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Thermodynamics of Barrow Holographic Dark Energy with Specific Cut-Off

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Abstract: Motivated by the work of Saridakis (*Phys. Rev. D* **102**, 123525 (2020)), the present study reports the cosmological consequences of Barrow holographic dark energy (HDE) and its thermodynamics. The literature demonstrates that dark energy (DE) may result from electroweak symmetry breaking that triggers a phase transition from early inflation to late-time acceleration. In the present study, we incorporated viscosity in the Barrow HDE. A reconstruction scheme is presented for the parameters associated with Barrow holographic dark energy under the purview of viscous cosmology. The equation of state (EoS) parameter is reconstructed in this scenario and quintessence behaviour is observed. Considering Barrow HDE as a specific case of Nojiri–Odintsov (NO) HDE, we have observed quintom behaviour of the EoS parameter and for some values of n the EoS has been observed to be very close to -1 for the current universe. The generalised second law of thermodynamics has come out to be valid in all the scenarios under consideration. Physical viability of considering Barrow HDE as a specific case of NO HDE is demonstrated in this study. Finally, it has been observed that the model under consideration is very close to Λ CDM and cannot go beyond it.

Keywords: holography; dark energy; bulk viscosity; thermodynamics



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1. Introduction

Gerard 't Hooft proposed the famous Holographic Principle (HP) inspired by black-hole thermodynamics [1,2]. HP states that all the information contained in a volume of space can be represented as a hologram, which corresponds to a theory located on the boundary of that space [3]. It is widely believed that HP is a fundamental principle of quantum gravity.

In the late 1990s, Reiss et al. [4] and Perlmutter et al. [5] independently reported that the current universe is passing through a phase of accelerated expansion. This started a new era in Modern Cosmology. The authors of [4,5] proved this by observational data. This was further supported by other observational studies [6–10]. Characterised by negative pressure to some exotic matter is thought to be responsible for this acceleration. The exotic matter is dubbed as “dark energy” (DE) [11,12]. It is described by an equation of state (EoS) parameter defined as $w = \frac{p}{\rho}$, where p is the pressure and ρ is the density due to DE. One can easily verify from Friedmann's equations that $w < \frac{-1}{3}$ is a necessary condition for the accelerated expansion of the universe. The simplest candidate of DE is cosmological constant (Λ), characterised by EoS parameter $w = -1$ [13]. Various DE models have been reviewed in the literature [11,12,14–26]. From References [27,28], currently the DE percentage is 68.3%. The remaining density is due to dark matter (DM), baryonic matter and radiation. The contributions of baryonic matter and radiation are negligible with respect to the total density of the universe. Dimopoulos and Markkanen [29] have demonstrated that

it is possible to obtain DE from the interplay of Higgs boson and inflation, and it has further been demonstrated that a key element for the same result is the electroweak symmetry breaking that can lead to a transition to inflation to late-time acceleration.

One of the broad types of DE candidates is Holographic DE (HDE), which is discussed in References [30–35]. The principle of HDE is HP. Its density is given by $\rho_\Lambda = 3c^2 M_p^2 L^{-2}$ [31,36,37], where c^2 represents a dimensionless constant, M_p is the reduced Planck mass and L stands for infrared (IR) cut-off. To date, there are different modifications in IR cut-off being made. HDE can be accounted for also using corrections without barotropic fluids, for example [38,39]. Observational constraints on DE and HDE can be also found in [40]. In this paper, we will study the Barrow Holographic DE.

In the Covid 19 pandemic, Barrow was very much inspired by its illustrations and deduced that intricate, fractal features on the black-hole structure may be introduced by the quantum-gravitational effects [41]. This complex structure leads to infinite/finite area but with finite volume. Therefore, the entropy expression to a deformed black-hole is [41]

$$S_B = \left(\frac{A}{A_0} \right)^{\Delta+1}, \quad (1)$$

where A is the standard horizon area and A_0 is the Planck area. The quantum gravitational deformation is quantified by Δ and $\Delta = 0$ corresponds to the standard Bekenstein–Hawking entropy. In addition, $\Delta = 1$ corresponds to the most intricate and fractal structure. Note that the usual “quantum-corrected” entropy with logarithmic corrections is very much different than the “above quantum-gravitationally corrected” entropy. No doubt, the involved foundation and physical principles are completely different but resemble Tsallis non-extensive entropy.

Gordon M. Barrow quoted “Thermodynamics should be built on energy not on heat and work” [42–46]. The standard HDE is given by $\rho_{DE} L^4 \leq S$, where L = horizon length and $S \propto A \propto L^2$. Therefore, using the Barrow entropy Equation (1) lead to

$$\rho_{DE} = CL^{-2(1-\Delta)}, \quad (2)$$

where C is the parameter with dimension $[L]^{-2(\Delta+1)}$. When $\Delta = 0$, the expression (2) will be standard HDE, i.e., $\rho_{DE} = 3c^2 M_p^2 L^{-2}$ (M_p is the Planck mass and L is IR cut-off) where $C = 3c^2 M_p^2$ and c^2 is the model parameter. When the deformation effects quantified by Δ , Barrow HDE will deviate from standard HDE and hence leading to different cosmological consequences. It is very interesting to note that in the limiting case of $\Delta \rightarrow 1$, the above expression becomes the constant, i.e., $\rho_{DE} = \text{constant}$.

The concept of viscosity has been analysed from different viewpoints in cosmology [47,48]. The universe consists of various components having different equations of state and cooling rate. Hence, there exist many contexts under which the bulk viscosity causes exponential decay of anisotropy [48]. The viscosity term in the viscosity model dominates the cosmic pressure and exceeds the pressure contribution from other cosmic matter contributions, which contradicts the traditional fluid theory [49]. In this context, let us refer to the extensive review of viscous cosmologies by Medina et al. [50]. In the study of [50], it was demonstrated that bulk viscosity is compatible with the FRW metric and our current study is in line with this. Furthermore, it has been shown by Murphy [51] that the bulk viscosity can lead to a non-singular universe and the consequences of the bulk universe have been discussed in many other studies. In the subsequent sections we are going to demonstrate Barrow HDE under the purview of bulk viscosity and as a specific case of the Nojiri–Odintsov cut-off.

