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Controllability of Semilinear Multi-Valued Differential Inclusions with Non-Instantaneous Impulses of Order $\alpha \in (1,2)$ without Compactness

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Abstract: Herein, we investigated the controllability of a semilinear multi-valued differential equation with non-instantaneous impulses of order $\alpha \in (1,2)$, where the linear part is a strongly continuous cosine family without compactness. We did not assume any compactness conditions on either the semi-group, the multi-valued function, or the inverse of the controllability operator, which is different from the previous literature.

Keywords: controllability problem; fractional differential inclusions; non-instantaneous impulsive; mild solutions

2010 Mathematics Subject Classification: primary 34A08; 26A33

Citation: Alsheekhhussain, Z.; Ibrahim, A.G. Controllability of Semilinear Multi-Valued Differential Equations with Non-Instantaneous Impulses of Order $\alpha \in (1,2)$ without Compactness. *Symmetry* **2021**, *13*, 566. https://doi.org/10.3390/sym13040566

Academic Editor: Dumitru Baleanu

Received: 2 March 2021
Accepted: 24 March 2021
Published: 29 March 2021

1. Introduction

Due to the multiple applications of fractional differential equations in science, many have authors studied various types of these applications, such as [1–4].

The motivation for considering nonlocal Cauchy problems is the physical problems. For instance, it has been used to determine the unknown physical parameters in some inverse heat condition problems.

Sometimes, an impulsive action continues to be active on a finite time interval. In this case, impulses are called non-instantaneous. In [5–8], there are many results concerning the existence of solutions of differential equations or inclusions with non-instantaneous impulses of fractional order $\gamma \in (0,1)$, while in [9–11], the authors considered second-order non-instantaneous impulsive differential equations.

Under different conditions, many authors have investigated the existence of solutions for differential equations or inclusions of order $\gamma \in (1,2)$; for example, Li et al. [12] considered an abstract Cauchy problem, He et al. [13] treated with nonlocal fractional evolution inclusions, and Wang et al. [14] generalized the work done by He et al. [13] to a case when there are non-instantaneous impulses.

On the contrary, it is known that controllability is a primary concept in control theory, which is important in both engineering and the sciences.

Recently, many researchers have studied controllability problems for different kinds of fractional differential equations or inclusions in infinite dimensional Banach spaces using different methods. In most of the existing works, different fixed point theorems and measures of non-compacntness have been employed to obtain a fixed point of the solution operator corresponding to the considered problem, and under restrictive hypotheses such as the compactness of the semi-group generated by the linear part (see [9]) or the nonlinear term (single-valued function or multi-valued function) satisfies a Lipschitz condition in the second variable (see [10,15–17]) or verifies a compact condition involving a measure of non-compactness (see [18–21]).

Consider the following non-instantaneous impulsive semilinear differential inclusion:

$$\begin{cases} {}^{c}D^{\alpha}_{s_j,\vartheta}x(\vartheta) \in A(x(\vartheta)) + F(\vartheta, x(\vartheta)) + \Upsilon(u(\vartheta)),\ a.e.\ \vartheta \in (s_j,\ \vartheta_{j+1}], j = 0,1,....,k, \\ x(\vartheta_j^+) = \sigma_j(\vartheta_j, x(\vartheta_j^-)), j = 1,......k, \\ x(\vartheta) = \sigma_j(\vartheta, x(\vartheta_j^-)), \vartheta \in (\vartheta_j\ s_j], j = 1,......k, \\ x(0) = x_0 - \sigma(x), x'(0) = x_1, \end{cases} \tag{1}$$

where $\mathcal{J} = [0,T], T > 0, \alpha \in (1,2)$, E is a real Banach space (the scalar field is $\mathbb{R}$), ${}^{c}D^{\alpha}_{s_j,\vartheta}$ is the Caputo derivative [22,23] of the order α with a lower limit at s_j, A is the infinitesimal generator of a strongly continuous cosine family of bounded linear operators $\{C(\vartheta) : \vartheta \in \mathbb{R}\}$in E, and $F : [0,T] \times E \rightarrow 2^E - \{\phi\}$ is a multi-function. Moreover, $0 = s_0 < \ \vartheta_1 \leq s_1 < \vartheta_2 \leq s_2 < \vartheta_3 \cdots < s_k < \vartheta_{k+1} = T,\ u(\vartheta_j^+), u(\vartheta_j^-)$ are the right and left limits of a function u at the point ϑ_j, respectively; $x_{0,}x_1 \in E$ are two fixed points; $\sigma_j : [\vartheta_j, s_j] \times E \longrightarrow E,\ j = 1,2,\ldots,k$, are continuous functions, such that for any $x \in E$, the function $\vartheta \mapsto \sigma_j(\vartheta, x)$ is differentiable; $\sigma : PC(\mathcal{J}, E) \rightarrow E$, where $PC(\mathcal{J}, E)$ will be specified later. Let $L^p(\mathcal{J}, X)$, $p > \frac{1}{\alpha}$, be a Banach space of admissible control functions, where X is a real Banach space. The control function u is in $L^p(\mathcal{J}, X)$, and Υ is a bounded linear operator from X into E.

Motivated by the works cited above, we prove, in this paper, without assuming that the semi-group $\{C(\vartheta) : \vartheta \in \mathbb{R}\}$ is compact or the multi-valued function F is Libschitz in the second variable or satisfies any condition involving a measure of non-compactness, and by using a fixed point theorem for weakly sequentially closed graph operators, the controllability of problem (1).

To clarify that our work improves the existing results: He et al. [13] showed the existence of mild solutions for (1) when $\Upsilon \equiv 0$, F satisfies a compactness condition, $\vartheta_j = s_j$ and $\sigma_j(\vartheta_j, x(\vartheta_j^-)) = x(\vartheta_j^-)$; Wang et al. [14] assumed a compactness condition on F and showed the compactness of the solution set for (1) when $\Upsilon \equiv 0$; Muslum et al. [16] discussed the controllability of (1), when F is a single-valued function satisfying a Lipschitz condition in the second variable and $\alpha = 2$; Li et al. [12] and Zhou et al. [17] discussed the controllability of (1) when F is a single-valued function satisfying a Lipschitz condition in the second variable or a compactness condition, $\sigma \equiv 0$, $\vartheta_j = s_j$ and $\sigma_j(\vartheta_j, x(\vartheta_j^-)) = x(\vartheta_j^-)$.

Moreover, there are results on the controllability without any compactness conditions of systems of fractional order $\delta \in (0,1)$, such as [24–26]. Furthermore, in [24], problem (1) was considered when $A \equiv 0$, and in [27], there were no impulse effects.

Finally, Sheng et al. [28] studied the controllability of nonlinear dynamical systems with a Mittag–Leffler kernel involving AB -derivative of order $\gamma \in (0,1)$ in the absence of impulse effects, where the linear part is a matrix operator.

We observed no study concerning the controllability of (1) without imposing any compactness condition on either the generating semi-group $\{C(\vartheta) : \vartheta \in \mathbb{R}\}$ or F. These are the main objectives of this paper.

This paper is organized as follows. Section 2 collects the known results, Section 3 contains the main results, and Section 4 provides examples to illustrate our theory.

2. Preliminaries and Notation

Let $P_{cl}(E) = \{S \subseteq E : S$ is non-empty, convex, and closed$\}$; $P_{cwk}(E) = \{S \subseteq E : S$ is non-empty, convex, and weakly compact$\}$; E_w is the space E endowed with weak topology. For set $D \subseteq E$, we denote by $\overline{D}^w$ the weak closure of D. For more information about the strongly cosine family, the reader can see [29], and for multi-valued function, [27].

Consider the Banach space:

$$\begin{aligned} PC(\mathcal{J}, E) &= \{x : \mathcal{J} \rightarrow E : x_{|_{\mathcal{J}_j}} \in C(\mathcal{J}_j, E), \\ \text{and } x(\vartheta_j^+) \text{ and } x(\vartheta_j^-) \text{ exist for each } j &= 1,2,....k\}, \end{aligned}$$

endowed with the norm:

$$||x||_{PC(\mathcal{J},E)} = \max\{||x(\vartheta)|| : \vartheta \in \mathcal{J}\},$$

where $\mathcal{J}_0 = [0, \vartheta_1]$ and $\mathcal{J}_j = [\vartheta_j, \vartheta_{j+1})$.

The following lemma is a particular case of Theorem 2.2 [30].

Lemma 1. *Let $Z \subseteq PC(\mathcal{J}, E)$ be weakly compact and convex, and $N : Z \to P_{cl}(Z)$(the family of non-empty closed and convex subsets of Z) be a multi-valued function with a weakly sequentially closed graph. Then, Γ has a fixed point.*

According definition 2.7 in [14], we give the following concept.

