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Midpoint Inequalities in Fractional Calculus Defined Using Positive Weighted Symmetry Function Kernels

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Abstract: The aim of our study is to establish, for convex functions on an interval, a midpoint version of the fractional HHF type inequality. The corresponding fractional integral has a symmetric weight function composed with an increasing function as integral kernel. We also consider a midpoint identity and establish some related inequalities based on this identity. Some special cases can be considered from our main results. These results confirm the generality of our attempt.

Keywords: symmetry; weighted fractional operators; convex functions; HHF type inequality



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1. Introduction

Let $\mathcal{J} \subset \mathcal{R}$ be an interval and let $u: \mathcal{J} \to \mathcal{R}$ be a continuous function. Then, the function u is called convex if it satisfies

$$\mathsf{u}(\kappa\,\mathbf{c}_1 + (1-\kappa)\mathbf{c}_2) \leq \kappa\,\mathsf{u}(\mathbf{c}_1) + (1-\kappa)\mathsf{u}(\mathbf{c}_2), \quad \forall\,\mathbf{c}_1,\mathbf{c}_2 \in \mathcal{J} \text{ and } \kappa \in [0,1]. \tag{1}$$

The function u is called concave whenever -u is convex.

For convex functions $u: \mathcal{J} \to \mathcal{R}$, there is an important integral inequality in the literature, namely the Hermite–Hadamard or, briefly, the HH integral inequality, which is given by [1]:

$$u\left(\frac{\mathbf{c}_1 + \mathbf{c}_2}{2}\right) \le \frac{1}{\mathbf{c}_2 - \mathbf{c}_1} \int_{\mathbf{c}_1}^{\mathbf{c}_2} \mathsf{u}(x) dx \le \frac{\mathsf{u}(\mathbf{c}_1) + \mathsf{u}(\mathbf{c}_2)}{2},\tag{2}$$

where $\mathbf{c}_1 < \mathbf{c}_2$ belong to \mathcal{J} . In the literature, one can observe that the HH integral inequality (2) has been applied to different classes of convexity such as GA–convexity [2], quasi-convexity [3,4], s–convexity [5], (α, m) –convexity [6], exponentially convexity [7,8], MT–convexity [9], and the readers can consult [10,11] to find other types.

As we know, fractional calculus is a generalized form of integer order calculus. Various forms of fractional derivatives including RL, Hadamard, Caputo, Caputo–Hadamard, Riesz, ψ –RL, Prabhakar, and weighted versions [12–16] have been developed to date. Most of these versions are described in the RL sense based on the corresponding fractional integral. Many integer-order integral inequalities such as Ostrowski [17], Simpson [18],

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Hardy [19], Olsen [20], Gagliardo–Nirenberg [21], Opial [22,23] and Rozanova [24] have been generalized and reformulated from the fractional point of view.

In addition, in 2013, the HH integral inequality (2) was generalized and reformulated by Sarikaya et al. [25] in terms of RL fractional integrals. Their result is given by:

$$\mathsf{u}\!\left(\frac{\mathbf{c}_1+\mathbf{c}_2}{2}\right) \leq \frac{\Gamma(\nu+1)}{2(\mathbf{c}_2-\mathbf{c}_1)^{\nu}}\!\left[{}^{RL}\mathcal{I}^{\nu}_{\mathbf{c}_1+}\mathsf{u}(\mathbf{c}_2) + {}^{RL}\mathcal{I}^{\nu}_{\mathbf{c}_2-}\mathsf{u}(\mathbf{c}_1)\right] \leq \frac{\mathsf{u}(\mathbf{c}_1)+\mathsf{u}(\mathbf{c}_2)}{2}, \tag{3}$$

where $u: \mathcal{J} \to \mathcal{R}$ is assumed to be a positive convex function, continuous on the closed interval $[\mathbf{c}_1, \mathbf{c}_2]$, and for Lebesgue, almost all $x \in [\mathbf{c}_1, \mathbf{c}_2]$ when $u(x) \in L^1[\mathbf{c}_1, \mathbf{c}_2]$ with $\mathbf{c}_1 < \mathbf{c}_2$, where ${}^{RL}\mathcal{I}^{\nu}_{\mathbf{c}_1+}$ and ${}^{RL}\mathcal{I}^{\nu}_{\mathbf{c}_2-}$ are the left- and right-sided RL fractional integrals of order $\nu > 0$, defined by [12]:

$${^{RL}}\mathcal{I}^{\nu}_{\mathbf{c}_{1}+}\mathsf{u}(x) = \frac{1}{\Gamma(\nu)} \int_{\mathbf{c}_{1}}^{x} (x-\kappa)^{\nu-1} \mathsf{u}(\kappa) d\kappa, \quad x > \mathbf{c}_{1};$$

$${^{RL}}\mathcal{I}^{\nu}_{\mathbf{c}_{2}-}\mathsf{u}(x) = \frac{1}{\Gamma(\nu)} \int_{x}^{\mathbf{c}_{2}} (\kappa-x)^{\nu-1} \mathsf{u}(\kappa) d\kappa, \quad x < \mathbf{c}_{2},$$

$$(4)$$

respectively.

The inequality (3) is also known as the endpoint HH inequality due to using the ends c_1 , c_2 of the interval.

On the other hand, the endpoint HH inequality (3) has been applied for various classes of convexity such as λ_{ψ} -convexity [26], F-convexity [27], (α, m) -convexity [28], MT-convexity [29]. The reader can find other types of convexity in the literature, which in particular, is true for [30]. In the mean time, applying the end-point HH inequality to other models of fractional calculus has received a huge amount of attention. For example, this is true for RL fractional models [31], conformable fractional models [32,33], generalized fractional models [34], ψ RL fractional models [35,36], tempered fractional models [37], and AB- and Prabhakar fractional models [38].

After extending the important field of the integral inequalities in (2) and (3), a new version of the endpoint HH inequality (3) was found by Sarikaya and Yildirim [39], namely the midpoint HH inequality due to using the midpoint $\frac{c_1+c_2}{2}$ of the interval, which is given by

$$u \left(\frac{\mathbf{c}_1 + \mathbf{c}_2}{2} \right) \leq \frac{2^{\nu - 1} \Gamma(\nu + 1)}{(\mathbf{c}_2 - \mathbf{c}_1)^{\nu}} \left\lceil RL \mathcal{I}_{\left(\frac{\mathbf{c}_1 + \mathbf{c}_2}{2}\right) +}^{\nu} u(\mathbf{c}_2) + \frac{RL}{2} \mathcal{I}_{\left(\frac{\mathbf{c}_1 + \mathbf{c}_2}{2}\right) -}^{\nu} u(\mathbf{c}_1) \right\rceil \leq \frac{u(\mathbf{c}_1) + u(\mathbf{c}_2)}{2}, \quad (5)$$

where the function $u:[c_1,c_2]\to \mathcal{R}$ is convex and continuous.

Definition 1 ([40]). Let $g:[c_1,c_2]\to [0,\infty)$ be a function. Then, we say g is symmetric with respect to $(c_1+c_2)/2$ if

$$g(c_1 + c_2 - x) = g(x), \quad \forall x \in [c_1, c_2]$$
 (6)

Based on above definition, in [41], Fejér found a new extension of the HH type inequality (2), namely the HHF type inequality, and the result is as follows:

$$u\left(\frac{\mathbf{c}_{1}+\mathbf{c}_{2}}{2}\right)\int_{\mathbf{c}_{1}}^{\mathbf{c}_{2}}g(x)dx \leq \int_{\mathbf{c}_{1}}^{\mathbf{c}_{2}}u(x)g(x)dx \leq \frac{u(\mathbf{c}_{1})+u(\mathbf{c}_{2})}{2}\int_{\mathbf{c}_{1}}^{\mathbf{c}_{2}}g(x)dx,\tag{7}$$

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where g is the integrable function, and Işcan [42] found the endpoint version of (7) in the sense of RL fractional integrals, which is also the extension of (3). The result is as follows:

where u is convex and continuous and the function g belongs to $L^1[\mathbf{c}_1, \mathbf{c}_2]$ and is symmetric (see Definition 1).

It is worth mentioning that the midpoint version of (8) has not been found yet, even though many related inequalities of midpoint type were obtained in [43].

Recently, Mohammed et al. [44] found a new endpoint HHF-inequality in terms of weighted fractional integrals with positive weighted symmetric function in a kernel, and their result is as follows:

$$\begin{aligned}
\mathbf{u} \left(\frac{\mathbf{c}_{1} + \mathbf{c}_{2}}{2} \right) & \left[\left(\varrho^{-1}(\mathbf{c}_{1}) + \mathcal{I}^{\nu:\varrho}(w \circ \varrho) \right) \left(\varrho^{-1}(\mathbf{c}_{2}) \right) + \left(\mathcal{I}^{\nu:\varrho}_{\varrho^{-1}(\mathbf{c}_{2}) -}(w \circ \varrho) \right) \left(\varrho^{-1}(\mathbf{c}_{1}) \right) \right] \\
& \leq w(\mathbf{c}_{2}) \left(\varrho^{-1}(\mathbf{c}_{1}) + \mathcal{I}^{\nu:\varrho}_{w \circ \varrho}(\mathsf{u} \circ \varrho) \right) \left(\varrho^{-1}(\mathbf{c}_{2}) \right) + w(\mathbf{c}_{1}) \left(w \circ_{\varrho} \mathcal{I}^{\nu:\varrho}_{\varrho^{-1}(\mathbf{c}_{2}) -}(\mathsf{u} \circ \varrho) \right) \left(\varrho^{-1}(\mathbf{c}_{1}) \right) \\
& \leq \frac{\mathsf{u}(\mathbf{c}_{1}) + \mathsf{u}(\mathbf{c}_{2})}{2} \left[\left(\varrho^{-1}(\mathbf{c}_{1}) + \mathcal{I}^{\nu:\varrho}(w \circ \varrho) \right) \left(\varrho^{-1}(\mathbf{c}_{2}) \right) \right. \\
& + \left(\mathcal{I}^{\nu:\varrho}_{\varrho^{-1}(\mathbf{c}_{2}) -}(w \circ \varrho) \right) \left(\varrho^{-1}(\mathbf{c}_{1}) \right) \right]. \quad (9)
\end{aligned}$$

Here, u is a convex and continuous function, $\varrho(x)$ a monotone increasing function from the interval $(\mathbf{c}_1, \mathbf{c}_2)$ onto itself with a continuous derivative $\varrho'(x)$ on the open interval $(\mathbf{c}_1, \mathbf{c}_2)$, and $w : [\mathbf{c}_1, \mathbf{c}_2] \to (0, \infty)$ is an integrable function, which is symmetric with respect to $(\mathbf{c}_1 + \mathbf{c}_2)/2$, where $\mathbf{c}_1 < \mathbf{c}_2$.

Definition 2. Let $(c_1, c_2) \subseteq \mathcal{R}$ and $\varrho(x)$ be an increasing positive and monotone function on the interval $(c_1, c_2]$ with a continuous derivative $\varrho'(x)$ on the open interval (c_1, c_2) . Then, the left-sided and right-sided the weighted fractional integrals of a function u according to another function $\varrho(x)$ on $[c_1, c_2]$ are defined by [15]:

$$\begin{pmatrix} c_{1} + \mathcal{I}_{w}^{\nu;\varrho} \mathsf{u} \end{pmatrix}(x) = \frac{[w(x)]^{-1}}{\Gamma(\nu)} \int_{c_{1}}^{x} \varrho'(\kappa) (\varrho(x) - \varrho(\kappa))^{\nu - 1} \mathsf{u}(\kappa) w(\kappa) d\kappa, \\
\left({}_{w} \mathcal{I}_{c_{2} -}^{\nu;\varrho} \mathsf{u} \right)(x) = \frac{[w(x)]^{-1}}{\Gamma(\nu)} \int_{x}^{c_{2}} \varrho'(\kappa) (\varrho(\kappa) - \varrho(x))^{\nu - 1} \mathsf{u}(\kappa) w(\kappa) d\kappa, \quad \nu > 0,
\end{pmatrix} (10)$$

for $[w(x)]^{-1} := \frac{1}{w(x)}$ such that $w(x) \neq 0$.

Remark 1. From Definition 2, we can obtain the following special cases.

- If $\varrho(x) = x$ and w(x) = 1, then the weighted fractional integrals (10) reduce to the classical RL fractional integrals (4).
- If w(x) = 1, we obtain the fractional integrals of the function u with respect to the function $\varrho(x)$, which is defined by [13,14]:

$$\begin{split} & \big(c_{1} + \mathcal{I}^{\nu:\varrho} \mathsf{u}\big)(x) = \frac{1}{\Gamma(\nu)} \int_{c_{1}}^{x} \varrho'(\kappa) (\varrho(x) - \varrho(\kappa))^{\nu - 1} \mathsf{u}(\kappa) d\kappa, \\ & \Big(\mathcal{I}_{c_{2} -}^{\nu:\varrho} \mathsf{u}\Big)(x) = \frac{1}{\Gamma(\nu)} \int_{x}^{c_{2}} \varrho'(\kappa) (\varrho(\kappa) - \varrho(x))^{\nu - 1} \mathsf{u}(\kappa) d\kappa, \quad \nu > 0. \end{split} \tag{11}$$

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In this article, we will investigate the midpoint version of (9) and some related HHF inequalities by using the weighted fractional integrals (10) with positive weighted symmetric functions in the kernel.

