symmetry

MDPI

Article

Fixed Point Theorems for Nonexpansive Type Mappings in Banach Spaces

Rajendra Pant [1,*,†], Prashant Patel [1,†], Rahul Shukla [1,†] and Manuel De la Sen [2,†]

1 Department of Mathematics & Applied Mathematics, Kingsway Campus, University of Johannesburg, Auckland Park 2006, South Africa; prashant.patel9999@gmail.com or 219127524@student.uj.ac.za (P.P.); rshukla.vnit@gmail.com or rshukla@uj.ac.za (R.S.)
2 Institute of Research and Development of Processes IIDP, Faculty of Science and Technology, University of the Basque Country, de Bilbao, Barrio Sarriena, 48940 Bizkaia, Spain; manuel.delasen@ehu.eus
* Correspondence: rpant@uj.ac.za or pant.rajendra@gmail.com
† These authors contributed equally to this work.

Abstract: In this paper, we present some fixed point results for a class of nonexpansive type and α-Krasnosel'skiĭ mappings. Moreover, we present some convergence results for one parameter nonexpansive type semigroups. Some non-trivial examples have been presented to illustrate facts.

Keywords: metric projection; condition (E); uniformly convex space

MSC: Primary: 47H10; Secondary: 54H25; 47H09

check for updates

Citation: Pant, R.; Patel, P.; Shukla, R.; De la Sen, M. Fixed Point Theorems for Nonexpansive Type Mappings in Banach Spaces. *Symmetry* **2021**, *13*, 585. https://doi.org/10.3390/sym13040585

Academic Editor: Wei-Shih Du, Alicia Cordero Barbero, Huaping Huang and Juan Ramón Torregrosa Sánchez

Received: 17 March 2021
Accepted: 28 March 2021
Published: 2 April 2021

1. Introduction and Preliminaries

Suppose Y is a nonempty subset of a Banach space X. A mapping $T : Y \to Y$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ and a point $z \in Y$ is a fixed point of T if $Tz = z$. It is well-known that a nonexpansive mapping need not have a fixed point in a general Banach space. However, by enriching the space with some geometric properties like uniform convexity or normal structure, it is possible to have fixed points of nonexpansive mappings. In 1965, the first existence results for nonexpansive mappings in Banach spaces was obtained by Browder [1], Göhde [2] and Kirk [3], independently. Since then, a number of generalizations and extensions of nonexpansive mappings and their results have been obtained by many authors. Some of the notable extensions and generalizations of nonexpansive mappings can be found in [4–14] and elsewhere.

In 2008, Suzuki [14] introduced a new class of nonexpansive type mappings known as mappings satisfying condition (C) and obtained some important fixed point results for these mappings. We note that, for this class of mappings, the nonexpansiveness condition need not hold for all points but for a certain point in the domain. Suzuki [14] also showed that these mappings need not to be continuous unlike the nonexpansive mappings. We call this class of mappings as Suzuki generalized nonexpansive mapping (SGNM for short). García-Falset et al. [5] considered a generalization of the SGNM, known as mappings satisfying condition (E). Pant and Shukla [13] combined the SGNM with Aoyama and Kohsaka [4] type α-nonexpansive mappings and introduced the notion of generalized α-nonexpansive mappings. Pandey et al. [12] combined the SGNM with other type of nonexpansive mappings and introduced a new class of mappings called α–Reich–Suzuki nonexpansive mappings. They also showed that the class of α–Reich–Suzuki nonexpansive mappings is contained in the class of mappings satisfying condition (E). Recently, Atailia et al. [15] combined the SGNM and Hardy and Rogers [16] type nonexpansive mappings and introduced a new class of mappings called as generalized contractions of Suzuki type. They obtained some fixed point results for their new class of nonexpansive type mappings. In this paper, we point out that the class of generalized contractions of Suzuki type mappings considered in [15] is contained properly in the class of α–Reich–Suzuki nonexpansive mappings considered in [12] which is a sub-class of mappings satisfying condition (E). Moreover, we show that results presented in [15] also hold for the class of mappings satisfying condition (E). Some non-trivial examples are also presented to illustrate facts. We also obtain a weak convergence theorem concerning the trajectory $(S(\zeta))_{\zeta>0}$ of a one parameter semigroup $\mathcal{S}$ of mappings satisfying condition (E). Finally, we consider the Halpern iteration for finding a common fixed point of a nonexpansive type semigroup and a countable family of mappings satisfying condition (E). In this way, results in [5,15,17–19] have been extended, generalized and complimented.

Now, we recall some useful notations, definitions and results from the literature. We denote $F(T)$ as the set of all fixed points of mapping T, i.e., $F(T) := \{z \in Y : \; Tz = z\}$. A Banach space X is said to be uniformly convex if, for each $\varepsilon \in (0,2]$ $\exists$ $\delta > 0$ such that $\left\|\frac{x+y}{2}\right\| \leq 1-\delta$ for all $x, y \in X$ with $\|x\| = \|y\| = 1$ and $\|x-y\| > \varepsilon$. A Banach space X is strictly convex if

$$\left\|\frac{x+y}{2}\right\| < 1,$$

whenever $x, y \in X$ with $\|x\| = \|y\| = 1$, $x \neq y$ [20].

Theorem 1. *[21]. Let X be a uniformly convex Banach space. Then for $\varepsilon, \Omega > 0$ and $x, y \in X$ with $\|x-y\| \geq \varepsilon$, $\|x\| \leq \Omega$, $\|y\| \leq \Omega$, there exists a $\delta > 0$ such that*

$$\left\|\frac{1}{2}(x+y)\right\| \leq \left[1-\delta\left(\frac{\varepsilon}{\Omega}\right)\right]\Omega.$$

Theorem 2. *[20]. Let X be a Banach space. The following conditions are equivalent:*

(i) *X is strictly convex.*
(ii) *If $x, y \in X$ and $\|x+y\| = \|x\| + \|y\|$, then $x = 0$ or $y = 0$ or $y = cx$ for some $c > 0$.*

Definition 1. *[22]. A Banach space X satisfies Opial property if, for every weakly convergent sequence (x_n) with weak limit $x \in X$, it holds that:*

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

for all $y \in X$ with $x \neq y$.

A Banach space which have a weakly sequentially continuous duality mapping also have the Opial property. All finite dimensional Banach spaces and Hilbert spaces have the Opial property. For $p \in (1,\infty)$, ℓ^p spaces have the Opial property. However, L_p $(1 < p < \infty,\; p \neq 2)$ does not have the Opial property [21].

Definition 2. *[23]. Suppose X is Banach space and Y is a nonempty subset of X. Suppose for every element $x \in X$, there exists a, $y \in Y$ such that for any $z \in Y$,*

$$\|y-x\| \leq \|z-x\|.$$

Then y is called a metric projection of x onto Y and is denoted by $P_Y(x)$. If $P_Y(x)$ exists and determined uniquely for all $x \in X$, then the mapping $P_Y : X \to Y$ is called the metric projection onto Y.

Definition 3. *[20]. A mapping $T : Y \to Y$ is said to be a quasi-nonexpansive if for all $x \in Y$ and $z \in \mathrm{F}(T)$,*

$$\|Tx - z\| \leq \|x - z\|.$$

It is well known that a nonexpansive mapping with a fixed point is quasi-nonexpansive. However, the converse need not be true.

Definition 4. *[14]. Suppose Y is a nonempty subset of a Banach space X. A mapping $T : Y \to Y$ satisfies condition (C) if*

$$\frac{1}{2}\|x - Tx\| \le \|x - y\| \text{ implies } \|Tx - Ty\| \le \|x - y\|$$

for all $x, y \in Y$.

Definition 5. *[5]. Suppose Y is a nonempty subset of a Banach space X. A mapping $T : Y \to Y$ satisfies condition (E_μ) on Y if there exists $\mu \ge 1$ such that*

$$\|x - Ty\| \le \mu\|x - Tx\| + \|x - y\|$$

for all $x, y \in Y$. The mapping T satisfies condition (E) on Y when T satisfies (E_μ) for some $\mu \ge 1$.

Proposition 1. *[5]. Suppose Y is a nonempty subset of a Banach space X and $T : Y \to Y$ satisfies condition (E) with $F(T) \neq \emptyset$. Then T is quasi-nonexpansive.*

Definition 6. *[12]. Suppose Y is a nonempty subset of a Banach space X. A mapping $T : Y \to Y$ is said to be a generalized α–Reich–Suzuki nonexpansive mapping if there exists an $\alpha \in [0, 1)$ such that*

$$\frac{1}{2}\|x - Tx\| \le \|x - y\| \text{ implies } \|Tx - Ty\| \le \max\{P(x, y), Q(x, y)\} \tag{1}$$

for all $x, y \in Y$, where

$$P(x, y) = \alpha\|Tx - x\| + \alpha\|Ty - y\| + (1 - 2\alpha)\|x - y\|$$
$$Q(x, y) = \alpha\|Tx - y\| + \alpha\|Ty - x\| + (1 - 2\alpha)\|x - y\|.$$

Definition 7. *[15]. Suppose Y is a nonempty convex subset of a Banach space X and $T : Y \to Y$ a mapping. A mapping $T_\alpha : Y \to Y$ is said to be an α-Krasnosel'skiĭ mapping associated with T if there exists $\alpha \in (0, 1)$ such that*

$$T_\alpha x = (1 - \alpha)x + \alpha Tx$$

for all $x \in Y$.