Definition 1. *By a mild solution of* (1), *we mean function $x(.,u) \in PC(\mathcal{J}, E)$ such that:*

$$x(\vartheta, u) = \begin{cases} C_q(\vartheta)(x_0 - \sigma(x)) + K_q(\vartheta)x_1 \\ + \int_0^{\vartheta}(\vartheta - s)^{q-1}P_q(\vartheta - s)(f(s) + \Upsilon(u(s))ds, \vartheta \in [0, \vartheta_1], \\ \sigma_j(\vartheta, x(\vartheta_j^-)), \vartheta \in (\vartheta_j, s_j], j = 1, 2, .., k, \\ C_q(\vartheta - s_j)\sigma_j(s_j, x(\vartheta_j^-)) + K_q(\vartheta - s_j)\sigma_j'(s_j, x(\vartheta_j^-)) \\ + \int_{s_j}^{\vartheta}(\vartheta - s)^{q-1}P_q(\vartheta - s)(f(s) + \Upsilon(u(s))ds, \vartheta \in [s_j, \vartheta_{j+1}], j = 1, 2, .., k, \end{cases} \tag{2}$$

where $f \in S^1_{F(.,x(.,u))}$,

$$C_q(\vartheta) = \int_0^{\infty} \xi_q(\theta)C(\vartheta^q\theta)d\theta, K_q(\vartheta) = \int_0^{\vartheta} C_q(s)ds, \ \vartheta \geq 0, q = \frac{\alpha}{2},$$

$$P_q(\vartheta) = \alpha\int_0^{\infty} \theta\xi_q(\theta)S(\vartheta^q\theta)d\theta, \vartheta \geq 0,$$

$$\xi_q(\theta) = \frac{1}{q}\theta^{-1-\frac{1}{q}}W_q(\theta^{-\frac{1}{q}}), \ \theta \in (0, \infty),$$

and:

$$W_q(\theta) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{-qn-1}\frac{\Gamma(nq+1)}{n!}\sin(n\pi q).$$

3. Results

In this section, we discuss the controllability of problem (1).

Definition 2. *System* (1) *is considered to be controllable on $\mathcal{J} = [0, T]$ if, for every $x_0, x_1, x_T \in E$, there exists a control function $u \in L^p(\mathcal{J}, X)$, $p > \frac{1}{q}(q = \frac{\alpha}{2})$, such that the corresponding mild solution satisfies $x(0) = x_0 - \sigma(x), x'(0) = x_1$, and $x(T) = x_T$.*

In order to establish the controllability of (1), we need the assumptions stated below:

(HA) $A : D(A) \subseteq E \to E$ is the infinitesimal generator of a strongly continuous cosine family of bounded linear operators $\{C(\vartheta) : \vartheta \in \mathbb{R}\}$ in E and there is $M > 0$ with $\sup_{\vartheta \geq 0} ||C(\vartheta)|| \leq M$.

(HF) $F : \mathcal{J} \times E \to 2^E - \{\phi\}$ is a multifunction with non-empty, convex, weakly compact values such that:

(i) For every $x \in E$, the multifunction $\vartheta \to F(\vartheta, x(\vartheta))$ has a measurable selection;

(ii) For any natural number n, there is function $\varphi_n \in L^p(\mathcal{J},\mathbb{R}^+)$, $p > \frac{1}{q}(q = \frac{\alpha}{2})$ such that $\sup_{||x||\leq n} ||F(\vartheta, x)|| \leq \varphi_n(\vartheta)$ for $a.e.\vartheta \in \mathcal{J}$, and:

$$\liminf_{n\to\infty} \frac{\|\varphi_n\|_{L^1(\mathcal{J},\,\mathbb{R}^+)}}{n} = 0. \tag{3}$$

(iii) For almost $\vartheta \in \mathcal{J}, x \to F(\vartheta, x)$ is upper semi-continuous from E_w to $E_w x$.
$(H\sigma)$ $\sigma : PC(\mathcal{J}, E) \to E$ is a function such that if $x_n \rightharpoonup x$ in $PC(\mathcal{J}, E)$, then $\sigma(x_n) \rightharpoonup \sigma(x)$, and there are two positive real numbers a, d, such that:

$$||\sigma(x)|| \leq a||x|| + d, \forall x \in PC(\mathcal{J}, E). \tag{4}$$

(H)For every $j = 1, 2, .., k$, the function $\sigma_j : [\vartheta_j\ s_j] \times E \to E$ is such that:

(i) For any $x \in E$, the function $\vartheta \to \sigma_j(\vartheta, x)$ is continuously differentiable;
(ii) There are positive real numbers $h_j, , j = 1, ..k$, such that $||\sigma_j(\vartheta, x)|| \leq h_j||x||, \vartheta \in [\vartheta_j, s_j], x \in E$;
(iii) There are positive real numbers $\eta_j, j = 1, ..k$, such that $||\sigma_j'(\vartheta, x)|| \leq \eta_j||x||, \vartheta \in [s_j, s_j], x \in E$;
(iv) For any $\vartheta \in \mathcal{J}$, $\sigma_j\ (\vartheta, .)$ and $\sigma_j'(\vartheta, .)$ are continuous from E_w to E_w .
(HW)The operator $W : L^p(\mathcal{J}, X) \to E$,

$$W(u) = \int_{s_k}^{T} (T - s)^{q-1} P_q(T - s)\Upsilon(u(s))ds.$$

has an inverse $W^{-1} : E \to L^p(\mathcal{J}, X)/Ker(W)$, such that there exists a $\varkappa > 0$ with $||W^{-1}|| \leq \varkappa$ and $||\Upsilon|| \leq \varkappa$.

Lemma 2 ([13]). *Condition (HA) implies that for any $(\vartheta, x) \in (\mathcal{J} \times E)$, we have*

$$||C_q(\vartheta)x|| \ \leq M||x||, ||K_q(\vartheta)x|| \ \leq \vartheta M||x||, \text{ and } ||P_q(\vartheta)x|| \ \leq \frac{M}{\Gamma(2q)}||x||.\vartheta^q. \tag{5}$$

Remark 1. *The operator W is well defined. In fact, from (iii) of Lemma 2, it follows that:*

$$\begin{aligned} ||W(u)|| &\leq \frac{M}{\Gamma(2q)} \int_{s_k}^{T} (T - s)^{2q-1} ||\Upsilon(u(s))||ds. \\ &\leq \frac{\varkappa M T^{2q-1}}{\Gamma(2q)} ||u||_{L^1(\mathcal{J},X)}. \end{aligned}$$

We recall this lemma.

Lemma 3 ([31]). *Assume that $F : \mathcal{J} \times E \to 2^E - \{\phi\}$ is a multifunction and satisfies (HF). Then, for any $x \in C(\mathcal{J}, E)$, the set $S^1_{F(.,x(.))} = \{f \in L^1(\mathcal{J}, E) : f(\vartheta) \in F(\vartheta, x(\vartheta)), a.e.\}$ is not empty.*

In the next proposition, we present a similar result to in Lemma 3, but in the space, $PC(\mathcal{J}, E)$.

Proposition 1. *If $F : \mathcal{J} \times E \to 2^E - \{\phi\}$ satisfies (HF), then for any $x \in PC(\mathcal{J}, E)$, the set:*

$$S^p_{F(.,x(.))} = \{f \in L^p(\mathcal{J}, E) : f(\vartheta) \in F(\vartheta, x(\vartheta)), a.e.\},$$

is not empty.

Proof. Let $x \in PC(\mathcal{J}, E)$. For every $j = 0, 1, 2, ..., k$, we define:

$$x_j^*(\vartheta) = \begin{cases} x(\vartheta); \vartheta \in \mathcal{J}_j, \\ x(\vartheta_j^+); \vartheta = \vartheta_j. \end{cases}$$

Then, $x_j^* \in C(\overline{\mathcal{J}_j}, E)$. By applying Lemma 3, there exist measurable functions $f_j^* : \overline{\mathcal{J}_j} \to E$, $j = 0, , ..., k$, such that $f_j^* \in L^1(\overline{\mathcal{J}_j}, E)$ and $f_j^*(\vartheta) \in F(\vartheta, x_j^*(\vartheta)), a.e. \vartheta \in \overline{\mathcal{J}_j}$.Define,$f : \mathcal{J} \to E, f(\vartheta) = f_j^*(\vartheta), \vartheta \in \mathcal{J}$. Obviously, $f \in L^1(\mathcal{J}, E)$ and $f(\vartheta) \in F(\vartheta, x(\vartheta)), a.e. \vartheta \in \mathcal{J}$. Moreover, by condition $(HF)(jj)$, there is $\varphi \in L^p(\mathcal{J}, \mathbb{R}^+)$, such that $||F(\vartheta, x)|| \leq \varphi(\vartheta)$, for *a.e.* $\vartheta \in \mathcal{J}$. Therefore, $||f(\vartheta)|| \leq \varphi(\vartheta)$, for $a.e. \vartheta \in \mathcal{J}$ .Hence, $f \in L^p(\mathcal{J}, E)$. $\square$

In the next theorem, we present the first controllability result for problem (1).