Nojiri and Odintsov [52] developed cosmological models, where the DE and DM were treated as imperfect fluids. Viscous fluids represent one particular case of what was presented in [52]. In the paper, we will incorporate the viscosity term in the various parameters of Barrow HDE. The paper is organised as follows: In Section 2, we will reconstruct the density, thermodynamic pressure of Barrow HDE. We will also reconstruct

effective pressure, effective EoS of Viscous Barrow HDE. We will also calculate viscous pressure of Barrow HDE. We will plot density versus Δ ; effective EoS versus redshift z and bulk viscous pressure of viscous Barrow HDE versus redshift z versus C . We will study accordingly. In Section 3, we will study the generalised second law of thermodynamics of viscous Barrow HDE using Barrow entropy. In Section 4, we will reconstruct the density, EoS parameter of Barrow HDE as a Specific NO HDE. We will plot EoS versus redshift z in this case and will study the outcomes. In Section 4.1, we will study the validity of the generalised second law of thermodynamics for Barrow HDE with NO cut-off. Here, we will plot the total entropy of the Barrow HDE with an NO cut-off against the cosmic time t . We give our conclusions in the Concluding Remarks.

2. Viscous Barrow Holographic Dark Energy

In this section we study the effect of viscosity in Barrow HDE. As it is known, viscosity refers to the resistance to flow. By considering many components in the cosmology, there is a contribution of bulk viscosity in the thermodynamic pressure [53], which also plays a very important and crucial role in accelerating the universe. The term bulk viscosity arises because of different cooling rates of the components. We can affirm that the bulk viscous pressure in cosmic media emerges as a result of coupling among the different component of the cosmic substratum [54–60]. In this context, we would like to mention that in recent years there has been an increased interest in studying cosmic fluid under the purview of bulk viscosity. In a recent work by [61], it has been demonstrated how bulk viscous modifications to the equation of state leads to physically viable results.

Here we will reconstruct the thermodynamic pressure of Barrow HDE with viscosity. Let us assume R_h is the radius of event horizon, then it is given by

$$\dot{R}_h = HR_h - 1, \quad (3)$$

or,

$$R_h \equiv a \int_t^\infty \frac{dt}{a} = a \int_a^\infty \frac{da}{Ha^2}. \quad (4)$$

Let us assume that infrared (IR) cut-off is the event horizon. Therefore replacing L in Equation (2) with R_h , we get the density of Barrow HDE ρ_{DE} as

$$\rho_{DE} = CR_h^{2(\Delta-1)}. \quad (5)$$

where R_h = radius of event horizon and C is constant. The deformation effect is quantified by Δ . As the DM is in the form of a dust particle, we can consider it as pressureless DM, i.e., $p_m = 0$. The two Friedmann equations are $3H^2 = \rho_{DE} + \rho_m$ and $6\frac{\ddot{a}}{a} = -(\rho_{DE} + \rho_m + 3(p_{DE} + \Pi))$, where $\Pi = \text{Viscous Pressure} = -3H\zeta$ and $\zeta = \zeta_0 + \zeta_1 H + \zeta_2(\dot{H} + H^2)$.

The conservation equation for pressureless DM is $\dot{\rho}_m + 3H\rho_m = 0$. By solving the expression, we get

$$\rho_m = \rho_{m0}a^{-3}. \quad (6)$$

Now we are introducing density parameters Ω_m and Ω_{DE} and they are given by

$$\Omega_m \equiv \frac{1}{3H^2}\rho_m, \quad (7)$$

and

$$\Omega_{DE} \equiv \frac{1}{3H^2}\rho_{DE}. \quad (8)$$

For $\Delta = 1$, the scenario coincides with Λ CDM cosmology with $\rho_{DE} = \text{constant} = \Lambda$. Using density parameters from expressions (7) and (8) in the expressions (4) and (5), we obtain

$$\int_x^\infty \frac{dx}{Ha} = \frac{1}{a} \left(\frac{C}{3H^2\Omega_{DE}} \right)^{\frac{1}{2(1-\Delta)}}. \quad (9)$$

Using ρ_m from Equation (6) in Equation (7), we obtain

$$\Omega_m = \Omega_{m0} \frac{H_0^2}{a^3 H^2}. \tag{10}$$

where $\Omega_{m0} H_0^2 = \frac{\rho_{m0}}{3}$. Now using the Friedmann Equation $\Omega_m + \Omega_{DE} = 1$ and also using Equations (8) and (10), we get

$$\frac{1}{aH} = \frac{\sqrt{a(1 - \Omega_{DE})}}{H_0 \sqrt{\Omega_{m0}}}. \tag{11}$$

Inserting Equation (11) into (9) results in

$$\int_x^\infty \frac{\sqrt{a(1 - \Omega_{DE})}}{H_0 \sqrt{\Omega_{m0}}} dx = \frac{1}{a} \left(\frac{C}{3H^2 \Omega_{DE}} \right)^{\frac{1}{2(1-\Delta)}}. \tag{12}$$

Differentiating Equation (12) with respect to $x = \ln a$, one gets

$$\frac{\Omega'_{DE}}{\Omega_{DE}(1-\Omega_{DE})} = 2\Delta + 1 + Q(1 - \Omega_{DE})^{\frac{\Delta}{2(\Delta-1)}} (\Omega_{DE})^{\frac{1}{2(1-\Delta)}} e^{\frac{3\Delta}{2(\Delta-1)}x}. \tag{13}$$

where $Q \equiv 2(1 - \Delta) \left(\frac{C}{3}\right)^{\frac{1}{2(\Delta-1)}} (H_0 \sqrt{\Omega_{m0}})^{\frac{\Delta}{1-\Delta}}$. Equation (13) is the evolution of Barrow HDE in a flat universe for dust matter. For $\Delta = 0$, it coincides with the usual HDE, i.e., $\Omega'_{DE}|_{\Delta=0} = \Omega_{DE}(1 - \Omega_{DE})(1 + 2\sqrt{\frac{3\Omega_{DE}}{C}})$. Now from Equation (11), we have

$$H = \frac{H_0 \sqrt{\Omega_{m0}}}{a \sqrt{a(1 - \Omega_{DE})}}. \tag{14}$$

From Equation (3) taking H from Equation (14), we get R_h as

$$R_h = \frac{a \sqrt{-a(\Omega_{DE} - 1)}}{H_0 \sqrt{\Omega_{m0}}} + e^{\frac{H_0 \sqrt{\Omega_{m0}} t}{a \sqrt{-a(\Omega_{DE} - 1)}}} C_1. \tag{15}$$

Now using this R_h in Equation (5), we obtain the reconstructed density of Barrow HDE $\rho_{DE,rec}$ as

$$\rho_{DE,rec} = C \left(C_1 e^{\frac{H_0 \sqrt{\Omega_{m0}} t}{a \sqrt{-a\Omega_{DE}}}} + \frac{a \sqrt{a - a\Omega_{DE}}}{H_0 \sqrt{\Omega_{m0}}} \right)^{2(-1+\Delta)}. \tag{16}$$