Definition 8. *[24]. Let Y be a nonempty subset of a Banach space X. A mapping $T : Y \to Y$ is called asymptotically regular if*

$$\lim_{n\to\infty} \|T^n x - T^{n+1}x\| = 0.$$

Theorem 3. (The Schauder fixed point theorem [20]). *Let Y be a nonempty compact convex subset of a Banach space X and a self-mapping T on Y. If T is continuous, then T has a fixed point in Y.*

Lemma 1. (*Demiclosedness principle*). *Let Y be a nonempty subset of a Banach space X which has an Opial property. Let $T : Y \to Y$ be a mapping satisfying condition (E). If (x_n) is a sequence in Y such that (x_n) converges weakly to x and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ then $Tx = x$. That is, $I - T$ is demiclosed at zero.*

Proof. The proof directly follows from [5] (Theorem 1). □

Song et al. [19] considered the following mappings:

Definition 9. *[19]. A mapping $T : Y \to Y$ is called α-nonexpansive if there is an $\alpha < 1$ such that for all $x, y \in Y$*

$$\|Tx - Ty\| \leq \alpha\|Tx - y\| + \alpha\|Ty - x\| + (1 - 2\alpha)\|x - y\|.$$

Definition 10. *[25]. An α-nonexpansive semigroup $\mathcal{S} = \{S(\zeta) : \zeta > 0\}$ is called uniformly asymptotically regular (or, u.a.r.) if for any positive s and any bounded subset K of $D(\mathcal{S})$,*

$$\lim_{\zeta\to\infty} \sup_{x\in K} \|S(s)(S(\zeta)x) - S(\zeta)x\| = 0.$$

Definition 11. *[19]. A family $\{S_n\}$ of α-nonexpansive mappings is said to be uniformly asymptotically regular if, for any bounded subset K of $\bigcap_{n=1}^{\infty} D(S_n)$ and for each positive integer m,*

$$\lim_{n\to\infty} \sup_{x\in K} \|S_m(S_n x) - S_n x\| = 0.$$

2. Previous Results and Discussions

Atailia et al. [15] considered the following type of nonexpansive mappings:

Definition 12. *Suppose Y is a subset of a Banach space X. A mapping $T : Y \to Y$ is called generalized contraction of Suzuki type if there exists $\beta \in (0, 1)$ and $\alpha_1, \alpha_2, \alpha_3 \in [0, 1]$ where $\alpha_1 + 2\alpha_2 + 2\alpha_3 = 1$ such that for all $x, y \in X$,*

$$\beta\|x - Tx\| \leq \|x - y\| \text{ implies } \|Tx - Ty\| \leq \alpha_1\|x - y\| + \alpha_2(\|x - Tx\| + \|y - Ty\|) + \alpha_3(\|x - Ty\| + \|y - Tx\|). \tag{2}$$

The following proposition illustrates that the mapping considered in Definition 12 is properly contained in the class of generalized α–Reich–Suzuki nonexpansive mappings.

Proposition 2. *Let Y be a subset of a Banach space X. If $T : Y \to Y$ is a generalized contraction of Suzuki type $\left(\text{with } \beta = \frac{1}{2}\right)$ then T is a generalized α–Reich–Suzuki nonexpansive mapping.*

Proof. Since T is a generalized contraction of Suzuki type, we have

$$\|Tx - Ty\| \leq \alpha_1\|x - y\| + \alpha_2(\|x - Tx\| + \|y - Ty\|) + \alpha_3(\|x - Ty\| + \|y - Tx\|). \tag{3}$$

We consider the following two cases.

Case (i) $(\|x - Tx\| + \|y - Ty\|) \geq (\|x - Ty\| + \|y - Tx\|)$. Then, (3) becomes

$$\|Tx - Ty\| \leq \alpha_1\|x - y\| + (\alpha_2 + \alpha_3)(\|x - Tx\| + \|y - Ty\|).$$

Take $\alpha_2 + \alpha_3 = \alpha$, since $\alpha_1 + 2\alpha_2 + 2\alpha_3 = 1$, $\alpha_1 = 1 - 2\alpha$, thus the above condition becomes

$$\|Tx - Ty\| \leq \alpha\|x - Tx\| + \alpha\|y - Ty\| + (1 - 2\alpha)\|x - y\|. \tag{4}$$

Case (ii) $(\|x - Tx\| + \|y - Ty\|) < (\|x - Ty\| + \|y - Tx\|)$. Then, (3) reduces to

$$\|Tx - Ty\| \leq \alpha_1\|x - y\| + (\alpha_2 + \alpha_3)(\|x - Ty\| + \|y - Tx\|).$$

Again, taking $\alpha_2 + \alpha_3 = \alpha$, and $\alpha_1 = 1 - 2\alpha$, we obtain

$$\|Tx - Ty\| \leq \alpha\|x - Ty\| + \alpha\|y - Tx\| + (1 - 2\alpha)\|x - y\|. \tag{5}$$

Let

$$\begin{aligned} P(x,y) &= \alpha\|Tx-x\| + \alpha\|Ty-y\| + (1-2\alpha)\|x-y\| \text{ and} \\ Q(x,y) &= \alpha\|Tx-y\| + \alpha\|Ty-x\| + (1-2\alpha)\|x-y\|. \end{aligned}$$

Then, in view of (4) and (5), we can conclude that

$$\|Tx-Ty\| \leq \max\{P(x,y), Q(x,y)\}.$$

Therefore, $T : X \to X$ is a generalized α–Reich–Suzuki nonexpansive mapping. □

The following example illustrates that the reverse inclusion need not be true.

Example 1. *Suppose* $X = \{(0,0),(1,0),(2,0),(3,0),(4,0),(0,5),(5,4)\}$ *is a subset of* $\mathbb{R}^2$ *with norm* $\|\cdot\| := \|(x_1,x_2)\| = |x_1| + |x_2|$. *Let* $T : X \to X$ *defined by*

$$T : \begin{pmatrix} (0,0),(1,0),(2,0),(3,0),(4,0),(0,5),(5,4) \\ (0,0),(0,0),(1,0),(2,0),(3,0),(0,0),(0,5) \end{pmatrix}.$$

It can be easily seen that, for all $x, y \in X$ *and* $\alpha \geq \frac{6}{10}$,

$$\|Tx-Ty\| \leq \max\{P(x,y), Q(x,y)\}.$$

Thus, T is generalized α*–Reich–Suzuki nonexpansive mapping.*
However, for $x = (4,0)$ *and* $y = (5,4)$

$$\beta\|x-Tx\| = \beta \leq \|x-y\| = 5, \text{and } \|Tx-Ty\| = 8.$$

Now, we have

$$\alpha_1\|x-y\| + \alpha_2(\|x-Tx\| + \|y-Ty\|) + \alpha_3(\|x-Ty\| + \|y-Tx\|) = 5 - 3\alpha_2 + 5\alpha_3.$$

Thus, for all $\alpha_1, \alpha_2, \alpha_3 \in [0,1]$

$$\|Tx-Ty\| > \alpha_1\|x-y\| + \alpha_2(\|x-Tx\| + \|y-Ty\|) + \alpha_3(\|x-Ty\| + \|y-Tx\|).$$

Hence T is not a generalized contraction of Suzuki type.

Atailia et al. [15] obtained the following lemma:

Lemma 2. *Suppose Y is a nonempty subset of a Banach space X,* $\beta \in \left[\frac{1}{2}, 1\right)$ *and* $T : Y \to Y$ *is a generalized contraction of Suzuki type. Then*

$$\|x-Ty\| \leq \left(\frac{2+\alpha_1+\alpha_2+3\alpha_3}{1-\alpha_2-\alpha_3}\right)\|x-Tx\| + \|x-y\|.$$

Proposition 3. *Suppose Y is a nonempty subset of a Banach space X and* $T : Y \to Y$ *is a generalized contraction of Suzuki type. Then T satisfies condition (E).*

Proof. If we take $\mu = \frac{2+\alpha_1+\alpha_2+3\alpha_3}{1-\alpha_2-\alpha_3} \geq 1$ in Lemma 2, then T satisfies the condition (E). □

The following example ensures that the reverse inclusion may not be true.

Example 2. *Suppose* $X = \left(\mathbb{R}^2, \|\cdot\|_{\frac{3}{2}}\right)$ *with the norm* $\|(x,y)\|_{\frac{3}{2}} = \left(\sum_{j=1}^{2} |x_j - y_j|^{\frac{3}{2}}\right)^{\frac{2}{3}}$, $x = (x_1,x_2), y = (y_1,y_2)$ *and* $Y = [-1,1]^2$ *be subset of X. Let* $T : Y \to Y$ *be defined by*

$$T(x_1, x_2) = \begin{cases} \left(\frac{|x_1|}{2}, x_2\right), & |x_1| < 1, \\ (-1, x_2), & x_1 = 1, \\ (1, x_2), & x_1 = -1. \end{cases}$$

First, we show that the mapping T satisfies condition (E). For this, the following cases are considered:

Case (a) $p = (x_1, x_2)$ *and* $q = (y_1, y_2)$ *with* $|x_1| < 1$ *and* $|y_1| < 1$. *Then,*

$$\|Tp - Tq\|_{\frac{3}{2}} \leq \|p - q\|_{\frac{3}{2}} \text{ and,}$$

$$\|p - Tq\|_{\frac{3}{2}} \leq \|p - Tp\|_{\frac{3}{2}} + \|Tp - Tq\|_{\frac{3}{2}} \leq \|p - Tp\|_{\frac{3}{2}} + \|p - q\|_{\frac{3}{2}}.$$

Case (b) $|x_1| < 1$ *and* $y_1 = 1$. *Then,*

$$\begin{aligned} \|p - Tq\|_{\frac{3}{2}} &= \left(|x_1 + 1|^{3/2} + |x_2 - y_2|^{3/2}\right)^{2/3}, \\ |p - q\|_{\frac{3}{2}} &= \left(|x_1 - y_1|^{3/2} + |x_2 - y_2|^{3/2}\right)^{2/3} \\ \text{and } \|p - Tp\|_{\frac{3}{2}} &= \left|\frac{2x_1 - |x_1|}{2}\right|. \end{aligned}$$

Since $|x_1| < 1,\ y_1 = 1$, *we have* $|x_1 + 1| \leq |x_1 - y_1| + 4\left|\frac{2x_1 - |x_1|}{2}\right|$. *Thus,*

$$\|p - Tq\|_{\frac{3}{2}} \leq 4\|p - Tp\|_{\frac{3}{2}} + \|p - q\|_{\frac{3}{2}}.$$

