symmetry

MDPI

Article

Tricomi's Method for the Laplace Transform and Orthogonal Polynomials

Paolo Emilio Ricci [1,*], Diego Caratelli [2,3] and Francesco Mainardi [4]

[1] Section of Mathematics, International Telematic University UniNettuno, Corso Vittorio Emanuele II 39, 00186 Rome, Italy
[2] Department of Research and Development, The Antenna Company, High Tech Campus 29, 5656 AE Eindhoven, The Netherlands; d.caratelli@tue.nl
[3] Department of Electrical Engineering, Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands
[4] Department of Physics and Astronomy, University of Bologna, Via Irnerio 46, 40126 Bologna, Italy; francesco.mainardi@bo.infn.it
* Correspondence: p.ricci@uninettunouniversity.net or paoloemilioricci@gmail.com

Abstract: Tricomi's method for computing a set of inverse Laplace transforms in terms of Laguerre polynomials is revisited. By using the more recent results about the inversion and the connection coefficients for the series of orthogonal polynomials, we find the possibility to extend the Tricomi method to more general series expansions. Some examples showing the effectiveness of the considered procedure are shown.

Keywords: resolvent; Laplace transform; Laguerre polynomials; connection coefficients

check for updates

Citation: Ricci, P.E.; Caratelli, D.; Mainardi, F. Tricomi's Method for the Laplace Transform and Orthogonal Polynomials. *Symmetry* **2021**, *13*, 589. https://doi.org/10.3390/sym13040589

Academic Editor: Dorian Popa

Received: 10 March 2021
Accepted: 1 April 2021
Published: 2 April 2021

Publisher's Note: MDPI stays neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Copyright: © 2021 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https://creativecommons.org/licenses/by/4.0/).

1. Introduction

In three notes presented to the Accademia dei Lincei in 1935 [1,2], Francesco G. Tricomi introduced a method for the computation of the Laplace transform of functions that can be developed in a series of Laguerre polynomials. Precisely, he proved the following proposition:

Proposition 1. *If the analytic function $F(s)$ is regular at infinity and we can find a real number h such that it can be represented with a series of the form*

$$F(s) = \frac{1}{s+h} \sum_{n=0}^{\infty} a_n \left(\frac{s+h-1}{s+h} \right)^n , \tag{1}$$

then it is the Laplace transform of the sum of the series of Laguerre polynomials

$$f(t) = e^{-ht} \sum_{n=0}^{\infty} a_n \, L_n(t) , \tag{2}$$

which is absolutely and uniformly convergent for $t > 0$.

In particular, for $h = 0$, that is, avoiding the shift that follows from a basic rule of the Laplace transform, Tricomi found the functions pair $(F(s), f(t))$:

$$F(s) = \frac{1}{s} \sum_{n=0}^{\infty} a_n \left(\frac{s-1}{s} \right)^n \quad \leftrightarrow \quad f(t) = \sum_{n=0}^{\infty} a_n \, L_n(t) . \tag{3}$$

We recall that the Laguerre polynomials are obtained by letting $\alpha = 0$ in the general formula

$$L_n^{(\alpha)}(x) = \frac{(\alpha+1)_n}{n!}\, {}_1F_1\left(\begin{matrix} -n \\ \alpha+1 \end{matrix}; x\right),$$

where $(a)_n = a(a+1)\cdots(a+n-1)$ denotes the rising factorial and, in particular, it results $(1)_n = n!$.

Tricomi says that the correspondence of the above functions was suggested to him by the series expansion of the Bessel function, in terms of Laguerre polynomials:

$$J_0(2\sqrt{x}) = \frac{1}{e}\sum_{k=0}^{\infty}\frac{L_k(x)}{k!}\,, \tag{4}$$

where

$$L_k(x) = \sum_{n=0}^{k}\binom{k}{n}\frac{(-1)^n x^n}{n!}\,. \tag{5}$$

More recently it was observed [3] that the same Bessel function admits another expansion, making use of a power series, the so-called Laguerre type exponential, namely:

$$J_0(2\sqrt{x}) = \sum_{k=0}^{\infty}\frac{(-1)^k x^k}{(k!)^2}\,. \tag{6}$$

In fact, it turns out that the expansions (4) and (6) are equivalent, since:

$$J_0(2\sqrt{x}) = \frac{1}{e}\sum_{k=0}^{\infty}\frac{1}{k!}\sum_{\ell=0}^{k}\binom{k}{\ell}\frac{(-1)^\ell x^\ell}{\ell!} = \frac{1}{e}\sum_{\ell=0}^{\infty}\sum_{k=\ell}^{\infty}\binom{k}{\ell}\frac{(-1)^\ell x^\ell}{k!\,\ell!} =$$

$$= \frac{1}{e}\sum_{\ell=0}^{\infty}\sum_{k=\ell}^{\infty}\frac{(-1)^\ell x^\ell}{(k-\ell)!\,(\ell!)^2} = \frac{1}{e}\sum_{\ell=0}^{\infty}\frac{(-1)^\ell x^\ell}{(\ell!)^2}\sum_{k=\ell}^{\infty}\frac{1}{(k-\ell)!} = \sum_{\ell=0}^{\infty}\frac{(-1)^\ell x^\ell}{(\ell!)^2}\,.$$

It is worth noting that the Laguerre-type exponentials were previously considered by Le Roi [4] and used in [5,6] in the framework of generalised Wright functions. Actually they are a particular case of more general special functions previously considered by V. Kiryakova [7].

As it is possible to transform the expansions in Laguerre polynomials, by using the inversion or the connection coefficients, into different expansions in terms of other polynomial sets, as it is shown in many articles [8–11], the above considerations suggest that we extend the Tricomi method in order to find different Laplace function pairs. This is done even on the basis of preceding results in [12] and the connection of the Laplace transform with orthogonal polynomials, reported in [13].

In what follows, we first apply the inversion coefficients in order to find the Laplace transform corresponding to a given power series, then we apply the same methodology to the series of orthogonal polynomials.

2. Tricomi LT of Laguerre Series

We use the classical notations. We denote the Laplace transform by *LT* by using for it the symbol $\mathcal{L}$.

$$\mathcal{L}(f(t)) := \int_0^{\infty} \exp^{-1}(s\,t) f(t)\,dt = F(s)\,, \tag{7}$$

and for the Laguerre polynomials of degree n the symbol L_n,

$$\mathcal{L}(L_n(t)) := \int_0^\infty \exp^{-1}(s\,t) L_n(t)\,dt = F(s) = \frac{1}{s}\left(\frac{s-1}{s}\right)^n. \tag{8}$$

Then the Proposition 1, asserts that under the conditions recalled in [1], assuming:

$$f(t) = e^{-h\,t}\sum_{n=0}^{\infty} a_n\,L_n(t)\,, \tag{9}$$

it results:

$$\mathcal{L}(f(t)) := \frac{1}{s+h}\sum_{n=0}^{\infty} a_n\left(\frac{s+h-1}{s+h}\right)^n. \tag{10}$$

3. LT of Power Series

The inversion coefficients of Laguerre polynomials are reported in [8,9], in the form:

$$t^n = \sum_{m=0}^{n} I_m(n) L_m(t)\,, \tag{11}$$

where

$$I_m(n) = (-n)_m \frac{1_n}{1_m} = (-1)^m \binom{n}{m} n!\,. \tag{12}$$

Then it results:

$$\begin{aligned}\sum_{n=0}^{\infty} a_n t^n &= \sum_{n=0}^{\infty}\sum_{m=0}^{n} a_n I_m(n) L_m(t) = \\ &= \sum_{m=0}^{\infty}\left(\sum_{n=m}^{\infty} a_n\,I_m(n)\right) L_m(t) = \sum_{m=0}^{\infty} b_m L_m(t)\,,\end{aligned} \tag{13}$$

where

$$b_m = \sum_{n=m}^{\infty} a_n\,I_m(n) \tag{14}$$

Then we can proclaim the result

Theorem 1. *If the analytic function $F(s)$ is regular at infinity, decreasing at least as 1/s and such that it can be represented with the power series:*

$$F(s) = \frac{1}{s+h}\sum_{m=0}^{\infty} b_m\left(\frac{s+h-1}{s+h}\right)^m, \tag{15}$$

then it is the Laplace transform of the power series

$$f(t) = e^{-h\,t}\sum_{n=0}^{\infty} a_n\,t^n\,, \tag{16}$$

where the coefficients a_n and b_m are linked by Equation (14), with the $I_m(n)$ in Equation (12).

Therefore, under the condition (14), we have found the pair:

$$F(s) = \frac{1}{s+h} \sum_{m=0}^{\infty} b_m \left(\frac{s+h-1}{s+h} \right)^m \quad \leftrightarrow \quad f(t) = e^{-ht} \sum_{n=0}^{\infty} a_n t^n . \tag{17}$$

4. LT of Orthogonal Polynomial Expansions

Suppose now that a function $f(t)$ satisfies the conditions:

1. it admits the Laplace transform,
2. by choosing a suitable h, it admits a series expansion with respect to a given orthogonal polynomial system $\{Q_n(t)\}$, of the type:

$$f(t) = e^{-ht} \sum_{n=0}^{\infty} c_n Q_n(t) , \tag{18}$$

where the coefficients c_n are known.

