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# A Note on Type-Two Degenerate Poly-Changhee Polynomials of the Second Kind

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**Abstract:** In this paper, we first define type-two degenerate poly-Changhee polynomials of the second kind by using modified degenerate polyexponential functions. We derive new identities and relations between type-two degenerate poly-Changhee polynomials of the second kind. Finally, we derive type-two degenerate unipoly-Changhee polynomials of the second kind and discuss some of their identities.

**Keywords:** modified degenerate polyexponential function; degenerate Changhee polynomials of the second kind; type-two degenerate poly-Changhee polynomials of the second kind; unipoly functions

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## 1. Introduction

As is well known, Changhee polynomials $Ch_n(x)$ are defined by means of the following generating function

$$\int_{\mathbb{Z}_p} (1+t)^{x+y} d_{\mu_{-1}} y = \frac{2}{2+t}(1+t)^x = \sum_{n=0}^{\infty} Ch_n(x)\frac{t^n}{n!} \tag{1}$$

(see [1,2]).

In the case when $x = 0$, $Ch_n(0) = Ch_n$ are called Changhee numbers.

The Euler polynomials are defined by the following generating function:

$$\int_{\mathbb{Z}_p} e^{(x+y)t} d_{\mu_{-1}} y = \frac{2}{e^t+1} e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} \tag{2}$$

(see [3]).

When $x = 0$, $E_n(0) = E_n$ are called the Euler numbers.

From (1) and (2), we note that

$$Ch_n(x) = \sum_{l=0}^{n} E_l(x) S_1(n,l),$$

and

$$E_n(x) = \sum_{l=0}^{n} Ch_n(x) S_2(n,l), (n \geq 0) \tag{3}$$

(see [1]).

For any non-zero $\lambda \in \mathbb{R}$, the degenerate exponential functions are defined by (see [4–6]):

$$e_\lambda^x(t) = (1+\lambda t)^{\frac{x}{\lambda}}, e_\lambda(t) = e_\lambda^1(t) = (1+\lambda t)^{\frac{1}{\lambda}}. \tag{4}$$

Here, we note that

$$e_\lambda^x(t) = \sum_{n=0}^{\infty} (x)_{n,\lambda} \frac{t^n}{n!}, \tag{5}$$

where $(x)_{0,\lambda} = 1, (x)_{n,\lambda} = x(x-\lambda)(x-2\lambda)\cdots(x-(n-1)\lambda), (n \geq 1)$.

In [7,8], Carlitz considered degenerate Bernoulli polynomials, which are given by

$$\frac{t}{e_\lambda(t)-1} e_\lambda^x(t) = \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1} (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \beta_{n,\lambda}(x) \frac{t^n}{n!}. \tag{6}$$

On setting $x = 0$, $\beta_{n,\lambda}(0) = \beta_{n,\lambda}$ are called degenerate Bernoulli numbers.

For $k \in \mathbb{Z}$, the modified degenerate polyexponential function [9] was defined by Kim and Kim to be

$$\mathrm{Ei}_{k,\lambda}(x) = \sum_{n=1}^{\infty} \frac{(1)_{n,\lambda} x^n}{(n-1)! n^k}, (|\ x\ | < 1). \tag{7}$$

Note that

$$\mathrm{Ei}_{1,\lambda}(x) = \sum_{n=1}^{\infty} \frac{(1)_{n,\lambda} x^n}{n!} = e_\lambda(x) - 1. \tag{8}$$

In [9], Kim et al. introduced degenerate poly-Genocchi polynomials, which are given by

$$\frac{\mathrm{Ei}_{k,\lambda}\left(\log_\lambda(1+t)\right)}{e_\lambda(t)+1} e_\lambda^x(t) = \sum_{n=0}^{\infty} G_{n,\lambda}^{(k)}(x) \frac{t^n}{n!}, (k \in \mathbb{Z}). \tag{9}$$

In the case when $x = 0$, $G_{n,\lambda}^{(k)}(0) = G_{n,\lambda}^{(k)}$ are called degenerate poly-Genocchi numbers.

Let $\lambda \in \mathbb{C}_p$ with $|\ \lambda\ | \leq 1$. The degenerate Changhee polynomials of the second kind $Ch_{n,\lambda}(x)$ are defined by

$$\begin{aligned} \int_{\mathbb{Z}_p} (1+\lambda \log(1+\lambda t))^{\frac{x+y}{\lambda}} d\mu_{-1} y &= \frac{2}{1+(1+\lambda \log(1+t))^{\frac{1}{\lambda}}} (1+\lambda \log(1+\lambda t))^{\frac{x}{\lambda}} \\ &= \sum_{n=0}^{\infty} Ch_{n,\lambda}(x) \frac{t^n}{n!} \end{aligned} \tag{10}$$

(see [10]).

When $x = 0$, $Ch_{n,\lambda}(0) = Ch_{n,\lambda}$ are called the degenerate Changhee numbers of the second kind.

In [11], the degenerate Daehee polynomials $D_{n,\lambda}(x)$ are defined by

$$\frac{\log_\lambda(1+t)}{t} (1+t)^x = \sum_{n=0}^{\infty} D_{n,\lambda}(x) \frac{t^n}{n!}, (\lambda \in \mathbb{R}). \tag{11}$$

For $x = 0$, $D_{n,\lambda}(0) = D_{n,\lambda}$ are called degenerate Daehee numbers.