The rest of our article is structured in the following way: In Section 2, we will prove the necessary and auxiliary lemmas, including the midpoint version of (9). In Section 3, we will prove our main results, including new midpoint fractional HHF integral inequalities with some related results. We will present some concluding remarks in Section 4.

2. Auxiliary Results

In this section, we prove analogues of the fractional HH inequalities (2)–(3) and HHF inequalities (7)–(8) for weighted fractional integral operators with positive weighted symmetric function kernels. Here, the main results are as follows: Theorem 1 (it is a generalisation of HH inequalities (2)–(3) and HHF inequality (7), and a reformulation of HHF inequality (8)) and Lemma 2 (it is a consequence of Theorem 1).

At first, we need the following lemma.

Lemma 1. Assume that $w: [c_1, c_2] \to (0, \infty)$ is an integrable function and symmetric with respect to $(c_1 + c_2)/2$, $c_1 < c_2$. Then,

(i) for each $\kappa \in [0,1]$, we have

$$w\left(\frac{\kappa}{2}c_1 + \frac{2-\kappa}{2}c_2\right) = w\left(\frac{2-\kappa}{2}c_1 + \frac{\kappa}{2}c_2\right). \tag{12}$$

(ii) For v > 0, we have

$$\begin{pmatrix}
e^{-1} \left(\frac{c_1 + c_2}{2}\right) + \mathcal{I}^{\nu:\varrho}(w \circ \varrho) \end{pmatrix} \left(\varrho^{-1}(c_2)\right) = \left(\mathcal{I}^{\nu:\varrho}_{\varrho^{-1} \left(\frac{c_1 + c_2}{2}\right) -}(w \circ \varrho)\right) \left(\varrho^{-1}(c_1)\right)$$

$$= \frac{1}{2} \left[\left(e^{-1} \left(\frac{c_1 + c_2}{2}\right) + \mathcal{I}^{\nu:\varrho}(w \circ \varrho)\right) \left(\varrho^{-1}(c_2)\right) + \left(\mathcal{I}^{\nu:\varrho}_{\varrho^{-1} \left(\frac{c_1 + c_2}{2}\right) -}(w \circ \varrho)\right) \left(\varrho^{-1}(c_1)\right) \right]. \quad (13)$$

Proof.

(i) Let $x = \frac{\kappa}{2}\mathbf{c}_1 + \frac{2-\kappa}{2}\mathbf{c}_2$. It is clear that $x \in [\mathbf{c}_1, \mathbf{c}_2]$ for each $\kappa \in [0, 1]$ and that $\mathbf{c}_1 + \mathbf{c}_2 - x = \frac{2-\kappa}{2}\mathbf{c}_1 + \frac{\kappa}{2}\mathbf{c}_2$. Then, by making use of the assumptions and Definition 1, we can obtain (12).

(ii) The symmetry property of w leads to

$$(w \circ \varrho)(\kappa) = w(\varrho(\kappa)) = w(\mathbf{c}_1 + \mathbf{c}_2 - \varrho(\kappa)), \quad \forall \kappa \in \left[\varrho^{-1}(\mathbf{c}_1), \varrho^{-1}(\mathbf{c}_2)\right].$$

From above and setting $\varrho(x) := \mathbf{c}_1 + \mathbf{c}_2 - \varrho(\kappa)$, it follows that

$$\begin{split} & \left(\varrho^{-1} \left(\frac{\mathbf{c}_1 + \mathbf{c}_2}{2} \right) + \mathcal{I}^{\nu : \varrho} (w \circ \varrho) \right) \left(\varrho^{-1} (\mathbf{c}_2) \right) \\ &= \frac{1}{\Gamma(\nu)} \int_{\varrho^{-1} \left(\frac{\mathbf{c}_1 + \mathbf{c}_2}{2} \right)}^{\varrho^{-1} (\mathbf{c}_2)} (\mathbf{c}_2 - \varrho(x))^{\nu - 1} (w \circ \varrho)(x) \varrho'(x) dx \\ &= \frac{1}{\Gamma(\nu)} \int_{\varrho^{-1} (\mathbf{c}_1)}^{\varrho^{-1} \left(\frac{\mathbf{c}_1 + \mathbf{c}_2}{2} \right)} (\varrho(\kappa) - \mathbf{c}_1)^{\nu - 1} w(\mathbf{c}_1 + \mathbf{c}_2 - \varrho(\kappa)) \varrho'(\kappa) d\kappa \end{split}$$

$$= \frac{1}{\Gamma(\nu)} \int_{\varrho^{-1}(\mathbf{c}_1)}^{\varrho^{-1}\left(\frac{\mathbf{c}_1+\mathbf{c}_2}{2}\right)} (\varrho(\kappa) - \mathbf{c}_1)^{\nu-1} (w \circ \varrho)(\kappa) \varrho'(\kappa) d\kappa$$

$$= \left(\mathcal{I}_{\varrho^{-1}\left(\frac{\mathbf{c}_1+\mathbf{c}_2}{2}\right)^{-}}^{\nu,\varrho} (w \circ \varrho)\right) \left(\varrho^{-1}(\mathbf{c}_1)\right),$$

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which completes the desired equality (13).

Remark 2. Throughout the present article, we denote $[w(x)]^{-1} = \frac{1}{w(x)}$ and $\varrho^{-1}(x)$ the inverse of the function $\varrho(x)$.

Theorem 1. Let $0 \le c_1 < c_2$, let $u : [c_1, c_2] \to \mathcal{R}$ be an L^1 convex function and $w : [c_1, c_2] \to \mathcal{R}$ be an integrable, positive and weighted symmetric function with respect to $\frac{c_1+c_2}{2}$. If, in addition, ϱ is an increasing and positive function from $[c_1, c_2)$ onto itself such that its derivative $\varrho'(x)$ is continuous on (c_1, c_2) , then for $\nu > 0$, the following inequalities are valid:

$$\begin{aligned}
\mathbf{u}\left(\frac{c_{1}+c_{2}}{2}\right) \left[\left(\varrho^{-1}\left(\frac{c_{1}+c_{2}}{2}\right)+\mathcal{I}^{\nu:\varrho}(w\circ\varrho)\right)\left(\varrho^{-1}(c_{2})\right)\right. \\
&+\left(\mathcal{I}^{\nu:\varrho}_{\varrho^{-1}\left(\frac{c_{1}+c_{2}}{2}\right)-}(w\circ\varrho)\right)\left(\varrho^{-1}(c_{1})\right)\right] \leq w(c_{2})\left(\varrho^{-1}\left(\frac{c_{1}+c_{2}}{2}\right)+\mathcal{I}^{\nu:\varrho}_{w\circ\varrho}(\mathbf{u}\circ\varrho)\right)\left(\varrho^{-1}(c_{2})\right) \\
&+w(c_{1})\left(w\circ\varrho\mathcal{I}^{\nu:\varrho}_{\varrho^{-1}\left(\frac{c_{1}+c_{2}}{2}\right)-}(\mathbf{u}\circ\varrho)\right)\left(\varrho^{-1}(c_{1})\right) \\
\leq \frac{\mathbf{u}(c_{1})+\mathbf{u}(c_{2})}{2}\left[\left(\varrho^{-1}\left(\frac{c_{1}+c_{2}}{2}\right)+\mathcal{I}^{\nu:\varrho}(w\circ\varrho)\right)\left(\varrho^{-1}(c_{2})\right) \\
&+\left(\mathcal{I}^{\nu:\varrho}_{\varrho^{-1}\left(\frac{c_{1}+c_{2}}{2}\right)-}(w\circ\varrho)\right)\left(\varrho^{-1}(c_{1})\right)\right]. \quad (14)
\end{aligned}$$

Proof. The convexity of u on $[c_1, c_2]$ gives

$$u\left(\frac{x+y}{2}\right) \le \frac{u(x) + u(y)}{2}$$
 for all $x, y \in [\mathbf{c}_1, \mathbf{c}_2]$.

So, for $x = \frac{\kappa}{2}\mathbf{c}_1 + \frac{2-\kappa}{2}\mathbf{c}_2$ and $y = \frac{2-\kappa}{2}\mathbf{c}_1 + \frac{\kappa}{2}\mathbf{c}_2$, $\kappa \in [0,1]$, it follows that

$$2\mathsf{u}\left(\frac{\mathbf{c}_1+\mathbf{c}_2}{2}\right) \le \mathsf{u}\left(\frac{\kappa}{2}\mathbf{c}_1 + \frac{2-\kappa}{2}\mathbf{c}_2\right) + \mathsf{u}\left(\frac{2-\kappa}{2}\mathbf{c}_1 + \frac{\kappa}{2}\mathbf{c}_2\right). \tag{15}$$

Multiplying both sides of (15) by $\kappa^{\nu-1}w(\frac{\kappa}{2}\mathbf{c}_1 + \frac{2-\kappa}{2}\mathbf{c}_2)$ and integrating the resulting inequality with respect to κ over [0,1],, we obtain

$$2\mathsf{u}\left(\frac{\mathbf{c}_{1}+\mathbf{c}_{2}}{2}\right)\int_{0}^{1}\kappa^{\nu-1}w\left(\frac{\kappa}{2}\mathbf{c}_{1}+\frac{2-\kappa}{2}\mathbf{c}_{2}\right)d\kappa$$

$$\leq \int_{0}^{1}\kappa^{\nu-1}\mathsf{u}\left(\frac{\kappa}{2}\mathbf{c}_{1}+\frac{2-\kappa}{2}\mathbf{c}_{2}\right)w\left(\frac{\kappa}{2}\mathbf{c}_{1}+\frac{2-\kappa}{2}\mathbf{c}_{2}\right)d\kappa$$

$$+\int_{0}^{1}\kappa^{\nu-1}\mathsf{u}\left(\frac{2-\kappa}{2}\mathbf{c}_{1}+\frac{\kappa}{2}\mathbf{c}_{2}\right)w\left(\frac{\kappa}{2}\mathbf{c}_{1}+\frac{2-\kappa}{2}\mathbf{c}_{2}\right)d\kappa. \quad (16)$$

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From the left-hand side of the inequality in (16), we use (13) to obtain

$$\begin{split} &\frac{2^{\nu-1}\Gamma(\nu)}{(\mathbf{c}_{2}-\mathbf{c}_{1})^{\nu}} \left[\left(\varrho^{-1} \left(\frac{\mathbf{c}_{1}+\mathbf{c}_{2}}{2} \right) + \mathcal{I}^{\nu:\varrho}(w \circ \varrho) \right) \left(\varrho^{-1}(\mathbf{c}_{2}) \right) + \left(\mathcal{I}^{\nu:\varrho}_{\varrho^{-1} \left(\frac{\mathbf{c}_{1}+\mathbf{c}_{2}}{2} \right) -}(w \circ \varrho) \right) \left(\varrho^{-1}(\mathbf{c}_{1}) \right) \right] \\ &= \frac{2^{\nu}\Gamma(\nu)}{(\mathbf{c}_{2}-\mathbf{c}_{1})^{\nu}} \left(\varrho^{-1} \left(\frac{\mathbf{c}_{1}+\mathbf{c}_{2}}{2} \right) + \mathcal{I}^{\nu:\varrho}(w \circ \varrho) \right) \left(\varrho^{-1}(\mathbf{c}_{2}) \right) \\ &= \frac{2^{\nu}}{(\mathbf{c}_{2}-\mathbf{c}_{1})^{\nu}} \int_{\varrho^{-1} \left(\frac{\mathbf{c}_{1}+\mathbf{c}_{2}}{2} \right)}^{\varrho^{-1}(\mathbf{c}_{2})} (\mathbf{c}_{2}-\varrho(x))^{\nu-1} (w \circ \varrho)(x) \varrho'(x) dx \\ &= \int_{\varrho^{-1} \left(\frac{\mathbf{c}_{1}+\mathbf{c}_{2}}{2} \right)}^{\varrho^{-1} \left(\frac{\mathbf{c}_{1}+\mathbf{c}_{2}}{2} \right)} \left(\frac{2(\mathbf{c}_{2}-\varrho(x))}{\mathbf{c}_{2}-\mathbf{c}_{1}} \right)^{\nu-1} (w \circ \varrho)(x) \varrho'(x) \frac{2dx}{\mathbf{c}_{2}-\mathbf{c}_{1}} \\ &= \int_{0}^{1} \kappa^{\nu-1} w \left(\frac{\kappa}{2} \mathbf{c}_{1} + \frac{2-\kappa}{2} \mathbf{c}_{2} \right) d\kappa, \quad \left[\text{denoting } \kappa := \frac{2(\mathbf{c}_{2}-\varrho(x))}{\mathbf{c}_{2}-\mathbf{c}_{1}} \right]. \end{split}$$