Case (c) $|x_1| < 1$ *and* $y_1 = -1$. *Then,*

$$\begin{aligned} \|p - Tq\|_{\frac{3}{2}} &= \left(|x_1 - 1|^{3/2} + |x_2 - y_2|^{3/2}\right)^{2/3}, \\ \|p - q\|_{\frac{3}{2}} &= \left(|x_1 - y_1|^{3/2} + |x_2 - y_2|^{3/2}\right)^{2/3} \\ \text{and } \|p - Tp\|_{\frac{3}{2}} &= \left|\frac{2x_1 - |x_1|}{2}\right|. \end{aligned}$$

Since $|x_1| < 1,\ y_1 = -1$, *we have* $|x_1 + 1| \leq |x_1 - y_1| + 4\left|\frac{2x_1 - |x_1|}{2}\right|$. *Thus,*

$$\|p - Tq\|_{\frac{3}{2}} \leq 4\|p - Tp\|_{\frac{3}{2}} + \|p - q\|_{\frac{3}{2}}.$$

Case (d) $x_1 = 1$ *and* $y_1 = 1$. *Then,*

$$\begin{aligned} \|p - Tq\|_{\frac{3}{2}} &= \left(|x_1 + 1|^{3/2} + |x_2 - y_2|^{3/2}\right)^{2/3}, \ \|p - Tp\|_{\frac{3}{2}} = |x_1 + 1| \\ \text{and } \|p - q\|_{\frac{3}{2}} &= \left(|x_1 - y_1|^{3/2} + |x_2 - y_2|^{3/2}\right)^{2/3}. \end{aligned}$$

Since $x_1 = 1$ *and* $y_1 = 1$, *we have* $|x_1 + 1| \leq 2|x_1 + 1| + |x_1 - y_1|$. *Thus,*

$$\|p - Tq\|_{\frac{3}{2}} \leq 2\|p - Tp\|_{\frac{3}{2}} + \|p - q\|_{\frac{3}{2}}.$$

Case (e) $x_1 = 1$ *and* $y_1 = -1$. *Then,*

$$\begin{aligned} \|p - Tq\|_{\frac{3}{2}} &= \left(|x_1 - 1|^{3/2} + |x_2 - y_2|^{3/2}\right)^{2/3}, \ \|p - Tp\|_{\frac{3}{2}} = |x_1 + 1| \\ \text{and } \|p - q\|_{\frac{3}{2}} &= \left(|x_1 - y_1|^{3/2} + |x_2 - y_2|^{3/2}\right)^{2/3}. \end{aligned}$$

Since $x_1 = 1$ *and* $y_1 = -1$*, we have* $|x_1 - 1| \leq |x_1 + 1| + |x_1 - y_1|$*. Thus,*

$$\|p - Tq\|_{\frac{3}{2}} \leq \|p - Tp\|_{\frac{3}{2}} + \|p - q\|_{\frac{3}{2}}.$$

Case (f) $x_1 = 1$ *and* $|y_1| < 1$*. Then,*

$$\|q - Tq\|_{\frac{3}{2}} = \left|\frac{2y_1 - |y_1|}{2}\right| \text{ and } \|p - Tp\|_{\frac{3}{2}} = |x_1 + 1|.$$

Since $x_1 = 1, |y_1| < 1$*, we have* $\|q - Tq\|_{\frac{3}{2}} \leq \|p - Tp\|_{\frac{3}{2}}$*. Thus,*

$$\|p - Tq\|_{\frac{3}{2}} \leq \|p - q\|_{\frac{3}{2}} + \|q - Tq\|_{\frac{3}{2}} \leq \|p - Tp\|_{\frac{3}{2}} + \|p - q\|_{\frac{3}{2}}.$$

Case (g) $x_1 = -1$ *and* $y_1 = -1$*. Then,*

$$\|p - Tq\|_{\frac{3}{2}} = \left(|x_1 - 1|^{3/2} + |x_2 - y_2|^{3/2}\right)^{2/3}, \ \|p - Tp\|_{\frac{3}{2}} = |x_1 - 1|$$
$$\text{and } \|p - q\|_{\frac{3}{2}} = \left(|x_1 - y_1|^{3/2} + |x_2 - y_2|^{3/2}\right)^{2/3}.$$

Since $x_1 = -1$ *and* $y_1 = -1$*, we have* $|x_1 - 1| \leq 2|x_1 - 1| + |x_1 - y_1|$*. Thus,*

$$\|p - Tq\|_{\frac{3}{2}} \leq 2\|p - Tp\|_{\frac{3}{2}} + \|p - q\|_{\frac{3}{2}}.$$

Case (h) $x_1 = -1$ *and* $y_1 = 1$*. Then,*

$$\|p - Tq\|_{\frac{3}{2}} = |x_2 - y_2|, \ \|p - Tp\|_{\frac{3}{2}} = |x_1 - 1|$$
$$\text{and } \|p - q\|_{\frac{3}{2}} = \left(|x_1 - y_1|^{3/2} + |x_2 - y_2|^{3/2}\right)^{2/3}.$$

Since $x_1 = -1$ *and* $y_1 = -1$*,*

$$\|p - Tq\|_{\frac{3}{2}} \leq \|p - Tp\|_{\frac{3}{2}} + \|p - q\|_{\frac{3}{2}}.$$

Case (i) $x_1 = -1$ *and* $|y_1| < 1$*. Then,*

$$\|q - Tq\|_{\frac{3}{2}} = \left|\frac{2y_1 - |y_1|}{2}\right| \text{ and } \|p - Tp\|_{\frac{3}{2}} = |x_1 - 1|.$$

Since $x_1 = -1, |y_1| < 1$*, we have* $\|q - Tq\|_{\frac{3}{2}} \leq \|p - Tp\|_{\frac{3}{2}}$*. Thus,*

$$\|p - Tq\|_{\frac{3}{2}} \leq \|p - q\|_{\frac{3}{2}} + \|q - Tq\|_{\frac{3}{2}} \leq \|p - Tp\|_{\frac{3}{2}} + \|p - q\|_{\frac{3}{2}}.$$

Therefore, in all the cases, T satisfies condition (E).

Furthermore, if $p = \left(\frac{1}{2}, x_2\right)$*,* $q = (1, x_2)$ *then* $\beta\|p - Tp\|_{\frac{3}{2}} = \frac{\beta}{4} \leq \|p - q\|_{\frac{3}{2}} = \frac{1}{2}$ *and* $\|Tp - Tq\|_{\frac{3}{2}} = \frac{10}{8}$ *and*

$$
\begin{aligned}
&\alpha_1\|p-q\|_{\frac{3}{2}}+\alpha_2\Big(\|p-Tp\|_{\frac{3}{2}}+\|q-Tq\|_{\frac{3}{2}}\Big)+\alpha_3\Big(\|p-Tq\|_{\frac{3}{2}}+\|q-Tp\|_{\frac{3}{2}}\Big)\\
&=\alpha_1\left\|\left(\frac{1}{2},x_2\right)-(1,x_2)\right\|_{\frac{3}{2}}+\alpha_2\left(\left\|\left(\frac{1}{2},x_2\right)-\left(\frac{1}{4},x_2\right)\right\|_{\frac{3}{2}}+\|(1,x_2)-(-1,x_2)\|_{\frac{3}{2}}\right)\\
&+\alpha_3\left(\left\|\left(\frac{1}{2},x_2\right)-(-1,x_2)\right\|_{\frac{3}{2}}+\left\|(1,x_2)-\left(\frac{1}{4},x_2\right)\right\|_{\frac{3}{2}}\right)\\
&=\frac{1}{2}\alpha_1+\alpha_2\left(\frac{1}{4}+2\right)+\alpha_3\left(\frac{3}{2}+\frac{3}{4}\right)\\
&=\frac{1}{2}\alpha_1+\frac{9}{4}(\alpha_2+\alpha_3)=\frac{1}{2}\alpha_1+\frac{9}{4}\left(\frac{1-\alpha_1}{2}\right)\\
&=\frac{9}{8}-\frac{5}{8}\alpha_1.
\end{aligned}
$$

Therefore, for all $\alpha_1,\alpha_2,\alpha_3\geq 0$, we have

$$\|Tp-Tq\|_{\frac{3}{2}}>\alpha_1\|p-q\|_{\frac{3}{2}}+\alpha_2\Big(\|p-Tp\|_{\frac{3}{2}}+\|q-Tq\|_{\frac{3}{2}}\Big)+\alpha_3\Big(\|p-Tq\|_{\frac{3}{2}}+\|q-Tp\|_{\frac{3}{2}}\Big).$$

Hence T is not a generalized contraction of Suzuki type.

3. α-Krasnosel'skiĭ Type Mappings

We prove some convergence results for mappings satisfying condition (E).