Note that $\lim_{\lambda \to 0} D_{n,\lambda}(x) = D_n(x), (n \geq 0)$ (see [12]).

The degenerate Stirling numbers of the first kind are defined by

$$\frac{1}{k!} (\log_\lambda(1+t))^k = \sum_{n=k}^{\infty} S_{1,\lambda}(n,k) \frac{t^n}{n!}, \quad (k \geq 0) \tag{12}$$

(see [6,13–19]).

Note here that $\lim_{\lambda \to 0} S_{1,\lambda}(n,k) = S_1(n,k)$, where $S_1(n,k)$ are the Stirling numbers of the first kind given by

$$\frac{1}{k!} (\log(1+t))^k = \sum_{n=k}^{\infty} S_1(n,k) \frac{t^n}{n!}, \quad (k \geq 0) \tag{13}$$

(see [5,20]).

The degenerate Stirling numbers of the second kind are given by

$$\frac{1}{k!}(e_\lambda(t)-1)^k=\sum_{n=l}^{\infty}S_{2,\lambda}(n,l)\frac{t^n}{n!} \tag{14}$$

(see [21]).

We note here that $\lim_{\lambda\to 0}S_{2,\lambda}(n,k)=S_2(n,k)$, where $S_2(n,k)$ are the Stirling numbers of the second kind given by

$$\frac{1}{k!}(e^t-1)^k=\sum_{n=l}^{\infty}S_2(n,l)\frac{t^n}{n!} \tag{15}$$

(see [1–30]).

In this article, we introduce type-two degenerate poly-Changhee polynomials of the second kind and derive explicit expressions and some identities of those polynomials. In addition, we introduce type-two degenerate unipoly-Changhee polynomials of the second kind and derive explicit multifarious properties.

## 2. Type-Two Degenerate Poly-Changhee Polynomials of the Second Kind

In this section, we define degenerate Changhee polynomials of the second kind by using the modified degenerate polyexponential function; these are called type-two degenerate poly-Changhee numbers and polynomials of the second kind in the following.

Let $\lambda\in\mathbb{C}$ and $k\in\mathbb{Z}$; we consider that the type-two degenerate poly-Changhee polynomials of the second kind are defined by

$$\frac{2\mathrm{Ei}_{k,\lambda}(\log_\lambda(1+t))}{t(1+(1+\lambda\log(1+t))^{\frac{1}{\lambda}})}(1+\lambda\log(1+t))^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}Ch_{n,\lambda}^{(k)}(x)\frac{t^n}{n!}. \tag{16}$$

In the special case, when $x=0$, $Ch_{n,\lambda}^{(k)}(0)=Ch_{n,\lambda}^{(k)}$ are called type-two degenerate poly-Changhee numbers of the second kind, where $\log_\lambda(t)=\frac{1}{\lambda}(t^\lambda-1)$ is the compositional inverse of $e_\lambda(t)$ that satisfies

$$\log_\lambda(e_\lambda(t))=e_\lambda(\log_\lambda(t))=t.$$

For $k=1$ in (16), we get

$$\frac{2}{1+(1+\lambda\log(1+t))^{\frac{1}{\lambda}}}(1+\lambda\log(1+t))^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}Ch_{n,\lambda}(x)\frac{t^n}{n!}, \tag{17}$$

where $Ch_{n,\lambda}(x)$ are called degenerate Changhee polynomials of the second kind (see Equation (10)).

Obviously,

$$\begin{aligned}\lim_{\lambda\longrightarrow 0}\left(\frac{2\mathrm{Ei}_{k,\lambda}(\log_\lambda(1+t))}{t(1+(1+\lambda\log(1+t))^{\frac{1}{\lambda}})}(1+\lambda\log(1+t))^{\frac{x}{\lambda}}\right)&=\sum_{n=0}^{\infty}\lim_{\lambda\longrightarrow 0}Ch_{n,\lambda}^{(k)}(x)\frac{t^n}{n!}\\&=\frac{2\mathrm{Ei}_k(\log(1+t))}{t(2+t)}(1+t)^x=\sum_{n=0}^{\infty}Ch_n^{(k)}(x)\frac{t^n}{n!},\end{aligned} \tag{18}$$

where $Ch_n^{(k)}(x)$ are called type-two poly-Changhee polynomials.

