \varkappa^*_{1[\alpha]}, \varkappa^*_{2[\alpha]})$. Then $f^{\rightarrow}_{L,A}(D)$ is pairwise RL-fuzzy semi-compact in $(B, \varkappa^*_1, \varkappa^*_2)$. □

6. Conclusions

The idea of RL-fuzzy bitopological spaces extends the idea of RL-fuzzy topological spaces and as well as the idea of L-fuzzy topological spaces in Kubiak-Šostak's sense. If we restrict the newly defined concepts by assuming that A equal to $\top_X$, we get L-fuzzy bitopological spaces. On the other hand, if we consider the case of $i = j$, we get L-fuzzy topological spaces in Kubiak-Šostak's sense [20,21].

In this paper, we initiated the idea of (i, j)-RL-semiopen gradation of L-fuzzy sets in RL-fuzzy bitopological spaces based on the concept of pseudo-complement. We studied different properties regarding the degree of (i, j)-RL-semiopenness of L-fuzzy set. Moreover, we elaborated pairwise RL-fuzzy semicontinuous and pairwise RL-fuzzy irresolute functions and discussed some of their elementary properties based on the (i, j)-RL-semiopen gradation. Further, the pairwise RL-fuzzy semi-compactness of an L-fuzzy set in RL-fuzzy bitopological spaces is defined and explained.

In the future, we are focusing on representing several kinds of openness as gradation in RL-fuzzy bitopology and use it to extend the corresponding kinds of continuity, separation, connectedness, and compactness.

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