As now we have $\rho_{DE,rec}$ (Equation (16)), H (Equation (14)) and let us take $p_{eff} = p_{DE} + \Pi$ and using these in the conservation equation $\dot{\rho}_{DE,rec} + 3H(\rho_{DE,rec} + p_{eff}) = 0$, we obtain

$$\frac{1}{3H_0 \sqrt{\Omega_{m0}}} a \sqrt{a(1 - \Omega_{DE})} \left(\frac{p_{eff} = 3CH_0 \left(C_1 e^{\frac{H_0 \sqrt{\Omega_{m0}} t}{a \sqrt{-a\Omega_{DE}}}} + \frac{a \sqrt{a - a\Omega_{DE}}}{H_0 \sqrt{\Omega_{m0}}} \right)^{2(-1+\Delta)} \sqrt{\Omega_{m0}}}{a \sqrt{a(1 - \Omega_{DE})}} - \frac{2CC_1 e^{\frac{H_0 \sqrt{\Omega_{m0}} t}{a \sqrt{-a\Omega_{DE}}}} H_0 \left(C_1 e^{\frac{H_0 \sqrt{\Omega_{m0}} t}{a \sqrt{-a\Omega_{DE}}}} + \frac{a \sqrt{a - a\Omega_{DE}}}{H_0 \sqrt{\Omega_{m0}}} \right)^{-3+2\Delta} \sqrt{\Omega_{m0}} (-1+\Delta)}{a \sqrt{a - a\Omega_{DE}}} \right). \tag{17}$$

Therefore, thermodynamic pressure $p_{DE} = p_{eff} - \Pi$. Hence,

$$p_{DE} = \left(\frac{1}{3a\sqrt{a-a\Omega_{DE}}} \right) \left(-aCH_0^2\sqrt{a-a\Omega_{DE}} \left(C_1 e^{\frac{H_0\sqrt{\Omega_{m0}}t}{a\sqrt{a-a\Omega_{DE}}}} + \frac{a\sqrt{a-a\Omega_{DE}}}{H_0\sqrt{\Omega_{m0}}} \right)^{2\Delta} \right. \\ \left. \Omega_{m0} \left(3a\sqrt{a-a\Omega_{DE}} + C_1 e^{\frac{H_0\sqrt{\Omega_{m0}}t}{a\sqrt{a-a\Omega_{DE}}}} H_0\sqrt{\Omega_{m0}}(1+2\Delta) \right) \right. \\ \left. \left(a\sqrt{a-a\Omega_{DE}} + C_1 e^{\frac{H_0\sqrt{\Omega_{m0}}t}{a\sqrt{a-a\Omega_{DE}}}} H_0\sqrt{\Omega_{m0}} \right)^{-3} + 9H_0\sqrt{\Omega_{m0}}\zeta \right). \quad (18)$$

which is the thermodynamic pressure of DE involving the viscous term ζ . As the viscous term is involved here, we can take $p_{DE} =$ effective pressure. Hence, effective pressure p_{eff} is Equation (18). As we know the effective EoS, $w_{eff} = \frac{p_{eff}}{\rho_{DE,rec}}$. Thereby, using p_{eff} from Equation (18) and $\rho_{DE,rec}$ from Equation (16), we get effective EoS as

$$w_{eff} = \frac{\left(\frac{H_0\sqrt{\Omega_{m0}}t}{C_1 e^{\frac{H_0\sqrt{\Omega_{m0}}t}{a\sqrt{a-a\Omega_{DE}}}} + \frac{a\sqrt{a-a\Omega_{DE}}}{H_0\sqrt{\Omega_{m0}}}} \right)^{2-2\Delta}}{3aC\sqrt{a-a\Omega_{DE}}} \left(-aCH_0^2\sqrt{a-a\Omega_{DE}} \left(C_1 e^{\frac{H_0\sqrt{\Omega_{m0}}t}{a\sqrt{a-a\Omega_{DE}}}} + \frac{a\sqrt{a-a\Omega_{DE}}}{H_0\sqrt{\Omega_{m0}}} \right)^{2\Delta} \right. \\ \left. \Omega_{m0} \left(3a\sqrt{a-a\Omega_{DE}} + C_1 e^{\frac{H_0\sqrt{\Omega_{m0}}t}{a\sqrt{a-a\Omega_{DE}}}} H_0\sqrt{\Omega_{m0}}(1+2\Delta) \right) \right. \\ \left. \left(a\sqrt{a-a\Omega_{DE}} + C_1 e^{\frac{H_0\sqrt{\Omega_{m0}}t}{a\sqrt{a-a\Omega_{DE}}}} H_0\sqrt{\Omega_{m0}} \right)^{-3} + 9H_0\sqrt{\Omega_{m0}}\zeta \right). \quad (19)$$

Now we will insert Δ in Π , to make it a viscous pressure in Barrow HDE. Now using $\rho_{DE,rec}$ from Equation (16), w_{eff} from Equation (19) in the conservation equation $\dot{\rho}_{DE,rec} + 3H\rho_{DE,rec}(1+w_{eff}) = 0$, we get H . Let us name it as H_{rec} , which is given by

$$H_{rec} = - \left(2CC_1 e^{\frac{H_0\sqrt{\Omega_{m0}}t}{a\sqrt{a-a\Omega_{DE}}}} H_0^3 \left(C_1 e^{\frac{H_0\sqrt{\Omega_{m0}}t}{a\sqrt{a-a\Omega_{DE}}}} + \frac{a\sqrt{a-a\Omega_{DE}}}{H_0\sqrt{\Omega_{m0}}} \right)^{2\Delta} \Omega_{m0}^{3/2} (-1+\Delta) \right) \\ \left(-9a^4(-1+\Omega_{DE})\sqrt{a-a\Omega_{DE}}\zeta - 27a^3 C_1 e^{\frac{H_0\sqrt{\Omega_{m0}}t}{a\sqrt{a-a\Omega_{DE}}}} H_0(-1+\Omega_{DE})\sqrt{\Omega_{m0}}\zeta + \right. \\ \left. 9C_1^3 e^{\frac{3H_0\sqrt{\Omega_{m0}}t}{a\sqrt{a-a\Omega_{DE}}}} H_0^3 \Omega_{m0}^{3/2} \zeta + aC_1 e^{\frac{H_0\sqrt{\Omega_{m0}}t}{a\sqrt{a-a\Omega_{DE}}}} H_0^2 \sqrt{a-a\Omega_{DE}} \Omega_{m0} \right. \\ \left. \left(-2C \left(C_1 e^{\frac{H_0\sqrt{\Omega_{m0}}t}{a\sqrt{a-a\Omega_{DE}}}} + \frac{a\sqrt{a-a\Omega_{DE}}}{H_0\sqrt{\Omega_{m0}}} \right)^{2\Delta} (-1+\Delta) + 27C_1 e^{\frac{H_0\sqrt{\Omega_{m0}}t}{a\sqrt{a-a\Omega_{DE}}}} \zeta \right) \right)^{-1}. \quad (20)$$