It follows that

$$2\mathsf{u}\left(\frac{\mathbf{c}_{1}+\mathbf{c}_{2}}{2}\right)\int_{0}^{1}\kappa^{\nu-1}w\left(\frac{\kappa}{2}\mathbf{c}_{1}+\frac{2-\kappa}{2}\mathbf{c}_{2}\right)d\kappa = \frac{2^{\nu}\Gamma(\nu)}{(\mathbf{c}_{2}-\mathbf{c}_{1})^{\nu}}\mathsf{u}\left(\frac{\mathbf{c}_{1}+\mathbf{c}_{2}}{2}\right) \times \left[\left(\varrho^{-1}\left(\frac{\mathbf{c}_{1}+\mathbf{c}_{2}}{2}\right)+\mathcal{I}^{\nu:\varrho}(w\circ\varrho)\right)\left(\varrho^{-1}(\mathbf{c}_{2})\right)+\left(\mathcal{I}^{\nu:\varrho}_{\varrho^{-1}\left(\frac{\mathbf{c}_{1}+\mathbf{c}_{2}}{2}\right)-}(w\circ\varrho)\right)\left(\varrho^{-1}(\mathbf{c}_{1})\right)\right]. \quad (17)$$

By evaluating the weighted fractional operators, we see that

$$\begin{split} &w(\mathbf{c}_2) \Biggl(_{\varrho^{-1}\left(\frac{\mathbf{c}_1+\mathbf{c}_2}{2}\right)+} \mathcal{I}_{w \circ \varrho}^{\nu;\varrho}(\mathbf{u} \circ \varrho) \Biggr) \Biggl(\varrho^{-1}(\mathbf{c}_2) \Biggr) + w(\mathbf{c}_1) \Biggl(_{w \circ \varrho} \mathcal{I}_{\varrho^{-1}\left(\frac{\mathbf{c}_1+\mathbf{c}_2}{2}\right)-}^{\nu;\varrho}(\mathbf{u} \circ \varrho) \Biggr) \Biggl(\varrho^{-1}(\mathbf{c}_1) \Biggr) \\ &= w(\mathbf{c}_2) \frac{(w \circ \varrho)^{-1}\left(\varrho^{-1}(\mathbf{c}_2)\right)}{\Gamma(\nu)} \int_{\varrho^{-1}\left(\frac{\mathbf{c}_1+\mathbf{c}_2}{2}\right)}^{\varrho^{-1}\left(\mathbf{c}_2\right)} (\mathbf{c}_2 - \varrho(x))^{\nu-1} (\mathbf{u} \circ \varrho)(x) (w \circ \varrho)(x) \varrho'(x) dx \\ &+ w(\mathbf{c}_1) \frac{(w \circ \varrho)^{-1}\left(\varrho^{-1}(\mathbf{c}_1)\right)}{\Gamma(\nu)} \int_{\varrho^{-1}\left(\mathbf{c}_1\right)}^{\varrho^{-1}\left(\frac{\mathbf{c}_1+\mathbf{c}_2}{2}\right)} (\varrho(x) - \mathbf{c}_1)^{\nu-1} (\mathbf{u} \circ \varrho)(x) (w \circ \varrho)(x) \varrho'(x) dx \\ &= \frac{(\mathbf{c}_2 - \mathbf{c}_1)^{\nu}}{2^{\nu}\Gamma(\nu)} \int_{\varrho^{-1}\left(\frac{\mathbf{c}_1+\mathbf{c}_2}{2}\right)}^{\varrho^{-1}\left(\frac{\mathbf{c}_1+\mathbf{c}_2}{2}\right)} \Biggl(\frac{2(\mathbf{c}_2 - \varrho(x))}{\mathbf{c}_2 - \mathbf{c}_1} \Biggr)^{\nu-1} (\mathbf{u} \circ \varrho)(x) (w \circ \varrho)(x) \varrho'(x) \frac{2dx}{\mathbf{c}_2 - \mathbf{c}_1} \\ &+ \frac{(\mathbf{c}_2 - \mathbf{c}_1)^{\nu}}{2^{\nu}\Gamma(\nu)} \int_{\varrho^{-1}\left(\mathbf{c}_1\right)}^{\varrho^{-1}\left(\frac{\mathbf{c}_1+\mathbf{c}_2}{2}\right)} \Biggl(\frac{2(\varrho(x) - \mathbf{c}_1)}{\mathbf{c}_2 - \mathbf{c}_1} \Biggr)^{\nu-1} (\mathbf{u} \circ \varrho)(x) (w \circ \varrho)(x) \varrho'(x) \frac{2dx}{\mathbf{c}_2 - \mathbf{c}_1}, \end{split}$$

where we used

$$\left[(w \circ \varrho)(\varrho^{-1}(y)) \right]^{-1} = \frac{1}{(w \circ \varrho)(\varrho^{-1}(y))} = \frac{1}{w(y)} \quad \text{for } y = \mathbf{c}_1, \, \mathbf{c}_2.$$
 (18)

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Setting $t_1 = \frac{2(\mathbf{c}_2 - \varrho(x))}{\mathbf{c}_2 - \mathbf{c}_1}$ and $t_2 = \frac{2(\varrho(x) - \mathbf{c}_1)}{\mathbf{c}_2 - \mathbf{c}_1}$, one can deduce that

$$\begin{split} & w(\mathbf{c}_2) \Bigg(_{\varrho^{-1}\left(\frac{\mathbf{c}_1+\mathbf{c}_2}{2}\right)+} \mathcal{I}^{\nu;\varrho}_{w \circ \varrho}(\mathbf{u} \circ \varrho) \Bigg) \Big(\varrho^{-1}(\mathbf{c}_2)\Big) + w(\mathbf{c}_1) \Bigg(_{w \circ \varrho} \mathcal{I}^{\nu;\varrho}_{\varrho^{-1}\left(\frac{\mathbf{c}_1+\mathbf{c}_2}{2}\right)-}(\mathbf{u} \circ \varrho) \Bigg) \Big(\varrho^{-1}(\mathbf{c}_1)\Big) \\ &= \frac{(\mathbf{c}_2 - \mathbf{c}_1)^{\nu}}{2^{\nu}\Gamma(\nu)} \Bigg[\int_0^1 t_1^{\nu-1} \mathbf{u} \bigg(\frac{t_1}{2}\mathbf{c}_1 + \frac{2-t_1}{2}\mathbf{c}_2\bigg) w \bigg(\frac{t_1}{2}\mathbf{c}_1 + \frac{2-t_1}{2}\mathbf{c}_2\bigg) dt_1 \\ &+ \int_0^1 t_2^{\nu-1} \mathbf{u} \bigg(\frac{2-t_2}{2}\mathbf{c}_1 + \frac{t_2}{2}\mathbf{c}_2\bigg) w \bigg(\frac{2-t_2}{2}\mathbf{c}_1 + \frac{t_2}{2}\mathbf{c}_2\bigg) dt_2 \\ &= \frac{(\mathbf{c}_2 - \mathbf{c}_1)^{\nu}}{2^{\nu}\Gamma(\nu)} \Bigg[\int_0^1 \kappa^{\nu-1} \mathbf{u} \bigg(\frac{\kappa}{2}\mathbf{c}_1 + \frac{2-\kappa}{2}\mathbf{c}_2\bigg) w \bigg(\frac{\kappa}{2}\mathbf{c}_1 + \frac{2-\kappa}{2}\mathbf{c}_2\bigg) d\kappa \\ &+ \int_0^1 \kappa^{\nu-1} \mathbf{u} \bigg(\frac{2-\kappa}{2}\mathbf{c}_1 + \frac{\kappa}{2}\mathbf{c}_2\bigg) \underbrace{w \bigg(\frac{\kappa}{2}\mathbf{c}_1 + \frac{2-\kappa}{2}\mathbf{c}_2\bigg)}_{\text{by using (12)}} d\kappa \Bigg]. \end{split}$$

It follows that

$$\begin{split} \int_0^1 \kappa^{\nu-1} \mathsf{u} \bigg(\frac{\kappa}{2} \mathbf{c}_1 + \frac{2-\kappa}{2} \mathbf{c}_2 \bigg) w \bigg(\frac{\kappa}{2} \mathbf{c}_1 + \frac{2-\kappa}{2} \mathbf{c}_2 \bigg) d\kappa \\ &+ \int_0^1 \kappa^{\nu-1} \mathsf{u} \bigg(\frac{2-\kappa}{2} \mathbf{c}_1 + \frac{\kappa}{2} \mathbf{c}_2 \bigg) w \bigg(\frac{\kappa}{2} \mathbf{c}_1 + \frac{2-\kappa}{2} \mathbf{c}_2 \bigg) d\kappa \end{split}$$

$$= \frac{2^{\nu}\Gamma(\nu)}{(\mathbf{c}_{2} - \mathbf{c}_{1})^{\nu}} \Big[w(\mathbf{c}_{2}) \left(e^{-1} \left(\frac{\mathbf{c}_{1} + \mathbf{c}_{2}}{2} \right) + \mathcal{I}_{w \circ \varrho}^{\nu : \varrho} (\mathbf{u} \circ \varrho) \right) \left(\varrho^{-1}(\mathbf{c}_{2}) \right) + w(\mathbf{c}_{1}) \left(w \circ_{\varrho} \mathcal{I}_{\varrho^{-1} \left(\frac{\mathbf{c}_{1} + \mathbf{c}_{2}}{2} \right) -}^{\nu : \varrho} (\mathbf{u} \circ \varrho) \right) \left(\varrho^{-1}(\mathbf{c}_{1}) \right) \Big]. \quad (19)$$

By making use of (17) and (19) in (16), we get

$$\begin{split} \mathbf{u} & \left(\frac{\mathbf{c}_{1} + \mathbf{c}_{2}}{2} \right) \left[\left(_{\varrho^{-1} \left(\frac{\mathbf{c}_{1} + \mathbf{c}_{2}}{2} \right) +} \mathcal{I}^{\nu : \varrho} (w \circ \varrho) \right) \left(\varrho^{-1} (\mathbf{c}_{2}) \right) \right. \\ & + \left(\mathcal{I}^{\nu : \varrho}_{\varrho^{-1} \left(\frac{\mathbf{c}_{1} + \mathbf{c}_{2}}{2} \right) -} (w \circ \varrho) \right) \left(\varrho^{-1} (\mathbf{c}_{1}) \right) \right] \leq w(\mathbf{c}_{2}) \left(_{\varrho^{-1} \left(\frac{\mathbf{c}_{1} + \mathbf{c}_{2}}{2} \right) +} \mathcal{I}^{\nu : \varrho}_{w \circ \varrho} (\mathbf{u} \circ \varrho) \right) \left(\varrho^{-1} (\mathbf{c}_{2}) \right) \\ & + w(\mathbf{c}_{1}) \left(_{w \circ \varrho} \mathcal{I}^{\nu : \varrho}_{\varrho^{-1} \left(\frac{\mathbf{c}_{1} + \mathbf{c}_{2}}{2} \right) -} (\mathbf{u} \circ \varrho) \right) \left(\varrho^{-1} (\mathbf{c}_{1}) \right). \end{split} \tag{20}$$

Thus, the proof of the first inequality of (14) is completed.

On the other hand, we can prove the second inequality of (14) by making use of the convexity of u to get

$$\mathsf{u}\left(\frac{\kappa}{2}\mathbf{c}_1 + \frac{2-\kappa}{2}\mathbf{c}_2\right) + \mathsf{u}\left(\frac{2-\kappa}{2}\mathbf{c}_1 + \frac{\kappa}{2}\mathbf{c}_2\right) \le \mathsf{u}(\mathbf{c}_1) + \mathsf{u}(\mathbf{c}_2). \tag{21}$$

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Multiplying both sides of (21) by $\kappa^{\nu-1}w\left(\frac{\kappa}{2}\mathbf{c}_1+\frac{2-\kappa}{2}\mathbf{c}_2\right)$ and integrating with respect to κ over [0,1] to get

$$\int_{0}^{1} \kappa^{\nu-1} \mathsf{u} \left(\frac{\kappa}{2} \mathbf{c}_{1} + \frac{2-\kappa}{2} \mathbf{c}_{2} \right) w \left(\frac{\kappa}{2} \mathbf{c}_{1} + \frac{2-\kappa}{2} \mathbf{c}_{2} \right) d\kappa
+ \int_{0}^{1} \kappa^{\nu-1} \mathsf{u} \left(\frac{2-\kappa}{2} \mathbf{c}_{1} + \frac{\kappa}{2} \mathbf{c}_{2} \right) w \left(\frac{\kappa}{2} \mathbf{c}_{1} + \frac{2-\kappa}{2} \mathbf{c}_{2} \right) d\kappa
\leq \left(\mathsf{u}(\mathbf{c}_{1}) + \mathsf{u}(\mathbf{c}_{2}) \right) \int_{0}^{1} \kappa^{\nu-1} w \left(\frac{\kappa}{2} \mathbf{c}_{1} + \frac{2-\kappa}{2} \mathbf{c}_{2} \right) d\kappa. \tag{22}$$