The polynomials $Q_n(t)$ can be represented in terms of a different orthogonal polynomials system $P_m(t)$ by means of the equation

$$Q_n(t) = \sum_{m=0}^{n} C_m(n) P_m(t) , \tag{19}$$

where the connection coefficients $C_m(n)$ are reported in [8].

In particular we can choose the basis of Laguerre polynomials $P_m(t) \equiv L_m(t)$, so that

$$\begin{aligned} f(t) = e^{-ht} \sum_{n=0}^{\infty} c_n \sum_{m=0}^{n} C_m(n) L_m(t) = e^{-ht} \sum_{m=0}^{\infty} \left(\sum_{n=m}^{\infty} c_n C_m(n) \right) L_m(t) = \\ = e^{-ht} \sum_{m=0}^{\infty} d_m L_m(t) , \end{aligned} \tag{20}$$

where

$$d_m = \sum_{n=m}^{\infty} c_n C_m(n) . \tag{21}$$

After this transformation, the function $f(t)$ is expressed in terms of Laguerre polynomials and the Tricomi method gives the result:

Theorem 2. *If the analytic function $F(s)$ is regular at infinity, decreasing at least as $1/s$ and such that, for a suitable choice of the parameter h, it can be represented with the power series:*

$$F(s) = \frac{1}{s+h} \sum_{m=0}^{\infty} d_m \left(\frac{s+h-1}{s+h} \right)^m , \tag{22}$$

then it is the Laplace transform of the power series

$$f(t) = e^{-ht} \sum_{n=0}^{\infty} c_n Q_n(t) , \tag{23}$$

where the coefficients c_n and d_m are linked by Equation (21), with the $C_m(n)$, reported in [8], depending on the considered polynomial set $Q_n(t)$.

Therefore, under the condition (21), we have found the pair:

$$F(s) = \frac{1}{s+h} \sum_{m=0}^{\infty} d_m \left(\frac{s+h-1}{s+h} \right)^m \quad \leftrightarrow \quad f(t) = e^{-ht} \sum_{n=0}^{\infty} c_n Q_n(t) . \tag{24}$$

Remark 1. *In [8] the inversion and connection coefficients are reported for many other polynomial sets, including the generalized Laguerre polynomials $L_n^{(\alpha)}(t)$; however, we could avoid the use of these polynomials since the result stated in this section can be applied even to them.*

Example 1. *As an example, assuming $Q_n(t) = H_n(t)$, that is the Hermite polynomials, according to [9], p. 687, it results:*

$$C_m(n) = \frac{1_n(-n)_m\, 2^n}{1_m}\, {}_2F_2\left(\begin{matrix}\frac{m-n}{2}, \frac{m-n+1}{2} \\ \frac{-n}{2}, \frac{-n+1}{2}\end{matrix}\; ; -\frac{1}{4}\right) \tag{25}$$

that is:

$$C_m(n) = (-1)^m\, 2^n n! \binom{n}{m} \sum_{k=0}^{\infty} \frac{(m-n)_k (m-n+1)_k}{(-n)_k (-n+1)_k} \frac{(-1)^k}{4^k\, k!}\,, \tag{26}$$

and consequently, by Theorem 2, it results:

$$\mathcal{L}\left(e^{-h\,t} \sum_{n=0}^{\infty} c_n H_n(t)\right) := \frac{1}{s+h} \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} c_n\, C_m(n) \left(\frac{s+h-1}{s+h}\right)^m, \tag{27}$$

where the connection coefficients $C_m(n)$ are explicitly given above.

5. Numerical Results

In what follows we show some numerical examples of the considered procedure, assuming $h = 0$, since a different choice does not affect the results.

The experiments have highlighted the benefit, in terms of numerical accuracy, arising from a direct numerical implementation rather than the use of automatic routines embedded in the Mathematica© computing system.

5.1. Example: LT of $e^{\mathrm{i}\pi t}$

Let us consider the function:

$$f(t) = e^{\mathrm{i}\pi t}\,, \tag{28}$$

whose Laplace transform can be trivially evaluated as:

$$F(s) = \mathcal{L}(f(t))(s) = \int_0^{\infty} \exp^{-1}(s\,t)\, f(t)\, dt = \frac{1}{s - \mathrm{i}\pi}\,. \tag{29}$$

The Laguerre polynomial series approximant of $f(t)$ is given by:

$$f_M(t) = \sum_{m=0}^{M} a_m\, L_m(t)\,, \tag{30}$$

with:

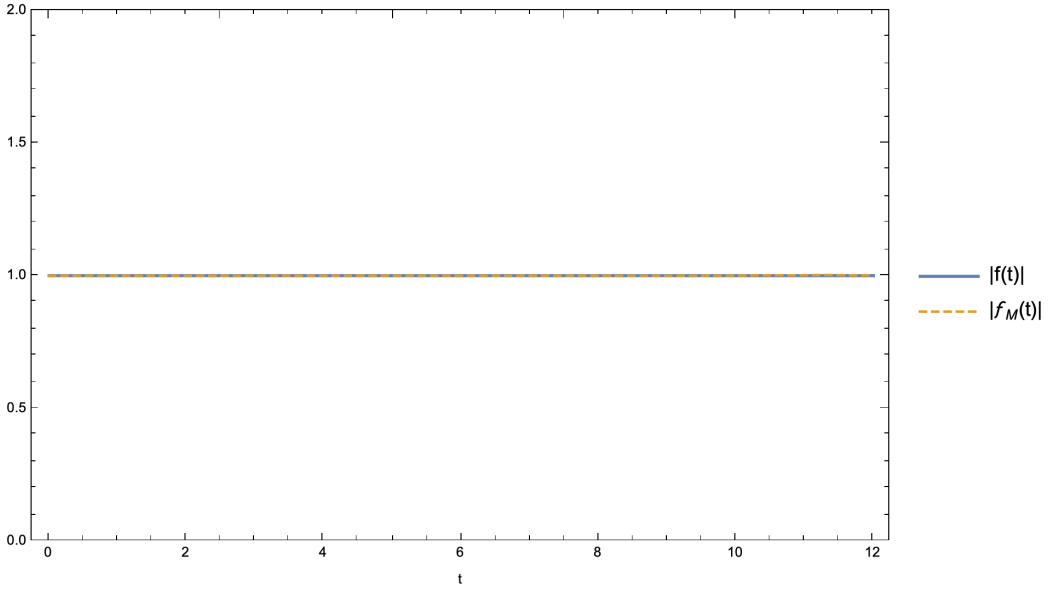
$$a_m = \int_0^{\infty} e^{-t}\, f(t)\, L_m(t)\, dt = \frac{\mathrm{i}}{\pi}\left(1 + \frac{\mathrm{i}}{\pi}\right)^{-m-1}. \tag{31}$$

Hence, using Tricomi's method, the Laplace transform $F(s)$ can be approximated as:

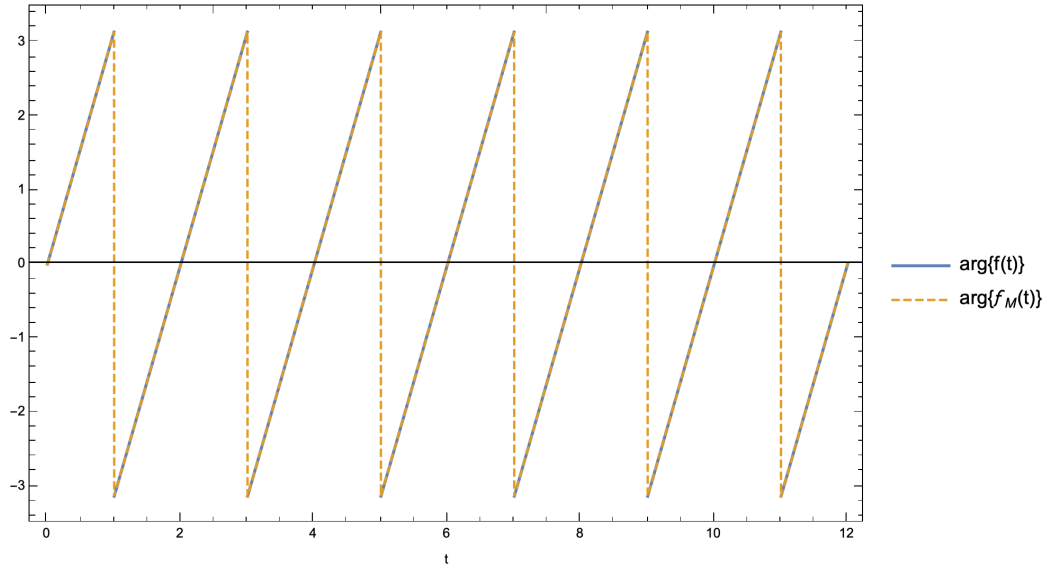
$$F_M(s) = \frac{1}{s} \sum_{m=0}^{M} a_m \left(\frac{s-1}{s}\right)^m. \tag{32}$$

Upon selecting the expansion order $M = 100$, one can readily verify that the approximant $f_M(t)$ in (30) and (31) is characterized by the distribution shown in Figure 1, whereas

the relevant LT in (32) shows, along the cut plane $\sigma = \mathrm{Re}\{s\} = 1$, the behavior plotted in Figure 2. It can be noticed that the agreement between the exact expression of $F(s)$ in (29) and its Tricomi's approximation $F_M(s)$ is excellent.