Theorem 1. *Suppose that $(HA), (HF), (H\sigma), (H)$, and (HW) are satisfied. Then, system* (1) *is controllable on $\mathcal{J}$ provided that:*

$$M(a + h + T\eta) + \frac{M^2 \varkappa^2}{\Gamma(2q)} \xi[\, h + T\eta] < 1, \tag{6}$$

where $h = \sum_{j=0}^{j=k} h_j, \eta = \sum_{j=0}^{j=k} \eta_j$ and $\xi = (\int_0^T (T-s)^{\frac{(2q-1)P}{p-1}} ds)^{\frac{P-1}{P}}$.

Proof. In view of Proposition 1, for any $x \in PC(\mathcal{J}, E)$, the set $S^p_{F(.,x(.))}$ is not empty. Then, for any $x \in PC(\mathcal{J}, E)$ and any $f \in S^p_{F(.,x(.))}$, we can define the control function $u_{x,f} \in L^p(\mathcal{J}, X)$ as:

$$\begin{aligned} u_{x,f} \quad &= \quad W^{-1}[x_T - C_q(T - s_k)\sigma_k(s_k, x(\vartheta_k^-)) - K_q(T - s_k)\sigma_k'(s_k, x(\vartheta_k^-)) \\ &\quad - \int_{s_k}^T (T-s)^{q-1} P_q(T-s) f(s) ds]. \end{aligned} \tag{7}$$

So, a multifunction $N : PC(\mathcal{J}, E) \to 2^{PC(\mathcal{J}, E)}$ can be defined as follows: For $x \in PC(\mathcal{J}, E)$, function $y \in N(x)$ if and only if:

$$y(\vartheta) = \begin{cases} C_q(\vartheta)(x_0 - \sigma(x)) + K_q(\vartheta)x_1 \\ + \int_0^\vartheta (\vartheta - s)^{q-1} P_q(\vartheta - s)(f(s) + \Upsilon(u_{x,f}(s))ds, \vartheta \in [0, \vartheta_1], \\ \sigma_j(\vartheta, x(\vartheta_j^-)), \vartheta \in (\vartheta_j, s_j], j = 1, 2, .., k \\ C_q(\vartheta - s_j)\sigma_j(s_j, x(\vartheta_j^-)) + K_q(\vartheta - s_j)\sigma_j'(s_j, x(\vartheta_j^-)) \\ + \int_{s_j}^\vartheta (\vartheta - s)^{q-1} P_q(\vartheta - s)(f(s) + \Upsilon(u_{x,f}(s))ds, \vartheta \in [s_j, \vartheta_{j+1}], j = 1, 2, .., k, \end{cases} \tag{8}$$

where $f \in S^1_{F(.,x(.))}$.

Using the control function, defined by (7), we prove that any fixed point for N is a mild solution for (1) , and such a solution satisfies $x(0) = x_0 - \sigma(x), x'(0) = x_1$ and $x(T) = x_T$. In fact, if x is a fixed point for N, then from (HW), (7) and (8), it yields:

$$\begin{aligned}
x(T) &= C_q(T-s_k)\sigma_k(s_k, x(\vartheta_k^-)) + K_q(T-s_k)\sigma_k'(s_k, x(\vartheta_k^-)) \\
&\quad + \int_{s_k}^{T}(T-s)^{q-1}P_q(T-s)f(s)ds \\
&\quad + W(u_{x,f}) \\
&= C_q(T-s_k)\sigma_k(s_k, x(\vartheta_k^-)) + K_q(T-s_k)\sigma_k'(s_k, x(\vartheta_k^-)) \\
&\quad + \int_{s_k}^{T}(T-s)^{q-1}P_q(T-s)f(s)ds \\
&\quad + x_T - C_q(T-s_k)\sigma_k(s_k, x(\vartheta_k^-)) \\
&\quad - K_q(T-s_k)\sigma_k'(s_k, x(\vartheta_k^-)) \\
&\quad - \int_{s_k}^{T}(T-s)^{q-1}P_q(T-s)f(s)ds \\
&= x_T.
\end{aligned}$$

Now, for any natural number k, set $D_k = \{x \in PC(\mathcal{J}, E) : \|x\|_{PC(\mathcal{J},E)} \leq k\}$.

Step 1. In this step, we assume that there is a natural number k_0, such that $N(D_{k_0}) \subseteq D_{k_0}$. Assume the opposite. So, for any natural number r, there are $x_r, y_r \in PC(\mathcal{J}, E)$ with $y_r \in N(x_r)$, $\|x_r\|_{PC(\mathcal{J},E)} \leq r$ and $\|y_r\|_{PC(\mathcal{J},E)} > r$. Then, there is $(f_r)_{j\geq 1} \in S^1_{F(.,x_r(.))}$, such that:

$$y_r(\vartheta) = \begin{cases} C_q(\vartheta)(x_0 - \sigma(x_r)) + K_q(\vartheta)x_1 + \int_0^{\vartheta}(\vartheta-s)^{q-1}P_q(\vartheta-s)f_j(s)ds \\ + \int_0^{\vartheta}(\vartheta-s)^{q-1}P_q(\vartheta-s)\Upsilon(u_{x_j,f_j}(s))ds, \vartheta \in [0,\vartheta_1], \\ \sigma_j(\vartheta, x_r(\vartheta_j^-)), \vartheta \in (\vartheta_j, s_j], j = 1,2,..,k \\ C_q(\vartheta - s_j)\sigma_j(s_j, x_r(\vartheta_j^-)) + K_q(\vartheta - s_j)\sigma_j'(s_j, x_r(\vartheta_j^-)) \\ + \int_{s_j}^{\vartheta}(\vartheta-s)^{q-1}P_q(\vartheta-s)\Upsilon(u_{x_r,f_r}(s))ds \\ + \int_{s_j}^{\vartheta}(\vartheta-s)^{q-1}P_q(\vartheta-s)f(s)ds, \vartheta \in [s_j, \vartheta_{j+1}], j = 1,2,..,k. \end{cases} \tag{9}$$

If $\vartheta \in [0, \vartheta_1]$, then:

$$\begin{aligned}
||y_r(\vartheta)|| &\leq M\,(||x_0|| + \, ||\sigma(x_r)||) + MT||x_1|| \\
&\quad + \frac{M}{\Gamma(2q)}\int_0^{\vartheta}(\vartheta-s)^{2q-1}\varphi_r(s)ds \\
&\quad + \frac{M\varkappa}{\Gamma(2q)}\int_0^{\vartheta}(\vartheta-s)^{2q-1}||u_{x_r,f_r}(s))||ds \\
&\leq M\,(||x_0|| + ar + d) + MT||x_1|| \\
&\quad + \frac{MT^{2q-1}}{\Gamma(2q)}||\varphi_r||_{L^1(\mathcal{J},\mathbb{R}^+)} \\
&\quad + \frac{M\varkappa}{\Gamma(2q)}\xi||u_{x_r,f_r}||_{L^p([\mathcal{J},E)}.
\end{aligned}$$

Notice that:

$$\begin{aligned}
& ||u_{x_r,f_r}||_{L^p(\mathcal{J},\, E)} \\
\leq \ & ||W^{-1}[x_T - C_q(T-s_k)\sigma_k(s_k, x_r(\vartheta_k^-)) - K_q(T-s_k)\sigma_k'(s_k, x_r(\vartheta_k^-)) \\
& - \int_{s_k}^{T} (T-s)^{q-1} P_q(T-s) f_r(s) ds]||_{L^p(\mathcal{J},\, E)} \\
\leq \ & ||W^{-1}||\ [||x_T|| + M\, hr + MT\eta\, r \\
& + \frac{M}{\Gamma(2q)} \int_{s_k}^{T} (T-s)^{2q-1} \varphi_r(s) ds \\
\leq \ & \varkappa[||x_T|| + M\, hr + MT\eta r + \ \frac{MT^{2q-1}}{\Gamma(2q)} ||\varphi_r||_{L^1(\mathcal{J},\mathbb{R}^+)}].
\end{aligned}$$

This implies that:

$$\begin{aligned}
||y_r(\vartheta)|| \ \leq \ & M\,(||x_0|| + ar + d) + MT||x_1|| \\
& + \frac{MT^{2q-1}}{\Gamma(2q)} ||\varphi_r||_{L^1(\mathcal{J},\mathbb{R}^+)} \\
& + \frac{M\varkappa^2}{\Gamma(2q)} \xi[||x_T|| + M\, hr + MT\eta r + \ \frac{MT^{2q-1}}{\Gamma(2q)} ||\varphi_r||_{L^1(\mathcal{J},\mathbb{R}^+)}].
\end{aligned}$$

If $\vartheta \in (\vartheta_j\ s_j]$, $j = 1, 2, \ldots, k$, then:

$$||y_r(\vartheta)|| \leq ||\sigma_j(\vartheta, x_r(\vartheta_j^-))|| \leq h\, r \leq Mr\, h$$