Let us assume the power-law form of scale factor as $a(t) = a_0(t-t_0)^n$. As we know that $H = \frac{\dot{a}}{a}$, by using the power-law form of scale factor, we get H . Let us denote this H by H_{recc} , which is given by

$$H_{recc} = \frac{n}{t-t_0}. \quad (21)$$

It is obvious that H_{recc} is defined for the case $t \neq t_0$, i.e., H will be having singularity at $t = t_0$. Now, by using H_{recc} in place of H in ζ i.e., $\zeta = \zeta_0 + \zeta_1 H_{recc} + \zeta_2 (\dot{H}_{recc} + H_{recc}^2)$. Then, using this ζ and H as H_{rec} from Equation (20) in $\Pi = -3H\zeta$, we obtain the viscous pressure in Barrow HDE as

$$\begin{aligned} \Pi = & \left(6CC_1 e^l H_0^3 \Omega_{m0}^{3/2} (C_1 e^l + j)^{2\Delta} (-1 + \Delta)(k + (-1 + n)n\zeta_2) \right) \\ & \left(9C_1^3 e^{3l} H_0^3 \Omega_{m0}^{3/2} (k + (-1 + n)n\zeta_2) - 27a_0^3 C_1 e^l H_0 (-1 + \Omega_{DE}) \sqrt{\Omega_{m0}} (t - t_0)^{3n} \right. \\ & \left. (k + (-1 + n)n\zeta_2) + 9a_0^3 (t - t_0)^{3n} (-a_0(-1 + \Omega_{DE})(t - t_0)^n)^{3/2} (k + (-1 + n)n\zeta_2) + \right. \\ & \left. a_0 C_1 e^{2l} H_0^2 \Omega_{m0} (t - t_0)^n \sqrt{-a_0(-1 + \Omega_{DE})(t - t_0)^n} \right. \\ & \left. (-2Ce^{-l} (C_1 e^l + j)^{2\Delta} (t - t_0)^2 (-1 + \Delta) + 27C_1 (k + (-1 + n)n\zeta_2)) \right)^{-1}, \end{aligned} \quad (22)$$

where, $l = -\frac{H_0(-1+\Omega_{DE})\sqrt{\Omega_{m0}}t}{(-a_0(-1+\Omega_{DE})(t-t_0)^n)^{3/2}}$, $k = (t - t_0)(t\zeta_0 - t_0\zeta_0 + n\zeta_1)$, $j = \frac{a_0(t-t_0)^n \sqrt{-a_0(-1+\Omega_{DE})(t-t_0)^n}}{H_0\sqrt{\Omega_{m0}}}$. Now we will reconstruct a thermodynamic DE pressure i.e., $p_{DE,rec}$ to make it thermodynamic pressure of viscous Barrow HDE. Thus, in the conservation equation $\dot{\rho}_{DE,rec} + 3H(\rho_{DE,rec} + p_{DE,rec} + \Pi) = 0$, using $\rho_{DE,rec}$ from Equation (16), Π from Equation (22), H as H_{rec} from Equation (20), we obtain $p_{DE,rec}$, which is a thermodynamic pressure of viscous Barrow HDE. Now using Taylor series expansion in the term $\sqrt{1 - \Omega_{DE}}$ of Equation (11) and ignoring higher order derivatives, we obtain $\frac{1}{aH} = \frac{\sqrt{a}}{H_0\sqrt{\Omega_{m0}}}(1 - \frac{1}{2}\Omega_{DE})$. From the above equation, we get Ω_{DE} as

$$\Omega_{DE} = 2 - \frac{2H_0\sqrt{\Omega_{m0}}}{\dot{a}\sqrt{a}}. \quad (23)$$

Now, using Ω_{DE} from Equation (23) in Equation (19), we get w_{eff} and plotted the evolution of effective EoS (19) of viscous Barrow HDE against the redshift z in Figure 1. In this figure, we have made the following choices of parameters: $a_0 = 0.2$, $C = 10^{45}$, $C_1 = 0.00015$, $\Omega_{m0} = 0.002$, $t_0 = 0.20$, $\zeta_0 = 0.5$, $\zeta_1 = 0.1$, $\zeta_2 = 0.92$, $\Delta = 0.04$. We would like to mention that following Elizalde et al. [62], we have chosen $H_0 = 73.393 \pm 0.1$ km/s/Mpc. In this context we would like to mention that the parameters are not chosen by any standard optimisation technique. Rather, we have confined ourselves to the ranges already mentioned in the existing literature.

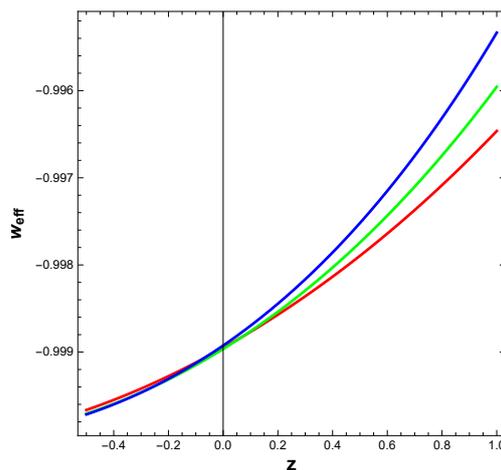


Figure 1. Evolution of effective equation of state (EoS) (Equation (19)) of viscous Barrow Holographic Dark Energy against redshift z . The red, green and blue lines correspond to $n = 0.7, 0.8, 0.9$, respectively.

From the figure we observe that behaviour of the effective EoS parameter w_{eff} (19) is quintessence. Now we will study the behaviour of $\rho_{DE,rec}$ (Equation (16)) when $\Delta \rightarrow -1$. We have plotted the reconstructed density of Barrow HDE against redshift z in Figure 2 for a range of values of Δ .