Then, by using (12) and (19) in (22), we get

$$\begin{split} w(\mathbf{c}_2) & \left(\varrho^{-1} \left(\frac{\mathbf{c}_1 + \mathbf{c}_2}{2}\right) + \mathcal{I}^{\nu;\varrho}_{w \circ \varrho}(\mathbf{u} \circ \varrho)\right) \left(\varrho^{-1}(\mathbf{c}_2)\right) \\ & + w(\mathbf{c}_1) \left(w \circ_{\varrho} \mathcal{I}^{\nu;\varrho}_{\varrho^{-1} \left(\frac{\mathbf{c}_1 + \mathbf{c}_2}{2}\right) -} (\mathbf{u} \circ \varrho)\right) \left(\varrho^{-1}(\mathbf{c}_1)\right) \\ & \leq \frac{\mathsf{u}(\mathbf{c}_1) + \mathsf{u}(\mathbf{c}_2)}{2} \left[\left(\varrho^{-1} \left(\frac{\mathbf{c}_1 + \mathbf{c}_2}{2}\right) + \mathcal{I}^{\nu;\varrho}(w \circ \varrho)\right) \left(\varrho^{-1}(\mathbf{c}_2)\right) \right. \\ & \left. + \left(\mathcal{I}^{\nu;\varrho}_{\varrho^{-1} \left(\frac{\mathbf{c}_1 + \mathbf{c}_2}{2}\right) -} (w \circ \varrho)\right) \left(\varrho^{-1}(\mathbf{c}_1)\right) \right]. \end{split}$$

This ends our proof. \Box

Remark 3. From Theorem 1, we can obtain some special cases as follows:

(i) If $\varrho(x) = x$, then inequality (14) becomes

where $_{c_1+}^{RL}\mathcal{I}_w^{\nu}$ and $_w^{RL}\mathcal{I}_{c_2-}^{\nu}$ are the left- and right-weighted RL fractional integrals, respectively, given by

$$\begin{split} & \left({_{c_1 + }^{RL}}\mathcal{I}_w^\nu \mathsf{u} \right)(x) = \frac{w^{-1}(x)}{\Gamma(\nu)} \int_{c_1}^x (x-\kappa)^{\nu-1} \mathsf{u}(\kappa) w(\kappa) d\kappa, \\ & \left({_{w}^{RL}}\mathcal{I}_{c_2 - }^\nu \mathsf{u} \right)(x) = \frac{w^{-1}(x)}{\Gamma(\nu)} \int_x^{c_2} (\kappa-x)^{\nu-1} \mathsf{u}(\kappa) w(\kappa) d\kappa, \quad \nu > 0. \end{split}$$

- (ii) If $\varrho(x) = x$ and $\nu = 1$, then inequality (14) becomes the inequality in (7).
- (iii) If $\varrho(x) = x$ and w(x) = 1, then inequality (14) becomes the inequality in (5).
- (iv) If $\varrho(x) = x$, w(x) = 1 and v = 1, then inequality (14) becomes the inequality in (2).

Lemma 2. Let $0 \le c_1 < c_2$, let $u : [c_1, c_2] \to \mathcal{R}$ be a continuous with a derivative $u' \in L^1[c_1, c_2]$ such that $u(x) = u(c_1) + \int_{c_1}^x u'(\kappa) d\kappa$ and let $w : [c_1, c_2] \to \mathcal{R}$ be an integrable, positive and weighted symmetric function with respect to $\frac{c_1+c_2}{2}$. If ϱ is a continuous increasing mapping from

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the interval $[c_1, c_2)$ onto itself with a derivative $\varrho'(x)$ which is continuous on (c_1, c_2) , then for $\nu > 0$, the following equality is valid:

$$\begin{split} \mathbf{u} & \left(\frac{c_{1} + c_{2}}{2} \right) \left[\left(e^{-1} \left(\frac{c_{1} + c_{2}}{2} \right) + \mathcal{I}^{\nu : \varrho} (w \circ \varrho) \right) \left(\varrho^{-1} (c_{2}) \right) \right. \\ & + \left(\mathcal{I}^{\nu : \varrho}_{\varrho^{-1} \left(\frac{c_{1} + c_{2}}{2} \right) -} (w \circ \varrho) \right) \left(\varrho^{-1} (c_{1}) \right) \right] \\ & - \left[w(c_{2}) \left(e^{-1} \left(\frac{c_{1} + c_{2}}{2} \right) + \mathcal{I}^{\nu : \varrho}_{w \circ \varrho} (\mathbf{u} \circ \varrho) \right) \left(\varrho^{-1} (c_{2}) \right) \right. \\ & + w(c_{1}) \left(e^{-1} \left(\frac{c_{1} + c_{2}}{2} \right) + \mathcal{I}^{\nu : \varrho}_{w \circ \varrho} (\mathbf{u} \circ \varrho) \right) \left(e^{-1} (c_{1}) \right) \right] \\ & = \frac{1}{\Gamma(\nu)} \int_{\varrho^{-1} (c_{1})}^{\varrho^{-1} \left(\frac{c_{1} + c_{2}}{2} \right)} \left[\int_{\varrho^{-1} (c_{1})}^{\kappa} \varrho'(x) (\varrho(x) - c_{1})^{\nu - 1} (w \circ \varrho)(x) dx \right] (\mathbf{u}' \circ \varrho) (\kappa) \varrho'(\kappa) d\kappa \\ & - \frac{1}{\Gamma(\nu)} \int_{\varrho^{-1} \left(\frac{c_{1} + c_{2}}{2} \right)}^{\varrho^{-1} (c_{2})} \left[\int_{\kappa}^{\varrho^{-1} (c_{2})} \varrho'(x) (c_{2} - \varrho(x))^{\nu - 1} (w \circ \varrho)(x) dx \right] (\mathbf{u}' \circ \varrho) (\kappa) \varrho'(\kappa) d\kappa. \end{split}$$
 (24)

Proof. Let us set

$$\begin{split} &\frac{1}{\Gamma(\nu)} \int_{\varrho^{-1}(\mathbf{c}_1)}^{\varrho^{-1}\left(\frac{\mathbf{c}_1+\mathbf{c}_2}{2}\right)} \left[\int_{\varrho^{-1}(\mathbf{c}_1)}^{\kappa} \varrho'(x) (\varrho(x) - \mathbf{c}_1)^{\nu-1} (w \circ \varrho)(x) dx \right] (\mathsf{u}' \circ \varrho)(\kappa) \varrho'(\kappa) d\kappa \\ &- \frac{1}{\Gamma(\nu)} \int_{\varrho^{-1}\left(\frac{\mathbf{c}_1+\mathbf{c}_2}{2}\right)}^{\varrho^{-1}(\mathbf{c}_2)} \left[\int_{\kappa}^{\varrho^{-1}(\mathbf{c}_2)} \varrho'(x) (\mathbf{c}_2 - \varrho(x))^{\nu-1} (w \circ \varrho)(x) dx \right] (\mathsf{u}' \circ \varrho)(\kappa) \varrho'(\kappa) d\kappa \\ &= \frac{1}{\Gamma(\nu)} \int_{\varrho^{-1}(\mathbf{c}_1)}^{\varrho^{-1}\left(\frac{\mathbf{c}_1+\mathbf{c}_2}{2}\right)} \left[\int_{\varrho^{-1}(\mathbf{c}_1)}^{\kappa} \varrho'(x) (\varrho(x) - \mathbf{c}_1)^{\nu-1} (w \circ \varrho)(x) dx \right] (\mathsf{u}' \circ \varrho)(\kappa) \varrho'(\kappa) d\kappa \\ &+ \frac{-1}{\Gamma(\nu)} \int_{\varrho^{-1}\left(\frac{\mathbf{c}_1+\mathbf{c}_2}{2}\right)}^{\varrho^{-1}(\mathbf{c}_2)} \left[\int_{\kappa}^{\varrho^{-1}(\mathbf{c}_2)} \varrho'(x) (\mathbf{c}_2 - \varrho(x))^{\nu-1} (w \circ \varrho)(x) dx \right] (\mathsf{u}' \circ \varrho)(\kappa) \varrho'(\kappa) d\kappa \\ &\coloneqq \Xi_1 + \Xi_2. \end{split}$$

By integrating by parts, using Lemma 1, and (10) and (11), we obtain

$$\begin{split} \Xi_1 &= \frac{1}{\Gamma(\nu)} \left(\int_{\varrho^{-1}(\mathbf{c}_1)}^{\kappa} \varrho'(x) (\varrho(x) - \mathbf{c}_1)^{\nu - 1} (w \circ \varrho)(x) dx \right) (\mathsf{u} \circ \varrho)(\kappa) d\kappa \bigg|_{\kappa = \varrho^{-1}(\mathbf{c}_1)}^{\varrho^{-1} \left(\frac{\mathbf{c}_1 + \mathbf{c}_2}{2} \right)} \\ &- \frac{1}{\Gamma(\nu)} \int_{\varrho^{-1}(\mathbf{c}_1)}^{\varrho^{-1} \left(\frac{\mathbf{c}_1 + \mathbf{c}_2}{2} \right)} \varrho'(\kappa) (\varrho(\kappa) - \mathbf{c}_1)^{\nu - 1} (w \circ \varrho)(\kappa) (\mathsf{u} \circ \varrho)(\kappa) d\kappa \\ &= \left(\frac{1}{\Gamma(\nu)} \int_{\varrho^{-1}(\mathbf{c}_1)}^{\varrho^{-1} \left(\frac{\mathbf{c}_1 + \mathbf{c}_2}{2} \right)} \varrho'(x) (\varrho(x) - \mathbf{c}_1)^{\nu - 1} (w \circ \varrho)(x) dx \right) \mathsf{u} \left(\frac{\mathbf{c}_1 + \mathbf{c}_2}{2} \right) \\ &- \underbrace{w(\mathbf{c}_1) \frac{(w \circ \varrho)^{-1} \left(\varrho^{-1}(\mathbf{c}_1)\right)}{\Gamma(\nu)}}_{\text{by using (18)}} \int_{\varrho^{-1}(\mathbf{c}_1)}^{\varrho^{-1} \left(\frac{\mathbf{c}_1 + \mathbf{c}_2}{2} \right)} \varrho'(\kappa) (\varrho(\kappa) - \mathbf{c}_1)^{\nu - 1} (w \circ \varrho)(\kappa) (\mathsf{u} \circ \varrho)(\kappa) d\kappa \end{split}$$

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$$\begin{split} &=\mathsf{u}\bigg(\frac{\mathbf{c}_1+\mathbf{c}_2}{2}\bigg)\bigg(\mathcal{I}^{\nu:\varrho}_{\varrho^{-1}\left(\frac{\mathbf{c}_1+\mathbf{c}_2}{2}\right)^{-}}(w\circ\varrho)\bigg)\bigg(\varrho^{-1}(\mathbf{c}_1)\bigg)\\ &-w(\mathbf{c}_1)\bigg({}_{w\circ\varrho}\mathcal{I}^{\nu:\varrho}_{\varrho^{-1}\left(\frac{\mathbf{c}_1+\mathbf{c}_2}{2}\right)^{-}}(\mathsf{u}\circ\varrho)\bigg)\bigg(\varrho^{-1}(\mathbf{c}_1)\bigg)\\ &=\frac{1}{2}\mathsf{u}\bigg(\frac{\mathbf{c}_1+\mathbf{c}_2}{2}\bigg)\bigg[\bigg({}_{\varrho^{-1}\left(\frac{\mathbf{c}_1+\mathbf{c}_2}{2}\right)^{+}}\mathcal{I}^{\nu:\varrho}(w\circ\varrho)\bigg)\bigg(\varrho^{-1}(\mathbf{c}_2)\bigg)\\ &+\bigg(\mathcal{I}^{\nu:\varrho}_{\varrho^{-1}\left(\frac{\mathbf{c}_1+\mathbf{c}_2}{2}\right)^{-}}(w\circ\varrho)\bigg)\bigg(\varrho^{-1}(\mathbf{c}_1)\bigg)\bigg]-w(\mathbf{c}_1)\bigg({}_{w\circ\varrho}\mathcal{I}^{\nu:\varrho}_{\varrho^{-1}\left(\frac{\mathbf{c}_1+\mathbf{c}_2}{2}\right)^{-}}(\mathsf{u}\circ\varrho)\bigg)\bigg(\varrho^{-1}(\mathbf{c}_1)\bigg). \end{split}$$