By using Equations (7), (10), and (16), we observe that

$$\begin{aligned}
\sum_{n=0}^{\infty} Ch_{n,\lambda}^{(k)} \frac{t^n}{n!} &= \frac{2\mathrm{Ei}_{k,\lambda}(\log_\lambda(1+t))}{t(1+(1+\lambda\log(1+t))^{\frac{1}{\lambda}})} \\
&= \frac{2}{1+(1+\lambda\log(1+t))^{\frac{1}{\lambda}}} \frac{1}{t} \sum_{m=0}^{\infty} \frac{(\log_\lambda(1+t))^{m+1}}{(m+1)!(m+1)^{k-1}} \\
&= \frac{2}{1+(1+\lambda\log(1+t))^{\frac{1}{\lambda}}} \frac{1}{t} \sum_{m=0}^{\infty} \frac{1}{(m+1)^{k-1}} \sum_{l=m+1}^{\infty} S_{1,\lambda}(l,m+1) \frac{t^l}{l!} \\
&= \frac{2}{1+(1+\lambda\log(1+t))^{\frac{1}{\lambda}}} \frac{1}{t} \sum_{m=0}^{\infty} \frac{1}{(m+1)^{k-1}} \sum_{l=m}^{\infty} S_{1,\lambda}(l+1,m+1) \frac{t^l}{(l+1)!} \\
&= \left(\sum_{s=0}^{\infty} Ch_{n,\lambda} \frac{t^n}{n!}\right) \left(\sum_{l=0}^{\infty} \sum_{m=0}^{l} \frac{1}{(m+1)^{k-1}} \frac{S_{1,\lambda}(l+1,m+1)}{l+1} \frac{t^l}{l!}\right) \\
L.H.S &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \sum_{m=0}^{l} \binom{n}{l} Ch_{n-l,\lambda} \frac{S_{1,\lambda}(l+1,m+1)}{l+1(m+1)^{k-1}}\right) \frac{t^n}{n!}. \qquad (19)
\end{aligned}$$

Therefore, using (19), we obtain the following theorem.

**Theorem 1.** *For $n \geq 0$ and $k \in \mathbb{Z}$, we have*

$$Ch_{n,\lambda}^{(k)} = \sum_{l=0}^{n} \sum_{m=0}^{l} \binom{n}{l} Ch_{n-l,\lambda} \frac{S_{1,\lambda}(l+1,m+1)}{l+1(m+1)^{k-1}}. \qquad (20)$$

**Corollary 1.** *For $n \geq 0$ and $k \in \mathbb{Z}$, we have*

$$Ch_{n,\lambda}^{(1)} = \sum_{l=0}^{n} \sum_{m=0}^{l} \binom{n}{l} Ch_{n-l,\lambda} \frac{S_{1,\lambda}(l+1,m+1)}{l+1}. \qquad (21)$$

From (16), we observe that

$$\begin{aligned}
&\sum_{n=0}^{\infty} Ch_{n,\lambda}^{(k)}(x) \frac{t^n}{n!} \\
&= \frac{2\mathrm{Ei}_{k,\lambda}(\log_\lambda(1+t))}{t(1+(1+\lambda\log(1+t))^{\frac{1}{\lambda}})} (1+\lambda\log(1+t))^{\frac{x}{\lambda}} \\
&= \left(\sum_{n=0}^{\infty} Ch_{n,\lambda}^{(k)} \frac{t^n}{n!}\right) \left(\sum_{m=0}^{\infty} \binom{\frac{x}{\lambda}}{m} \lambda^m (\log(1+t))^m\right) \\
&= \left(\sum_{n=0}^{\infty} Ch_{n,\lambda}^{(k)} \frac{t^n}{n!}\right) \left(\sum_{m=0}^{\infty} (x)_{m,\lambda} \sum_{l=m}^{\infty} S_1(l,m) \frac{t^s}{s!}\right) \\
&= \left(\sum_{n=0}^{\infty} Ch_{n,\lambda}^{(k)} \frac{t^n}{n!}\right) \left(\sum_{l=0}^{\infty} \sum_{m=0}^{l} (x)_{m,\lambda} S_1(l,m) \frac{t^l}{l!}\right) \\
\sum_{n=0}^{\infty} Ch_{n,\lambda}^{(k)}(x) \frac{t^n}{n!} &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \sum_{m=0}^{l} \binom{n}{l} Ch_{n-l,\lambda}^{(k)}(x)_{m,\lambda} S_1(l,m)\right) \frac{t^n}{n!}. \qquad (22)
\end{aligned}$$

By comparing the coefficients on both sides of (22), we obtain the following theorem.

**Theorem 2.** *Let $n \geq 0$ and $k \in \mathbb{Z}$. Then, we have*

$$Ch_{n,\lambda}^{(k)}(x) = \sum_{l=0}^{n} \sum_{m=0}^{l} \binom{n}{l} Ch_{n-l,\lambda}^{(k)}(x)_{m,\lambda} S_1(l,m). \tag{23}$$

In [4], the degenerate Bernoulli polynomials of the second kind are defined by

$$\frac{t}{\log_\lambda(1+t)}(1+t)^x = \sum_{n=0}^{\infty} b_{n,\lambda}(x)\frac{t^n}{n!} \tag{24}$$

(see [30]).