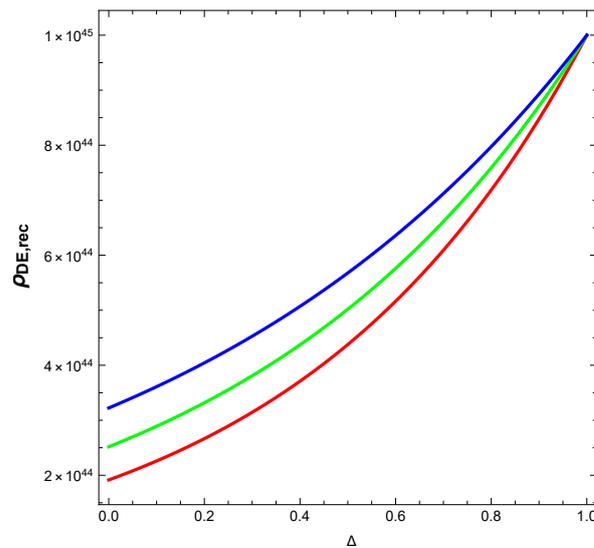


Figure 2. Evolution of density of Barrow holographic dark energy (HDE) (Equation (16)) against Δ . The red, green and blue line corresponds to $z = -0.1, 0, 0.1$, respectively.

In Figure 2 it is observed that density of Barrow HDE has an increasing tendency when the deformation quantifying factor Δ tends to 1. Moreover, at higher values of Δ , the density converges to a point for early current and future universe. This indicates that at that point we can study the evolution of the universe at large for different phases of evolution.

Using the expression of Ω_{DE} from Equation (23) in the expression of bulk viscous pressure of Barrow HDE Π , i.e., Equation (22), we have plotted Π versus redshift z versus C in Figure 3. This figure shows that the constant of Barrow HDE has a significant role to play in the bulk viscous pressure. The effect of bulk viscosity is more for the higher values of C than the lower values. It is further observed that under the current framework the effect of bulk viscosity has a decaying pattern with redshift.

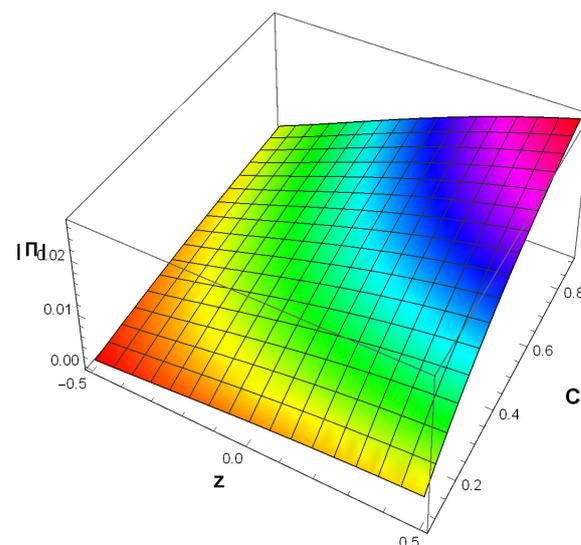


Figure 3. Evolution of bulk viscous pressure of Barrow HDE (Equation (22)) against redshift z and against C , as shown in Equation (2).

3. Generalised Second Law of Thermodynamics of Viscous Barrow HDE

In this section, we will study the generalised second law of thermodynamics using barrow entropy [46]. We consider the universe horizon to be the boundary of the thermo-

dynamical system. We can take it as an apparent horizon, as it is the most appropriate one. There are many choices in the literature and we chose here the apparent horizon [42–44]. The apparent horizon is given by

$$\tilde{r}_A = \frac{1}{\sqrt{H^2 + \frac{k}{a^2}}}. \quad (24)$$

where k quantifies the spatial curvature and hence $k = 0$, as we considered the universe to be flat. Therefore Equation (24) becomes

$$\tilde{r}_A = \frac{1}{H}. \quad (25)$$

From the first Friedmann equation $3H^2 = \rho_m + \rho_{DE}$ and Equation (25), we get

$$\frac{1}{\tilde{r}_A^2} = \frac{1}{3}(\rho_m + \rho_{DE}). \quad (26)$$

Using $\rho_m = \rho_{m0}a^{-3}$, $a = a_0(t - t_0)^n$ and $\rho_{DE,rec}$ from Equation (16) in place of ρ_{DE} in Equation (26), we get apparent horizon \tilde{r}_A .

Now we will check whether the total entropy of the system, i.e., sum of the entropy enclosed by the apparent horizon plus entropy of the apparent horizon of the system is a non-decreasing function of time or not. The apparent horizon \tilde{r}_A is dependent on time. Therefore, changes in apparent horizon $d\tilde{r}_A$ in time interval dt will contribute a change in volume dV . Hence, the energy and entropy of the system will change by dE and dS , respectively. The first law of thermodynamics is $TdS = dE + PdV$. Therefore, the dark energy entropy and dark matter entropy will be [45]:

$$dS_{DE} = \frac{1}{T}(P_{DE}dV + dE_{DE}), \quad (27)$$

$$dS_m = \frac{1}{T}(P_m dV + dE_m). \quad (28)$$

where dS_{DE} = DE entropy, dS_m = DM entropy, P_{DE} = DE pressure, P_m = DM pressure. V is the universe volume bounded by apparent horizon and is given by $V = \frac{4\pi\tilde{r}_A^3}{3}$. Therefore, $dV = 4\pi\tilde{r}_A^2 d\tilde{r}_A$. We assume the system to be in equilibrium, so we can consider the temperature of the universe fluids to be same. Dividing Equations (27) and (28) by t , we get

$$\dot{S}_{DE} = \frac{1}{T}(P_{DE}4\pi\tilde{r}_A^2\dot{\tilde{r}}_A + \dot{E}_{DE}), \quad (29)$$

$$\dot{S}_m = \frac{1}{T}(P_m4\pi\tilde{r}_A^2\dot{\tilde{r}}_A + \dot{E}_m). \quad (30)$$