(**a**)



(**b**)

Figure 1. Magnitude (**a**) and argument (**b**) of the function $f(t) = e^{\mathrm{i}\pi t}$ as compared to the relevant Laguerre polynomial series approximant $f_M(t)$ of order $M = 100$.

5.2. Example: LT of $J_0\left(2\sqrt{t}\right)$

Let us consider the function:

$$f(t) = J_0\left(2\sqrt{t}\right), \tag{33}$$

whose Laplace transform can be evaluated as:

$$F(s) = \mathcal{L}(f(t))(s) = \int_0^\infty \exp^{-1}(s\,t)\, f(t)\, dt = \frac{e^{-1/s}}{s}. \tag{34}$$

The Laguerre polynomial series approximant of $f(t)$ is given by:

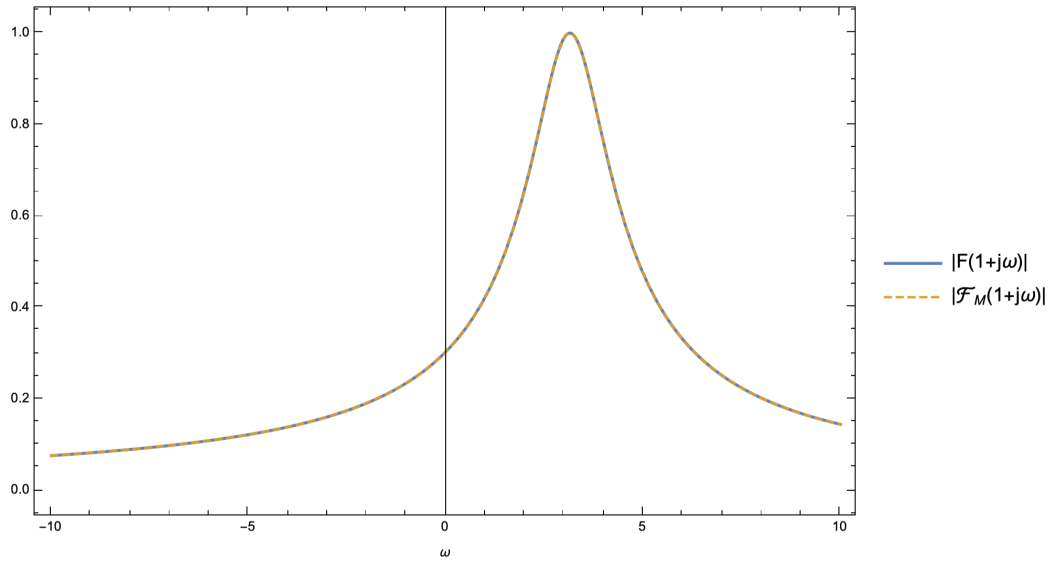
$$f_M(t) = \sum_{m=0}^{M} a_m \, L_m(t) \,, \tag{35}$$

with:

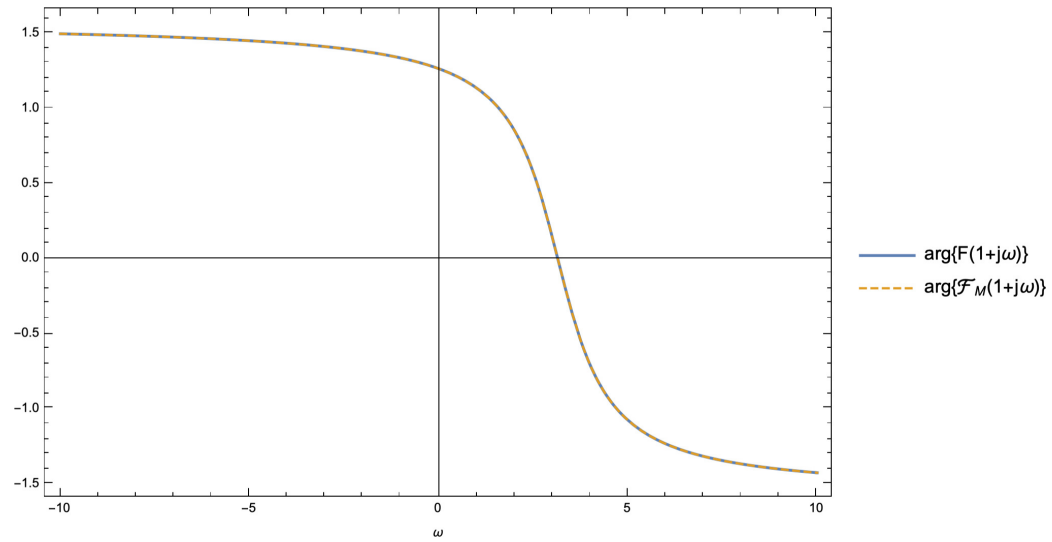
$$a_m = \int_0^\infty e^{-t} \, f(t) \, L_m(t) \, dt = \frac{1}{e \, m!} \,. \tag{36}$$

Hence, using Tricomi's method, the Laplace transform $F(s)$ can be approximated as:

$$F_M(s) = \frac{1}{s} \sum_{m=0}^{M} a_m \left(\frac{s-1}{s} \right)^m . \tag{37}$$



(**a**)



(**b**)

Figure 2. Magnitude (**a**) and argument (**b**) of the Laplace transform relevant to $f(t) = e^{\mathrm{i}\pi t}$ as a function of the complex variable $s = \sigma + \mathrm{i}\,\omega$ for $\sigma = 1$ when computed using the exact integral expression $F(s)$ and the corresponding Tricomi's series approximant $F_M(t)$ of order $M = 100$.

Upon selecting the expansion order $M = 100$, one can readily verify that the approximant $f_M(t)$ in (35) and (36) is characterized by the distribution shown in Figure 3, whereas the relevant LT in (37) shows, along the cut plane $\omega = \mathrm{Im}\{s\} = 1$, the behavior plotted in Figure 4. It can be noticed that the agreement between the exact expression of $F(s)$ in (34) and its Tricomi's approximation $F_M(s)$ is excellent.

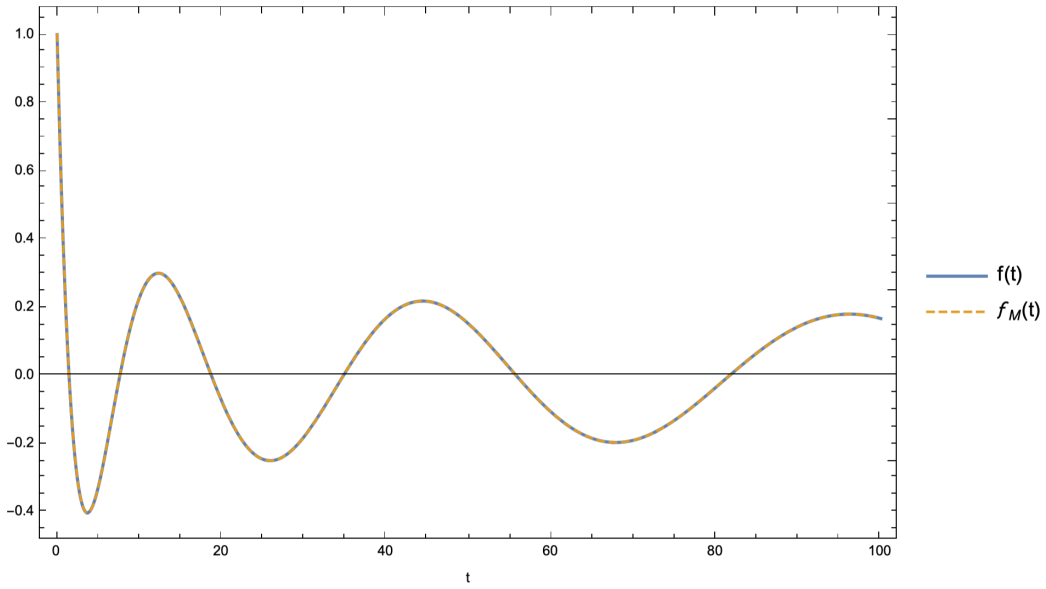


Figure 3. Distribution of the function $f(t) = J_0\left(2\sqrt{t}\right)$ as compared to the relevant Laguerre polynomial series approximant $f_M(t)$ of order $M = 100$.