For $x = 0$, $b_{n,\lambda}(0) = b_{n,\lambda}$ are called degenerate Bernoulli numbers of the second kind. From (7), we note that

$$\frac{d}{dx}\mathrm{Ei_{k,\lambda}}(\log_\lambda(1+x)) = \frac{\mathrm{d}}{\mathrm{dx}} \sum_{\mathrm{n=1}}^{\infty} \frac{(1)_{\mathrm{n,\lambda}}(\log_\lambda(1+\mathrm{x}))^{\mathrm{n}}}{(\mathrm{n}-1)!\mathrm{n}^{\mathrm{k}}}$$

$$= \frac{(1+x)^{\lambda-1}}{\log_\lambda(1+x)} \sum_{n=1}^{\infty} \frac{(1)_{n,\lambda}(\log_\lambda(1+x))^n}{(n-1)!n^{k-1}} = \frac{(1+x)^{\lambda-1}}{\log_\lambda(1+x)}\mathrm{Ei_{k-1,\lambda}}(\log_\lambda(1+x)). \tag{25}$$

Thus, from (16) and (25), we have

$$\sum_{n=0}^{\infty} Ch_{n,\lambda}^{(k)}\frac{x^n}{n!} = \frac{2}{x(1+(1+\lambda\log(1+x))^{\frac{1}{\lambda}})}\mathrm{Ei}_{k,\lambda}(\log_\lambda(1+x))$$

$$= \frac{2}{x(1+(1+\lambda\log(1+x))^{\frac{1}{\lambda}})} \int_0^x \underbrace{\frac{(1+t)^{\lambda-1}}{\log_\lambda(1+t)} \int_0^t \cdots \frac{(1+t)^{\lambda-1}}{\log_\lambda(1+t)} \int_0^t}_{(k-2)-\text{times}} \frac{(1+t)^{\lambda-1}}{\log_\lambda(1+t)} t dt \cdots dt$$

$$= \frac{2}{1+(1+\lambda\log(1+x))^{\frac{1}{\lambda}}} \sum_{m=0}^{\infty} \sum_{m_1+\cdots+m_{k-1}=m} \binom{m}{m_1+\cdots+m_{k-1}}$$

$$\times \frac{b_{m_1,\lambda}(\lambda-1)}{m_1+1} \frac{b_{m_2,\lambda}(\lambda-1)}{m_1+m_2+1} \cdots \frac{b_{m_{k-1},\lambda}(\lambda-1)}{m_1+\cdots+m_{k-1}+1} \frac{x^m}{m!}$$

$$\sum_{n=0}^{\infty} Ch_{n,\lambda}^{(k)}\frac{x^n}{n!} = \frac{1}{2}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m} \sum_{m_1+\cdots+m_{k-1}=m} \binom{m}{m_1+\cdots+m_{k-1}}$$

$$\times \frac{b_{m_1,\lambda}(\lambda-1)}{m_1+1} \frac{b_{m_2,\lambda}(\lambda-1)}{m_1+m_2+1} \cdots \frac{b_{m_{k-1},\lambda}(\lambda-1)}{m_1+\cdots+m_{k-1}+1} Ch_{n-m,\lambda}\frac{x^n}{n!}. \tag{26}$$

Therefore, using (26), we obtain the following theorem.

**Theorem 3.** *For $n \geq 0$, we have*

$$Ch_{n,\lambda}^{(k)} = \frac{1}{2}\sum_{m=0}^{n}\binom{n}{m} \sum_{m_1+\cdots+m_{k-1}=m} \binom{m}{m_1+\cdots+m_{k-1}}$$

$$\times \frac{b_{m_1,\lambda}(\lambda-1)}{m_1+1} \frac{b_{m_2,\lambda}(\lambda-1)}{m_1+m_2+1} \cdots \frac{b_{m_{k-1},\lambda}(\lambda-1)}{m_1+\cdots+m_{k-1}+1} Ch_{n-m,\lambda}. \tag{27}$$

**Corollary 2.** *For $n \geq 0$, we have*

$$Ch_{n,\lambda}^{(2)} = \frac{1}{2}\sum_{m=0}^{n}\binom{n}{m}\frac{b_{m,\lambda}(\lambda-1)}{m+1} Ch_{n-m,\lambda}.$$

Let $k \geq 1$ be an integer. For $s \in \mathbb{C}$, we define the function $\eta_{k,\lambda}(s)$ as

$$\begin{aligned}\eta_{k,\lambda}(s) &= \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{t(1+(1+\lambda \log(1+t))^{\frac{1}{\lambda}})} 2\mathrm{Ei}_{k,\lambda}\left(\log_\lambda(1+t)\right) dt \\ &= \frac{1}{\Gamma(s)} \int_0^1 \frac{t^{s-1}}{t(1+(1+\lambda \log(1+t))^{\frac{1}{\lambda}})} 2\mathrm{Ei}_{k,\lambda}\left(\log_\lambda(1+t)\right) dt \\ &+ \frac{1}{\Gamma(s)} \int_1^\infty \frac{t^{s-1}}{t(1+(1+\lambda \log(1+t))^{\frac{1}{\lambda}})} 2\mathrm{Ei}_{k,\lambda}\left(\log_\lambda(1+t)\right) dt. \end{aligned} \tag{28}$$

The second integral converges absolutely for any $s \in \mathbb{C}$, and hence, the second term on the right-hand side vanishes at non-positive integers. That is,

$$\lim_{s\to -m} \left| \frac{1}{\Gamma(s)} \int_1^\infty \frac{t^{s-1}}{t(1+(1+\lambda \log(1+t))^{\frac{1}{\lambda}})} 2\mathrm{Ei}_{k,\lambda}\left(\log_\lambda(1+t)\right) dt \right| \leq \frac{1}{\Gamma(-m)} M = 0. \tag{29}$$

On the other hand, for $\Re(s) > 0$, the first integral in (29) can be written as

$$\frac{1}{\Gamma(s)} \sum_{l=0}^{\infty} \frac{Ch_{l,\lambda}^{(k)}}{l!} \frac{1}{s+l},$$

which defines an entire function of $s$. Thus, we may conclude that $\eta_{k,\lambda}(s)$ can be continued to an entire function of $s$.