Let $\vartheta \in (s_j, \vartheta_{j+1}]$. By applying the arguments used in the case $\vartheta \in [0, \vartheta_1]$, it yields:

$$\begin{aligned}
& ||y_r(\vartheta)|| \leq ||C_q(\vartheta - s_j)\sigma_j(s_j, x(\vartheta_j^-))|| + ||K_q(\vartheta - s_j)\sigma_j'(s_j, x(\vartheta_j^-))|| \\
& + \frac{MT^{2q-1}}{\Gamma(2q)} ||\varphi_r||_{L^1(\mathcal{J},\mathbb{R}^+)} \\
& + \frac{M\varkappa^2}{\Gamma(2q)} \xi[||x_T|| + M\, hr + MT\eta r + \ \frac{M}{\Gamma(2q)} ||\varphi_r||_{L^p(\mathcal{J},\mathbb{R}^+)}] \\
\leq \ & Mr\, h + MT\eta r + \frac{MT^{2q-1}}{\Gamma(2q)} ||\varphi_r||_{L^1(\mathcal{J},\mathbb{R}^+)}. \\
& + \frac{M\varkappa^2}{\Gamma(2q)} \xi[||x_T|| + M\, hr + MT\eta r + \ \frac{M}{\Gamma(2q)} ||\varphi_r||_{L^p(\mathcal{J},\mathbb{R}^+)}].
\end{aligned}$$

Then:

$$\begin{aligned}
\|y_r\|_{PC(\mathcal{J},E)} \ \leq \ & M\,(||x_0|| + ar + d) + MT||x_1|| \\
& + M\, hr + MT\eta r + \frac{MT^{2q-1}}{\Gamma(2q)} ||\varphi_r||_{L^1(\mathcal{J},\mathbb{R}^+)} \\
& + \frac{M\varkappa^2}{\Gamma(2q)} \xi[||x_T|| + M\, hr + MT\eta r + \ \frac{MT^{2q-1}}{\Gamma(2q)} ||\varphi_r||_{L^1(\mathcal{J},\mathbb{R}^+)}].
\end{aligned}$$

By dividing both sides by r and taking the limit as $r \to \infty$,we get:

$$1 < M(a + h + T\eta) + \frac{M\varkappa^2}{\Gamma(2q)} \xi[M\, h + MT\eta],$$

which contradicts (6) and our claim in this step is completed.

Now, let k_0 be $N(D_{k_0}) \subseteq D_{k_0}$.

Step 2. The restriction of N on D_{k_0} has a weakly sequentially closed graph.

Consider a sequence $\{x_r\}_{r\geq 1}$ with $x_r \rightharpoonup x$ in D_{k_0}, $y_r \in N(x_r)$ with $y_r \rightharpoonup y$ in $PC(\mathcal{J}, E)$ and $f_r \in S^{P}_{F(.,x_r(.))}$, such that y_r satisfies (9). Notice that the set $||x_r(\vartheta)|| \leq k_0, \forall r \geq 1$, and $\forall \vartheta \in \mathcal{J}$. Hence, by (ii) of (HF), there is $\varphi_{k_0} \in L^p(\mathcal{J}, \mathbb{R}^+)$ such that:

$$||f_r(\vartheta)|| \leq \varphi_{r_0}(\vartheta), \forall r \geq 1, \text{ and } a.e. \vartheta \in \mathcal{J}, \tag{10}$$

which implies with the fact $P > \frac{1}{q} > 1$, that there exists a subsequence of (f_n), which we denote again by (f_n), such that $f_n \rightharpoonup h \in L^p(\mathcal{J}, E)$.

On the contrary, it is known that linear bounded operators on normed spaces map a weakly convergent sequence into a weakly convergent sequence, and hence, by $(H\sigma)$ and $(H)(jv)$, we get, for any $\vartheta \in \mathcal{J}$:

$$C_q(\vartheta)(x_0 - \sigma(x_r)) \rightharpoonup C_q(\vartheta)(u_0 - \sigma(x)),$$

$$C_q(\vartheta - s_j)\sigma_j(s_j, x_r(\vartheta_j^-)) \rightharpoonup C_q(\vartheta - s_j)\sigma_j(s_j, x(\vartheta_j^-)),$$

and:

$$K_q(\vartheta - s_j)\sigma_j'(s_j, x_r(\vartheta_j^-)) \rightharpoonup K_q(\vartheta - s_j)\sigma_j'(s_j, x(\vartheta_j^-)).$$

Next, for any $j = 0, 1, 2, .., k$, consider the operator $R_j : L^p([s_j, \vartheta_{j+1}], E) \to C([s_j, \vartheta_{j+1}], E)$, defined by:

$$R_j(f)(\vartheta) = \int_{s_j}^{\vartheta} (\vartheta - s)^{q-1} P_q(\vartheta - s) f(s) ds. \tag{11}$$

Clearly, $R_j, j = 0, 1, 2, ..., k$ are linear. In addition, using Holder's inequality, we get:

$$\begin{aligned} ||R_j(f)(\vartheta)|| &\leq \frac{M}{\Gamma(2q)} \int_{s_j}^{\vartheta} (\vartheta - s)^{2q-1} ||f(s)|| ds \\ &\leq \frac{M}{\Gamma(2q)} \left(\int_{s_j}^{\vartheta} (\vartheta - s)^{\frac{(2q-1)P}{p-1}} ds\right)^{\frac{P-1}{P}} ||f||_{L^p([s_j, \vartheta_{j+1}], E)} \\ &= \frac{M\xi}{\Gamma(2q)} ||f||_{L^p([s_j, \vartheta_{j+1}], E)}. \end{aligned}$$

which means that R_j is bounded and, hence, continuous. In order to prove that $R_j(f_r)(\vartheta) \rightharpoonup R_j(h)(\vartheta)$ in E, $\vartheta \in (s_j, \vartheta_{j+1}]$, $j = 0, 1, 2, .., k$, suppose that $v : E \to \mathbb{R}$ is a linear continuous functional and ϑ is a fixed point in$[s_j, \vartheta_{j+1}]$. By the linearity and continuity of R_j, the operator $f \to v(R_j(f)(\vartheta)$ is linear and continuously functional on $L^p(\mathcal{J}, E)$, and hence, $R_j(f_r)(\vartheta) \rightharpoonup R_j(h)(\vartheta)$. Then, for any $\vartheta \in [s_j, \vartheta_{j+1}]$, $j = 0, 1, 2, .., k$:

$$\int_{s_j}^{\vartheta} (\vartheta - s)^{q-1} P_q(\vartheta - s) f_r(s) ds,$$

converges weakly to:

$$\int_{s_j}^{\vartheta} (\vartheta - s)^{q-1} P_q(\vartheta - s) h(s) ds.$$

Next, for any $j = 1, 2, .., k.$,let $S_j : E \to C([s_j, T], E)$:

$$S_j\ (x)(\vartheta) = \int_{s_j}^{\vartheta} (\vartheta - s)^{q-1} P_q(\vartheta - s) \Upsilon(W^{-1}(x)(s)) ds. \tag{12}$$

By the linearity of the integral operator and of the operators $P_q(.)$, Υ and W^{-1}, one can easily see that $S_j, j = 0,1,2,...,k$ are linear. Moreover, applying to Holder's inequality gives:

$$\begin{aligned} & ||S_j\ (x)(\vartheta)|| \\ \leq\ & \frac{\varkappa M}{\Gamma(q)} \int_{s_j}^{\vartheta} (\vartheta - s)^{2q-1} ||W^{-1}(x)(s)||_X ds \\ \leq\ & \frac{\varkappa M \xi}{\Gamma(\alpha)} ||W^{-1}(x)||_{L^p([\mathcal{J},X],E)} \\ \leq\ & \frac{\varkappa^2 M \xi}{\Gamma(\alpha)} ||x||. \end{aligned}$$

This shows that $S_j, j = 1,2,..,k$ are bounded and, hence, continuous. By arguing as above, one can show that:

$$S_j(x_T - C_q(T - s_k)\sigma_k(s_k, x_r(\vartheta_k^-)) - K_q(T - s_k)\sigma_k'(s_k, x_r(\vartheta_k^-)))$$

converges weakly convergent to:

$$S_j(x_T - C_q(T - s_k)\sigma_k(s_k, x(\vartheta_k^-)) - K_q(T - s_k)\sigma_k'(s_k, x(\vartheta_k^-))).$$

Now, for any $j = 1,2,..,m.$,let $\vartheta_j : L^p([s_j, T], E) \to C([s_j, T], E)$ be defined as:

$$\begin{aligned} \vartheta_j\ (f)(\vartheta) &= \int_{s_j}^{\vartheta} (\vartheta - s)^{q-1} P_q(\vartheta - s) \times \\ & \Upsilon(W^{-1}[-\int_{s_m}^{T} (T - \tau)^{q-1} P_q(T - \tau) f(\tau) d\tau](s)) ds. \end{aligned} \tag{13}$$