5.3. Example: LT of $e^t\,\Gamma(t)$

Let us consider the function:

$$f(t) = e^t\,\Gamma(t)\,, \tag{38}$$

whose Laplace transform can be evaluated as:

$$F(s) = \mathcal{L}(f(t))(s) = \int_0^\infty \exp^{-1}(s\,t)\,f(t)\,dt = \frac{\log(s)}{s-1}\,. \tag{39}$$

The Laguerre polynomial series approximant of $f(t)$ is given by:

$$f_M(t) = \sum_{m=0}^{M} a_m\,L_m(t)\,, \tag{40}$$

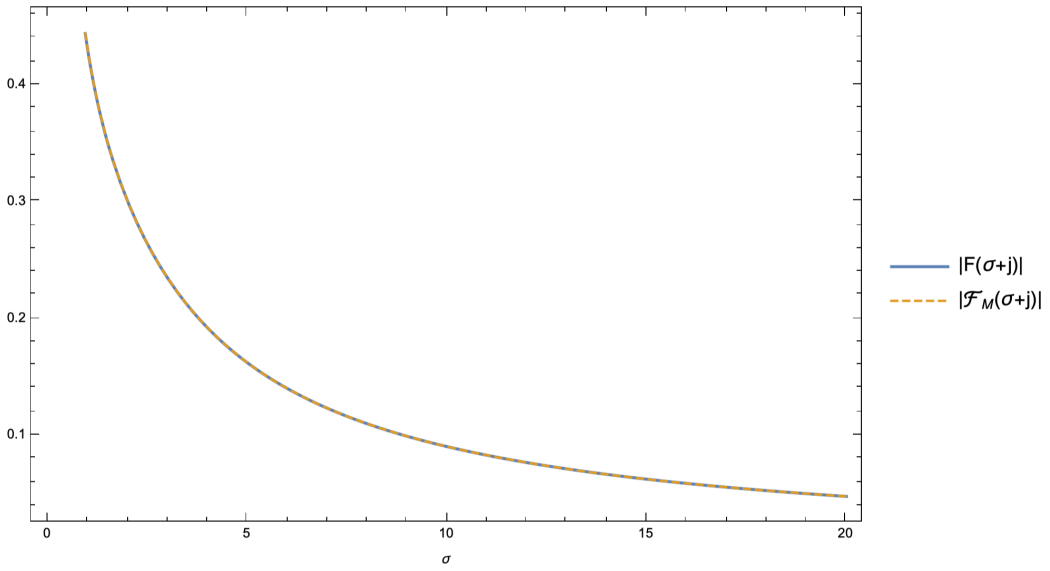
with:

$$a_m = \int_0^\infty e^{-t}\,f(t)\,L_m(t)\,dt = \frac{1}{m+1}\,. \tag{41}$$

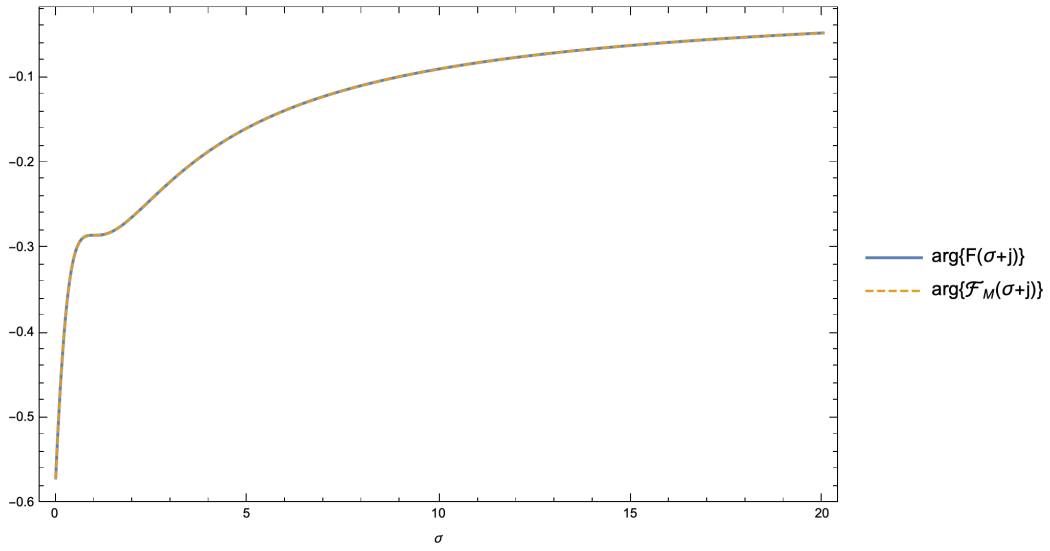
Hence, using Tricomi's method, the Laplace transform $F(s)$ can be approximated as:

$$F_M(s) = \frac{1}{s}\sum_{m=0}^{M} a_m\left(\frac{s-1}{s}\right)^m. \tag{42}$$

Upon selecting the expansion order $M = 200$, one can readily verify that the approximant $f_M(t)$ in (40) and (41) is characterized by the distribution shown in Figure 5, whereas the relevant LT in (42) shows, along the cut plane $\sigma = \mathrm{Re}\{s\} = 1$, the behavior plotted in Figure 6. It can be noticed that the agreement between the exact expression of $F(s)$ in (39) and its Tricomi's approximation $F_M(s)$ is excellent.



(**a**)



(**b**)

Figure 4. Magnitude (**a**) and argument (**b**) of the Laplace transform relevant to $f(t) = J_0\left(2\sqrt{t}\right)$ as a function of the complex variable $s = \sigma + \mathrm{i}\,\omega$ for $\omega = 1$ when computed using the exact integral expression $F(s)$ and the corresponding Tricomi's series approximant $F_M(t)$ of order $M = 100$.

5.4. Example: LT of $\sin(t)$

Let us consider the function:

$$f(t) = \sin(t)\,, \tag{43}$$

whose Laplace transform can be trivially evaluated as:

$$F(s) = \mathcal{L}(f(t))(s) = \int_0^\infty \exp^{-1}(s\,t)\,f(t)\,dt = \frac{1}{s^2+1}\,. \tag{44}$$

The Maclaurin series approximant of $f(t)$ is given by:

$$f_M(t) = \sum_{m=0}^{M} b_m\,t^m\,, \tag{45}$$

where:

$$b_m = \frac{f^{(m)}(0)}{m!} = \frac{1}{m!}\sin\left(\frac{m\pi}{2}\right), \tag{46}$$

with $f^{(m)}(0)$ denoting the m-th derivative of $f(t)$ in the origin. In the light of (11) and (12), the expansion (45) can be recast in terms of Laguerre polynomials as follows:

$$f_M(t) = \sum_{m=0}^{M} a_m\, L_m(t)\,, \tag{47}$$

where:

$$a_m = \sum_{n=m}^{M} b_n\, I_m(n)\,. \tag{48}$$

Hence, using Tricomi's method, the Laplace transform $F(s)$ can be approximated as:

$$F_M(s) = \frac{1}{s}\sum_{m=0}^{M} a_m \left(\frac{s-1}{s}\right)^m. \tag{49}$$

Upon selecting the expansion order $M = 100$, one can readily verify that the approximant $f_M(t)$ in (47) and (48) is characterized by the distribution shown in Figure 7, whereas the relevant LT in (49) shows, along the cut plane $\omega = \mathrm{Im}\{s\} = 1$, the behavior plotted in Figure 8. It can be noticed that the agreement between the exact expression of $F(s)$ in (44) and its Tricomi's approximation $F_M(s)$ is excellent.

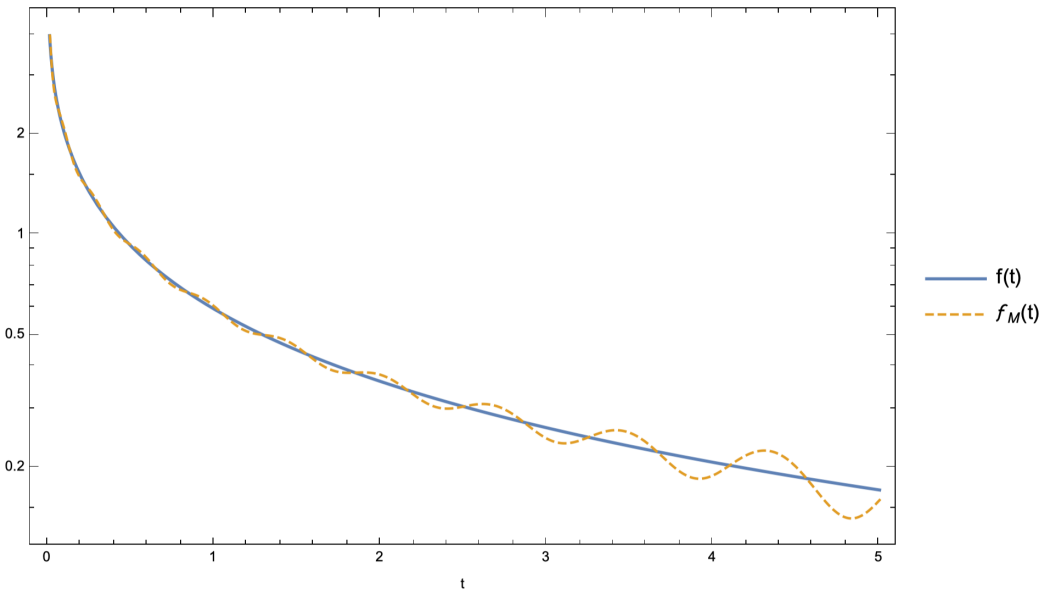


Figure 5. Distribution of the function $f(t) = e^t\,\Gamma(t)$ as compared to the relevant Laguerre polynomial series approximant $f_M(t)$ of order $M = 200$.