To consider the relationship between thermodynamical quantities \dot{E}_{DE} and \dot{E}_m with cosmological quantities ρ_{DE} and ρ_m , we use

$$E_{DE} = \frac{4\pi}{3}\tilde{r}_A^3\rho_{DE}, \quad (31)$$

$$E_m = \frac{4\pi}{3}\tilde{r}_A^3\rho_m. \quad (32)$$

Now we have \tilde{r}_A , so we can find $\dot{\tilde{r}}_A$, E_{DE} from Equation (31), E_m from Equation (32) and hence \dot{E}_{DE} and \dot{E}_m . We consider $T \approx$ horizon temperature (T_h) = $\frac{1}{2\pi\tilde{r}_A}$. Therefore, we can calculate \dot{S}_{DE} and \dot{S}_m from Equations (29) and (30), respectively. Now we will calculate horizon entropy \dot{S}_h . Applying entropy expression to a deformed black hole Equation (1)

with standard horizon area $A = 4\pi\tilde{r}_A^2$, we get $S_h = \gamma\tilde{r}_A^{2(\Delta+1)}$, where $\gamma \equiv \left(\frac{4\pi}{A_0}\right)^{1+\Delta}$. Therefore, horizon entropy is given by

$$\dot{S}_h = \gamma 2(1 + \Delta)\tilde{r}_A^{2\Delta+1}\dot{\tilde{r}}_A. \tag{33}$$

Therefore, $\dot{S}_{total} = \dot{S}_{DE} + \dot{S}_m + \dot{S}_h$. After calculating \dot{S}_{total} , we plotted it in Figure 4. In this figure, we have considered $a_0 = 0.001$, $C = 0.09$, $C_1 = 0.00015$, $H_0 = 73.32$, $\Omega_{m0} = 0.002$, $t_0 = 0.20$, $\xi_0 = 0.000005$, $\xi_1 = 0.00001$, $\xi_2 = 0.92$, $\rho_{m0} = 0.32$, $A_0 = 0.00905$. From the figure we have seen that \dot{S}_{total} is positive and is non-decreasing. Hence, it satisfies the second law of thermodynamics. This implies the validity of the generalised second law of thermodynamics in the case of viscous Barrow HDE. It is further observed that with evolution of the universe \dot{S}_{total} is increasing. This indicates that the validity of the generalised second law of thermodynamics is expected to occur with the evolution of the universe in case of viscous Barrow HDE.

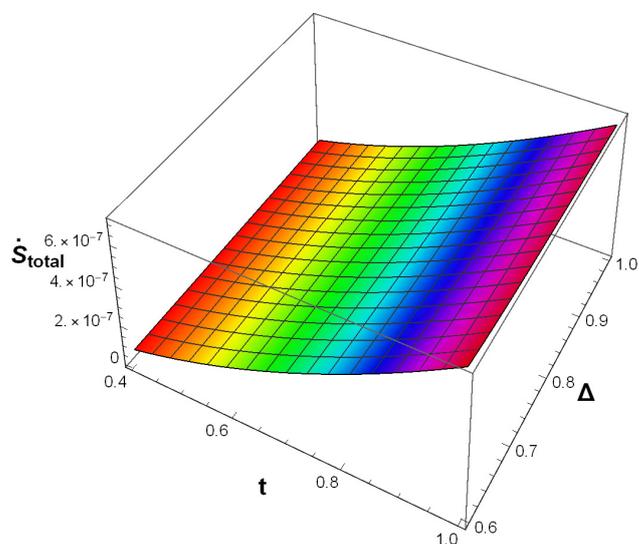


Figure 4. Plot of \dot{S}_{total} of viscous Barrow HDE against the cosmic time t and Δ .

4. Barrow HDE as a Specific NO HDE

In this section, we consider Barrow HDE as a particular case of NO HDE. The NO HDE was proposed in the work of Nojiri and Odintsov [34]. This was further studied in [11]. The DE density for NO HDE is defined as

$$\rho_{NO} = \frac{3c^2}{L^2}, \tag{34}$$

with

$$\frac{c}{L} = \frac{1}{R_h}(\alpha_0 + \alpha_1 R_h + \alpha_2 R_h^2), \tag{35}$$

where R_h is the future event horizon discussed in Equations (3) and (4). For the choice of power law form of scale factor $a(t) = a_0(t - t_0)^n$, we have IR cut off L as

$$L = \frac{c}{\frac{(n-1)\alpha_0}{t+C_2(n-1)(t-t_0)^n-t_0} + \alpha_1 + \frac{(t-t_0)\alpha_2}{n-1} + C_2(t-t_0)^n\alpha_2}. \tag{36}$$

At this juncture, before passing on to the reconstruction approach let us have an overview of the background of the NO HDE. In this context, it may be noted that Nojiri and Odintsov [34] demonstrated a unifying approach to the early and late-time universe through a phantom cosmology. They considered a gravity-scalar system containing the

usual potential and scalar coupling function within the kinetic term. Their study [34] resulted in the possibility of a phantom–non-phantom transition in such a manner that the universe could have the phantom EoS in the early as well as in the late-time. Contrary to the study of [34], our work, a specific case of NO HDE, has led to a quintessence behaviour with no crossing of the phantom boundary; see Figure 1. In this connection, we further note that the generalised HDE with NO cut-off, as proposed in [34], suggested a unified cosmological scenario for tachyon phantoms and for time-dependent phantom EoS. We further take into account the study of Nojiri and Odintsov [63], where a generalised HDE was proposed with infrared cut-off identified with the combination of the FRW universe parameters. Their study took into account the Hubble rate $H(t) = f_0|t_s - t|^\alpha$. However, in our study, we have taken into consideration a Hubble rate $H = \frac{n}{t-t_0}$, for which we could get a universe where the generalised second law of thermodynamics has come out to be valid. Hence, we can state that the Barrow HDE, a specific case of more general NO HDE can lead to a universe where the generalised second law of thermodynamics is valid. Nojiri et al. [64] established that at late times, the effective fluid can act as the driving force behind the accelerated expansion in the absence of a cosmological constant. Consistent with the findings of [64] in our work on a specific form of NO HDE, the generalised second law appeared to be valid without any cosmological constant. In this context let us mention the work of Nojiri et al. [65], who applied the HP at early times to realise the bounce scenario. The current study with a specific NO HDE cut-off can be further extended to check the realisation of holographic bounce and to study the mechanism of holographic preheating [65] under this framework. Lastly, let us mention the study of Nojiri et al. [66], which confronted the cosmological scenario arising from the application of non-extensive thermodynamics with varying exponents. Their study could provide a description of both inflation and late-time acceleration with the same choices of parameters. We further reiterate that the current Barrow HDE can be examined for its realisation for early inflation and late-time acceleration as a specific case of NO HDE.