Further, from (28) and (29), we obtain

$$\begin{aligned}\eta_{k,\lambda}(-m) &= \lim_{s\to -m} \frac{1}{\Gamma(s)} \int_0^1 \frac{t^{s-1}}{t(1+(1+\lambda \log(1+t))^{\frac{1}{\lambda}})} 2\mathrm{Ei}_{k,\lambda}\left(\log_\lambda(1+t)\right) dt \\ &= \lim_{s\to -m} \frac{1}{\Gamma(s)} \int_0^1 t^{s-1} \sum_{l=0}^{\infty} \frac{Ch_{l,\lambda}^{(k)} t^l}{l!} dt = \lim_{s\to -m} \frac{1}{\Gamma(s)} \sum_{l=0}^{\infty} \frac{Ch_{l,\lambda}^{(k)}}{s+l} \frac{1}{l!} \\ &= \cdots + 0 + \cdots + 0 + \lim_{s\to -m} \frac{1}{\Gamma(s)} \frac{1}{s+m} \frac{Ch_{m,\lambda}^{(k)}}{m!} + 0 + 0 + \cdots \\ &= \lim_{s\to -m} \frac{\left(\frac{\Gamma(1-s)\sin \pi s}{\pi}\right)}{s+m} \frac{Ch_{m,\lambda}^{(k)}}{m!} = \Gamma(1+m)\cos(\pi m) \frac{Ch_{m,\lambda}^{(k)}}{m!} \\ &= (-1)^m Ch_{m,\lambda}^{(k)}. \end{aligned} \tag{30}$$

Therefore, using (30), we obtain the following theorem.

**Theorem 4.** *Let $k \geq 1$ and $m \in \mathbb{N} \bigcup \{0\}$, $s \in \mathbb{C}$; we have*

$$\eta_{k,\lambda}(-m) = (-1)^m Ch_{m,\lambda}^{(k)}.$$

From (16), we note that

$$\begin{aligned}
2\mathrm{Ei}_{k,\lambda}(\log_\lambda(1+t)) &= t(1+(1+\lambda\log(1+t))^{\frac{1}{\lambda}})\sum_{n=0}^{\infty} Ch_{n,\lambda}^{(k)}\frac{t^n}{n!} \\
&= t\left(\sum_{n=0}^{\infty} Ch_{n,\lambda}^{(k)}\frac{t^n}{n!}\right)\left(1+\sum_{m=0}^{\infty}\binom{\frac{1}{\lambda}}{m}\lambda^m(\log(1+t))^m\right) \\
&= t\left(\sum_{n=0}^{\infty} Ch_{n,\lambda}^{(k)}\frac{t^n}{n!}\right)\left(1+\sum_{m=0}^{\infty}(1)_{m,\lambda}\sum_{l=m}^{\infty}S_1(l,m)\frac{t^l}{l!}\right) \\
&= t\left(\sum_{n=0}^{\infty} Ch_{n,\lambda}^{(k)}\frac{t^n}{n!}\right)\left(1+\sum_{l=0}^{\infty}\left(\sum_{m=0}^{l}(1)_{m,\lambda}S_1(l,m)\right)\frac{t^l}{l!}\right) \\
&= t\sum_{n=0}^{\infty}\left(Ch_{n,\lambda}^{(k)}+\sum_{l=0}^{n}\sum_{m=0}^{l}\binom{n}{l}(1)_{m,\lambda}S_1(l,m)Ch_{n-l,\lambda}^{(k)}\right)\frac{t^n}{n!} \\
&= \sum_{n=1}^{\infty} n\left(Ch_{n-1,\lambda}^{(k)}+\sum_{l=0}^{n-1}\sum_{m=0}^{l}\binom{n}{l}(1)_{m,\lambda}S_1(l,m)Ch_{n-1-l,\lambda}^{(k)}\right)\frac{t^n}{n!}. \qquad (31)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
2\mathrm{Ei}_{k,\lambda}(\log_\lambda(1+t)) &= 2\sum_{m=1}^{\infty}\frac{(1)_{m,\lambda}(\log_\lambda(1+t))^m}{(m-1)!m^k} \\
&= 2\sum_{m=1}^{\infty}\frac{(1)_{m,\lambda}(\log_\lambda(1+t))^m}{(m-1)!m^k}\frac{m!}{m!} \\
&= 2\sum_{m=1}^{\infty}\frac{(1)_{m,\lambda}}{m^{k-1}}\sum_{n=m}^{\infty}S_{1,\lambda}(n,m)\frac{t^n}{n!} \\
L.H.S &= 2\sum_{n=1}^{\infty}\left(\sum_{m=1}^{n}\frac{(1)_{m,\lambda}S_{1,\lambda}(n,m)}{m^{k-1}}\right)\frac{t^n}{n!}. \qquad (32)
\end{aligned}$$

Therefore, using (31) and (32), we obtain the following theorem.