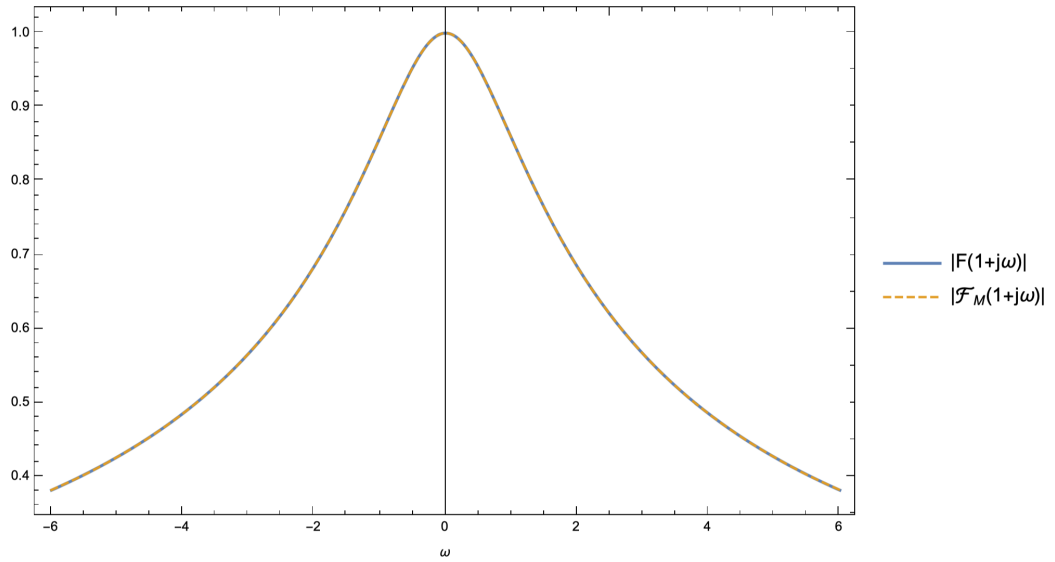
5.5. Example: LT of e^{-t^2}

Let us consider the function:

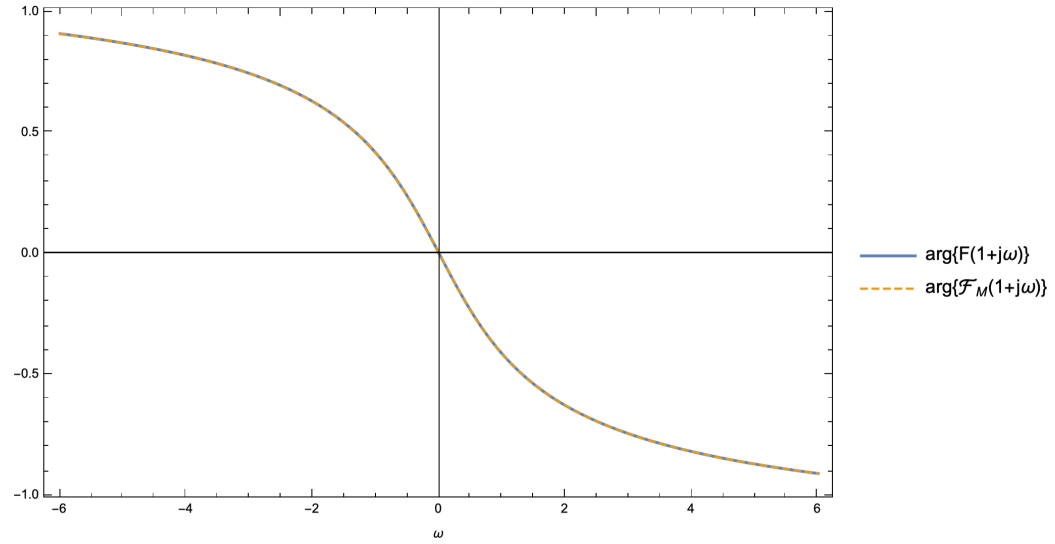
$$f(t) = e^{-t^2}\,, \tag{50}$$

whose Laplace transform can be easily evaluated as:

$$F(s) = \mathcal{L}(f(t))(s) = \int_0^\infty \exp^{-1}(s\,t)\, f(t)\, dt = \frac{1}{2}\sqrt{\pi}e^{\frac{s^2}{4}}\,\mathrm{erfc}\left(\frac{s}{2}\right). \tag{51}$$



(**a**)



(**b**)

Figure 6. Magnitude (**a**) and argument (**b**) of the Laplace transform relevant to $f(t) = e^t\,\Gamma(t)$ as a function of the complex variable $s = \sigma + \mathrm{i}\,\omega$ for $\sigma = 1$ when computed using the exact integral expression $F(s)$ and the corresponding Tricomi's series approximant $F_M(t)$ of order $M = 100$.

The Hermite series approximant of $f(t)$ is given by:

$$f_M(t) = \sum_{m=0}^{M} b_m\, H_m(t)\,, \tag{52}$$

where:

$$b_m = \frac{1}{2^m\, m!\, \sqrt{\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-t^2} H_m(t)\, dt\,. \tag{53}$$

The general polynomial $H_n(t)$ can be represented in terms of $L_m(t)$ $(m = 0, 1, \ldots, n)$ by means of the equation:

$$H_n(t) = \sum_{m=0}^{n} C_m(n)\, L_m(t)\,, \tag{54}$$

where the connection coefficients $C_m(n)$ are computed using (25) or (26). In this way, the expansion (52) can be recast as follows:

$$f_M(t) = \sum_{m=0}^{M} a_m \, L_m(t) \, , \tag{55}$$

where:

$$a_m = \sum_{n=m}^{M} b_n \, C_m(n) \, . \tag{56}$$

Hence, using Tricomi's method, the Laplace transform $F(s)$ can be approximated as:

$$F_M(s) = \frac{1}{s} \sum_{m=0}^{M} a_m \left(\frac{s-1}{s} \right)^m . \tag{57}$$

Upon selecting the expansion order $M = 16$, one can readily verify that the approximant $f_M(t)$ in (55) and (56) is characterized by the distribution shown in Figure 9, whereas the relevant LT in (57) shows, along the cut plane $\sigma = \mathrm{Re}\{s\} = 5$, the behavior plotted in Figure 10. It can be noticed that the agreement between the exact expression of $F(s)$ in (51) and its Tricomi's approximation $F_M(s)$ is rather good.

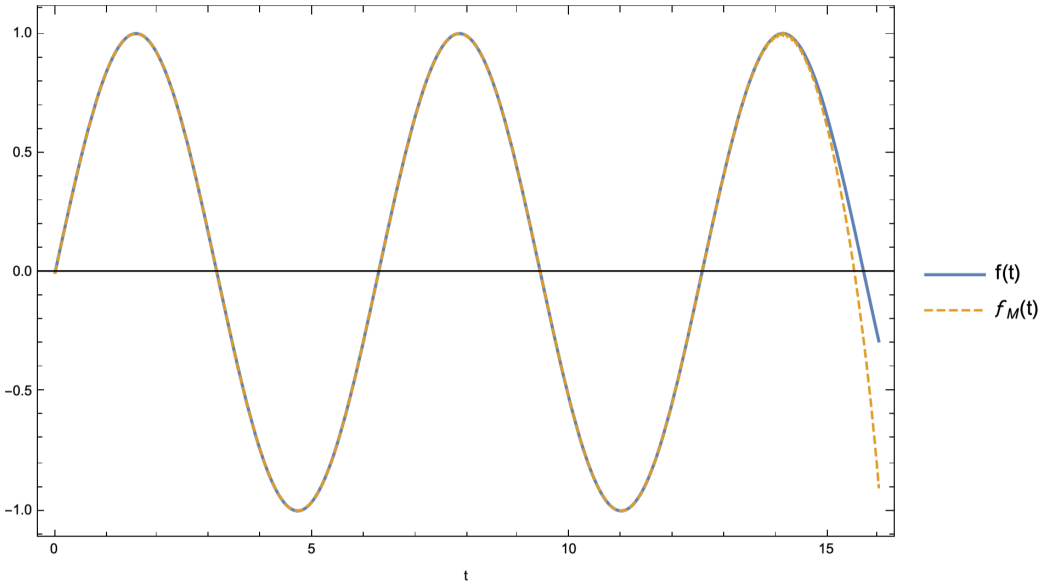


Figure 7. Distribution of the function $f(t) = \sin(t)$ as compared to the relevant Maclaurin series approximant $f_M(t)$ of order $M = 100$.