Now we demonstrate Barrow HDE as a specific case of NO HDE. In Equations (34)–(36), c , α_0 , α_1 and α_2 are numerical constants and C_2 is the constant of integration. Equation (36) represents the NO cut-off as a function of cosmic time t . Now, we consider this NO cut-off as the cut-off for Barrow HDE and from this consideration, we get Barrow HDE generalised by NO HDE and hence, we get the density for Barrow HDE generalised through NO cut-off $\rho_{BarrowHDE}$, which is

$$\rho_{BarrowHDE} = C \left(\frac{c}{\frac{(n-1)\alpha_0}{t+C_2(n-1)(t-t_0)^{n-t_0}} + \alpha_1 + \frac{(t-t_0)\alpha_2}{n-1} + C_2(t-t_0)^n\alpha_2} \right)^{2(-1+\Delta)}. \quad (37)$$

We will find the thermodynamic pressure for Barrow HDE generalised through NO cut-off, i.e., $p_{BarrowHDE}$ from the conservation equation $\dot{\rho}_{BarrowHDE} + 3H(\rho_{BarrowHDE} + p_{BarrowHDE}) = 0$, we get $p_{BarrowHDE}$. Hence, the EoS parameter for Barrow HDE generalised through NO cut-off i.e., $w_{BarrowHDE}$ can be calculated by using $\rho_{BarrowHDE}$ from Equation (37) and $p_{BarrowHDE}$ on equation $w_{BarrowHDE} = \frac{p_{BarrowHDE}}{\rho_{BarrowHDE}}$. In Figure 5, we have plotted the EoS parameter for Barrow HDE as a specific case of NO HDE. In this figure, the evolution of the reconstructed EoS parameter is demonstrated for $\Delta = 0.4$ and the range of values of $1.5 \leq n \leq 2$. It is apparent from this figure that for smaller values of n , the transition from quintessence to phantom is happening at an earlier stage of the universe. However, for $n \approx 2$, the transition is happening at a later stage. Therefore, in general we can say that the EoS parameter for Barrow HDE reconstructed through NO HDE is characterise by quintom behaviour. Moreover, for $n \approx 1.56$, we have $w_{BarrowHDE} \approx -1$ for $z = 0$ and hence, it is consistent with the observation. Hence, we can conclude that as the IR cut-off for Barrow HDE is reconstructed through NO HDE, the transition from quintessence to phantom is available. It further indicates that under this reconstruction scheme the universe may end with a Big-Rip in the future.

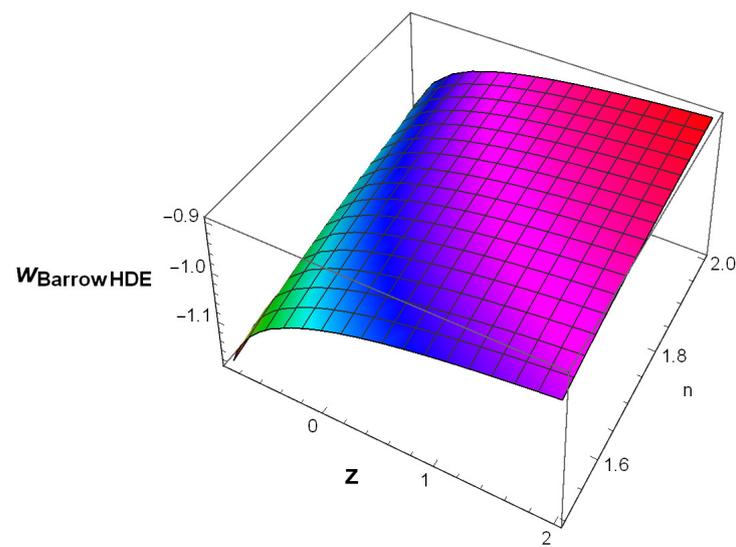


Figure 5. Evolution of the EoS parameter for Barrow HDE generalised through the Nojiri–Odintsov (NO) cut-off, i.e., $w_{BarrowHDE}$ against the redshift z and against n .

4.1. Generalised Second Law of Thermodynamics for Barrow HDE with NO Cut-Off

In this subsection, we have studied the generalised second law of thermodynamics for Barrow HDE with NO cut-off using Barrow entropy as in Section 3. We have proceeded similarly as Section 3 just by taking the scale factor as $a(t) = a_0(t - t_0)^n$, with $n > 0$. We have calculated the total entropy of the Barrow HDE with NO cut-off, i.e., $\dot{S}_{total,BarrowHDE}$, and plotted this in Figure 6 against the cosmic time t . In this figure, we have considered $a_0 = 0.001$, $c = 0.06$, $\alpha_0 = 0.004$, $\alpha_1 = 0.005$, $\alpha_2 = 0.0003$, $C = 0.00015$, $C_2 = 0.09$, $t_0 = 0.20$, $\Delta = 0.04$, $\rho_{m0} = 0.32$, $A_0 = 0.00905$. Figure 6 indicates that $\dot{S}_{total,BarrowHDE}$ is positive and non-decreasing. Therefore, we have observed the validity of the generalised second law of thermodynamics when Barrow HDE is considered as a specific case of NO HDE.

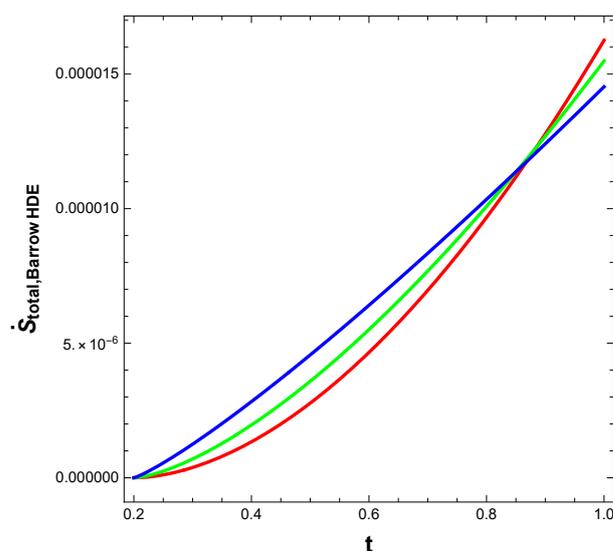


Figure 6. Plot of $\dot{S}_{total,BarrowHDE}$ of Barrow HDE with NO cut-off against the cosmic time t . The red, green and blue lines correspond to $n = 0.9$, 0.8 and 0.7 , respectively.