**Theorem 5.** *Let $k \geq 1$ and $m \in \mathbb{N}\bigcup\{0\}$, $s \in \mathbb{C}$; we have*

$$\begin{aligned}
&\sum_{m=1}^{n}\frac{(1)_{m,\lambda}S_{1,\lambda}(n,m)}{m^{k-1}} \\
&= \frac{n}{2}\left(Ch_{n-1,\lambda}^{(k)}+\sum_{l=0}^{n-1}\sum_{m=0}^{l}\binom{n}{l}(1)_{m,\lambda}S_1(l,m)Ch_{n-1-l,\lambda}^{(k)}\right).
\end{aligned}$$

For $k = 1$ in Theorem 5, we get the following corollary.

**Corollary 3.** *For $m \in \mathbb{N}\bigcup\{0\}$, $s \in \mathbb{C}$, we have*

$$\begin{aligned}
&\sum_{m=1}^{n}(1)_{m,\lambda}S_{1,\lambda}(n,m) \\
&= \frac{n}{2}\left(Ch_{n-1,\lambda}+\sum_{l=0}^{n-1}\sum_{m=0}^{l}\binom{n}{l}(1)_{m,\lambda}S_1(l,m)Ch_{n-1-l,\lambda}\right).
\end{aligned}$$

From (16), we note that

$$\begin{aligned}
&\sum_{n=0}^{\infty}\left(Ch_{n,\lambda}^{(k)}(x+1)+Ch_{n,\lambda}^{(k)}(x)\right)\frac{t^n}{n!}\\
&=\frac{2\mathrm{Ei}_{k,\lambda}(\log_\lambda(1+t))}{t(1+(1+\lambda\log(1+t))^{\frac{1}{\lambda}})}(1+\lambda\log(1+t))^{\frac{x}{\lambda}}\left((1+\lambda\log(1+t))^{\frac{1}{\lambda}}\right)\\
&=\left(\sum_{n=0}^{\infty}Ch_{n,\lambda}^{(k)}(x)\frac{t^n}{n!}\right)\left(\sum_{m=0}^{\infty}\binom{\frac{1}{\lambda}}{m}\lambda^m(\log(1+t))^m\right)\\
&=\left(\sum_{n=0}^{\infty}Ch_{n,\lambda}^{(k)}(x)\frac{t^n}{n!}\right)\left(\sum_{m=0}^{\infty}(1)_{m,\lambda}\sum_{l=m}^{\infty}S_1(l,m)\frac{t^l}{l!}\right)\\
&=\left(\sum_{n=0}^{\infty}Ch_{n,\lambda}^{(k)}(x)\frac{t^n}{n!}\right)\left(\sum_{l=0}^{\infty}\left(\sum_{m=0}^{l}(1)_{m,\lambda}S_1(l,m)\right)\frac{t^l}{l!}\right)\\
&=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{m=0}^{l}\binom{n}{l}(1)_{m,\lambda}S_1(l,m)Ch_{n-l,\lambda}^{(k)}(x)\right)\frac{t^n}{n!}. \qquad (33)
\end{aligned}$$

By comparing the coefficients on both sides of (33), we get the following theorem.

**Theorem 6.** *Let $k\in\mathbb{Z}$ and $n\geq 0$; we have*

$$Ch_{n,\lambda}^{(k)}(x+1)+Ch_{n,\lambda}^{(k)}(x)=\sum_{l=0}^{n}\sum_{m=0}^{l}\binom{n}{l}(1)_{m,\lambda}S_1(l,m)Ch_{n-l,\lambda}^{(k)}(x).$$

## 3. Type-Two Degenerate Unipoly-Changhee Polynomials of the Second Kind

The unipoly function $u_k(x|p)$ is defined by Kim and Kim to be (see [20]):

$$u_k(x|p)=\sum_{n=1}^{\infty}\frac{p(n)}{n^k}x^n,(k\in\mathbb{Z}), \qquad (34)$$

where $p$ is any arithmetic function that is a real or complex valued function defined on the set of positive integers $\mathbb{N}$.

Moreover,

$$u_k(x|1)=\sum_{n=1}^{\infty}\frac{x^n}{n^k}=\mathrm{Li}_k(x) \qquad (35)$$

(see [22,23,28]) is the ordinary polylogarithm function.

In this paper, we consider the degenerate unipoly function attached to polynomials $p(x)$ as follows:

$$u_{k,\lambda}(x|p)=\sum_{i=1}^{\infty}p(i)\frac{(1)_{i,\lambda}x^i}{i^k}. \qquad (36)$$

It is worth noting that

$$u_{k,\lambda}\left(x|\frac{1}{\Gamma}\right)=\mathrm{Ei}_{k,\lambda}(x) \qquad (37)$$

is the modified degenerate polyexponential function.