5.6. Example: LT of $\frac{t+1}{t^2+1}$

Let us consider the function:

$$f(t) = \frac{t+1}{t^2+1} \, , \tag{58}$$

whose Laplace transform can be easily evaluated as:

$$F(s) = \mathcal{L}(f(t))(s) = \int_0^\infty \exp^{-1}(s\,t)\, f(t)\, dt = \\ = \mathrm{Ci}(s)(\sin(s) - \cos(s)) + \frac{1}{2}(\pi - 2\mathrm{Si}(s))(\sin(s) + \cos(s)) \, . \tag{59}$$

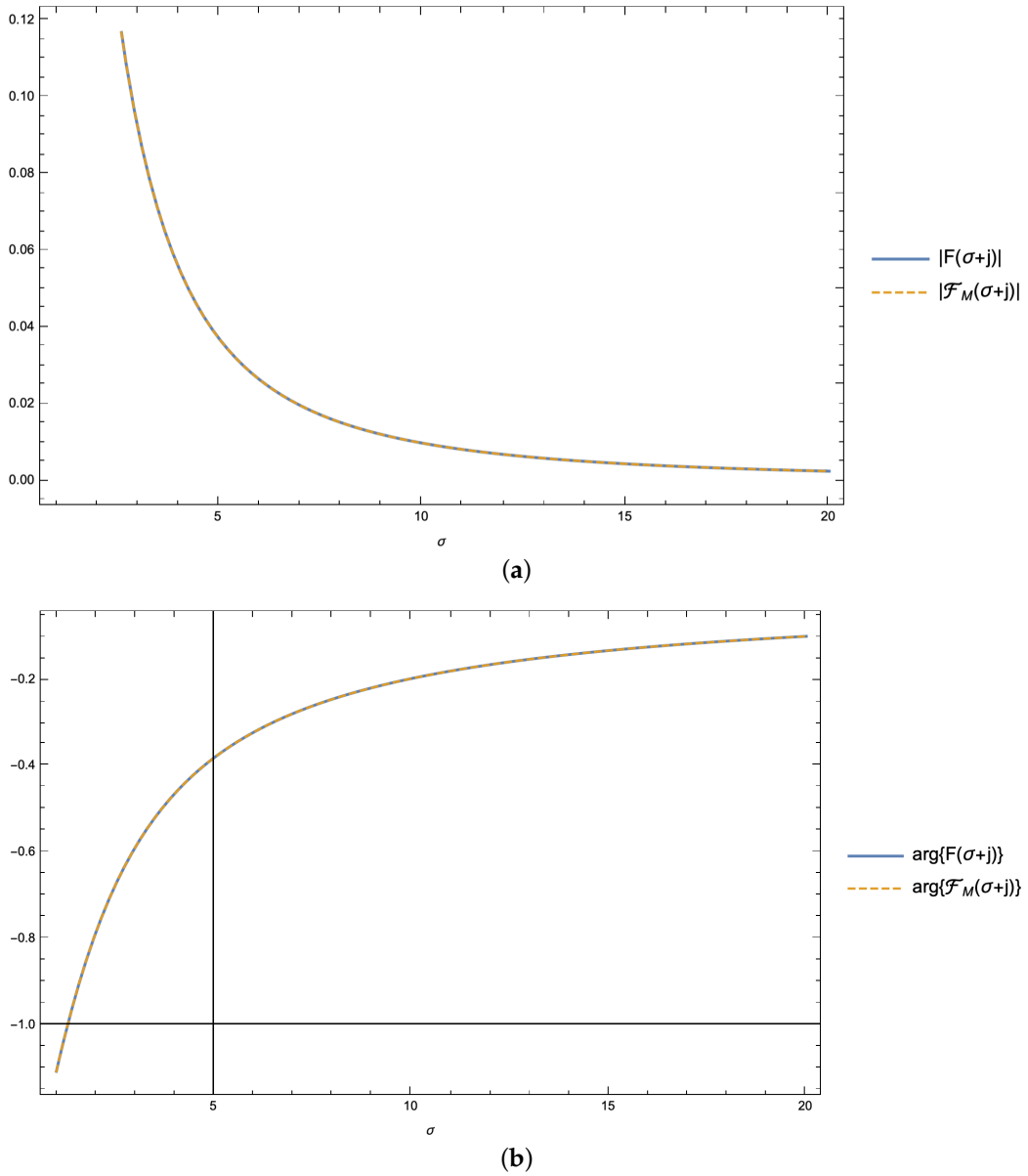


Figure 8. Magnitude (**a**) and argument (**b**) of the Laplace transform relevant to $f(t) = \sin(t)$ as a function of the complex variable $s = \sigma + \mathrm{i}\,\omega$ for $\omega = 1$ when computed using the exact integral expression $F(s)$ and the Tricomi's series approximant $F_M(t)$ as derived from the truncated Maclaurin expansion of $f(t)$ with order $M = 100$.

The Hermite series approximant of $f(t)$ is given by:

$$f_M(t) = \sum_{m=0}^{M} b_m \, H_m(t) \,, \tag{60}$$

where:

$$b_m = \frac{1}{2^m \, m! \, \sqrt{\pi}} \int_{-\infty}^{\infty} f(t) \, e^{-t^2} H_m(t) \, dt \,. \tag{61}$$

The expansion (60) can be recast in terms of Laguerre polynomials as follows:

$$f_M(t) = \sum_{m=0}^{M} a_m \, L_m(t) \,, \tag{62}$$

with:

$$a_m = \sum_{n=m}^{M} b_n \, C_m(n) \, . \tag{63}$$

where the connection coefficients $C_m(n)$ are computed using (25) or (26). Hence, using Tricomi's method, the Laplace transform $F(s)$ can be approximated as:

$$F_M(s) = \frac{1}{s} \sum_{m=0}^{M} a_m \left(\frac{s-1}{s} \right)^m . \tag{64}$$

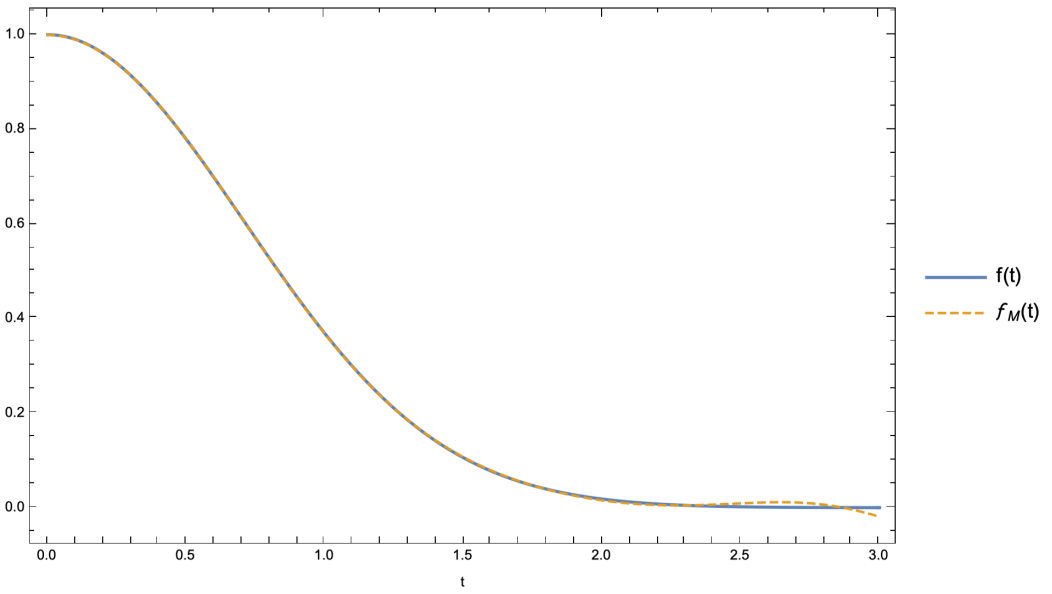
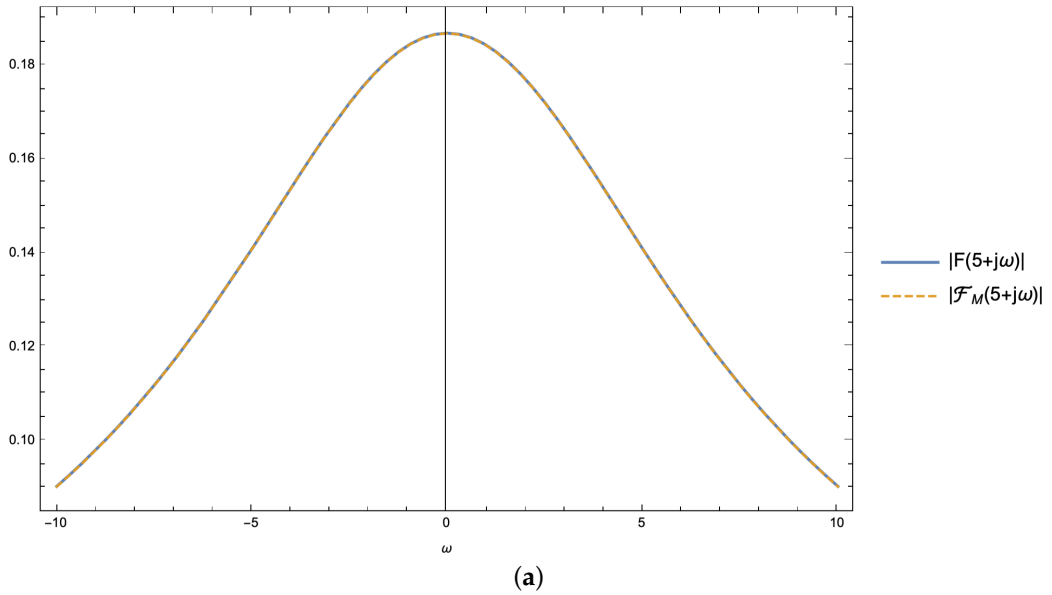
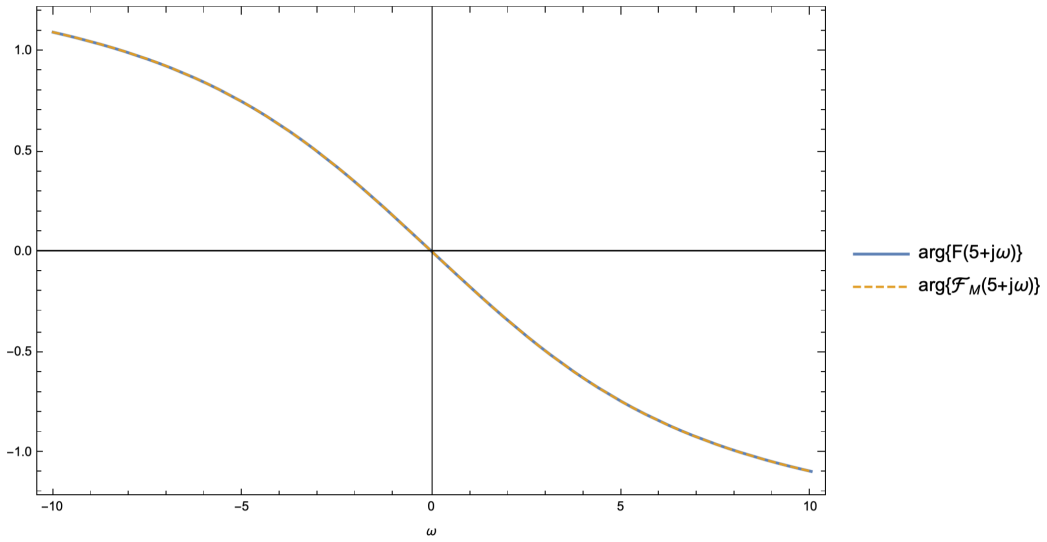