By the linearity of the integral operator and of the operators $P_q(.)$, Υ and W^{-1}, one can easily see that $\vartheta_j, j = 0,1,2,...,m$ are linear. Moreover, applying to Holder's inequality gives:

$$\begin{aligned} & ||\vartheta_j\ (f)(\vartheta)|| \\ \leq\ & \frac{\varkappa M}{\Gamma(q)} \int_{s_j}^{\vartheta} (\vartheta - s)^{2q-1} \times \\ & ||W^{-1}[-\int_{s_k}^{T} (T - \tau)^{q-1} P_q(T - \tau) f(\tau) d\tau](s)||_X ds \\ \leq\ & \frac{\varkappa M \xi}{\Gamma(2q)} ||W^{-1}[-\int_{s_k}^{T} (T - \tau)^{q-1} P_q(T - \tau) f(\tau) d\tau]||_{L^p(\mathcal{J},X)} \\ \leq\ & \frac{\varkappa^2 M \xi}{\Gamma(2q)} ||\int_{s_k}^{T} (T - \tau)^{q-1} P_q(T - \tau) f(\tau) d\tau||_E \\ \leq\ & \frac{\varkappa^2 M^2 \xi}{\Gamma(2q)^2} \int_{s_k}^{T} (T - \tau)^{2q-1} f(\tau) d\tau] \\ \leq\ & \frac{\varkappa^2 M^2 \xi 2}{\Gamma(2q)^2}\ ||f\ ||_{L^p([s_k,T],E)} \\ \leq\ & \frac{\varkappa^2 M^2 \xi 2}{\Gamma(2q)^2}\ ||f\ ||_{L^p([s_j,T],E)}. \end{aligned}$$

This shows that $\vartheta_j, j = 1,2,..,k$ are linear and bounded and, hence, continuous. Then, by applying the same arguments used above, we can clearly that:

$$\int_{s_j}^{\vartheta} (\vartheta - s)^{q-1} P_q(\vartheta - s) \Upsilon(W^{-1}[-\int_{s_k}^{T} (T - \tau)^{q-1} P_q(T - \tau) f_r(\tau) d\tau](s)) ds,$$

converges weakly to:

$$\int_{s_j}^{\vartheta}(\vartheta-s)^{q-1}P_q(\vartheta-s)\Upsilon(W^{-1}[-\int_{s_k}^{T}(T-\tau)^{q-1}P_q(T-\tau)h(\tau)d\tau](s))ds,$$

From the argument above, we get $y_r(\vartheta) \rightharpoonup v(\vartheta)$, $\forall\vartheta \in \mathcal{J}$, and $y_r(\vartheta_j^+) \rightharpoonup v(\vartheta_j^+)$, $j = 0,1,2,..,k$, where:

$$v(\vartheta)=\begin{cases} C_q(\vartheta)(x_0-\sigma(x))+K_q(\vartheta)x_1+\int_0^{\vartheta}(\vartheta-s)^{q-1}P_q(\vartheta-s)h(s)ds \\ +\int_0^{\vartheta}(\vartheta-s)^{q-1}P_q(\vartheta-s)\Upsilon(u_{x,h}(s))ds, \vartheta\in[0,\vartheta_1], \\ \sigma_j(\vartheta,x(\vartheta_j^-)), \vartheta\in(\vartheta_j,s_j], j=1,2,..,k \\ C_q(\vartheta-s_j)\sigma_j(s_j,x(\vartheta_j^-))+K_q(\vartheta-s_j)\sigma_j'(s_j,x(\vartheta_j^-)) \\ +\int_{s_j}^{\vartheta}(\vartheta-s)^{q-1}P_q(\vartheta-s)\Upsilon(u_{x,f}(s))ds \\ +\int_{s_j}^{\vartheta}(\vartheta-s)^{q-1}P_q(\vartheta-s)h(s)ds, \vartheta\in[s_j,\vartheta_{j+1}], j=1,2,..,k. \end{cases} \tag{14}$$

Furthermore, by following the arguments used in the first step, we can show that the sequence $\{y_r\}_{r\geq1}$ is bounded in $PC(\mathcal{J},E)$. Then, by Lemma 2.5 in [24], $y_r \rightharpoonup w$ in $PC(\mathcal{J},E)$. By the uniqueness of the weak limit, we get $y(\vartheta) = v(\vartheta), \vartheta \in \mathcal{J}$.

Next, we demonstrate that $h(\vartheta) \in F(\vartheta, x(\vartheta))$for $a.e.\vartheta \in \mathcal{J}$. From the weak convergence of (f_r) toward h, the Mazur's Lemma ensures the existence of a sequence, (h_r), of convex combinations of (f_r) with $h_r(\vartheta) \to h(\vartheta)$, for $a.e.\vartheta \in \mathcal{J}$. Let $\vartheta_0 \in \mathcal{J}$ be such that $h_r(\vartheta_0) \to h(\vartheta_0)$, $f_r(\vartheta_0) \in F(\vartheta_0, x_r(\vartheta_0))$,$\forall r \geq 1$, $F(\vartheta,.)$ is upper semicontinuous from E_w to E_w and $h(\vartheta_0) \notin F(\vartheta_0, x(\vartheta_0))$. In view of the Hahn Banach theorem, there is an open convex set Ω, such that $F(\vartheta_0, x(\vartheta_0)) \subseteq \Omega$ and $h(\vartheta_0) \notin \overline{\Omega}$. Notice that Ω is weakly open (first statement in Remark 2.7 [24]), then by the upper semi-continuity of $F(\vartheta_0,.)$ at $x(\vartheta_0)$, there is a weak neighborhood U for $x(\vartheta_0)$, such that if $z \in U$, then $F(\vartheta_0, z) \subseteq \Omega$. Because $x_r(\vartheta_0) \rightharpoonup x(\vartheta_0)$, it follows, by the second assertion of Remark 2.7 in [24], that there exists a natural number k_0 with $x_r(\vartheta_0) \in U$, $\forall r \geq k_0$, and hence $f_r(\vartheta_0) \subseteq \Omega$, $\forall r \geq k_0$. Since Ω is convex, $h_r(\vartheta_0) \in \Omega$, $\forall r \geq k_0$, which implies that $h(\vartheta_0) \in \overline{\Omega}$, and this contradicts the fact that $h(\vartheta_0) \notin \overline{\Omega}$. Therefore, $h(\vartheta) \in F(\vartheta, x(\vartheta))$ for $a.e.$ $\vartheta \in \mathcal{J}$.

Step 3. In this step, we prove that $N(D_{k_0})$ is relatively weakly compact.

Let $y_r \in N(x_r)$ and $x_r \in D_{k_0}$. This implies that, for any $r \geq 1$, there is $f_r \in S^1_{F(.,x_r(.))}$, such that y_r satisfies (9). By using the same arguments employed in the previous step, there is a subsequence of (f_r), denoted again by (f_r), with $f_r \rightharpoonup h \in L^p(\mathcal{J},E)$ and $y_r \rightharpoonup v$, where v is given by (14). Then, $N(D_{k_0})$ is relatively weakly compact.

Step 4. Let $Z_{k_0} = \overline{co}(\overline{N(D_{k_0})}^{w}$. From Step 3, $(\overline{N(D_{k_0})}^{w}$ is weakly compact compact, and hence, Z_{k_0} is convex and weakly compact. Furthermore, since D_{k_0} is convex and closed, and using the first statement in Remark 2.7 in [26], we have $\overline{D_{k_0}}^{w} = D_{k_0}$. Then, by step 1, one has

$$N(Z_{k_0}) = N(\overline{co}(\overline{N(D_{k_0})}^{w}) \subseteq N(\overline{co}(\overline{D_{k_0})}^{w}) = N(\overline{co}(D_{k_0})) = N(D_{k_0}) \subseteq Z_{k_0}.$$

By noting that Z_{k_0} is convex and weakly compact and by applying Lemma 1, N has a fixed point. This completes the proof. □

In the following, we give another controllability result for(1) under a less restrictive growth assumption.