By using (36), we define type-two degenerate unipoly-Changhee polynomials of the second kind by

$$\frac{2u_{k,\lambda}(\log_\lambda(1+t)|p)}{t(1+(1+\lambda\log(1+t))^{\frac{1}{\lambda}})}(1+\lambda\log(1+t))^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}Ch_{n,\lambda,p}^{(k)}(x)\frac{t^n}{n!}. \qquad (38)$$

In the case when $x=0, Ch_{n,\lambda,p}^{(k)}(0)=Ch_{n,\lambda,p}^{(k)}$ are called type-two degenerate unipoly-Changhee numbers of the second kind. Let us take $p(n)=\frac{1}{\Gamma\lambda}$. Then, we have

$$\begin{aligned}\sum_{n=0}^{\infty} Ch^{(k)}_{n,\lambda,\frac{1}{\Gamma}}(x)\frac{t^n}{n!} &= \frac{2u_{k,\lambda}(\log_\lambda(1+t)|\frac{1}{\Gamma}p)}{t(1+(1+\lambda\log(1+t))^{\frac{1}{\lambda}})}(1+\lambda\log(1+t))^{\frac{x}{\lambda}}\\ &= \frac{2}{t(1+(1+\lambda\log(1+t))^{\frac{1}{\lambda}})}\sum_{m=1}^{\infty}\frac{(\log_\lambda(1+t))^m}{m^k(m+1)!}(1+\lambda\log(1+t))^{\frac{x}{\lambda}}\\ &= \frac{2\mathrm{Ei}_{k,\lambda}(\log_\lambda(1+t))}{t(1+(1+\lambda\log(1+t))^{\frac{1}{\lambda}})}(1+\lambda\log(1+t))^{\frac{x}{\lambda}}\\ &= \sum_{n=0}^{\infty} Ch^{(k)}_{n,\lambda}(x)\frac{t^n}{n!}. \end{aligned} \tag{39}$$

Thus, using (39), we have the following theorem.

**Theorem 7.** *Let $n \geq 0$ and $k \in \mathbb{Z}$, and let $\Gamma n$ be a Gamma function. Then, we have*

$$Ch^{(k)}_{n,\lambda,\frac{1}{\Gamma}}(x) = Ch^{(k)}_{n,\lambda}(x). \tag{40}$$

From (38), we get

$$\begin{aligned}\sum_{n=0}^{\infty} Ch^{(k)}_{n,\lambda,p}\frac{t^n}{n!} &= \frac{2u_{k,\lambda}(\log_\lambda(1+t)|p)}{t(1+(1+\lambda\log(1+t))^{\frac{1}{\lambda}})}\\ &= \frac{2}{t(1+(1+\lambda\log(1+t))^{\frac{1}{\lambda}})}\sum_{m=1}^{\infty}\frac{p(m)(1)_{m,\lambda}}{m^k}(\log_\lambda(1+t))^m\\ &= \frac{2}{t(1+(1+\lambda\log(1+t))^{\frac{1}{\lambda}})}\sum_{m=0}^{\infty}\frac{p(m+1)(1)_{m+1,\lambda}(m+1)!}{(m+1)^k}\sum_{l=m+1}^{\infty}S_{1,\lambda}(m+1,l)\frac{t^l}{l!}\\ &= \left(\sum_{j=0}^{\infty}Ch_{j,\lambda}\frac{t^j}{j!}\right)\left(\sum_{m=0}^{\infty}\sum_{l=0}^{m}\frac{p(m+1)(1)_{m+1,\lambda}(m+1)!}{(m+1)^k}\frac{S_{1,\lambda}(m+1,l+1)}{l+1}\frac{t^l}{l!}\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{\infty}\sum_{m=0}^{l}\binom{n}{l}\frac{p(m+1)(1)_{m+1,\lambda}(m+1)!S_{1,\lambda}(m+1,l+1)Ch_{n-l,\lambda}}{(m+1)^k(l+1)}\right)\frac{t^n}{n!}. \end{aligned} \tag{41}$$

Therefore, by comparing the coefficients on both sides of (41), we obtain the following theorem.

**Theorem 8.** *Let $n \in \mathbb{N}$ and $k \in \mathbb{Z}$. Then, we have*

$$Ch^{(k)}_{n,\lambda,p} = \sum_{l=0}^{\infty}\sum_{m=0}^{l}\binom{n}{l}\frac{p(m+1)(1)_{m+1,\lambda}(m+1)!S_{1,\lambda}(m+1,l+1)Ch_{n-l,\lambda}}{(m+1)^k(l+1)}. \tag{42}$$

In particular,

$$Ch^{(k)}_{n,\lambda,\frac{1}{\Gamma}} = Ch^{(k)}_{n,\lambda} = \sum_{l=0}^{\infty}\sum_{m=0}^{l}\binom{n}{l}\frac{S_{1,\lambda}(m+1,l+1)Ch_{n-l,\lambda}}{(m+1)^{k-1}(l+1)}. \tag{43}$$