Analogously, we get

$$\begin{split} \Xi_2 &= \frac{-1}{\Gamma(\nu)} \Biggl(\int_{\kappa}^{\varrho^{-1}(\mathbf{c}_2)} \varrho'(x) (\mathbf{c}_2 - \varrho(x))^{\nu - 1} (w \circ \varrho)(x) dx \Biggr) (\mathsf{u} \circ \varrho)(\kappa) d\kappa \Biggl|_{t = \varrho^{-1}\left(\frac{\mathbf{c}_1 + \mathbf{c}_2}{2}\right)}^{\varrho^{-1}(\mathbf{c}_2)} \\ &- \frac{1}{\Gamma(\nu)} \int_{\varrho^{-1}\left(\frac{\mathbf{c}_1 + \mathbf{c}_2}{2}\right)}^{\varrho^{-1}(\mathbf{c}_2)} \varrho'(\kappa) (\mathbf{c}_2 - \varrho(\kappa))^{\nu - 1} (w \circ \varrho)(\kappa) (\mathsf{u} \circ \varrho)(\kappa) d\kappa \\ &= \Biggl(\frac{1}{\Gamma(\nu)} \int_{\varrho^{-1}\left(\frac{\mathbf{c}_1 + \mathbf{c}_2}{2}\right)}^{\varrho^{-1}(\mathbf{c}_2)} \varrho'(x) (\mathbf{c}_2 - \varrho(x))^{\nu - 1} (w \circ \varrho)(x) dx \Biggr) \mathsf{u} \Biggl(\frac{\mathbf{c}_1 + \mathbf{c}_2}{2} \Biggr) \\ &- \underbrace{w(\mathbf{c}_2) \frac{(w \circ \varrho)^{-1} \left(\varrho^{-1}(\mathbf{c}_2)\right)}{\Gamma(\nu)}}_{\text{by using (18)}} \int_{\varrho^{-1}\left(\frac{\mathbf{c}_1 + \mathbf{c}_2}{2}\right)}^{\varrho^{-1}(\mathbf{c}_2)} \varrho'(\kappa) (\mathbf{c}_2 - \varrho(\kappa))^{\nu - 1} (w \circ \varrho)(\kappa) (\mathsf{u} \circ \varrho)(\kappa) d\kappa \end{aligned}$$

$$\begin{split} &=\mathsf{u}\bigg(\frac{\mathbf{c}_1+\mathbf{c}_2}{2}\bigg) \bigg(_{\varrho^{-1}\left(\frac{\mathbf{c}_1+\mathbf{c}_2}{2}\right)+} \mathcal{I}^{\nu:\varrho}(w \circ \varrho)\bigg) \Big(\varrho^{-1}(\mathbf{c}_2)\Big) \\ &-w(\mathbf{c}_2) \bigg(_{\varrho^{-1}\left(\frac{\mathbf{c}_1+\mathbf{c}_2}{2}\right)+} \mathcal{I}^{\nu:\varrho}_{w \circ \varrho}(\mathsf{u} \circ \varrho)\bigg) \Big(\varrho^{-1}(\mathbf{c}_2)\Big) \\ &=\frac{1}{2} \mathsf{u}\bigg(\frac{\mathbf{c}_1+\mathbf{c}_2}{2}\bigg) \bigg[\bigg(_{\varrho^{-1}\left(\frac{\mathbf{c}_1+\mathbf{c}_2}{2}\right)+} \mathcal{I}^{\nu:\varrho}(w \circ \varrho)\bigg) \Big(\varrho^{-1}(\mathbf{c}_2)\Big) \\ &+ \Big(\mathcal{I}^{\nu:\varrho}_{\varrho^{-1}(\mathbf{c}_2)-}(w \circ \varrho)\Big) \Big(\varrho^{-1}(\mathbf{c}_1)\Big) \bigg] - w(\mathbf{c}_2) \bigg(_{\varrho^{-1}\left(\frac{\mathbf{c}_1+\mathbf{c}_2}{2}\right)+} \mathcal{I}^{\nu:\varrho}_{w \circ \varrho}(\mathsf{u} \circ \varrho)\bigg) \Big(\varrho^{-1}(\mathbf{c}_2)\Big). \end{split}$$

Thus, we deduce:

$$\begin{split} \Xi_1 + \Xi_2 &= \mathsf{u}\bigg(\frac{\mathbf{c}_1 + \mathbf{c}_2}{2}\bigg) \Bigg[\Bigg(_{\varrho^{-1}\left(\frac{\mathbf{c}_1 + \mathbf{c}_2}{2}\right) +} \mathcal{I}^{\nu;\varrho}(w \circ \varrho) \Bigg) \Big(\varrho^{-1}(\mathbf{c}_2)\Big) \\ &+ \Bigg(\mathcal{I}^{\nu;\varrho}_{\varrho^{-1}\left(\frac{\mathbf{c}_1 + \mathbf{c}_2}{2}\right) -} (w \circ \varrho) \Bigg) \Big(\varrho^{-1}(\mathbf{c}_1)\Big) \Bigg] - \Bigg[w(\mathbf{c}_2) \Bigg(_{\varrho^{-1}\left(\frac{\mathbf{c}_1 + \mathbf{c}_2}{2}\right) +} \mathcal{I}^{\nu;\varrho}_{w \circ \varrho}(\mathsf{u} \circ \varrho) \Bigg) \Big(\varrho^{-1}(\mathbf{c}_2)\Big) \\ &+ w(\mathbf{c}_1) \Bigg(_{w \circ \varrho} \mathcal{I}^{\nu;\varrho}_{\varrho^{-1}\left(\frac{\mathbf{c}_1 + \mathbf{c}_2}{2}\right) -} (\mathsf{u} \circ \varrho) \Bigg) \Big(\varrho^{-1}(\mathbf{c}_1)\Big) \Bigg], \end{split}$$

which completes the proof of Lemma 2. \Box

Remark 4. From Lemma 2, we can obtain some special cases as follows:

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(i) If $\varrho(x) = x$, then equality (24) becomes

where $\binom{c_1+c_2}{2}+T_w^{\nu}$ and $\binom{RL}{w}T_{\binom{c_1+c_2}{2}}^{\nu}$ are as defined in Remark 3.

(ii) If $\varrho(x) = x$ and w(x) = 1, then equality (24) becomes

$$\begin{split} \frac{2^{\nu-1}\Gamma(\nu+1)}{(c_2-c_1)^{\nu}} \left[\frac{c_1+c_2}{2} + \mathcal{I}^{\nu} \mathsf{u}(c_2) + \mathcal{I}^{L} \mathcal{I}^{\nu}_{\left(\frac{c_1+c_2}{2}\right)-} \mathsf{u}(c_1) \right] - \mathsf{u}\left(\frac{c_1+c_2}{2}\right) &= \frac{c_2-c_1}{4} \\ \times \left[\int_0^1 \kappa^{\nu} \mathsf{u}' \left(\frac{\kappa}{2} c_1 + \frac{2-\kappa}{2} c_2\right) d\kappa - \int_0^1 \kappa^{\nu} \mathsf{u}' \left(\frac{2-\kappa}{2} c_1 + \frac{\kappa}{2} c_2\right) d\kappa \right], \end{split}$$

which is already obtained in ([39] [Lemma 3]).

(iii) If $\varrho(x) = x$, w(x) = 1 and v = 1, then equality (24) becomes

$$\frac{1}{c_2 - c_1} \int_{c_1}^{c_2} \mathsf{u}(x) dx - \mathsf{u}\left(\frac{c_1 + c_2}{2}\right) = \frac{c_2 - c_1}{4} \left[\int_0^1 \kappa \mathsf{u}'\left(\frac{\kappa}{2}c_1 + \frac{2 - \kappa}{2}c_2\right) d\kappa - \int_0^1 \kappa \mathsf{u}'\left(\frac{2 - \kappa}{2}c_1 + \frac{\kappa}{2}c_2\right) d\kappa \right], \quad (26)$$

which is already obtained in ([39] [Corollary 1]).

3. Main Results

By the help of Lemma 2, we can deduce the following HHF inequalities.

Theorem 2. Let $0 \le c_1 < c_2$, let $u : [c_1, c_2] \subseteq [0, \infty) \to \mathcal{R}$ be a (continuously) differentiable function on the interval $[c_1, c_2]$ such that $u(x) = u(c_1) + \int_{c_1}^x u'(\kappa) d\kappa$, and let $w : [c_1, c_2] \to \mathcal{R}$ be an integrable, positive and weighted symmetric function with respect to $\frac{c_1+c_2}{2}$. If, in addition, |u'| is convex on $[c_1, c_2]$, and ϱ is an increasing and positive function from $[c_1, c_2)$ onto itself such that its derivative $\varrho'(x)$ is continuous on (c_1, c_2) , then for v > 0 the following inequalities are valid:

$$\begin{split} |\Xi_1+\Xi_2| &= \left|\frac{1}{\Gamma(\nu)}\int_{\varrho^{-1}(c_1)}^{\varrho^{-1}\left(\frac{c_1+c_2}{2}\right)} \left[\int_{\varrho^{-1}(c_1)}^{\kappa} \varrho'(x)(\varrho(x)-c_1)^{\nu-1}(w\circ\varrho)(x)dx\right] \\ &\qquad \qquad \times (\mathsf{u}'\circ\varrho)(\kappa)\varrho'(\kappa)d\kappa \\ &-\frac{1}{\Gamma(\nu)}\int_{\varrho^{-1}\left(\frac{c_1+c_2}{2}\right)}^{\varrho^{-1}(c_2)} \left[\int_{\kappa}^{\varrho^{-1}(c_2)} \varrho'(x)(c_2-\varrho(x))^{\nu-1}(w\circ\varrho)(x)dx\right] (\mathsf{u}'\circ\varrho)(\kappa)\varrho'(\kappa)d\kappa \end{split}$$

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$$\leq \frac{(c_{2}-c_{1})^{\nu+1}}{2^{\nu+2}\Gamma(\nu+3)} \Big\{ \|w\|_{\left[c_{1},\frac{c_{1}+c_{2}}{2}\right],\infty} \left[(\nu+3)|u'(c_{1})| + (\nu+1)|u'(c_{2})| \right] \\
+ \|w\|_{\left[\frac{c_{1}+c_{2}}{2},c_{2}\right],\infty} \left[(\nu+1)|u'(c_{1})| + (\nu+3)|u'(c_{2})| \right] \Big\} \\
\leq \frac{(c_{2}-c_{1})^{\nu+1}\|w\|_{\left[c_{1},c_{2}\right],\infty}}{2^{\nu+1}\Gamma(\nu+2)} \left[|u'(c_{1})| + |u'(c_{2})| \right]. \tag{27}$$

Proof. By making use of Lemma 2 and properties of the modulus, we obtain

$$\begin{split} &|\Xi_{1}+\Xi_{2}|\\ &=\left|\frac{1}{\Gamma(\nu)}\int_{\varrho^{-1}(\mathbf{c}_{1})}^{\varrho^{-1}\left(\frac{\mathbf{c}_{1}+\mathbf{c}_{2}}{2}\right)}\left[\int_{\varrho^{-1}(\mathbf{c}_{1})}^{\kappa}\varrho'(x)(\varrho(x)-\mathbf{c}_{1})^{\nu-1}(w\circ\varrho)(x)dx\right](\mathsf{u}'\circ\varrho)(\kappa)\varrho'(\kappa)d\kappa\\ &-\frac{1}{\Gamma(\nu)}\int_{\varrho^{-1}\left(\frac{\mathbf{c}_{1}+\mathbf{c}_{2}}{2}\right)}^{\varrho^{-1}(\mathbf{c}_{2})}\left[\int_{\kappa}^{\varrho^{-1}(\mathbf{c}_{2})}\varrho'(x)(\mathbf{c}_{2}-\varrho(x))^{\nu-1}(w\circ\varrho)(x)dx\right](\mathsf{u}'\circ\varrho)(\kappa)\varrho'(\kappa)d\kappa\\ &\leq\frac{1}{\Gamma(\nu)}\int_{\varrho^{-1}(\mathbf{c}_{1})}^{\varrho^{-1}\left(\frac{\mathbf{c}_{1}+\mathbf{c}_{2}}{2}\right)}\left|\int_{\varrho^{-1}(\mathbf{c}_{1})}^{\kappa}\varrho'(x)(\varrho(x)-\mathbf{c}_{1})^{\nu-1}(w\circ\varrho)(x)dx\right||(\mathsf{u}'\circ\varrho)(\kappa)|\varrho'(\kappa)d\kappa\\ &+\frac{1}{\Gamma(\nu)}\int_{\varrho^{-1}\left(\frac{\mathbf{c}_{1}+\mathbf{c}_{2}}{2}\right)}^{\varrho^{-1}(\mathbf{c}_{2})}\left|\int_{\kappa}^{\varrho^{-1}(\mathbf{c}_{2})}\varrho'(x)(\mathbf{c}_{2}-\varrho(x))^{\nu-1}(w\circ\varrho)(x)dx\right|\\ &\times|(\mathsf{u}'\circ\varrho)(\kappa)|\varrho'(\kappa)d\kappa. \quad (28) \end{split}$$

Since |u'| is convex on $[c_1, c_2]$, we get for $\kappa \in [\varrho^{-1}(c_1), \varrho^{-1}(c_2)]$:

$$\begin{aligned} \left| (\mathbf{u}' \circ \varrho)(\kappa) \right| &= \left| \mathbf{u}' \left(\frac{\mathbf{c}_2 - \varrho(\kappa)}{\mathbf{c}_2 - \mathbf{c}_1} \mathbf{c}_1 + \frac{\varrho(\kappa) - \mathbf{c}_1}{\mathbf{c}_2 - \mathbf{c}_1} \mathbf{c}_2 \right) \right| \\ &\leq \frac{\mathbf{c}_2 - \varrho(\kappa)}{\mathbf{c}_2 - \mathbf{c}_1} |\mathbf{u}'(\mathbf{c}_1)| + \frac{\varrho(\kappa) - \mathbf{c}_1}{\mathbf{c}_2 - \mathbf{c}_1} |\mathbf{u}'(\mathbf{c}_2)|. \end{aligned} (29)$$