5. Concluding Remarks

Motivated by the work of Saridakis [41], the present study attempts to probe the cosmological consequences of Barrow HDE and its thermodynamics. In the first phase

of the study, we have studied the effect of bulk viscosity in the presence of Barrow HDE. We have reconstructed the density of Barrow HDE as $\rho_{DE,rec}$ in Equation (16). We also found effective pressure p_{eff} of viscous Barrow HDE as in Equation (18). After finding p_{eff} (Equation (18)), we have derived effective EoS of viscous Barrow HDE w_{eff} as in Equation (19). Thereafter, we calculated viscous pressure Π in Equation (22) and we also reconstructed thermodynamic pressure $p_{DE,rec}$ of viscous Barrow HDE. In Figure 1, we have plotted w_{eff} (Equation (19)) versus redshift z . From Figure 1, we observed that the behaviour of w_{eff} (Equation (19)) is quintessence. Next, we studied the behaviour of $\rho_{DE,rec}$ (Equation (16)) as $\Delta \rightarrow 1$ (see Figure 2). It is apparent from this figure that there is an increasing tendency of $\rho_{DE,rec}$ (Equation (16)) as the deformation quantifying factor $\Delta \rightarrow 1$, which indicates that we can study the evolution of the universe in its different phases. In addition, we have studied the behaviour of the bulk viscous pressure Π (Equation (22)) under the purview of Barrow HDE with the evolution of the universe for a range of values of C in Figure 3. The study demonstrated above shows the decaying effect of bulk viscous pressure with the evolution of the universe. Furthermore, the positive impact of the deformation quantifying factor on the bulk viscous pressure is understandable from this figure. This is in contrast with the finding of [67], where the effect of bulk viscosity was found to have an increasing pattern under the purview of holographic Ricci DE.

In Section 3, we have demonstrated the generalised second law of thermodynamics under the purview of the bulk-viscosity of the Barrow HDE. Here, for the study we have taken apparent horizon as the enveloping horizon of the universe. We have calculated the total entropy \dot{S}_{total} of the system. The \dot{S}_{total} has been plotted in Figure 4, which shows that \dot{S}_{total} corresponding to the viscous Barrow HDE is increasing and is staying at a positive level. Therefore, we conclude that the generalised second law of thermodynamics is obeyed by this model [68]. This finding is consistent with the study of [68], where the validity of generalised second law of thermodynamics was examined in the presence of viscous DE and it was observed that the generalised second law of thermodynamics is fulfilled in the presence of bulk viscosity. However, the approach of the current study differs from Setare and Sheykhi [68] in the sense that the standard Eckart approach is adopted here.

In Section 4, we have demonstrated reconstructed schemes of Barrow HDE as a specific NO HDE. We have reconstructed the density, i.e., $\rho_{BarrowHDE}$ in Equation (37) for Barrow HDE generalised through NO cut-off. We have also reconstructed the EoS parameter $w_{BarrowHDE}$ for Barrow HDE generalised through NO cut-off and plotted it in Figure 5. This figure shows the quintom behaviour of $w_{BarrowHDE}$. Moreover, $w_{BarrowHDE} \approx -1$ at $z = 0$ for some values of n . It also suggests that the universe may end with a Big-Rip in the future. Finally, for this reconstructed Barrow HDE we have demonstrated the generalised second law of thermodynamics. For the Barrow HDE with NO cut-off it is observed that (see Figure 6) the time derivative of the total entropy is staying at a positive level and hence, it is concluded that the generalised second law holds if we consider Barrow HDE as a specific case of NO HDE.

While concluding, let us have a look into the reconstructed Barrow HDE for its attainability of a Λ CDM fixed point. It is done through statefinder parameters [69,70] $r = \frac{\ddot{a}}{aH^3}$, $s = \frac{r-1}{3(q-\frac{1}{2})}$, where q is the deceleration parameter given by $q = \frac{-\ddot{a}}{aH^2}$. The trajectories in the $\{r-s\}$ plane can exhibit different behaviours for different models. The deviation from the point $(1,0)$ indicates the departure of a model from Λ CDM model. In the present study, the reconstructed Barrow HDE is tested for its deviation from Λ CDM model through the statefinder trajectory plotted in Figure 7 for different values of n . It is observed that this reconstructed Barrow HDE can attain $r \approx 1$, $s \approx 0$. Hence, we can conclude that this model is very close to Λ CDM. Furthermore, it is also observed that the statefinder trajectories cannot go beyond the Λ CDM fixed point. In this context, let us comment on the departure of the model from Λ CDM. The attainment of Λ CDM leads us to interpret that Λ CDM-type cosmology can be reconstructed by Barrow HDE with the Nojiri–Odintsov cut-off in the present formulation that included dark matter. The asymptotic behaviour of density at late times has been observed under this formulation

and hence we can interpret that even without a cosmological constant we can reproduce Λ CDM cosmology under the current framework. Hence, we can finally comment that although the current model is deviated from Λ CDM, with appropriate formulation the model can reproduce Λ CDM at late times. However, as the statefinder trajectories could not go beyond the Λ CDM fixed point, we can say that the current model cannot interpolate between dust and Λ CDM.

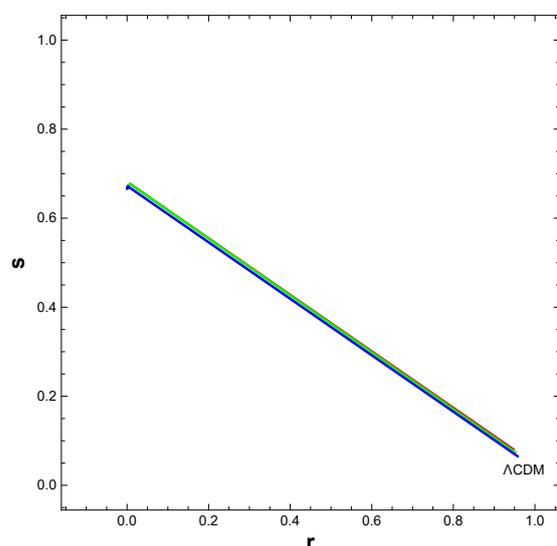


Figure 7. The statefinder trajectory for the reconstructed Barrow HDE. The Λ CDM fixed point is found to be attainable by the model.

We propose to carry out a similar viscous cosmology under the purview of modified theories of gravity in future with the background evolution as Barrow HDE. While concluding we would like to draw the attention of the readers to the previously published two works [22,23] by the authors of the present paper, where the inflationary cosmology was demonstrated through a scalar field model and a generalised version of HDE. In a similar manner we propose to investigate the slow roll parameters under the consideration of Barrow HDE with NO cut-off as the fluid responsible for the inflationary expansion.

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