Theorem 4. *Let Y be a nonempty convex subset of a uniformly convex Banach space X and a mapping $T:Y\to Y$ satisfies condition (E) with $F(T)\neq\emptyset$. Then the α-Krasnosel'skiĭ mapping T_α for $\alpha\in(0,1)$ is asymptotically regular.*

Proof. Let $y_0\in Y$. For each $n\in\mathbb{N}\cup\{0\}$, define $y_{n+1}=T_\alpha y_n$. Thus,

$$T_\alpha y_n=y_{n+1}=(1-\alpha)y_n+\alpha Ty_n$$

and

$$T_\alpha y_n-y_n=T_\alpha y_n-T_\alpha y_{n-1}=\alpha(Ty_n-y_n).$$

Now, to prove T_α is asymptotically regular, it is sufficient to show that $\lim_{n\to\infty}\|Ty_n-y_n\|=0$. By Lemma (1) for all $x_0\in F(T)$, we have

$$\|x_0-Ty_n\|\leq\|x_0-y_n\|, \tag{6}$$

and

$$
\begin{aligned}
\|x_0-y_{n+1}\|=\|x_0-T_\alpha y_n\|&=\|x_0-(1-\alpha)y_n-\alpha Ty_n\|\\
&\leq(1-\alpha)\|x_0-y_n\|+\alpha\|x_0-Ty_n\|\\
&=(1-\alpha)\|x_0-y_n\|+\alpha\|x_0-y_n\|\\
&=\|x_0-y_n\|.
\end{aligned}\tag{7}
$$

Therefore, the sequence $(\|x_0-y_n\|)$ is bounded by $u_0=\|x_0-y_0\|$. If $y_{n_0}=x_0$ for any $n_0\in\mathbb{N}$ then from (7), $y_n\to x_0$ as $n\to\infty$. If $y_n\neq x_0$ for all $n\in\mathbb{N}$, take

$$z_n=\frac{x_0-y_n}{\|x_0-y_n\|}\text{ and }z_n'=\frac{x_0-Ty_n}{\|x_0-y_n\|}. \tag{8}$$

If $\alpha \leq \frac{1}{2}$ and from (8), we have

$$\begin{aligned}\|x_0 - y_{n+1}\| = \|x_0 - T_\alpha y_n\| &= \|x_0 - (1-\alpha)y_n - \alpha T y_n\| \\ &= \|x_0 - y_n + \alpha y_n - \alpha T y_n - 2\alpha x_0 + 2\alpha x_0 + \alpha y_n - \alpha y_n\| \\ &= \|(1-2\alpha)x_0 - (1-2\alpha)y_n + (2\alpha x_0 - \alpha y_n - \alpha T y_n)\| \\ &\leq (1-2\alpha)\|x_0 - y_n\| + \alpha\|2x_0 - y_n - Ty_n\| \\ &= 2\alpha\|x_0 - y_n\|\frac{\|z_n + z_n^{'}\|}{2} + (1-2\alpha)\|x_0 - y_n\|. \end{aligned} \tag{9}$$

Using the uniform convexity of the space X with $\|z_n\| \leq 1$, $\|z_n^{'}\| \leq 1$ and $\|z_n - z_n^{'}\| = \frac{\|y_n - Ty_n\|}{\|x_0 - y_n\|} \geq \frac{\|y_n - Ty_n\|}{u_0} = \varepsilon$ (say) (noting that modulus of convexity, $\delta(\varepsilon)$, is a non-decreasing function of ε), we obtain

$$\frac{\|z_n + z_n^{'}\|}{2} \leq 1 - \delta\left(\frac{\|y_n - Ty_n\|}{u_0}\right). \tag{10}$$

From (9) and (10),

$$\begin{aligned}\|x_0 - y_{n+1}\| &\leq \left(2\alpha\left(1 - \delta\left(\frac{\|y_n - Ty_n\|}{u_0}\right)\right) + (1-2\alpha)\right)\|x_0 - y_n\| \\ &= \left(1 - 2\alpha\delta\left(\frac{\|y_n - Ty_n\|}{u_0}\right)\right)\|x_0 - y_n\|.\end{aligned}$$

Using induction in the above inequality, it follows that

$$\|x_0 - y_{n+1}\| \leq \prod_{j=0}^{n}\left(1 - 2\alpha\delta\left(\frac{\|y_j - Ty_j\|}{u_0}\right)\right)u_0. \tag{11}$$

We shall prove that $\lim_{n\to\infty}\|Ty_n - y_n\| = 0$. On the other hand, consider that $\|Ty_n - y_n\|$ does not converge to zero. Then, there exists a subsequence (y_{n_k}) of (y_n) such that $\|Ty_{n_k} - y_{n_k}\|$ converges to $\eta > 0$. Since $\delta(\cdot) \in [0,1]$ is non decreasing and $\alpha \leq \frac{1}{2}$, we have $1 - 2\alpha\delta\left(\frac{\|y_k - Ty_k\|}{u_0}\right) \in [0,1]$ for all $k \in \mathbb{N}$. Since $\|Ty_{n_k} - y_{n_k}\| \to \eta$ so, for sufficiently large k, $\|Ty_{n_k} - y_{n_k}\| \geq \frac{\eta}{2}$, from (11), we have

$$\|x_0 - y_{n_{k+1}}\| \leq \left(1 - 2\alpha\delta\left(\frac{\eta}{2u_0}\right)\right)^{(n_{k+1})}u_0. \tag{12}$$

Making $k \to \infty$, it follows that $y_{n_k} \to x_0$. By (6), we get $Ty_{n_k} \to x_0$ and $\|y_{n_k} - Ty_{n_k}\| \to 0$ as $k \to \infty$, which is a contradiction.
If $\alpha > \frac{1}{2}$, then $1 - \alpha < \frac{1}{2}$ because $\alpha \in (0,1)$. Now,

$$\begin{aligned}\|x_0 - y_{n+1}\| &= \|x_0 - (1-\alpha)y_n - \alpha Ty_n\| \\ &= \|x_0 - y_n + \alpha y_n - \alpha Ty_n + (2-2\alpha)x_0 - (2-2\alpha)x_0 + Ty_n - Ty_n \\ &+ \alpha Ty_n - \alpha Ty_n\| \\ &= \|(2x_0 - y_n - Ty_n) - \alpha(2x_0 - y_n - Ty_n) + 2\alpha(x_0 - Ty_n) - (x_0 - Ty_n)\| \\ &\leq (1-\alpha)\|2x_0 - y_n - Ty_n\| + (2\alpha - 1)\|x_0 - y_n\| \\ &\leq 2(1-\alpha)\|x_0 - y_n\|\frac{\|z_n + z_n^{'}\|}{2} + (2\alpha - 1)\|x_0 - y_n\|\end{aligned}$$

and, by the uniform convexity of X, we obtain

$$\|x_0 - y_{n+1}\| \leq \left(2(1-\alpha) - 2(1-\alpha)\delta\left(\frac{\|y_n - Ty_n\|}{u_0}\right) + (2\alpha - 1)\right)\|x_0 - y_n\|.$$

Using induction in the above inequality, we get

$$\|x_0 - y_{n+1}\| \leq \prod_{j=0}^{n}\left(1 - 2(1-\alpha)\delta\left(\frac{\|y_j - Ty_j\|}{u_0}\right)\right)u_0.$$

Using the similar argument as in the previous case, it can be easily shown that $\|Ty_n - y_n\| \to 0$ as $n \to \infty$. Therefore, in both cases, T_α is asymptotically regular and this completes the proof. □

Theorem 5. *Suppose Y is a nonempty and closed subset of a Banach space X. Let $T : Y \to Y$ be a mapping satisfying condition (E) with $F(T) \neq \emptyset$. Then:*

(i) *$F(T)$ is closed in Y.*

(ii) *If the subset Y is convex and space X is strictly convex then $F(T)$ is convex.*

(iii) *If the subset Y is convex compact and space X is strictly convex. If T is continuous, then, for any $w_0 \in Y$, $\alpha \in (0,1)$, the α-Krasnosel'skiĭ process $(T_\alpha^n(w_0))$ converges to some $w^* \in F(T)$.*

Proof. (i) Let $(w_n) \subseteq F(T)$ such that $w_n \to w \in Y$ as $n \to \infty$. Thus, $Tw_n = w_n$, we show that $Tw = w$. Since T is quasi-nonexpansive, we get

$$\|w_n - Tw\| \leq \|w_n - w\|.$$

This implies that $Tw = w$ and $F(T)$ is closed.

(ii) Since X is strictly convex, Y is convex, fix $\beta \in (0,1)$ and $x, y \in F(T)$ such that $x \neq y$, take $w = \beta x + (1-\beta)y \in Y$. Since mapping T satisfies condition (E),

$$\|x - Tw\| \leq \|x - Tx\| + \|x - w\| = \|x - w\|.$$

Similarly,

$$\|y - Tw\| \leq \|y - w\|.$$

From strict convexity of X, there is a $\mu \in [0,1]$ in such a way that $Tw = \mu x + (1-\mu)y$

$$(1-\mu)\|x - y\| = \|Tx - Tw\| \leq \|x - w\| = (1-\beta)\|x - y\|, \tag{13}$$

and

$$\mu\|x - y\| = \|Ty - Tw\| \leq \|y - w\| = \beta\|x - y\|. \tag{14}$$

From (13) and (14), we obtain

$$1 - \mu \leq 1 - \beta \text{ and } \mu \leq \beta \ \Rightarrow \mu = \beta.$$

Hence, $Tw = w$ and $w \in F(T)$.