From (38), we observe that

$$\sum_{n=0}^{\infty} C_{n,\lambda}^{(k,p)}(x)\frac{t^n}{n!} = \frac{2u_{k,\lambda}(\log_\lambda(1+t)|p)}{t(1+(1+\lambda\log(1+t))^{\frac{1}{\lambda}})}(1+\lambda\log(1+t))^{\frac{x}{\lambda}}$$
$$= \frac{2u_{k,\lambda}(\log_\lambda(1+t)|p)}{t(1+(1+\lambda\log(1+t))^{\frac{1}{\lambda}})}\sum_{m=0}^{\infty}\binom{\frac{x}{\lambda}}{m}\lambda^m(\log(1+\lambda t))^m$$
$$= \left(\sum_{n=0}^{\infty} Ch_{n,\lambda,p}^{(k)}\frac{t^n}{n!}\right)\left(\sum_{m=0}^{\infty}(x)_{m,\lambda}\sum_{l=m}^{\infty}S_1(l,m)\frac{t^l}{l!}\right)$$
$$= \left(\sum_{n=0}^{\infty} Ch_{n,\lambda,p}^{(k)}\frac{t^n}{n!}\right)\left(\sum_{l=0}^{\infty}\sum_{m=0}^{l}(x)_{m,\lambda}S_1(l,m)\frac{t^l}{l!}\right)$$
$$L.H.S = \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{m=0}^{l}\binom{n}{l}Ch_{n-l,\lambda,p}^{(k)}(x)_{m,\lambda}S_1(l,m)\right)\frac{t^n}{n!}. \tag{44}$$

From (44), we obtain the following theorem.

**Theorem 9.** *Let $n \geq 0$ and $k \in \mathbb{Z}$. Then, we have*

$$Ch_{n,\lambda,p}^{(k)}(x) = \sum_{l=0}^{n}\sum_{m=0}^{l}\binom{n}{l}Ch_{n-l,\lambda,p}^{(k)}(x)_{m,\lambda}S_1(l,m). \tag{45}$$

From (38), we observe that

$$\sum_{n=0}^{\infty} Ch_{n,\lambda,p}^{(k)}\frac{t^n}{n!} = \frac{2u_{k,\lambda}(\log_\lambda(1+t)|p)}{t(1+(1+\lambda\log(1+t))^{\frac{1}{\lambda}})}$$
$$= \frac{2}{t(1+(1+\lambda\log(1+t))^{\frac{1}{\lambda}})}\sum_{m=0}^{\infty}\frac{p(m+1)(1)_{m+1,\lambda}m!}{(m+1)^k m!}(\log_\lambda(1+t))^{m+1}$$
$$= \frac{2\log_\lambda(1+t)}{t(1+(1+\lambda\log(1+t))^{\frac{1}{\lambda}})}\sum_{m=0}^{\infty}\frac{p(m+1)(1)_{m+1,\lambda}m!}{(m+1)^k m!}(\log_\lambda(1+t))^m$$
$$= \frac{\log_\lambda(1+t)}{t}\frac{2}{1+(1+\lambda\log(1+t))^{\frac{1}{\lambda}}}\sum_{m=0}^{\infty}\frac{p(m+1)(1)_{m+1,\lambda}m!}{(m+1)^k}\sum_{l=m}^{\infty}S_{1,\lambda}(l,m)\frac{t^l}{l!}$$
$$= \left(\sum_{s=0}^{\infty}D_{s,\lambda}\frac{t^s}{s!}\right)\left(\sum_{a=0}^{\infty}Ch_{a,\lambda}\frac{t^a}{a!}\right)\left(\sum_{l=0}^{\infty}\sum_{m=0}^{n}\frac{p(m+1)(1)_{m+1,\lambda}m!}{(m+1)^k}S_{1,\lambda}(l,m)\frac{t^l}{l!}\right)$$
$$= \left(\sum_{b=0}^{\infty}\sum_{a=0}^{b}\binom{b}{a}D_{b-a,\lambda}Ch_{a,\lambda}\frac{t^b}{b!}\right)\left(\sum_{l=0}^{\infty}\sum_{m=0}^{n}\frac{p(m+1)(1)_{m+1,\lambda}m!}{(m+1)^k}S_{1,\lambda}(l,m)\frac{t^l}{l!}\right)$$
$$L.H.S = \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{a=0}^{n-l}\sum_{m=0}^{l}\binom{n}{l}D_{n-l-a,\lambda}C_{a,\lambda}\frac{p(m+1)(1)_{m+1,\lambda}m!}{(m+1)^k}S_{1,\lambda}(l,m)\right)\frac{t^n}{n!}. \tag{46}$$

By comparing the coefficients on both sides of (46), we obtain the following theorem.

**Theorem 10.** *Let $n \geq 0$ and $k \in \mathbb{Z}$. Then, we have*

$$Ch_{n,\lambda,p}^{(k)} = \sum_{l=0}^{n}\sum_{a=0}^{n-l}\sum_{m=0}^{l}\binom{n}{l}D_{n-l-a,\lambda}Ch_{a,\lambda}\frac{p(m+1)(1)_{m+1,\lambda}m!}{(m+1)^k}S_{1,\lambda}(l,m). \tag{47}$$

## 4. Conclusions

In this article, we introduced type-two degenerate poly-Changhee polynomials of the second kind and derived some beautiful identities and relations between type-two degenerate poly-Changhee numbers of the second kind and Stirling numbers of first and second kind. In addition, we gave the relation between degenerate Bernoulli polynomials of the second kind and type-two degenerate poly-Changhee numbers of the second kind. Again, we defined type-two degenerate unipoly-Changhee polynomials of the the second kind and obtained some properties and relationships of degenerate unipoly-Changhee numbers of the second kind and the Daehee numbers.

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