Hence, we obtain

$$\begin{aligned} |\Xi_{1} + \Xi_{2}| &\leq \frac{\|w\| \left[\mathbf{c}_{1}, \frac{\mathbf{c}_{1} + \mathbf{c}_{2}}{2} \right], \infty}{(\mathbf{c}_{2} - \mathbf{c}_{1})\Gamma(\nu)} \int_{\varrho^{-1}(\mathbf{c}_{1})}^{\varrho^{-1}\left(\frac{\mathbf{c}_{1} + \mathbf{c}_{2}}{2}\right)} \left| \int_{\varrho^{-1}(\mathbf{c}_{1})}^{\kappa} \varrho'(x)(\varrho(x) - \mathbf{c}_{1})^{\nu-1} dx \right| \\ &\times \left[(\mathbf{c}_{2} - \varrho(\kappa)) |\mathbf{u}'(\mathbf{c}_{1})| + (\varrho(\kappa) - \mathbf{c}_{1}) |\mathbf{u}'(\mathbf{c}_{2})| \right] \varrho'(\kappa) d\kappa \\ &+ \frac{\|w\| \left[\frac{\mathbf{c}_{1} + \mathbf{c}_{2}}{2}, \mathbf{c}_{2} \right], \infty}{(\mathbf{c}_{2} - \mathbf{c}_{1})\Gamma(\nu)} \int_{\varrho^{-1}\left(\frac{\mathbf{c}_{1} + \mathbf{c}_{2}}{2}\right)}^{\varrho^{-1}(\mathbf{c}_{2})} \left| \int_{\kappa}^{\varrho^{-1}(\mathbf{c}_{2})} \varrho'(x)(\mathbf{c}_{2} - \varrho(x))^{\nu-1} dx \right| \\ &\times \left[(\mathbf{c}_{2} - \varrho(\kappa)) |\mathbf{u}'(\mathbf{c}_{1})| + (\varrho(\kappa) - \mathbf{c}_{1}) |\mathbf{u}'(\mathbf{c}_{2})| \right] \varrho'(\kappa) d\kappa \\ &= \frac{(\mathbf{c}_{2} - \mathbf{c}_{1})^{\nu+1}}{2^{\nu+2}\Gamma(\nu+3)} \left\{ \|w\| \left[\mathbf{c}_{1}, \frac{\mathbf{c}_{1} + \mathbf{c}_{2}}{2} \right], \infty \left[(\nu+3) |\mathbf{u}'(\mathbf{c}_{1})| + (\nu+1) |\mathbf{u}'(\mathbf{c}_{2})| \right] \right\} \\ &\leq \frac{(\mathbf{c}_{2} - \mathbf{c}_{1})^{\nu+1} \|w\| \left[\mathbf{c}_{1}, \mathbf{c}_{2} \right], \infty}{2^{\nu+1}\Gamma(\nu+2)} \left[|\mathbf{u}'(\mathbf{c}_{1})| + |\mathbf{u}'(\mathbf{c}_{2})| \right], \quad (30) \end{aligned}$$

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where

$$\int_{\varrho^{-1}(\mathbf{c}_{1})}^{\kappa} \varrho'(x) (\varrho(x) - \mathbf{c}_{1})^{\nu - 1} dx = \frac{(\varrho(\kappa) - \mathbf{c}_{1})^{\nu}}{\nu};$$

$$\int_{\kappa}^{\varrho^{-1}(\mathbf{c}_{2})} \varrho'(x) (\mathbf{c}_{2} - \varrho(x))^{\nu - 1} dx = \frac{(\mathbf{c}_{2} - \varrho(\kappa))^{\nu}}{\nu};$$

$$\int_{\varrho^{-1}\left(\frac{\mathbf{c}_{1}+\mathbf{c}_{2}}{2}\right)}^{\varrho^{-1}\left(\frac{\mathbf{c}_{1}+\mathbf{c}_{2}}{2}\right)}(\varrho(\kappa)-\mathbf{c}_{1})^{\nu+1}\varrho'(\kappa)d\kappa = \int_{\varrho^{-1}\left(\frac{\mathbf{c}_{1}+\mathbf{c}_{2}}{2}\right)}^{\varrho^{-1}\left(\mathbf{c}_{2}\right)}(\mathbf{c}_{2}-\varrho(\kappa))^{\nu+1}\varrho'(\kappa)d\kappa = \frac{(\mathbf{c}_{2}-\mathbf{c}_{1})^{\nu+2}}{2^{\nu+2}(\nu+2)};$$

$$\begin{split} \int_{\varrho^{-1}\left(\mathbf{c}_{1}\right)}^{\varrho^{-1}\left(\frac{\mathbf{c}_{1}+\mathbf{c}_{2}}{2}\right)} (\varrho(\kappa)-\mathbf{c}_{1})^{\nu}(\mathbf{c}_{2}-\varrho(\kappa))\varrho'(\kappa)d\kappa \\ &= \int_{\varrho^{-1}\left(\frac{\mathbf{c}_{1}+\mathbf{c}_{2}}{2}\right)}^{\varrho^{-1}\left(\mathbf{c}_{2}\right)} (\mathbf{c}_{2}-\varrho(\kappa))^{\nu}(\varrho(\kappa)-\mathbf{c}_{1})\varrho'(\kappa)d\kappa = \frac{(\mathbf{c}_{2}-\mathbf{c}_{1})^{\nu+2}(\nu+3)}{2^{\nu+2}(\nu+1)(\nu+2)}. \end{split}$$

This completes our proof. \Box

Remark 5. From Theorem 2, we can obtain some special cases as follows:

(i) If $\varrho(x) = x$, then inequality (27) becomes

$$\left| \mathsf{u} \left(\frac{c_{1} + c_{2}}{2} \right) \left[\left(\frac{c_{1} + c_{2}}{2} \right)^{+} \mathcal{I}^{\nu} w(c_{2}) + \mathcal{I}^{L} \mathcal{I}^{\nu}_{\left(\frac{c_{1} + c_{2}}{2} \right) -} w(c_{1}) \right] \right. \\
\left. - \left[w(c_{2}) \left(\left(\frac{c_{1} + c_{2}}{2} \right)^{+} \mathcal{I}^{\nu}_{w} \mathsf{u} \right) (c_{2}) + w(c_{1}) \left(\mathcal{I}^{L}_{w} \mathcal{I}^{\nu}_{\left(\frac{c_{1} + c_{2}}{2} \right) -} \mathsf{u} \right) (c_{1}) \right] \right| \\
\leq \frac{(c_{2} - c_{1})^{\nu + 1}}{2^{\nu + 2} \Gamma(\nu + 3)} \left\{ \left\| w \right\|_{\left[c_{1}, \frac{c_{1} + c_{2}}{2}\right], \infty} \left[(\nu + 3) | \mathsf{u}'(c_{1})| + (\nu + 1) | \mathsf{u}'(c_{2})| \right] \right. \\
+ \left\| w \right\|_{\left[\frac{c_{1} + c_{2}}{2}, c_{2} \right], \infty} \left[(\nu + 1) | \mathsf{u}'(c_{1})| + (\nu + 3) | \mathsf{u}'(c_{2})| \right] \right\} \\
\leq \frac{(c_{2} - c_{1})^{\nu + 1} \| w \|_{\left[c_{1}, c_{2}\right], \infty}}{2^{\nu + 1} \Gamma(\nu + 2)} \left[\left| \mathsf{u}'(c_{1}) \right| + \left| \mathsf{u}'(c_{2}) \right| \right]. \tag{31}$$

(ii) If $\varrho(x) = x$ and w(x) = 1, then inequality (27) becomes

$$\left| \frac{2^{\nu-1}\Gamma(\nu+1)}{(c_{2}-c_{1})^{\nu}} \left[\frac{c_{1}+c_{2}}{2} + \mathcal{I}^{\nu} \mathsf{u}(c_{2}) + \mathcal{I}^{RL} \mathcal{I}^{\nu}_{\left(\frac{c_{1}+c_{2}}{2}\right)-} \mathsf{u}(c_{1}) \right] - \mathsf{u}\left(\frac{c_{1}+c_{2}}{2}\right) \right| \\
\leq \frac{(c_{2}-c_{1})^{\nu+1}}{2^{\nu+2}\Gamma(\nu+3)} \left\{ \left[(\nu+3)|\mathsf{u}'(c_{1})| + (\nu+1)|\mathsf{u}'(c_{2})| \right] \\
+ \left[(\nu+1)|\mathsf{u}'(c_{1})| + (\nu+3)|\mathsf{u}'(c_{2})| \right] \right\} \leq \frac{(c_{2}-c_{1})^{\nu+1}}{2^{\nu+1}\Gamma(\nu+2)} \left[|\mathsf{u}'(c_{1})| + |\mathsf{u}'(c_{2})| \right], \quad (32)$$

which is already obtained in ([39] [Theorem 5]).

(iii) If $\varrho(x) = x$, w(x) = 1 and v = 1, then inequality (27) becomes

$$\left| \frac{1}{c_2 - c_1} \int_{c_1}^{c_2} \mathsf{u}(x) dx - \mathsf{u}\left(\frac{c_1 + c_2}{2}\right) \right| \le \frac{c_2 - c_1}{8} \left[|\mathsf{u}'(c_1)| + |\mathsf{u}'(c_2)| \right], \tag{33}$$

which is already obtained in ([45] [Theorem 2.2]).

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Theorem 3. Let $0 \le c_1 < c_2$, let $u : [c_1, c_2] \subseteq [0, \infty) \to \mathcal{R}$ be a (continuously) differentiable function on the interval $[c_1, c_2]$ such that $u(x) = u(c_1) + \int_{c_1}^x u'(\kappa) d\kappa$, and let $w : [c_1, c_2] \to \mathcal{R}$ be an integrable, positive and weighted symmetric function with respect to $\frac{c_1+c_2}{2}$. If, in addition, $|u'|^q$ is convex on $[c_1, c_2]$ with $q \ge 1$, and ϱ is an increasing and positive function from $[c_1, c_2)$ onto itself such that its derivative $\varrho'(x)$ is continuous on (c_1, c_2) , then for v > 0, the following inequalities are valid:

$$\begin{split} |\Xi_{1} + \Xi_{2}| &\leq \frac{(c_{2} - c_{1})^{\nu+1}}{2^{\nu+1+\frac{1}{q}}(\nu+2)^{\frac{1}{q}}\Gamma(\nu+2)} \\ & \times \left\{ \|w\|_{\left[c_{1}, \frac{c_{1}+c_{2}}{2}\right], \infty} \left[(\nu+3)|u'(c_{1})|^{q} + (\nu+1)|u'(c_{2})|^{q} \right]^{\frac{1}{q}} \right. \\ & + \|w\|_{\left[\frac{c_{1}+c_{2}}{2}, c_{2}\right], \infty} \left[(\nu+1)|u'(c_{1})|^{q} + (\nu+3)|u'(c_{2})|^{q} \right]^{\frac{1}{q}} \right\} \\ & \leq \frac{(c_{2} - c_{1})^{\nu+1} \|w\|_{\left[c_{1}, c_{2}\right], \infty}}{2^{\nu+1+\frac{1}{q}}(\nu+2)^{\frac{1}{q}}\Gamma(\nu+2)} \left\{ \left[(\nu+3)|u'(c_{1})|^{q} + (\nu+1)|u'(c_{2})|^{q} \right]^{\frac{1}{q}} + \left[(\nu+1)|u'(c_{1})|^{q} + (\nu+3)|u'(c_{2})|^{q} \right]^{\frac{1}{q}} \right\}. \quad (34) \end{split}$$

Proof. Since $|\mathbf{u}'|^q$ is convex on $[\mathbf{c}_1, \mathbf{c}_2]$, we get for $\kappa \in [\varrho^{-1}(\mathbf{c}_1), \varrho^{-1}(\mathbf{c}_2)]$:

$$\begin{aligned} \left| (\mathbf{u}' \circ \varrho)(\kappa) \right|^{q} &= \left| \mathbf{u}' \left(\frac{\mathbf{c}_{2} - \varrho(\kappa)}{\mathbf{c}_{2} - \mathbf{c}_{1}} \mathbf{c}_{1} + \frac{\varrho(\kappa) - \mathbf{c}_{1}}{\mathbf{c}_{2} - \mathbf{c}_{1}} \mathbf{c}_{2} \right) \right|^{q} \\ &\leq \frac{\mathbf{c}_{2} - \varrho(\kappa)}{\mathbf{c}_{2} - \mathbf{c}_{1}} |\mathbf{u}'(\mathbf{c}_{1})|^{q} + \frac{\varrho(\kappa) - \mathbf{c}_{1}}{\mathbf{c}_{2} - \mathbf{c}_{1}} |\mathbf{u}'(\mathbf{c}_{2})|^{q}. \end{aligned} (35)$$