(iii) Let us define (w_n) by $w_n = T_\alpha^n w_0, w_0 \in Y$, where $T_\alpha w_0 = (1-\alpha)w_0 + \alpha T w_0, \alpha \in (0,1)$. Since Y is compact, then there exists a subsequence (w_{n_k}) of (w_n) that converges to some $w^* \in Y$. Since T is continuous, by the Schauder theorem, we have $F(T) \neq \emptyset$. Now, we show that $w^* \in F(T)$. Let $y_0 \in F(T)$

$$\begin{aligned}\|w_n - y_0\| &= \|T_\alpha^n w_0 - y_0\| \\ &\leq \|T_\alpha^{n-1} w_0 - y_0\| = \|w_{n-1} - y_0\|.\end{aligned}$$

Therefore, $(\|w_n - y_0\|)$ is decreasing sequence which bounded below by 0. So, it converges. Furthermore, since T_α is continuous,

$$
\begin{aligned}
\|w^* - y_0\| &= \lim_{k\to\infty} \|w_{n_{k+1}} - y_0\| \le \lim_{k\to\infty} \|w_{n_k+1} - y_0\| \\
&= \lim_{k\to\infty} \|T_\alpha w_{n_k} - y_0\| \\
&= \|T_\alpha w^* - y_0\| \\
&= \|(1-\alpha)w^* + \alpha T w^* - y_0\| \\
&\le (1-\alpha)\|w^* - y_0\| + \alpha\|Tw^* - y_0\|.
\end{aligned} \tag{15}
$$

Since $\alpha \neq 0$, it implies that

$$\|w^* - y_0\| \le \|Tw^* - y_0\|.$$

Since T is quasi-nonexpansive,

$$\|Tw^* - y_0\| \le \|w^* - y_0\|.$$

From the above two equations, we obtain

$$\|w^* - y_0\| = \|Tw^* - y_0\|. \tag{16}$$

In addition, from (15), we have

$$
\begin{aligned}
\|w^* - y_0\| &\le \|(1-\alpha)w^* + \alpha T w^* - y_0\| \\
&\le (1-\alpha)\|w^* - y_0\| + \alpha\|Tw^* - y_0\| = \|w^* - y_0\|.
\end{aligned}
$$

This follows that

$$\|(1-\alpha)w^* + \alpha T w^* - y_0\| = (1-\alpha)\|w^* - y_0\| + \alpha\|Tw^* - y_0\|.$$

Since X is strictly convex, either $Tw^* - y_0 = c(w^* - y_0)$ for some $c > 0$ or $w^* = y_0$. From (16), it follows that $c = 1$, then, $Tw^* = w^*$ and $w^* \in F(T)$. Since $\lim_{n\to\infty} \|w_n - y_0\|$ exists and (w_{n_k}) converges strongly to w^*, (w_n) converges strongly to $w^* \in F(T)$. □

Theorem 6. *Let Y be a nonempty closed convex subset of a uniformly convex Banach space X and $T : Y \to Y$ a mapping satisfying condition (E) with $F(T) \neq \emptyset$. Suppose P is the metric projection from X into $F(T)$. Then, for each $x \in Y$, the sequence $(PT^n x)$ converges to some $y \in F(T)$.*

Proof. Let $x \in Y$, for $n, m \in \mathbb{N}$ such that $n \ge m$. Then

$$\|PT^n x - T^n x\| \le \|PT^m x - T^n x\|. \tag{17}$$

Since $PT^n x \in F(T)$ for all $n \in \mathbb{N}$ and T is quasi-nonexpansive

$$\|PT^m x - T^n x\| = \|PT^m x - T(T^{n-1}x)\| \le \|T^{n-1}x - PT^m x\|.$$

Therefore, for $n \ge m$, it follows that

$$\|PT^m x - T^n x\| \le \|T^m x - PT^m x\|. \tag{18}$$

From (17) and (18), $n \ge m$, we get

$$\|PT^n x - T^n x\| \le \|PT^m x - T^m x\|,$$

it implies that $\lim\limits_{n\to\infty} \|PT^n x - T^n x\|$ exists. Take $\lim\limits_{n\to\infty} \|PT^n x - T^n x\| = \theta$.

If $\theta = 0$, then, for all $\varepsilon > 0$, there exists an integer $n_0(\varepsilon)$ such that

$$\|PT^n x - T^n x\| < \frac{\varepsilon}{4} \tag{19}$$

for all $n \geq n_0$. Therefore, if $n \geq m \geq n_0$ and using (18) and (19), it follows that

$$\begin{aligned}
\|PT^n x - PT^m x\| &\leq \|PT^n x - PT^{n_0} x\| + \|PT^{n_0} x - PT^m x\| \\
&\leq \|PT^n x - T^n x\| + \|T^n x - PT^{n_0} x\| + \|PT^m x - T^m x\| \\
&\quad + \|T^m x - PT^{n_0} x\| \\
&\leq \|PT^n x - T^n x\| + \|T^{n_0} x - PT^{n_0} x\| + \|PT^m x - T^m x\| \\
&\quad + \|T^{n_0} x - PT^{n_0} x\| \\
&\leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon.
\end{aligned}$$

Thus, $(PT^n x)$ is a Cauchy sequence in $F(T)$, which is closed (by Theorem (5)) and X is complete, then $(PT^n x)$ must converge to a point in $F(T)$. Now, letting $\theta > 0$, we claim that the sequence $(PT^n x)$ is a Cauchy sequence in X. On the other hand, there exists an $\varepsilon_0 > 0$ such that, for every $n_0 \in \mathbb{N}$, there exists $r_0, s_0 \geq n_0$ such that

$$\|PT^{r_0} x - PT^{s_0} x\| \geq \varepsilon_0,$$

we choose $\Theta_0 > 0$ small enough so that

$$(\theta + \Theta_0)\left(1 - \delta\left(\frac{\varepsilon_0}{\theta + \Theta_0}\right)\right) < \theta,$$

and m_0 sufficiently large so that, for all $p \geq m_0$,

$$\theta \leq \|PT^p x - T^p x\| \leq \theta + \Theta_0.$$

For this m_0, there exist p_1, p_2 such that $p_1, p_2 > m_0$ and

$$\|PT^{p_1} x - PT^{p_2} x\| \geq \varepsilon_0.$$

Thus, for $p_0 \geq \max(p_1, p_2)$, we have

$$\|PT^{p_1} x - T^{p_0} x\| \leq \|PT^{p_1} x - T^{p_1} x\| < \theta + \Theta_0$$

and

$$\|PT^{p_2} x - T^{p_0} x\| \leq \|PT^{p_2} x - T^{p_2} x\| < \theta + \Theta_0.$$

Since X is uniformly convex, we get

$$\begin{aligned}
\theta \leq \|PT^{p_0} x - T^{p_0} x\| &\leq \left\| \frac{PT^{p_1} x + PT^{p_2} x}{2} - T^{p_0} x \right\| \\
&\leq (\theta + \Theta_0)\left(1 - \delta\left(\frac{\varepsilon_0}{\theta + \Theta_0}\right)\right) \\
&< \theta
\end{aligned}$$

which is a contradiction and it completes the proof. □

4. One Parameter E_μ Nonexpansive Semigroup

In this section, first we coin the definition of one parameter E_μ nonexpansive semigroup.

Definition 13. *Suppose Y is a closed and convex subset of a Banach space X and $\mathcal{S} = \{S(\zeta) : \zeta > 0\}$ is a family of mappings with domain $D(\mathcal{S}) = \bigcap_{\zeta>0} D(S(\zeta))$ and range $R(\mathcal{S})$, where $D(S(\zeta)), R(\mathcal{S}) \subseteq Y$. A one parameter E_μ nonexpansive semigroup is a family $\mathcal{S} = \{S(\zeta) : \zeta > 0\}$ of mappings satisfying the following conditions:*

1. *For each $\zeta > 0$, $S(\zeta)$ is a mapping satisfying condition (E), i.e., there exists $\mu \geq 1$ and for all $x, y \in D(\mathcal{S})$*
$$\|x - S(\zeta)y\| \leq \mu\|x - S(\zeta)x\| + \|x - y\|; \tag{20}$$
2. *$S(0)x = x$ for all $x \in D(\mathcal{S})$;*
3. *$S(\zeta + \xi)x = S(\zeta) \cdot S(\xi)x$ for all $\zeta, \xi > 0$ and $x \in D(\mathcal{S})$.*

The weak convergence of trajectories of one parameter semigroups of nonexpansive mappings was studied by many mathematicians, especially by Baillon [26], Bruck [27], Pazy [28], Miyadera [29], and Reich [30]. Motivated by the [17] (Theorem 1′) and [18] (Theorem 3.2), we present our next result. Now, present a weak convergence theorem concerning the trajectory $(S(\zeta))_{\zeta>0}$ of a one parameter semigroup $\mathcal{S}$ of mappings satisfying condition (E).

Theorem 7. *Suppose Y is a closed convex subset of a uniformly convex Banach space X having the Opial property. Let $S = (S(\zeta) : \zeta > 0)$ be a semigroup of E_μ-nonexpansive mappings on Y. Then, for each $x \in Y$, $(S(\zeta)x)_{\zeta>0}$ converges weakly to a common fixed point of $\mathcal{S}$, provided that $(S(\zeta + \beta)x - S(\zeta)x)_{\zeta>0}$ converges strongly to 0 for all $\beta > 0$, and $(S(\zeta)x)_{\zeta>0}$ is bounded.*

Proof. Since $(S(\zeta)x)_{\zeta>0}$ is bounded, there is a subsequence $(S(\zeta_i)x)_{i>0}$ of $(S(\zeta)x)_{\zeta>0}$ such that $S(\zeta_i)x \rightharpoonup u^\dagger$, where $\zeta_i \to \infty$ as $i \to \infty$. Since, for all $\beta > 0$, $(S(\zeta + \beta)x - S(\zeta)x)_{\zeta>0}$ converges strongly to 0, we have that $S(\zeta_i + \xi)x \rightharpoonup u^\dagger$, where $\zeta_i \to \infty$ as $i \to \infty$ for any $\xi > 0$. By Opial property, we get
$$\begin{aligned} r_{\xi+\zeta} &= \limsup_{i\to\infty} \|S(\zeta_i + \xi + \zeta)x - u^\dagger\| \\ &\leq \limsup_{i\to\infty} \|S(\zeta_i + \xi + \zeta)x - S(\zeta)u^\dagger\|. \end{aligned} \tag{21}$$

Now, by the triangle inequality and (20), we obtain
$$\begin{aligned} \|S(\zeta_i + \xi + \zeta)x - S(\zeta)u^\dagger\| &\leq \|S(\zeta_i + \xi + \zeta)x - S(\zeta_i + \xi)x\| + \|S(\zeta_i + \xi)x - S(\zeta)u^\dagger\| \\ &\leq \|S(\zeta_i + \xi + \zeta)x - S(\zeta_i + \xi)x\| \\ &\quad + \mu\|S(\zeta) \cdot S(\zeta_i + \xi)x - S(\zeta_i + \xi)x\| + \|S(\zeta_i + \xi)x - u^\dagger\| \\ &= (\mu + 1)\|S(\zeta_i + \xi + \zeta)x - S(\zeta_i + \xi)x\| + \|S(\zeta_i + \xi)x - u^\dagger\|. \end{aligned}$$