Figure 9. Distribution of the function $f(t) = e^{-t^2}$ as compared to the relevant Hermite series approximant $f_M(t)$ of order $M = 16$.



(a)

Figure 10. *Cont.*



(**b**)

Figure 10. Magnitude (**a**) and argument (**b**) of the Laplace transform relevant to $f(t) = e^{-t^2}$ as a function of the complex variable $s = \sigma + \mathrm{i}\,\omega$ for $\sigma = 5$ when computed using the exact integral expression $F(s)$ and the Tricomi's series approximant $F_M(t)$ as derived from the truncated Hermite expansion of $f(t)$ with order $M = 16$.

Upon selecting the expansion order $M = 16$, one can readily verify that the approximant $f_M(t)$ in (62)–(63) is characterized by the distribution shown in Figure 11, whereas the relevant LT in (64) shows, along the cut plane $\omega = \mathrm{Im}\{s\} = 1$, the behavior plotted in Figure 12. It can be noticed that the agreement between the exact expression of $F(s)$ in (59) and its Tricomi's approximation $F_M(s)$ is rather good.

From the visual inspection of Figures 12 and 13, one can readily notice that the derivation of Tricomi's series approximant directly from the truncated Laguerre expansion of $f(t)$ with a fixed order M, as per the expression (3), enables a more accurate representation of the Laplace transform $F(s) = \mathcal{L}(f(t))$ when compared to the relevant Tricomi's approximant derived from a truncated Hermite expansion of $f(t)$ with the same order M.

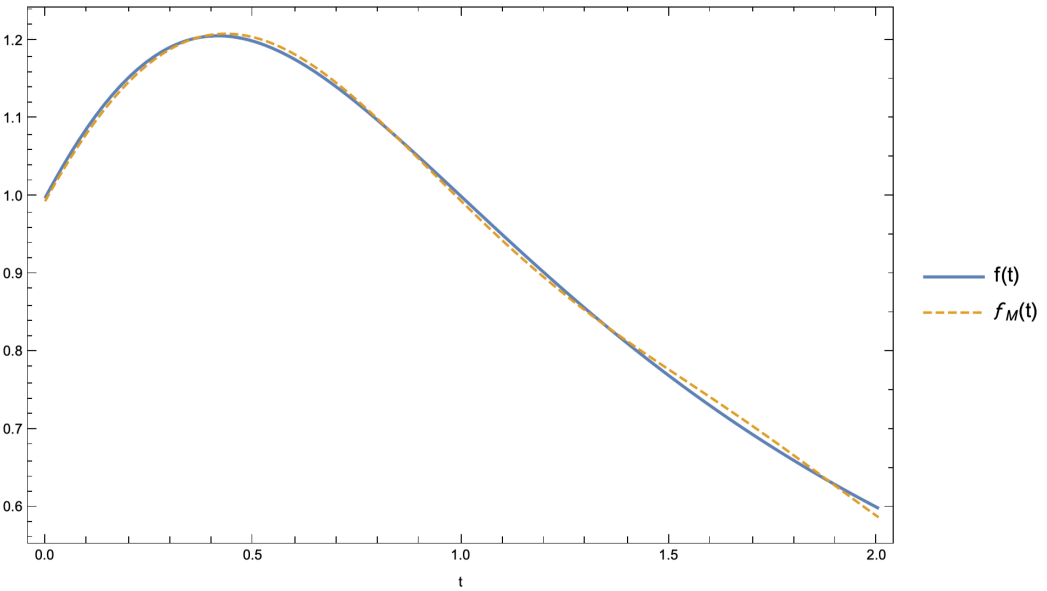


Figure 11. Distribution of the function $f(t) = \frac{t+1}{t^2+1}$ as compared to the relevant Hermite series approximant $f_M(t)$ of order $M = 16$.