By making use of Lemma 2, power mean inequality and convexity of $|u'|^q$, we get

$$\begin{split} &|\Xi_{1}+\Xi_{2}|\\ &\leq \frac{1}{\Gamma(\nu)}\int_{\varrho^{-1}(\mathbf{c}_{1})}^{\varrho^{-1}\left(\frac{\mathbf{c}_{1}+\mathbf{c}_{2}}{2}\right)} \bigg| \int_{\varrho^{-1}(\mathbf{c}_{1})}^{\kappa} \varrho'(x)(\varrho(x)-\mathbf{c}_{1})^{\nu-1}(w\circ\varrho)(x)dx \bigg| |(\mathbf{u}'\circ\varrho)(\kappa)|\varrho'(\kappa)d\kappa \\ &+\frac{1}{\Gamma(\nu)}\int_{\varrho^{-1}\left(\frac{\mathbf{c}_{1}+\mathbf{c}_{2}}{2}\right)}^{\varrho^{-1}(\mathbf{c}_{2})} \bigg| \int_{\kappa}^{\varrho^{-1}(\mathbf{c}_{2})} \varrho'(x)(\mathbf{c}_{2}-\varrho(x))^{\nu-1}(w\circ\varrho)(x)dx \bigg| |(\mathbf{u}'\circ\varrho)(\kappa)|\varrho'(\kappa)d\kappa \\ &\leq \frac{1}{\Gamma(\nu)}\left(\int_{\varrho^{-1}(\mathbf{c}_{1})}^{\varrho^{-1}\left(\frac{\mathbf{c}_{1}+\mathbf{c}_{2}}{2}\right)} \bigg| \int_{\varrho^{-1}(\mathbf{c}_{1})}^{\kappa} \varrho'(x)(\varrho(x)-\mathbf{c}_{1})^{\nu-1}(w\circ\varrho)(x)dx \bigg| \varrho'(\kappa)d\kappa \right)^{1-\frac{1}{q}} \\ &\times \left(\int_{\varrho^{-1}\left(\mathbf{c}_{1}\right)}^{\varrho^{-1}\left(\mathbf{c}_{2}\right)} \bigg| \int_{\varrho^{-1}(\mathbf{c}_{1})}^{\kappa} \varrho'(x)(\varrho(x)-\mathbf{c}_{1})^{\nu-1}(w\circ\varrho)(x)dx \bigg| |(\mathbf{u}'\circ\varrho)(\kappa)|^{q}\varrho'(\kappa)d\kappa \right)^{\frac{1}{q}} \\ &+\frac{1}{\Gamma(\nu)}\left(\int_{\varrho^{-1}\left(\frac{\mathbf{c}_{1}+\mathbf{c}_{2}}{2}\right)}^{\varrho^{-1}(\mathbf{c}_{2})} \bigg| \int_{\kappa}^{\varrho^{-1}(\mathbf{c}_{2})} \varrho'(x)(\mathbf{c}_{2}-\varrho(x))^{\nu-1}(w\circ\varrho)(x)dx \bigg| \varrho'(\kappa)d\kappa \right)^{1-\frac{1}{q}} \\ &\times \left(\int_{\varrho^{-1}\left(\frac{\mathbf{c}_{1}+\mathbf{c}_{2}}{2}\right)}^{\varrho^{-1}(\mathbf{c}_{2})} \bigg| \int_{\kappa}^{\varrho^{-1}(\mathbf{c}_{2})} \varrho'(x)(\mathbf{c}_{2}-\varrho(x))^{\nu-1}(w\circ\varrho)(x)dx \bigg| |(\mathbf{u}'\circ\varrho)(\kappa)|^{q}\varrho'(\kappa)d\kappa \right)^{1-\frac{1}{q}} \\ &\times \left(\int_{\varrho^{-1}\left(\frac{\mathbf{c}_{1}+\mathbf{c}_{2}}{2}\right)} \bigg| \int_{\kappa}^{\varrho^{-1}(\mathbf{c}_{2})} \varrho'(x)(\mathbf{c}_{2}-\varrho(x))^{\nu-1}(w\circ\varrho)(x)dx \bigg| |(\mathbf{u}'\circ\varrho)(\kappa)|^{q}\varrho'(\kappa)d\kappa \right)^{1-\frac{1}{q}} \end{aligned}$$

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$$\leq \frac{\|w\|_{\left[c_{1},\frac{c_{1}+c_{2}}{2}\right],\infty}}{\Gamma(\nu)} \\ \times \left(\int_{\varrho^{-1}(c_{1})}^{\varrho^{-1}(c_{1})} \int_{\varrho^{-1}(c_{1})}^{\kappa} \varrho'(x)(\varrho(x)-c_{1})^{\nu-1}dx \Big| \varrho'(\kappa)d\kappa\right)^{1-\frac{1}{q}} \\ \times \left(\int_{\varrho^{-1}(c_{1})}^{\varrho^{-1}(c_{2})} \int_{\varrho^{-1}(c_{1})}^{\kappa} \varrho'(x)(\varrho(x)-c_{1})^{\nu-1}dx \Big| |(u'\circ\varrho)(\kappa)|^{q}\varrho'(\kappa)d\kappa\right)^{\frac{1}{q}} \\ \times \left(\int_{\varrho^{-1}(\frac{c_{1}+c_{2}}{2})}^{\varrho^{-1}(c_{2})} \int_{\kappa}^{\kappa} \varrho'(x)(c_{2}-\varrho(x))^{\nu-1}dx \Big| |(u'\circ\varrho)(\kappa)|^{q}\varrho'(\kappa)d\kappa\right)^{\frac{1}{q}} \\ \times \left(\int_{\varrho^{-1}(\frac{c_{1}+c_{2}}{2})}^{\varrho^{-1}(c_{2})} \int_{\kappa}^{\kappa} \varrho'(x)(c_{2}-\varrho(x))^{\nu-1}dx \Big| |(u'\circ\varrho)(\kappa)|^{q}\varrho'(\kappa)d\kappa\right)^{\frac{1}{q}} \\ \times \left(\int_{\varrho^{-1}(\frac{c_{1}+c_{2}}{2})}^{\varrho^{-1}(c_{2})} \int_{\kappa}^{\kappa} \varrho'(x)(c_{2}-\varrho(x))^{\nu-1}dx \Big| |(u'\circ\varrho)(\kappa)|^{q}\varrho'(\kappa)d\kappa\right)^{\frac{1}{q}} \\ \leq \frac{\|w\|_{\left[c_{1},\frac{c_{1}+c_{2}}{2}\right],\infty}}{\Gamma(\nu)} \left(\int_{\varrho^{-1}(c_{1})}^{\varrho^{-1}(c_{2})} \int_{\kappa}^{\kappa} \varrho'(x)(\varrho(x)-c_{1})^{\nu-1}dx \Big| |(u'\circ\varrho)(\kappa)|^{q}\varrho'(\kappa)d\kappa\right)^{\frac{1}{q}} \\ \times \left[\int_{\varrho^{-1}(c_{1})}^{\varrho^{-1}(\frac{c_{1}+c_{2}}{2})} \int_{\varrho^{-1}(c_{1})}^{\kappa} \varrho'(x)(\varrho(x)-c_{1})^{\nu-1}dx \Big| |(u'\circ\varrho)(\kappa)|^{\frac{1}{q}}\varrho'(\kappa)d\kappa\right)^{\frac{1}{q}} \\ \times \left[\int_{\varrho^{-1}(c_{1})}^{\varrho^{-1}(c_{2})} \left(\int_{\varrho^{-1}(c_{1}+c_{2})}^{\varrho^{-1}(c_{2})} \left(\int_{\varrho^{-1}(c_{1}+c_{2})}^{\varrho^{-1}(c_{2})} \left(\int_{\varrho^{-1}(c_{1}+c_{2})}^{\varrho^{-1}(c_{2})} \left(\int_{\varrho^{-1}(c_{1}+c_{2})}^{\varrho^{-1}(c_{2})} \left(\int_{\varrho^{-1}(c_{2})}^{\varrho^{-1}(c_{2})} \left(\int_$$

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$$\leq \frac{(\mathbf{c}_{2} - \mathbf{c}_{1})^{\nu+1} \|w\|_{\left[\mathbf{c}_{1}, \mathbf{c}_{2}\right], \infty}}{2^{\nu+1+\frac{1}{q}} (\nu+2)^{\frac{1}{q}} \Gamma(\nu+2)} \left\{ \left[(\nu+3) |\mathbf{u}'(\mathbf{c}_{1})|^{q} + (\nu+1) |\mathbf{u}'(\mathbf{c}_{2})|^{q} \right]^{\frac{1}{q}} + \left[(\nu+1) |\mathbf{u}'(\mathbf{c}_{1})|^{q} + (\nu+3) |\mathbf{u}'(\mathbf{c}_{2})|^{q} \right]^{\frac{1}{q}} \right\}, \quad (36)$$

where it is easily seen that

$$\begin{split} \int_{\varrho^{-1}\left(\mathbf{c}_{1}\right)}^{\varrho^{-1}\left(\frac{\mathbf{c}_{1}+\mathbf{c}_{2}}{2}\right)} \left| \int_{\varrho^{-1}\left(\mathbf{c}_{1}\right)}^{\kappa} \varrho'(x)(\varrho(x)-\mathbf{c}_{1})^{\nu-1}dx \right| \varrho'(\kappa)d\kappa \\ &= \int_{\varrho^{-1}\left(\frac{\mathbf{c}_{1}+\mathbf{c}_{2}}{2}\right)}^{\varrho^{-1}\left(\mathbf{c}_{2}\right)} \left| \int_{\kappa}^{\varrho^{-1}\left(\mathbf{c}_{2}\right)} \varrho'(x)(\mathbf{c}_{2}-\varrho(x))^{\nu-1}dx \right| \varrho'(\kappa)d\kappa = \frac{(\mathbf{c}_{2}-\mathbf{c}_{1})^{\nu+1}}{2^{\nu+1}\nu(\nu+1)}. \end{split}$$

Hence, the proof is completed. \Box

Remark 6. From Theorem 3, we can obtain some special cases as follows:

(i) If $\varrho(x) = x$, then inequality (34) becomes

$$\leq \frac{(c_{2}-c_{1})^{\nu+1}}{2^{\nu+1+\frac{1}{q}}(\nu+2)^{\frac{1}{q}}\Gamma(\nu+2)} \times \left\{ \|w\|_{\left[c_{1},\frac{c_{1}+c_{2}}{2}\right],\infty}\left[(\nu+3)|u'(c_{1})|^{q}+(\nu+1)|u'(c_{2})|^{q}\right]^{\frac{1}{q}} + \|w\|_{\left[\frac{c_{1}+c_{2}}{2},c_{2}\right],\infty}\left[(\nu+1)|u'(c_{1})|^{q}+(\nu+3)|u'(c_{2})|^{q}\right]^{\frac{1}{q}} \right\} \\
\leq \frac{(c_{2}-c_{1})^{\nu+1}\|w\|_{\left[c_{1},c_{2}\right],\infty}}{2^{\nu+1+\frac{1}{q}}(\nu+2)^{\frac{1}{q}}\Gamma(\nu+2)} \left\{ \left[(\nu+3)|u'(c_{1})|^{q}+(\nu+1)|u'(c_{2})|^{q}\right]^{\frac{1}{q}} + \left[(\nu+1)|u'(c_{1})|^{q}+(\nu+3)|u'(c_{2})|^{q}\right]^{\frac{1}{q}} \right\}. \quad (37)$$

(ii) If $\varrho(x) = x$ and w(x) = 1, then inequality (34) becomes

$$\left| \frac{2^{\nu-1}\Gamma(\nu+1)}{(c_{2}-c_{1})^{\nu}} \left[\frac{c_{1}+c_{2}}{2} + \mathcal{I}^{L}\mathcal{I}^{\nu} \mathsf{u}(c_{2}) + \mathcal{I}^{L}\mathcal{I}^{\nu} \left(\frac{c_{1}+c_{2}}{2} \right) - \mathsf{u}(c_{1}) \right] - \mathsf{u}\left(\frac{c_{1}+c_{2}}{2} \right) \right| \\
\leq \frac{(c_{2}-c_{1})^{\nu+1}}{2^{\nu+1+\frac{1}{q}}(\nu+2)^{\frac{1}{q}}\Gamma(\nu+2)} \left\{ \left[(\nu+3)|\mathsf{u}'(c_{1})|^{q} + (\nu+1)|\mathsf{u}'(c_{2})|^{q} \right]^{\frac{1}{q}} + \left[(\nu+1)|\mathsf{u}'(c_{1})|^{q} + (\nu+3)|\mathsf{u}'(c_{2})|^{q} \right]^{\frac{1}{q}} \right\}, \quad (38)$$

which is already obtained in ([39] [Theorem 5]).