Thus,
$$\begin{aligned} \limsup_{i\to\infty} \|S(\zeta_i + \xi + \zeta)x - S(\zeta)u^\dagger\| \leq{}& \limsup_{i\to\infty}(\mu + 1)\|S(\zeta_i + \xi + \zeta)x - S(\zeta_i + \xi)x\| \\ &+ \limsup_{i\to\infty} \|S(\zeta_i + \xi)x - u^\dagger\|. \end{aligned}$$

Since $(S(\zeta + \beta)x - S(\zeta)x)_{\zeta>0}$ converges strongly to 0 for all $\beta > 0$, we have
$$\limsup_{i\to\infty} \|S(\zeta_i + \xi + \zeta)x - S(\zeta)u^\dagger\| \leq \limsup_{i\to\infty} \|S(\zeta_i + \xi)x - u^\dagger\|,$$

and from (21)
$$r_{\xi+\zeta} \leq r_\xi$$

for all $\xi, \zeta > 0$. Thus, (r_ξ) is a monotone decreasing and convergent to $r = \inf\{r_\xi : \xi > 0\}$. If $r = 0$, then there is a sequence $(S(p_i)x)$ with $p_i \to \infty$, which converges strongly to $u^\dagger$. Furthermore,

$$\begin{aligned} \lim_{i\to\infty} S(p_i+\zeta)x &= \lim_{i\to\infty} S(\zeta)\cdot S(p_i)x \\ &= \lim_{i\to\infty} S(\zeta)u^\dagger \end{aligned} \tag{22}$$

for all $\zeta > 0$. Now, we show that $S(p_i+\zeta)x \to u^\dagger$ as $p_i \to \infty$.

$$\|S(p_i+\zeta)x - u^\dagger\| \le \|S(p_i+\zeta)x - S(p_i)x\| + \|S(p_i)x - u^\dagger\| \to 0.$$

Thus, $S(p_i+\zeta)x \to u^\dagger$. From (22), it follows that $S(\zeta)u^\dagger = u^\dagger$ for all $\zeta > 0$. Again, let $r \neq 0$ and assume that, for some $\varepsilon > 0$ and $\zeta_0 > 0$, $\|S(\zeta_0)u^\dagger - u^\dagger\| \ge \varepsilon$. We can find an $\varepsilon_1 > 0$ such that $(r+\varepsilon_1)\left[1 - \delta\left(\frac{\varepsilon}{r+\varepsilon_1}\right)\right] < r$, where δ is the modulus of convexity of the norm. Choose $\xi_0 > 0$ such that $r_{\xi-\zeta_0} \le r + \varepsilon_1$ for all $\xi \ge \xi_0$. Then,

$$\limsup_{i\to\infty} \|S(\zeta_i+\xi-\zeta_0)x - u^\dagger\| \le r + \varepsilon_1. \tag{23}$$

By the triangle inequality and (20), we get

$$\begin{aligned} \limsup_{i\to\infty} \|S(\zeta_i+\xi)x - S(\zeta_0)u^\dagger\| &\le \limsup_{i\to\infty} \|S(\zeta_i+\xi)x - S(\zeta_i+\xi-\zeta_0)x\| \\ &\quad + \limsup_{i\to\infty} \|S(\zeta_i+\xi-\zeta_0)x - S(\zeta_0)u^\dagger\| \\ &\le (\mu+1)\limsup_{i\to\infty} \|S(\zeta_i+\xi)x - S(\zeta_i+\xi-\zeta_0)x\| \\ &\quad + \limsup_{i\to\infty} \|S(\zeta_i+\xi-\zeta_0)x - u^\dagger\|. \end{aligned}$$

Since $(S(\zeta+\beta)x - S(\zeta)x)_{\zeta>0}$ converges strongly to 0 and by (23), we obtain for all $\xi > \xi_0$

$$\limsup_{i\to\infty} \|S(\zeta_i+\xi)x - S(\zeta_0)u^\dagger\| \le r + \varepsilon_1. \tag{24}$$

Since $S(\zeta_i+\xi)x \rightharpoonup u^\dagger$ and in view of the Opial property, we get

$$\limsup_{i\to\infty} \|S(\zeta_i+\xi)x - u^\dagger\| \le r + \varepsilon_1$$

for all $\xi \ge \xi_0$. Since X is uniformly convex and X has the Opial property, we have that, for each $\xi \ge \xi_0$,

$$\begin{aligned} r &\le \limsup_{i\to\infty} \|S(\zeta_i+\xi)x - u^\dagger\| \\ &\le \limsup_{i\to\infty} \left\| S(\zeta_i+\xi)x - \left(\frac{S(\zeta_0)u^\dagger + u^\dagger}{2}\right)\right\| \\ &\le (r+\varepsilon_1)\left[1 - \delta\left(\frac{\varepsilon}{r+\varepsilon_1}\right)\right] < r, \end{aligned}$$

a contradiction. This implies that $u^\dagger$ is a common fixed point of $\mathcal{S}$. Moreover, we claim that there exists a $v^\dagger \in F(\mathcal{S})$ such that $\lim_{\zeta\to\infty} P_{F(\mathcal{S})}S(\zeta)x = v^\dagger$, where $P_{F(\mathcal{S})}$ is the metric projection on $F(\mathcal{S})$. From Theorem 5, we see that $F(\mathcal{S})$ is a convex and closed subset of Y, thus the metric projection $P_{F(\mathcal{S})}$ is well defined in $F(\mathcal{S})$. Take

$$\sigma(\zeta+\xi) = \|S(\zeta+\xi)x - P_{F(\mathcal{S})}S(\zeta+\xi)x\|.$$

By the definition of metric projection,

$$\|S(\zeta+\xi)x - P_{F(\mathcal{S})}S(\zeta+\xi)x\| \le \|S(\zeta+\xi)x - P_{F(\mathcal{S})}S(\zeta)x\|.$$

Since $P_{F(\mathcal{S})}S(\zeta)x \in F(\mathcal{S})$ and $S(\zeta)$ is a mapping satisfying condition (E), we have

$$\begin{aligned}\|S(\zeta+\xi)x - P_{F(\mathcal{S})}S(\zeta)x\| &\le \mu\|P_{F(\mathcal{S})}S(\zeta)x - S(\zeta)P_{F(\mathcal{S})}S(\zeta)x\| + \|S(\zeta)x - P_{F(\mathcal{S})}S(\zeta)x\| \\ &= \|S(\zeta)x - P_{F(\mathcal{S})}S(\zeta)x\| = \sigma(\zeta). \end{aligned} \tag{25}$$

Therefore, for all $\xi,\zeta > 0$, $\sigma(\zeta+\xi) \le \sigma(\zeta)$. This follows that $(\sigma(\zeta))_{\zeta>0}$ is monotonically decreasing and converging to $\sigma = \inf(\sigma(\zeta) : \zeta > 0)$. Let $\sigma = 0$. Thus, for $\xi,\zeta > 0$, using triangle inequality and (25), we have

$$\begin{aligned}\|P_{F(\mathcal{S})}S(\zeta+\xi)x - P_{F(\mathcal{S})}S(\zeta)x\| &\le \|P_{F(\mathcal{S})}S(\zeta+\xi)x - S(\zeta+\xi)x\| \\ &\quad + \|S(\zeta+\xi)x - P_{F(\mathcal{S})}S(\zeta)x\| \\ &\le \|P_{F(\mathcal{S})}S(\zeta+\xi)x - S(\zeta+\xi)x\| + \|S(\zeta)x - P_{F(\mathcal{S})}S(\zeta)x\| \\ &= \sigma(\zeta+\xi) + \sigma(\zeta).\end{aligned}$$

Since $\lim_{\zeta\to\infty} \sigma(\zeta) = \sigma = 0$, it implies that $(P_{F(\mathcal{S})}S(\zeta)x)_{\zeta>0}$ is convergent to some point $v^\dagger \in F(\mathcal{S})$ (here $F(\mathcal{S})$ is a closed subset of Y). Again, let $\sigma > 0$. If $(P_{F(\mathcal{S})}S(\zeta)x)_{\zeta>0}$ does not converge strongly, then there is a sequence $(P_{F(\mathcal{S})}S(\zeta_i)x)$ with $i \to \infty$, for given $\varepsilon > 0$, the following holds: for all $j, i \ge 1, j \ne i$,

$$\|P_{F(\mathcal{S})}S(\zeta_j)x - P_{F(\mathcal{S})}S(\zeta_i)x\| \ge \varepsilon. \tag{26}$$

We can choose $\varepsilon_2 > 0$ such that $(\sigma+\varepsilon_2)\left[1-\delta\left(\frac{\varepsilon}{\sigma+\varepsilon_2}\right)\right] < \sigma$ and $\overline{\zeta} > 0$ such that $\sigma(\zeta) \le \sigma+\varepsilon_2$ for all $\zeta \ge \overline{\zeta}$. Now, for all $\zeta_j > \zeta_i \ge \overline{\zeta}$, we have

$$\sigma(\zeta_i) = \|S(\zeta_i)x - P_{F(\mathcal{S})}S(\zeta_i)x\| \le \sigma+\varepsilon_2, \tag{27}$$

$$\sigma(\zeta_j) = \|S(\zeta_j)x - P_{F(\mathcal{S})}S(\zeta_j)x\| \le \sigma+\varepsilon_2. \tag{28}$$

Since $\zeta_j > \zeta_i$ and from (25), we have

$$\|S(\zeta_j)x - P_{F(\mathcal{S})}S(\zeta_i)x\| \le \|S(\zeta_i)x - P_{F(\mathcal{S})}S(\zeta_i)x\|.$$

From above inequality and (27), we have

$$\|S(\zeta_j)x - P_{F(\mathcal{S})}S(\zeta_i)x\| \le \sigma+\varepsilon_2. \tag{29}$$