Theorem 2. *Under the assumptions of Theorem 1 after replacing $(HF)(ii)$ by the following condition:*
$(H_2)^*$ there exists $\varphi \in L^p(\mathcal{J},\mathbb{R}^+)$, $p > \frac{1}{q}$, such that for any $x \in E$:

$$||F(\vartheta,x)|| \leq \varphi(\vartheta)(1+||x||), \textit{for a.e.} \vartheta \in \mathcal{J}, \tag{15}$$

then, system (1) *is controllable on* $\mathcal{J}$ *provided that:*

$$M(a+h+T\eta)+\frac{MT^{2q-1}}{\Gamma(2q)}||\varphi||_{L^1(\mathcal{J},\mathbb{R}^+)} + \frac{M^2\varkappa^2}{\Gamma(2q)}\xi[\,h+T\eta+\frac{T^{2q-1}}{\Gamma(2q)}||\varphi||_{L^1(\mathcal{J},\mathbb{R}^+)}] < 1. \tag{16}$$

Proof. We only need to prove that there is a natural number k_0, such that $N(D_{k_0}) \subseteq D_{k_0}$. Assume that there are $x_r, y_r \in PC(\mathcal{J}, E)$ with $y_r \in N(x_r)$, $\|x_r\|_{PC(\mathcal{J},E)} \leq r$,$\|y_r\|_{PC(\mathcal{J},E)} > r$ and $(f_r)_{r\geq 1} \in S^1_{F(.,x_r(.))}$, such that:

$$y_r(\vartheta) = \begin{cases} C_q(\vartheta)(x_0-\sigma(x_r)) + K_q(\vartheta)x_1 + \int_0^\vartheta(\vartheta-s)^{q-1}P_q(\vartheta-s)f_r(s)ds \\ +\int_0^\vartheta(\vartheta-s)^{q-1}P_q(\vartheta-s)\Upsilon(u_{x_r,f_r}(s))ds, \vartheta\in[0,\vartheta_1], \\ \sigma_j(\vartheta, x_r(\vartheta_j^-)), \vartheta\in(\vartheta_j, s_j], j=1,2,..,k \\ C_q(\vartheta-s_j)\sigma_j(s_j, x_r(\vartheta_j^-)) + K_q(\vartheta-s_j)\sigma_j'(s_j, x_r(\vartheta_j^-)) \\ +\int_{s_j}^\vartheta(\vartheta-s)^{q-1}P_q(\vartheta-s)\Upsilon(u_{x_r,f_r}(s))ds \\ +\int_{s_j}^\vartheta(\vartheta-s)^{q-1}P_q(\vartheta-s)f(s)ds, \vartheta\in[s_j,\vartheta_{j+1}], j=1,2,..,k. \end{cases} \tag{17}$$

Note that, by (15):

$$||f_r(\vartheta)|| \leq (1+r)\varphi(\vartheta), a.e.\vartheta\in\mathcal{J}. \tag{18}$$

If $\vartheta \in [0,\vartheta_1]$, then from (17) and (18), it yields:

$$\begin{aligned} ||y_r(\vartheta)|| &\leq M\,(||x_0|| + \,||\sigma(x_r)||) + MT||x_1|| \\ &\quad + \frac{M(1+r)}{\Gamma(2q)}\int_0^\vartheta(\vartheta-s)^{2q-1}\varphi(s)ds \\ &\quad + \frac{M\varkappa}{\Gamma(2q)}\int_0^\vartheta(\vartheta-s)^{2q-1}||u_{x_r,f_r}(s))||ds \\ &\leq M\,(||x_0|| + ar + d) + MT||x_1|| \\ &\quad + \frac{(1+r)MT^{2q-1}}{\Gamma(2q)}||\varphi||_{L^1(\mathcal{J},\mathbb{R}^+)} \\ &\quad + \frac{M\varkappa}{\Gamma(2q)}\xi||u_{x_r,f_r}||_{L^p([\mathcal{J},E)} \end{aligned}$$

Notice that:

$$\begin{aligned} & ||u_{x_r,f_r}||_{L^p(\mathcal{J},\,E)} \\ \leq\; & ||W^{-1}[x_T - C_q(T-s_k)\sigma_k(s_k, x_r(\vartheta_k^-)) - K_q(T-s_k)\sigma_k'(s_k, x_r(\vartheta_k^-)) \\ & -\int_{s_k}^T(T-s)^{q-1}P_q(T-s)f_r(s)ds]||_{L^p(\mathcal{J},\,E)} \\ \leq\; & ||W^{-1}||\;[||x_T|| + r\,hM + MT\eta\; r \\ & + \frac{M(1+r)}{\Gamma(2q)}\int_{s_m}^T(T-s)^{2q-1}\varphi(s)ds] \\ \leq\; & \varkappa[||x_T|| + M\,hr + MT\eta r + \frac{MT^{2q-1}(1+r)}{\Gamma(2q)}||\varphi||_{L^1(\mathcal{J},\mathbb{R}^+)}]. \end{aligned}$$

It follows that:

$$\begin{aligned} ||y_r(\vartheta)|| &\leq M\,(||x_0|| + ar + d) + MT||u_1|| \\ &+ \frac{MT^{2q-1}(r+1)}{\Gamma(2q)}||\varphi||_{L^1(\mathcal{J},\mathbb{R}^+)} \\ &+ \frac{M\varkappa^2}{\Gamma(2q)}\xi[||x_T|| + M\,hr + MT\eta r + \frac{MT^{2q-1}}{\Gamma(2q)}||\varphi||_{L^1(\mathcal{J},\mathbb{R}^+)}]. \end{aligned}$$

If $\vartheta \in (\vartheta_j\ s_j], j = 1, 2, \ldots, m$, then:

$$||y_r(\vartheta)|| \leq ||\sigma_j(\vartheta, x_r(\vartheta_j^-))|| \leq h\,r \leq Mr\,h$$

Let $\vartheta \in (s_j, \vartheta_{j+1}]$. As in the case $\vartheta \in [0, \vartheta_1]$, we get:

$$\begin{aligned} & ||y_r(\vartheta)|| \\ \leq\ & Mr\,h + MT\eta r + \frac{MT^{2q-1}(r+1)}{\Gamma(2q)}||\varphi||_{L^1(\mathcal{J},\mathbb{R}^+)} \\ & + \frac{M\varkappa^2}{\Gamma(2q)}\xi[||x_T|| + M\,hr + MT\eta r + \frac{(1+r)MT^{2q-1}}{\Gamma(2q)}||\varphi||_{L^1(\mathcal{J},\mathbb{R}^+)}\xi]. \end{aligned}$$

Then:

$$\begin{aligned} r &< \|y_r\|_{PC(\mathcal{J},E)} \leq M\,(||x_0|| + ar + d) + MT||x_1|| \\ & + M\,hr + MT\eta r + \frac{MT^{2q-1}(r+1)}{\Gamma(2q)}||\varphi||_{L^1(\mathcal{J},\mathbb{R}^+)} \\ & + \frac{M\varkappa^2}{\Gamma(2q)}\xi[||x_T|| + M\,hr + MT\eta r + \frac{(1+r)MT^{2q-1}}{\Gamma(2q)}||\varphi||_{L^1(\mathcal{J},\mathbb{R}^+)}]. \end{aligned}$$

By dividing both sides by r and taking the limit as $r \to \infty$, we get:

$$\begin{aligned} 1 &< M(a + h + T\eta) + \frac{MT^{2q-1}}{\Gamma(2q)}||\varphi||_{L^1(\mathcal{J},\mathbb{R}^+)} \\ & + \frac{M^2\varkappa^2}{\Gamma(2q)}\xi[\,h + T\eta + \frac{T^{2q-1}}{\Gamma(2q)}||\varphi||_{L^1(\mathcal{J},\mathbb{R}^+)}], \end{aligned}$$

which contradicts (16). □

The next theorem gives another controllability result for (1) under a less restrictive growth assumption.

Theorem 3. *Under the assumptions of Theorem 1 after replacing $(HF)(ii)$ by the following condition:* $(H)^{**}$ *there exists $\varphi \in L^p(\mathcal{J}, \mathbb{R}^+)$, $p > \frac{1}{q}$, and a nondecreasing function $\beta : [0, \infty) \to (0, \infty)$, such that for any $x \in E$:*

$$||F(\vartheta, x)|| \leq \varphi(\vartheta)\beta(||x||), \text{for a.e.} \vartheta \in \mathcal{J}, \tag{19}$$

then, the system (1) is controllable on $\mathcal{J}$, provided that there is $r > 0$, such that:

$$\begin{aligned} r &< M\,(||x_0|| + ar + d) + MT||x_1|| \\ & + M\,hr + MT\eta r + \frac{MT^{2q-1}\beta(r)}{\Gamma(2q)}||\varphi||_{L^1(\mathcal{J},\mathbb{R}^+)} \\ & + \frac{M\varkappa^2}{\Gamma(2q)}\xi[||x_T|| + M\,hr + MT\eta r + \frac{\beta(r)MT^{2q-1}}{\Gamma(2q)}||\varphi||_{L^1(\mathcal{J},\mathbb{R}^+)}]. \end{aligned} \tag{20}$$

Proof. It is sufficient to prove that there is a natural number k_0, such that $N(D_{k_0}) \subseteq D_{k_0}$. Assume that there are $x_r, y_r \in PC(\mathcal{J}, E)$ with $y_r \in N(x_r)$, $\|x_r\|_{PC(\mathcal{J},E)} \leq r$,$\|y_r\|_{PC(\mathcal{J},E)} > r$ and $(f_r)_{r\geq 1} \in S^1_{F(.,x_r(.))}$, such that y_r satisfies (17) Notice that, by (19):

$$||f_r(\vartheta)|| \leq \varphi(\vartheta)\beta(r), a.e. \vartheta \in \mathcal{J}.$$