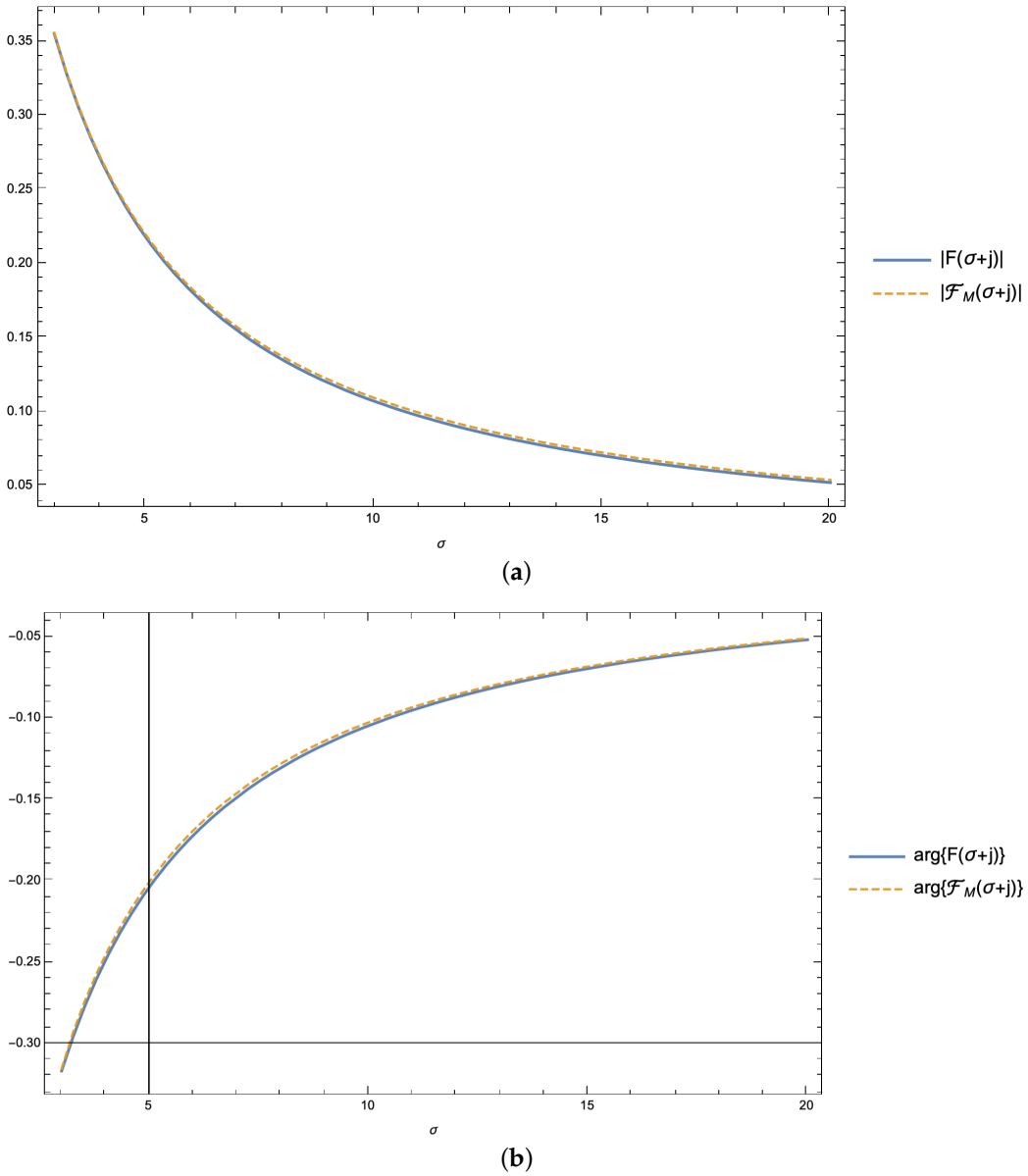
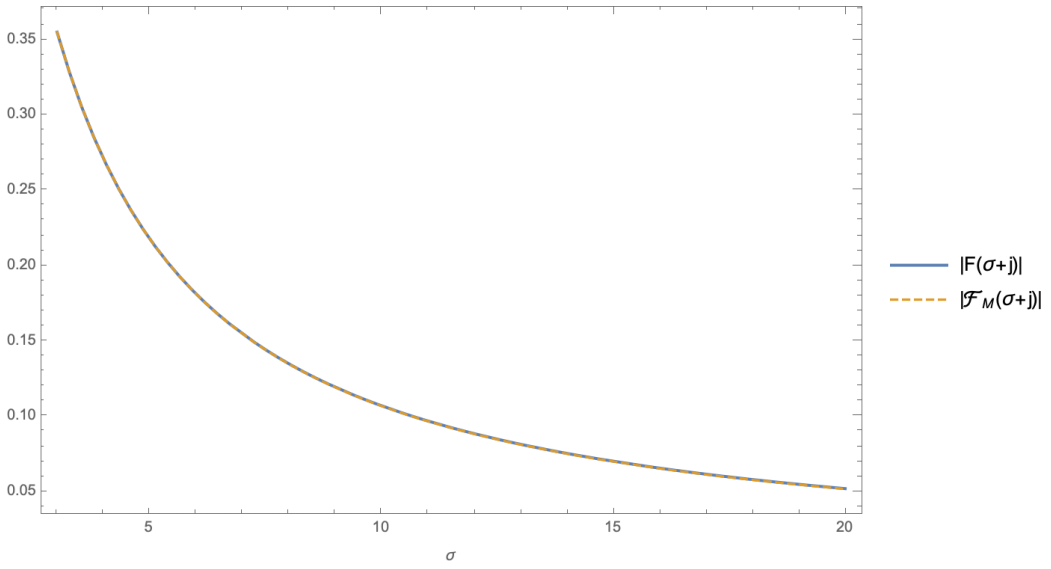
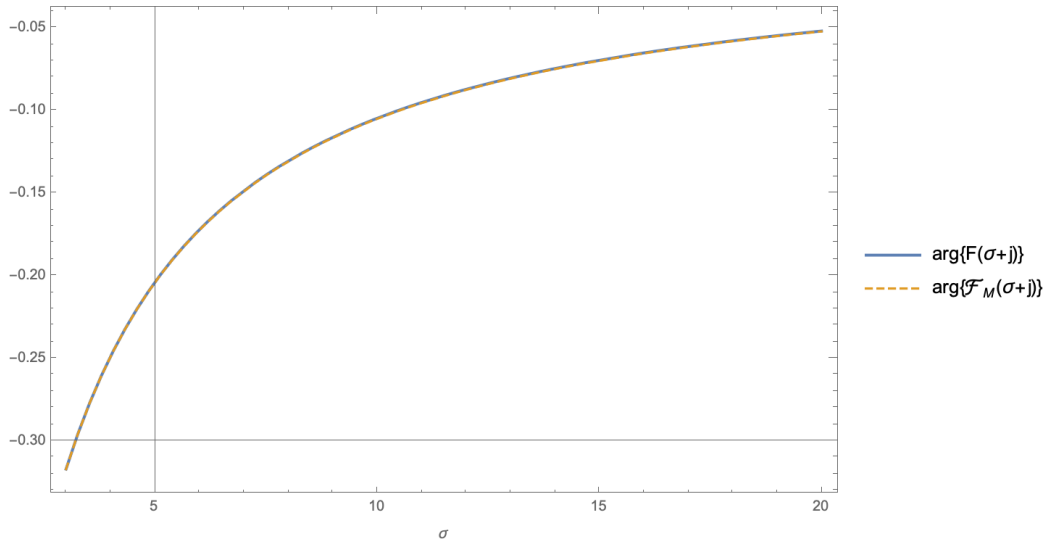


Figure 12. Magnitude (**a**) and argument (**b**) of the Laplace transform relevant to $f(t) = \frac{t+1}{t^2+1}$ as a function of the complex variable $s = \sigma + \mathrm{i}\,\omega$ for $\omega = 1$ when computed using the exact integral expression $F(s)$ and the Tricomi's series approximant $F_M(t)$ as derived from the truncated Hermite expansion of $f(t)$ with order $M = 16$.



(**a**)



(**b**)

Figure 13. Magnitude (**a**) and argument (**b**) of the Laplace transform relevant to $f(t) = \frac{t+1}{t^2+1}$ as a function of the complex variable $s = \sigma + \mathrm{i}\,\omega$ for $\omega = 1$ when computed using the exact integral expression $F(s)$ and the Tricomi's series approximant $F_M(t)$ as derived directly from the truncated Laguerre expansion of $f(t)$ with order $M = 16$.

6. Conclusions

Starting from a classical result by F.G. Tricomi, showing the possibility to use expansions in terms of Laguerre polynomials for obtaining the corresponding Laplace transforms, we have exploited the link between different polynomial sets, via inversion or connection coefficients, in order to apply the Tricomi method to more general expansions.

It has been shown that the described procedure works and that from the numerical point of view, the direct expansion in Laguerre polynomials guarantees a tremendously greater numerical stability of the Tricomi method. This confirms the special role played by Laguerre polynomials in the framework of the Laplace transform, as had been highlighted by the pioneering work of Tricomi.

Author Contributions: P.E.R., D.C., F.M. have contributed equally to this work. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The research work of F.M. has been carried out in the framework of the activities of the National Group of Mathematical Physics (GNFM, INdAM), Italy.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Tricomi, F.G. Trasformazione di Laplace e polinomi di Laguerre, I. Inversione della trasformazione; II. Alcune nuove formule sui polinomi di Laguerre. *Rend. Lincei* **1935**, *21*, 232–239. 332–335.
2. Tricomi, F.G. Ancora sull'inversione della trasformazione di Laplace. *Rend. Lincei* **1935**, *21*, 420–426.
3. Dattoli, G.; Ricci, P.E. Laguerre-type exponentials, and the relevant L-circular and L-hyperbolic functions. *Georgian Math. J.* **2003**, *10*, 481–494. [CrossRef]
4. Le Roy, È. Valeurs asymptotiques de certaines séries procédant suivant les puissances entières et positives d'une variable réelle. (French). *Darboux Bull.* **1899**, *4*, 245–268.
5. Garra, R.; Polito, F. On Some Operators Involving Hadamard Derivatives. *Integral Transforms Spec. Funct.* **2013**, *24*, 773–782. [CrossRef]
6. Garrappa, R.; Rogosin, S.; Mainardi, F. On a generalized three-parameter Wright function of Le Roy type. *Fract. Calc. Appl. Anal.* **2017**, *20*, 1196–1215. [CrossRef]
7. Kiryakova, V. *Generalized Fractional Calculus and Applications*; Pitman Research Notes in Mathematics Series 301; Longman: Harlow, UK, 1994.
8. Ben Cheikh, Y.; Chaggara, H. Connection coefficients via lowering operators. *J. Comput. Appl. Math.* **2005**, *178*, 45–61. [CrossRef]
9. Ben Cheikh, Y.; Chaggara, H. Connection coefficients between Boas-Buck polynomial sets. *J. Math. Anal. Appl.* **2006**, *319*, 665–689. [CrossRef]
10. Maroni, P.; da Rocha, Z. Connection coefficients between orthogonal polynomials and the canonical sequence: An approach based on symbolic computation. *Numer. Algor.* **2008**, *47*, 291–314. [CrossRef]
11. Ronveaux, A. Orthogonal polynomials: Connection and linearization coefficients. In Proceedings of the International Workshop on Orthogonal Polynomials in Mathematical Physics, Leganés, Spain, 24–26 June 1996; Alfaro, M., Alvarez-Nodarse, R., López Lagomasino, G., Marcellán, F., Eds.; Universidad Carlos III de Madrid: Leganes, Spain, 1997.
12. Ward, E.E. The calculation of transients in dynamical systems. *Camb. Philos. Soc.* **1954**, *50*, 49–54. [CrossRef]
13. Lee, S.L. Fourier-Laplace transforms and orthogonal polynomials. *J. Approx. Theory* **2020**, *256*, 105436. [CrossRef]