Since X is uniformly convex and, from (26), (28), and (29), we have that, for all $\zeta_j > \zeta_i \ge \overline{\zeta}$,

$$\begin{aligned}\sigma &\le \left\| S(\zeta_j)x - \left(\frac{P_{F(\mathcal{S})}S(\zeta_j)x + P_{F(\mathcal{S})}S(\zeta_i)x}{2}\right)\right\| \\ &\le (\sigma+\varepsilon_2)\left[1-\delta\left(\frac{\varepsilon}{\sigma+\varepsilon_2}\right)\right] < \sigma,\end{aligned}$$

a contradiction. Thus, $(P_{F(\mathcal{S})}S(\zeta)x)_{\zeta>0}$ converges strongly to some point $v^\dagger \in F(\mathcal{S})$. Next, we show that $(S(\zeta)x)_{\zeta>0}$ converges weakly to $v^\dagger = \lim_{\zeta\to\infty} P_{F(\mathcal{S})}S(\zeta)x$. We have shown that $S(\zeta_i)x \rightharpoonup u^\dagger$ where $\zeta_i \to \infty$ as $i \to \infty$ and $u^\dagger \in F(\mathcal{S})$. If $u^\dagger \ne v^\dagger$, then, by the Opial property, we get

$$\begin{aligned}\limsup_{i\to\infty}\|S(\zeta_i)x - v^\dagger\| &= \limsup_{i\to\infty}\|S(\zeta_i)x - P_{F(\mathcal{S})}S(\zeta_i)x\| \\ &\le \limsup_{i\to\infty}\|S(\zeta_i)x - u^\dagger\| \\ &< \limsup_{i\to\infty}\|S(\zeta_i)x - v^\dagger\|,\end{aligned}$$

a contradiction. Therefore, $u^\dagger = v^\dagger$ and this completes the proof. □

In 2018, Song et al. [19] considered the α-nonexpansive mapping semigroups and obtained a common fixed point of this class of semigroup using the Halpern iteration process [31]. They considered the one-parameter α-nonexpansive semigroup as follows:

Definition 14. *A one-parameter α-nonexpansive semigroup is a family $\mathcal{S} = \{S(\zeta) : \zeta > 0\}$ of mappings with domain $D(\mathcal{S}) = \bigcap_{\zeta>0} D(S(\zeta))$ and range $R(\mathcal{S})$ such that:*

1. *For each $\zeta > 0$, $S(\zeta)$ is α-nonexpansive, that is, there exists $\alpha < 1$ and for all $x, y \in D(\mathcal{S})$,*

$$\|S(\zeta)x - S(\zeta)y\| \le \alpha\|S(\zeta)x - y\| + \alpha\|S(\zeta)y - x\| + (1-2\alpha)\|x - y\|;$$

2. *$S(0)x = x$ for all $x \in D(\mathcal{S})$;*
3. *For all $\zeta, s > 0$ and $x \in D(\mathcal{S})$, $S(\zeta + s)x = S(\zeta)S(s)x$.*

Song et al. [19] proved the following lemma:

Lemma 3. *Let Y be a convex closed subset of a Hilbert space M. Let $T : Y \to Y$ be an α-nonexpansive mapping. Then, for all $x, y \in Y$, we have*

$$\|Tx - Ty\| \le \|x - y\| + \left(\frac{2|\alpha|}{(1-\alpha)}\right)\|Tx - x\|. \tag{30}$$

Proposition 4. *Suppose Y is a nonempty subset of a Hilbert space M and $T : Y \to Y$ is an α-nonexpansive mapping. Then T satisfies condition (E).*

Proof. By the triangle inequality and (30), we get

$$\begin{aligned}\|x - Ty\| &\le \|x - Tx\| + \|Tx - Ty\| \\ &\le \|x - Tx\| + \|x - y\| + \left(\frac{2|\alpha|}{(1-\alpha)}\right)\|x - Tx\| \\ &\le \left(1 + \frac{2|\alpha|}{(1-\alpha)}\right)\|x - Tx\| + \|x - y\|.\end{aligned}$$

Take $\mu = \left(1 + \frac{2|\alpha|}{(1-\alpha)}\right)$; then, T is a mapping satisfying condition (E). □

The following example demonstrates that the inclusion in the above proposition is strict.

Example 3. *Suppose $\mathbb{R}$ is the set of real numbers with the standard norm and $Y = [0, 4]$ a subset of $\mathbb{R}$. Let $T : Y \to Y$ defined as*

$$Tx = \begin{cases} 0, \text{ if } x \neq 4, \\ 3, \text{ if } x = 4. \end{cases}$$

First, we show that T satisfies condition (E). We consider three nontrivial cases:

Case (1) *$x \le 3$ and $y = 4$. Then*

$$\begin{aligned}\|x - Ty\| = \|3 - x\| &\le \|x\| + \|4 - x\| \\ &= \|x - Tx\| + \|y - x\|.\end{aligned}$$

Case (2) *$x > 3$ and $y = 4$. Then*

$$\begin{aligned}\|x - Ty\| = \|x - 3\| &\le 1 + \|4 - x\| \\ &\le \|x\| + \|4 - x\| \\ &\le \|x - Tx\| + \|y - x\|.\end{aligned}$$

Case (3) *If $x = 4, y \ne 4$, then*

$$\begin{aligned}\|x - Ty\| = \|4\| &\le 4\|1\| + \|4 - y\| \\ &\le 4\|x - Tx\| + \|x - y\|.\end{aligned}$$

Moreover, for $x = 3$, $y = 4$ and for any, $\alpha < 1$

$$\begin{aligned}\|Tx - Ty\| = 3 > 1 + 2\alpha &= 4\alpha + 1 - 2\alpha \\ &= \alpha\|3 - 3\| + \alpha\|4 - 0\| + (1 - 2\alpha)\|4 - 3\| \\ &= \alpha\|x - Ty\| + \alpha\|y - Tx\| + (1 - 2\alpha)\|x - y\|.\end{aligned}$$

Hence, T is not an α-nonexpansive considered by Song et al. [19] or in Definition 9.

Remark 1. *From proposition 4, we see that the class of a one-parameter α-nonexpansive semigroup contained in the class of a one-parameter E_μ nonexpansive semigroup.*

Song et al. presented the following theorem as the main result in [19].

Theorem 8. *Let Y be a nonempty convex closed subset of a Hilbert space M. Let $\mathcal{S} = \{S(\zeta) : \zeta > 0\}$ be the u.a.r. semigroup of α-nonexpansive mappings from Y into itself with $F(\mathcal{S}) \ne \emptyset$. For a fixed $u, x_0 \in Y$, and for each $n \in \mathbb{N}$ the sequence defined by*

$$x_{n+1} = \gamma_n u + (1 - \gamma_n)S(\zeta_n)x_n, \tag{31}$$

where $\gamma_n \in (0, 1)$, $\zeta_n > 0$ and the following assumptions hold:

$$\lim_{n\to\infty} \gamma_n = 0, \ \sum_{n=1}^{\infty} \gamma_n = \infty, \ \lim_{n\to\infty} \zeta_n = \infty. \tag{32}$$

Then, the sequence (x_n) converges strongly to $u^\dagger = P_{F(\mathcal{S})}u$.

Now, we extend Theorem 8 for the class of one-parameter E_μ-nonexpansive semigroups.

Theorem 9. *Let Y and M be defined as in Theorem 8. Let $\mathcal{S} = \{S(\zeta) : \zeta > 0\}$ be the u.a.r. semigroup of E_μ-nonexpansive mappings from Y into itself with $F(\mathcal{S}) \ne \emptyset$. For a fixed $u, x_0 \in Y$, and, for each $n \in \mathbb{N}$, the sequence defined by*

$$x_{n+1} = \gamma_n u + (1 - \gamma_n)S(\zeta_n)x_n, \tag{33}$$

where γ_n and ζ_n are the same as in Theorem 8. Then, the sequence (x_n) converges strongly to $u^\dagger = P_{F(\mathcal{S})}u$.

Proof. Let $v^\dagger \in F(\mathcal{S})$. From Proposition 1, we have

$$\|S(\zeta)x - v^\dagger\| \le \|x - v^\dagger\|$$

for all $x \in Y$ and $\zeta > 0$. Thus, from (33), we get

$$\begin{aligned}\|x_{n+1} - v^\dagger\| &= \|\gamma_n(u - v^\dagger) + (1-\gamma_n)(S(\zeta_n)x_n - v^\dagger)\| \\ &\leq \gamma_n\|u - v^\dagger\| + (1-\gamma_n)\|S(\zeta_n)x_n - v^\dagger\| \\ &\leq \gamma_n\|u - v^\dagger\| + (1-\gamma_n)\|x_n - v^\dagger\| \\ &\leq \max\{\|u - v^\dagger\|, \|x_n - v^\dagger\|\}.\end{aligned}$$

Consequently, the sequence (x_n) is bounded. Since $\|S(\zeta_n)x_n - v^\dagger\| \leq \|x_n - v^\dagger\|$, the sequence $(S(\zeta_n)x_n)$ is bounded. From (32) and (33), we have

$$\lim_{n\to\infty}\|x_{n+1} - S(\zeta_n)x_n\| = \lim_{n\to\infty}\gamma_n\|u - S(\zeta_n)x_n\| = 0. \tag{34}$$

Let C be a bounded subset of Y containing (x_n) and $(S(\zeta_n)x_n)$. Since $\mathcal{S} = (S(\zeta))$ is u.a.r. E_μ-nonexpansive semigroup and from (32), i.e., $\lim\limits_{n\to\infty}\zeta_n = \infty$, we have that, for any $\xi > 0$,

$$\lim_{n\to\infty}\|S(\xi)S(\zeta_n)x_n - S(\zeta_n)x_n\| \leq \lim_{n\to\infty}\sup_{x\in C}\|S(\xi)S(\zeta_n)x - S(\zeta_n)x\| = 0. \tag{35}$$