So, if $\vartheta \in [0, \vartheta_1]$, then:

$$\begin{aligned} ||y_r(\vartheta)|| &\leq M\,(||x_0|| + \,||\sigma(x_r)||) + MT||x_1|| \\ &\quad + \frac{M\beta(r)}{\Gamma(2q)} \int_0^{\vartheta} (\vartheta - s)^{2q-1} \varphi(s) ds \\ &\quad + \frac{M\varkappa}{\Gamma(2q)} \int_0^{\vartheta} (\vartheta - s)^{2q-1} ||u_{x_r,f_r}(s))|| ds \\ &\leq M\,(||x_0|| + ar + d) + MT||x_1|| \\ &\quad + \frac{\beta(r) MT^{2q-1}}{\Gamma(2q)} ||\varphi||_{L^1(\mathcal{J},\mathbb{R}^+)} \\ &\quad + \frac{M\varkappa}{\Gamma(2q)} \xi ||u_{x_r,f_r}||_{L^p([\mathcal{J},E)}. \end{aligned}$$

Notice that:

$$\begin{aligned} & ||u_{x_r,f_r}||_{L^p(\mathcal{J},\,E)} \\ \leq\ & ||W^{-1}[x_T - C_q(T - s_m)\sigma_m(s_m, x_r(\vartheta_m^-)) - K_q(T - s_m)\sigma'_m(s_m, x_r(\vartheta_m^-)) \\ & - \int_{s_m}^{T} (T - s)^{q-1} P_q(T - s) f_r(s) ds]||_{L^p(\mathcal{J},\,E)} \\ \leq\ & ||W^{-1}||\ [||x_T|| + M\,hr + MT\eta\ r \\ & + \frac{M\beta(r)}{\Gamma(2q)} \int_{s_m}^{T} (T - s)^{2q-1} \varphi(s) ds \\ \leq\ & \varkappa[||x_T|| + M\,hr + MT\eta r + \ \frac{MT^{2q-1}\beta(r)}{\Gamma(2q)} ||\varphi||_{L^1(\mathcal{J},\mathbb{R}^+)}]. \end{aligned}$$

It follows that:

$$\begin{aligned} ||y_r(\vartheta)|| &\leq M\,(||x_0|| + ar + d) + MT||u_1|| \\ &\quad + \frac{MT^{2q-1}\beta(r)}{\Gamma(2q)} ||\varphi||_{L^1(\mathcal{J},\mathbb{R}^+)} \\ &\quad + \frac{M\varkappa^2}{\Gamma(2q)} \xi [||x_T|| + M\,hr + MT\eta r + \ \frac{\beta(r) MT^{2q-1}}{\Gamma(2q)} ||\varphi||_{L^1(\mathcal{J},\mathbb{R}^+)}]. \end{aligned}$$

If $\vartheta \in (\vartheta_j\ s_j], j = 1, 2, \ldots, m$, then:

$$||y_r(\vartheta)|| \leq ||\sigma_j(\vartheta, x_r(\vartheta_j^-))|| \leq h\ r \leq Mr\ h$$

Let $\vartheta \in (s_j, \vartheta_{j+1}]$. As in the case $\vartheta \in [0, \vartheta_1]$, we get:

$$\begin{aligned} & ||y_r(\vartheta)|| \\ \leq\ & Mr\ h + MT\eta r + \frac{MT^{2q-1}(r+1)}{\Gamma(2q)} ||\varphi||_{L^1(\mathcal{J},\mathbb{R}^+)} \\ & + \frac{M\varkappa^2}{\Gamma(2q)} \xi [||x_T|| + M\,hr + MT\eta r + \ \frac{(1 + r) MT^{2q-1}}{\Gamma(2q)} ||\varphi||_{L^1(\mathcal{J},\mathbb{R}^+)}]. \end{aligned}$$

Then:

$$\begin{aligned} r \quad < \quad & \|y_r\|_{PC(\mathcal{J},E)} \leq M\,(||x_0|| + ar + d) + MT||x_1|| \\ & + M\,hr + MT\eta r + \frac{MT^{2q-1}\beta(r)}{\Gamma(2q)}||\varphi||_{L^1(\mathcal{J},\mathbb{R}^+)} \\ & + \frac{M\varkappa^2}{\Gamma(2q)}\xi[||x_T|| + M\,hr + MT\eta r + \frac{\beta(r)MT^{2q-1}}{\Gamma(2q)}||\varphi||_{L^1(\mathcal{J},\mathbb{R}^+)}], \end{aligned}$$

which contradicts with (20). □

4. Examples

Example 1. *Let* $E = L^2(0,\pi)$*,* $\mathcal{J} = [0,1]$*,* $s_0 = 0$*,* $\vartheta_1 = \frac{1}{4}, s_1 = \frac{1}{2}, \vartheta_2 = T = 1, p = 2$*,* $x_0, x_1 \in E$ *are two fixed elements in E and* c_1, c_1 *are two real numbers. For any* $x : \mathcal{J} \to E$*, we denote to the values of* $x(\vartheta)$ *at* (y) *by* $x(\vartheta, y); \vartheta \in \mathcal{J}$ *and* $y \in (0,\pi)$*. Consider the fractional partial differential equation:*

$$\begin{cases} {}^cD^{\frac{3}{2}}_{0,\vartheta}x(\vartheta,y) = \partial_{yy}x(\vartheta,y) + \lambda_0\vartheta x(\vartheta,y)) + b(y)Z(x(\vartheta,y)), \\ \;a.e.\ \vartheta \in (0,\frac{1}{4}] \cup (\frac{1}{2},1], y \in [0,\pi], \\ x(\vartheta,y) = \sin(\vartheta)x(\vartheta_1^-,y), \vartheta \in (\vartheta_1\ s_1], y \in (0,\pi), \\ x(0,y) = x_0(y) - c_1x(\vartheta_1,y) - c_2x(\vartheta_2,y),\ y \in (0,\pi), \\ \partial_y x(0,y) = x_1(y),\ y \in (0,\pi), \end{cases} \tag{21}$$

where $0 \leq \lambda_0 \leq c_4 < 1$*. We define an operator* $A : D(A) \subset E \to E$ *as follows:* $Az = z''$ *with:*

$$D(A) = \{z \in L^2(0,\pi) : z_{yy} \in L^2(0,\pi),\ z(0) = z(\pi) = 0\}.$$

Note that the operator A has the representation ([32], p. 1307):

$$Ax = \sum_{m=1}^{\infty} -m^2 < x,\ x_m > x_m, x \in D(A), \tag{22}$$

where $x_m(y) = \sqrt{2}\ \sin my, m = 1,2,...,$ is the orthonormal set of eigenfunctions of A. In addition, A is the infinitesimal generator of a strongly continuous cosine family, $C(\vartheta)_{\vartheta\in\mathbb{R}}$, which is given by:

$$C(\vartheta)(x) = \sum_{m=1}^{\infty} \cos m\vartheta < x\ , x_m > x_m, x \in E,$$

and the associated sine family, $S(\vartheta)_{\vartheta\in\mathbb{R}}$, is written as:

$$S(\vartheta)(x) = \sum_{m=1}^{\infty} \frac{\sin m\vartheta}{m} < x\ ,\ x_m > x_m, x \in E.$$

It is known that $||C(\vartheta)|| \leq e^{-\pi^2 t}$ and $||S(\vartheta)|| \leq e^{-\pi^2\vartheta}$ for $t \geq 0$ [32]. Then, $||C(\vartheta)|| \leq M = e^{-\pi^2}, \vartheta \geq 0$. Let $\sigma : PC(\mathcal{J}, E) \to E$ be such that:

$$\sigma(x) = \sum_{j=1}^{j=2} c_j(x(\vartheta_j)), \tag{23}$$

where $c_j, j = 1,2$ are positive real numbers. Notice that:

$$||\sigma(x)|| \leq ||x|| \sum_{j=1}^{j=2} c_j.$$

If $x_n \rightharpoonup x$ in $PC(\mathcal{J}, E)$, then according to [24] (Lemma 2.5) $x_n(\vartheta_j) \rightharpoonup x(\vartheta_j)$ in E, and hence $\sigma(x_n) \rightharpoonup \sigma(x)$ in E. This shows that $(H\sigma)$ is satisfied with $a = c_1 + c_2$.

Furthermore, let $\sigma_1 : [\vartheta_1, s_1] \times E \to E$, defined as follows:

$$\sigma_1(\vartheta, x)(y) = c_3(\sin\vartheta)x(\vartheta_1^-, y); \vartheta \in [\frac{1}{4}, \frac{1}{2}], y \in (0, \pi). \tag{24}$$

Then, (H) is satisfied with $h_1 = \eta_1 = c_3$. So, $h = \eta = c_3$.