Thus, for all $\xi > 0$, from the triangle inequality and (20), we get

$$\begin{aligned}\|x_{n+1} - S(\xi)x_{n+1}\| &\leq \|x_{n+1} - S(\zeta_n)x_n\| + \|S(\zeta_n)x_n - S(\xi)x_{n+1}\| \\ &\leq \|x_{n+1} - S(\zeta_n)x_n\| + \mu\|S(\zeta_n)x_n - S(\xi)S(\zeta_n)x_n\| + \|x_{n+1} - S(\zeta_n)x_n\| \\ &= 2\|x_{n+1} - S(\zeta_n)x_n\| + \mu\|S(\zeta_n)x_n - S(\xi)S(\zeta_n)x_n\|.\end{aligned}$$

From (34) and (35), it implies that, for all $\xi > 0$,

$$\lim_{n\to\infty}\|x_{n+1} - S(\xi)x_{n+1}\| = 0. \tag{36}$$

Since the sequence (x_n) is bounded in Y, it has a subsequence (x_{n_j}) such that $x_{n_j} \rightharpoonup w^\dagger$, for some $w^\dagger \in Y$. Moreover, for all $\xi > 0$, from (36),

$$\lim_{j\to\infty}\|x_{n_j} - S(\xi)x_{n_j}\| = 0.$$

By the demiclosedness principle for mapping $S(\xi)$, we have $w^\dagger \in F(S(\xi))$. Since ξ is an arbitrary, $w^\dagger \in F(\mathcal{S})$. From Theorem 5, it implies that $F(\mathcal{S})$ is closed and convex subset of Y. Therefore, metric projection $P_{F(\mathcal{S})} : M \to F(\mathcal{S})$ is well defined. Now, it remains to prove that (x_n) converges strongly to $u^\dagger = P_{F(\mathcal{S})}u$. The rest of the proof directly follows from [19] (Theorem 3.3). $\square$

Now, we extend [19] (Theorem 3.4) from a family of u.a.r. α-nonexpansive mappings to a family of u.a.r. E_μ-nonexpansive mappings.

Theorem 10. *Let Y and M be defined as in Theorem 8. Suppose $\{S_n\}$ is a family of u.a.r. E_μ-nonexpansive mappings on Y such that $\mathfrak{F} = \bigcap\limits_{n=1}^{\infty} F(S_n) \neq \emptyset$. For fixed $u, x_0 \in Y$ and $n \in \mathbb{N}$, define the sequence (x_n) by*

$$x_{n+1} = \gamma_n u + (1-\gamma_n)S_n x_n,$$

where γ_n is same as in Theorem 8. Then, the sequence (x_n) converges strongly to $u^\dagger = P_{\mathfrak{F}}u$.

Proof. By replacing $S(\zeta_n)$ and $S(\xi)$ with S_n and S_m, respectively in Theorem 9, we can easily obtain the desired conclusion. $\square$

5. Conclusions

In this paper, we showed that the class mapping considered in [15] is properly contained in the class of generalized α–Reich–Suzuki nonexpansive mappings. We also showed that a generalized contraction of Suzuki type mapping satisfies the condition (E) but not conversely. Finally, we obtained some new fixed point results for α-Krasnosel'skiĭ mappings and one parameter nonexpansive type semigroups.

Author Contributions: Supervision, R.P.; Validation, R.S.; Writing—original draft, P.P.; Writing—review and editing, R.P. and M.D.l.S. All authors have read and agreed to the published version of the manuscript.

Funding: The authors thanks the Basque Government for its support through Grant IT1207-19.

Acknowledgments: We are very thankful to the reviewers for their constructive comments and suggestions that have been useful for the improvement of this paper. The second and third authors acknowledge the support from the GES 4.0 fellowship, University of Johannesburg, South Africa.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Browder, F.E. Fixed-point theorems for noncompact mappings in Hilbert space. *Proc. Natl. Acad. Sci. USA* **1965**, *53*, 1272–1276. [CrossRef]
2. Göhde, D. Zum Prinzip der kontraktiven Abbildung. *Math. Nachr.* **1965**, *30*, 251–258. [CrossRef]
3. Kirk, W.A. A fixed point theorem for mappings which do not increase distances. *Am. Math. Mon.* **1965**, *72*, 1004–1006. [CrossRef]
4. Aoyama, K.; Kohsaka, F. Fixed point theorem for α-nonexpansive mappings in Banach spaces. *Nonlinear Anal.* **2011**, *74*, 4387–4391. [CrossRef]
5. García-Falset, J.; Llorens-Fuster, E.; Suzuki, T. Fixed point theory for a class of generalized nonexpansive mappings. *J. Math. Anal. Appl.* **2011**, *375*, 185–195. [CrossRef]
6. Goebel, K.; Pineda, M.J. A new type of nonexpansiveness. In Proceedings of the 8-th International Conference on Fixed Point Theory and Applications, Chiang Mai, Thailand, 16–22 July 2007.
7. Goebel, K.; Kirk, W.A. A fixed point theorem for asymptotically nonexpansive mappings. *Proc. Am. Math. Soc.* **1972**, *35*, 171–174. [CrossRef]
8. Kannan, R. Fixed point theorems in reflexive Banach spaces. *Proc. Am. Math. Soc.* **1973**, *38*, 111–118. [CrossRef]
9. Kirk, W.A.; Xu, H.K. Asymptotic pointwise contractions. *Nonlinear Anal.* **2008**, *69*, 4706–4712. [CrossRef]
10. Llorens-Fuster, E.; Moreno Gálvez, E. The fixed point theory for some generalized nonexpansive mappings. *Abstr. Appl. Anal.* **2011**, *2011*, 1–15. [CrossRef]
11. Nicolae, A. Generalized asymptotic pointwise contractions and nonexpansive mappings involving orbits. *Fixed Point Theory Appl.* **2009**, *2010*, 1–19. [CrossRef]
12. Pandey, R.; Pant, R.; Rakočević, V.; Shukla, R. Approximating fixed points of a general class of nonexpansive mappings in Banach spaces with applications. *Results Math.* **2019**, *74*, 1–24. [CrossRef]
13. Pant, R.; Shukla, R. Approximating fixed points of generalized α-nonexpansive mappings in Banach spaces. *Numer. Funct. Anal. Optim.* **2017**, *38*, 248–266. [CrossRef]
14. Suzuki, T. Fixed point theorems and convergence theorems for some generalized nonexpansive mappings. *J. Math. Anal. Appl.* **2008**, *340*, 1088–1095. [CrossRef]
15. Atailia, S.; Redjel, N.; Dehici, A. Some fixed point results for generalized contractions of Suzuki type in Banach spaces. *J. Fixed Point Theory Appl.* **2019**, *21*, 1–16. [CrossRef]
16. Hardy, G.E.; Rogers, T.D. A generalization of a fixed point theorem of Reich. *Can. Math. Bull.* **1973**, *16*, 201–206. [CrossRef]
17. Emmanuele, G. Asymptotic behavior of iterates of nonexpansive mappings in Banach spaces with Opial's condition. *Proc. Am. Math. Soc.* **1985**, *94*, 103–109.
18. Hirano, N. Nonlinear ergodic theorems and weak convergence theorems. *J. Math. Soc. Jpn.* **1982**, *34*, 35–46. [CrossRef]
19. Song, Y.; Muangchoo-in, K.; Kumam, P.; Cho, Y.J. Successive approximations for common fixed points of a family of α-nonexpansive mappings. *J. Fixed Point Theory Appl.* **2018**, *20*, 10–13. [CrossRef]
20. Kirk, W.A.; Sims, B. *Handbook of Metric Fixed Point Theory*; Springer Science & Business Media: Berlin/Heidelberg, Germany, 2013.
21. Chidume, C. *Geometric Properties of Banach Spaces and Nonlinear Iterations*, *Lecture Notes in Mathematics*; Springer: London, UK, 2009; Volume 1965.
22. Opial, Z. Weak convergence of the sequence of successive approximations for nonexpansive mappings. *Bull. Am. Math. Soc.* **1967**, *73*, 591–597. [CrossRef]
23. Cegielski, A. *Iterative Methods for Fixed Point Problems in Hilbert Spaces*, *Lecture Notes in Mathematics*; Springer: Berlin/Heidelberg, Germany, 2012; Volume 2057.

24. Browder, F.E.; Petryshyn, W.V. The solution by iteration of nonlinear functional equations in Banach spaces. *Bull. Am. Math. Soc.* **1966**, *72*, 571–575. [CrossRef]
25. Benavides, T.D.; Acedo, G.L.; Xu, H.K. Construction of sunny nonexpansive retractions in Banach spaces. *Bull. Aust. Math. Soc.* **2002**, *66*, 9–16. [CrossRef]
26. Baillon, J.B. Quelques propriétés de convergence asymptotique pour les semi-groupes de contractions impaires. *C. R. Acad. Sci. Paris Sér. A-B* **1976**, *283*, 75–78.
27. Bruck, R.E. On the almost-convergence of iterates of a nonexpansive mapping in Hilbert space and the structure of the weak ω-limit set. *Isr. J. Math.* **1978**, *29*, 1–16. [CrossRef]
28. Pazy, A. On the asymptotic behavior of semigroups of nonlinear contractions in Hilbert space. *J. Funct. Anal.* **1978**, *27*, 292–307. [CrossRef]
29. Miyadera, I. Asymptotic behavior of iterates of nonexpansive mappings in Banach spaces. *Proc. Jpn. Acad. Ser. A Math. Sci.* **1978**, *54*, 212–214. [CrossRef]
30. Reich, S. Asymptotic behavior of semigroups of nonlinear contractions in Banach spaces. *J. Math. Anal. Appl.* **1976**, *53*, 277–290. [CrossRef]
31. Halpern, B. Fixed points of nonexpanding maps. *Bull. Am. Math. Soc.* **1967**, *73*, 957–961. [CrossRef]