Let $F : \mathcal{J} \times L^2(0, \pi) \to 2^{L^2(0,\pi)} - \{\phi\}$:

$$F(\vartheta, \psi) = \{z \in L^2(0, \pi) : z(y) = \lambda\vartheta\psi(y), 0 \le \lambda \le c_4 < 1, \vartheta \in \mathcal{J}, y \in (0, \pi)\}.$$

This multi-valued function has a non-empty convex weakly compact values and:

$$||F(\vartheta, h)|| = \sup\{||z|| : z \in F(t.h)\} \le c_4\vartheta||h||, \vartheta \in \mathcal{J}, h \in L^2(0, \pi)$$

which yields that F satisfies (15) with $\varphi(\vartheta) = c_4\vartheta; \vartheta \in \mathcal{J}$. Obviously, F verifies condition (i) and (iii) of(HF).

Assume that $\Upsilon : L^2[0, 1] \to L^2[0, 1]$ is a bounded linear operator, such that the operator $W : L^2(\mathcal{J}, L^2(\mathcal{J})) \to L^2(\mathcal{J})$, which is defined by:

$$W(u) := \int_{\frac{1}{2}}^{1} (1-s)^{\frac{-1}{4}} P_q(T-s)\Upsilon u(s)ds.$$

is linear and bounded and has an inverse, such that there is $\varkappa > 0$ with $||W^{-1}|| \le \varkappa$ and $||\Upsilon|| \le \varkappa$. Notice that $\xi = (\int_0^1 (1-s)ds)^{\frac{1}{2}} = \frac{1}{\sqrt{2}}$ and $||\varphi||_{L^1(\mathcal{J},\mathbb{R}^+)} = \frac{c_4}{2}$. By applying Theorem 2, problem (1) is controllable if:

$$\begin{aligned} &e^{-\pi^2}(c_1 + c_2 + 2c_3) + \frac{e^{-\pi^2}}{\Gamma(\frac{3}{2})}\frac{c_4}{2} \\ &+ \frac{e^{-2\pi^2}\varkappa^2}{\sqrt{2}\Gamma(\frac{3}{2})}[\,2c_3 + \frac{c_4}{2\Gamma(\frac{3}{2})}] < 1. \end{aligned} \tag{25}$$

By choosing small enough c_1, c_2, c_3, and c_4, we can arrive at (25).

Example 2. *Let E A,* $\mathcal{J} = [0, 1]$ *be as in the previous and Z be a convex, weakly compact subset of E with* $Sup\{\ ||z||\ :\ z \in Z\} \le \lambda$, *for some* $\lambda > 0$. *Let* $s_0 = 0, s_j = \frac{2j}{9},, \vartheta_i = \frac{2j-1}{9}, j = 1, 2, 3, 4, \vartheta_5 = 1$. *Consider,* $F : \mathcal{J} \times E \to P_{ck}(E)$ *as a multi-valued function defined by:*

$$F(\vartheta, \psi) = \{z \in E : z(y) = \frac{ye^{-r\vartheta}\sqrt{||\psi||}}{\lambda\,(1 + \sqrt{||\psi||})}Z\}, \tag{26}$$

where $r \in (1, \infty)$ *Then, for every* $\psi \in E, \vartheta \to F(\vartheta, x)$ *is strongly measurable, and for any* $\vartheta \in \mathcal{J}$, $F(\vartheta,)$ *is upper semicontinuous from* E_w *to* E_w. *Moreover, for any natural number n, one obtains:*

$$\sup_{||\psi|| \le n} ||F(\vartheta, \psi)|| \le \frac{\pi}{\sqrt{2}}e^{-r}\sqrt{n} = \varphi_n(\vartheta); \vartheta \in \mathcal{J}.$$

It follows that $\lim\limits_{n\to\infty}\inf\frac{\|\varphi_n\|_{L^1(\mathcal{J},\,\mathbb{R}^+)}}{n} = 0$. So, (HF) is verified.

Let $\sigma : PC(\mathcal{J}, E) \to E$ with:

$$\sigma(x) = \sum_{j=1}^{j=4} c_i x(\vartheta_i), \tag{27}$$

where c_i are positive real numbers. We have:

$$||\sigma(x)|| \leq ||x||_{PC(\mathcal{J},E)} \sum_{j=1}^{j=4} c_i.$$

Then, (H_σ) is realized with $a = \sum_{j=1}^{j=4} c_i$. Furthermore, for any $j = 1, 2, 3, 4$, let σ_i : $[\vartheta_i, s_i] \times E \to E$ be such that:

$$\sigma_j(\vartheta, x)(y) = \omega(\sin j\vartheta)x(y), \tag{28}$$

where $\omega > 0$ is a real number. Clearly, $\frac{d}{dt}(\sigma_j(\vartheta, x)) = (i \cos i\vartheta)(x)$. This proves that (H) is satisfied with $h_j = \omega, \eta_j = j\omega, j = 1, 2, 3, 4$. That is, $h = 4\omega$ and $\eta = \frac{16}{9}\omega$.

Finally, let $\Upsilon : E \to E$, with $(\Upsilon u(\vartheta))(y) = m(\vartheta)(u(\vartheta)(y)) = m(\vartheta)u(\vartheta, y); \vartheta \in \mathcal{J}, u \in L^2(\mathcal{J}, E)$,where $m : \mathcal{J} \to \mathbb{R}$ is continuous. Let $\varkappa > 0$ with $||W^{-1}|| \leq \varkappa$ and $||\Upsilon|| \leq \varkappa$.

Then, by applying Theorem 1, the following non-instantaneous impulsive fractional differential inclusion:

$$\begin{cases} {}^cD^{\frac{3}{2}}_{0,t}x(\vartheta, y) \in \partial_{yy}x(\vartheta, y) + \frac{ye^{-r\vartheta}}{\lambda} \frac{\sqrt{||x(\vartheta)||}}{(1+\sqrt{||x(\vartheta)||})} Z + m(t)u(\vartheta, y), \\ a.e.\ t \in \cup_{j=0}^{j=4}(\frac{2j}{9}, \frac{2j+1}{9}], y \in (0, \pi), \\ x(\vartheta, y) = \omega(\sin j\vartheta)x(v_j^-, y), \vartheta \in (\frac{2j-1}{9}, \frac{2j}{9}], j = 1, 2, 3, 4, \\ x(0, y) = x_0 - \sum_{j=1}^{j=4} c_i x(\vartheta_i, y), y \in (0, \pi) \\ x'(0, y) = x_1(y), y \in (0, \pi). \end{cases} \tag{29}$$

is controllable if:

$$M(a + h + \eta) + \frac{M\varkappa^2}{\Gamma(2q)}\xi[M\,h + MT\eta] < 1.$$

That is:

$$e^{-\pi^2}(\sum_{j=1}^{j=4} c_i + 4\omega + \frac{16}{9}\omega) + \frac{e^{-2\pi^2}\varkappa^2}{\sqrt{2}\Gamma(3)}[\,4\omega + \frac{16}{9}\omega] < 1. \tag{30}$$

By choosing small enough c_1, c_2, c_3, c_4, and ω , we can arrive at (30).

5. Discussion and Conclusions

In recent years, the controllability of different kinds of fractional differential equations and inclusions have been considered by using various types of approaches. In order to ensure that the system is controllable, usually, a suitable fixed point is applied to prove the existence of a fixed point for the solution operator corresponding to the considered system. In the majority of the existing results concerning the controllability, authors have assumed that the semi-group generated the system is compact [15] or the non-linear term is Lipschitz in the second term [10,16,17], or verifies a condition expressed in terms of a measure on non-compactness [18–21]. Moreover, many authors have studied the controllability of systems in the absence of impulse effects [15,17,28]. Unlike the works conducted in [10,15,21], this paper established results concerning the controllability of semilinear differential inclusions of order $\alpha \in (1, 2)$ in the presence of non-instantaneous impulses (problem (1)), without hypotheses of compactness on the semi-group $\{C(\vartheta) : \vartheta \in \mathbb{R}\}$ or any condition on the multi-valued function F involving a measure of noncompactness. We applied a fixed point theorem for weakly sequentially closed graph multivalued operators. Therefore, this work generalized many recent works, such as in [10,15–21]. Moreover, our technique can be used to extend the considered problems in [24–26] to the case when the order of the system is $\alpha \in (1, 2)$. We think that studying the controllability of some fractional differential equations or inclusions with non-instantaneous impulses by using numerical approach is a good future research direction, as in [33].

6. Materials and Methods

Our technique is based on fixed point theorems for weakly sequentially closed multi-valued functions.

Author Contributions: Both the authors contributed equality and read and approved the final version of the manuscript. Both the authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: This research has been funded by the Scientific Research Deanship at University of Ha'il–Kingdom of Saudi Arabia through project number BA-2012.

Conflicts of Interest: The authors declare no conflict of interest.

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