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(iii) If $\varrho(x) = x$, w(x) = 1 and $\nu = 1$, then inequality (34) becomes

$$\left| \frac{1}{c_{2} - c_{1}} \int_{c_{1}}^{c_{2}} \mathsf{u}(x) dx - \mathsf{u}\left(\frac{c_{1} + c_{2}}{2}\right) \right| \\
\leq \frac{c_{2} - c_{1}}{8\sqrt[q]{3}} \left\{ \left[|\mathsf{u}'(c_{1})|^{q} + 2|\mathsf{u}'(c_{2})|^{q} \right]^{\frac{1}{q}} + \left[2|\mathsf{u}'(c_{1})|^{q} + |\mathsf{u}'(c_{2})|^{q} \right]^{\frac{1}{q}} \right\}. \tag{39}$$

Theorem 4. Let $0 \le c_1 < c_2$, let $u: [c_1, c_2] \subseteq [0, \infty) \to \mathcal{R}$ be a (continuously) differentiable function on the interval $[c_1, c_2]$ such that $u(x) = u(c_1) + \int_{c_1}^x u'(\kappa) d\kappa$, and let $w: [c_1, c_2] \to \mathcal{R}$ be an integrable, positive and weighted symmetric function with respect to $\frac{c_1+c_2}{2}$. If, in addition, $|u'|^q$ is convex on $[c_1, c_2]$ with $\frac{1}{p} + \frac{1}{q} = 1$ and q > 1, and ϱ is an increasing and positive function from $[c_1, c_2)$ onto itself such that its derivative $\varrho'(x)$ is continuous on (c_1, c_2) , then for v > 0 the following inequalities are valid:

$$\begin{split} |\Xi_{1} + \Xi_{2}| &\leq \frac{(c_{2} - c_{1})^{\nu+1}}{2^{\nu+1+\frac{2}{q}}(p\nu+1)^{\frac{1}{p}}\Gamma(\nu+1)} \Big\{ \|w\|_{\left[c_{1}, \frac{c_{1}+c_{2}}{2}\right], \infty} \\ &\times \left[3|u'(c_{1})|^{q} + |u'(c_{2})|^{q} \right]^{\frac{1}{q}} + \|w\|_{\left[\frac{c_{1}+c_{2}}{2}, c_{2}\right], \infty} \left[|u'(c_{1})|^{q} + 3|u'(c_{2})|^{q} \right]^{\frac{1}{q}} \Big\} \\ &\leq \frac{(c_{2} - c_{1})^{\nu+1} \|w\|_{\left[c_{1}, c_{2}\right], \infty}}{2^{\nu+1+\frac{2}{q}}(p\nu+1)^{\frac{1}{p}}\Gamma(\nu+1)} \\ &\times \Big\{ \left[3|u'(c_{1})|^{q} + |u'(c_{2})|^{q} \right]^{\frac{1}{q}} + \left[|u'(c_{1})|^{q} + 3|u'(c_{2})|^{q} \right]^{\frac{1}{q}} \Big\}. \end{split}$$
(40)

Proof. Since $|u'|^q$ is convex on $[\mathbf{c}_1, \mathbf{c}_2]$, we get for $\kappa \in [\varrho^{-1}(\mathbf{c}_1), \varrho^{-1}(\mathbf{c}_2)]$:

$$\begin{split} \left| (\mathbf{u}' \circ \varrho)(\kappa) \right|^q &= \left| \mathbf{u}' \left(\frac{\mathbf{c}_2 - \varrho(\kappa)}{\mathbf{c}_2 - \mathbf{c}_1} \mathbf{c}_1 + \frac{\varrho(\kappa) - \mathbf{c}_1}{\mathbf{c}_2 - \mathbf{c}_1} \mathbf{c}_2 \right) \right|^q \\ &\leq \frac{\mathbf{c}_2 - \varrho(\kappa)}{\mathbf{c}_2 - \mathbf{c}_1} |\mathbf{u}'(\mathbf{c}_1)|^q + \frac{\varrho(\kappa) - \mathbf{c}_1}{\mathbf{c}_2 - \mathbf{c}_1} |\mathbf{u}'(\mathbf{c}_2)|^q. \end{split}$$

By using Lemma 2, Hölder's inequality, convexity of $|\mathbf{u}'|^q$ and properties of modulus, we have

$$\begin{split} |\Xi_1 + \Xi_2| &\leq \frac{1}{\Gamma(\nu)} \int_{\varrho^{-1}(\mathbf{c}_1)}^{\varrho^{-1}\left(\frac{\mathbf{c}_1 + \mathbf{c}_2}{2}\right)} \left| \int_{\varrho^{-1}(\mathbf{c}_1)}^{\kappa} \varrho'(x) (\varrho(x) - \mathbf{c}_1)^{\nu - 1} (w \circ \varrho)(x) dx \right| \\ & \times |(\mathbf{u}' \circ \varrho)(\kappa)| \varrho'(\kappa) d\kappa \\ &+ \frac{1}{\Gamma(\nu)} \int_{\varrho^{-1}\left(\frac{\mathbf{c}_1 + \mathbf{c}_2}{2}\right)}^{\varrho^{-1}(\mathbf{c}_2)} \left| \int_{\kappa}^{\varrho^{-1}(\mathbf{c}_2)} \varrho'(x) (\mathbf{c}_2 - \varrho(x))^{\nu - 1} (w \circ \varrho)(x) dx \right| |(\mathbf{u}' \circ \varrho)(\kappa)| \varrho'(\kappa) d\kappa \\ &\leq \frac{1}{\Gamma(\nu)} \left(\int_{\varrho^{-1}(\mathbf{c}_1)}^{\varrho^{-1}\left(\frac{\mathbf{c}_1 + \mathbf{c}_2}{2}\right)} \left| \int_{\varrho^{-1}(\mathbf{c}_1)}^{\kappa} \varrho'(x) (\varrho(x) - \mathbf{c}_1)^{\nu - 1} (w \circ \varrho)(x) dx \right|^p \varrho'(\kappa) d\kappa \right)^{\frac{1}{p}} \end{split}$$

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$$\begin{split} &\times \left(\int_{\varrho^{-1}(\mathbf{c}_{1})}^{e^{-1}\left(\frac{c_{1}+c_{2}}{2}\right)}\left|(\mathbf{u}'\circ\varrho)(\kappa)\right|^{q}\varrho'(\kappa)d\kappa\right)^{\frac{1}{q}} \\ &+ \frac{1}{\Gamma(\nu)}\left(\int_{\varrho^{-1}\left(\frac{c_{1}+c_{2}}{2}\right)}^{e^{-1}(\mathbf{c}_{2})}\left|\int_{\kappa}^{e^{-1}(\mathbf{c}_{2})}\varrho'(x)(\mathbf{c}_{2}-\varrho(x))^{\nu-1}(w\circ\varrho)(x)dx\right|^{p}\varrho'(\kappa)d\kappa\right)^{\frac{1}{p}} \\ &\times \left(\int_{\varrho^{-1}\left(\frac{c_{1}+c_{2}}{2}\right)}^{e^{-1}(\mathbf{c}_{2})}\left|(\mathbf{u}'\circ\varrho)(\kappa)\right|^{q}\varrho'(\kappa)d\kappa\right)^{\frac{1}{q}} \\ &\times \left(\int_{\varrho^{-1}\left(\frac{c_{1}+c_{2}}{2}\right)}^{e^{-1}\left(\frac{c_{1}+c_{2}}{2}\right)}\left|\int_{\kappa}^{\kappa}\varrho'(x)(\varrho(x)-\mathbf{c}_{1})^{\nu-1}dx\right|^{p}\varrho'(\kappa)d\kappa\right)^{\frac{1}{p}} \\ &\times \left(\int_{\varrho^{-1}(\mathbf{c}_{1})}^{e^{-1}\left(\frac{c_{1}+c_{2}}{2}\right)}\left|\int_{\kappa}^{\kappa}\varrho'(x)(\varrho(x)-\mathbf{c}_{1})^{\nu-1}dx\right|^{p}\varrho'(\kappa)d\kappa\right)^{\frac{1}{p}} \\ &\times \left(\int_{\varrho^{-1}\left(\frac{c_{1}+c_{2}}{2}\right)}^{e^{-1}(\mathbf{c}_{2})}\left|\int_{\kappa}^{\ell}\varrho'(x)(e_{2}-\varrho(x))^{\nu-1}dx\right|^{p}\varrho'(\kappa)d\kappa\right)^{\frac{1}{p}} \\ &\times \left(\int_{\varrho^{-1}\left(\frac{c_{1}+c_{2}}{2}\right)}^{e^{-1}(\mathbf{c}_{2})}\left|\int_{\kappa}^{\ell}\varrho'(x)(e_{2}-\varrho(x))^{\nu-1}dx\right|^{p}\varrho'(\kappa)d\kappa\right)^{\frac{1}{p}} \\ &\times \left(\int_{\varrho^{-1}\left(\frac{c_{1}+c_{2}}{2}\right)}^{e^{-1}(\mathbf{c}_{2})}\left|\int_{\varrho^{-1}(\mathbf{c}_{1})}^{\kappa}\varrho'(x)(\varrho(x)-\mathbf{c}_{1})^{\nu-1}dx\right|^{p}\varrho'(\kappa)d\kappa\right)^{\frac{1}{p}} \\ &\times \left(\int_{\varrho^{-1}\left(\frac{c_{1}+c_{2}}{2}\right)}^{e^{-1}(\mathbf{c}_{2})}\left|\int_{\varrho^{-1}(\mathbf{c}_{1})}^{\kappa}\varrho'(x)(\varrho(x)-\mathbf{c}_{1})^{\nu-1}dx\right|^{p}\varrho'(\kappa)d\kappa\right)^{\frac{1}{p}} \\ &\times \left[\int_{\varrho^{-1}\left(\frac{c_{1}+c_{2}}{2}\right)}^{e^{-1}(\mathbf{c}_{2})}\left|\int_{\varrho^{-1}(\mathbf{c}_{1})}^{\kappa}\varrho'(x)(\varrho(x)-\mathbf{c}_{1})^{\nu-1}dx\right|^{p}\varrho'(\kappa)d\kappa\right)^{\frac{1}{p}} \\ &\times \left[\int_{\varrho^{-1}\left(\frac{c_{1}+c_{2}}{2}\right)}^{e^{-1}(\mathbf{c}_{2})}\left|\int_{\varrho^{-1}\left(\mathbf{c}_{1}\right)}^{\ell}\varrho'(x)(e_{2}-\varrho(x))^{\nu-1}dx\right|^{p}\varrho'(\kappa)d\kappa\right)^{\frac{1}{p}} \\ &\times \left[\int_{\varrho^{-1}\left(\frac{c_{1}+c_{2}}{2}\right)}^{\varrho^{-1}\left(\frac{c_{1}+c_{2}}{2}\right)}\left|\int_{\kappa}^{\ell}\varrho'(x)(e_{2}-\varrho(x))^{\nu-1}dx\right|^{p}\varrho'(\kappa)d\kappa\right)^{\frac{1}{p}} \\ &\times \left[\int_{\varrho^{-1}\left(\frac{c_{1}+c_{2}}{2}\right)}^{\ell}\left|\int_{\varrho^{-1}\left(\frac{c_{1}+c_{2}}{2}\right)}^{\ell}\varrho'(x)(e_{2}-\varrho(x))^{\nu-1}dx\right|^{p}\varrho'(\kappa)d\kappa\right)^{\frac{1}{p}} \\ &\times \left[\int_{\varrho^{-1}\left(\frac{c_{1}+c_{2}}{2}\right)}^{\ell}\left|\int_{\kappa}^{\ell}\varrho'(x)(e_{2}-\varrho(\kappa))^{\nu-1}dx\right|^{p}\varrho'(\kappa)d\kappa\right)^{\frac{1}{p}} \\ &\times \left[\int_{\varrho^{-1}\left(\frac{c_{1}+c_{2}}{2}\right)}^{\ell}\left|\int_{\kappa}^{\ell}\varrho'(x)(e_{2}-\varrho(\kappa))^{\nu-1}dx\right|^{p}\varrho'(\kappa)d\kappa\right)^{\frac{1}{p}} \\ &\times \left[\int_{\varrho^{-1}\left(\frac{c_{1}+c_{2}}{2}\right)}^{\ell}\left|\int_{\kappa}^{\ell}\varrho'(x)(e_{2}-\varrho(\kappa))^{\nu-1}dx\right|^{p}\varrho'(\kappa)d\kappa\right)^{\frac{1}{p}} \\ &\times \left[\int_{\ell}^{\ell}\varrho'(x)(e_{2}-\varrho'(x))^{\nu-1}dx\right|^{p}\varrho'(\kappa)d\kappa\right)^{\frac{1}{p}} \\ &\times \left[\int_{\ell}^{\ell$$

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$$\begin{split} &= \frac{(\mathbf{c}_2 - \mathbf{c}_1)^{\nu+1}}{2^{\nu+1+\frac{2}{q}}(p\nu+1)^{\frac{1}{p}}\Gamma(\nu+1)} \Big\{ \|w\|_{\left[\mathbf{c}_1,\frac{\mathbf{c}_1+\mathbf{c}_2}{2}\right],\infty} \big[3|\mathbf{u}'(\mathbf{c}_1)|^q + |\mathbf{u}'(\mathbf{c}_2)|^q \big]^{\frac{1}{q}} \\ &\quad + \|w\|_{\left[\frac{\mathbf{c}_1+\mathbf{c}_2}{2},\mathbf{c}_2\right],\infty} \big[|\mathbf{u}'(\mathbf{c}_1)|^q + 3|\mathbf{u}'(\mathbf{c}_2)|^q \big]^{\frac{1}{q}} \\ &\quad \leq \frac{(\mathbf{c}_2 - \mathbf{c}_1)^{\nu+1} \|w\|_{\left[\mathbf{c}_1,\mathbf{c}_2\right],\infty}}{2^{\nu+1+\frac{2}{q}}(p\nu+1)^{\frac{1}{p}}\Gamma(\nu+1)} \\ &\quad \times \Big\{ \big[3|\mathbf{u}'(\mathbf{c}_1)|^q + |\mathbf{u}'(\mathbf{c}_2)|^q \big]^{\frac{1}{q}} + \big[|\mathbf{u}'(\mathbf{c}_1)|^q + 3|\mathbf{u}'(\mathbf{c}_2)|^q \big]^{\frac{1}{q}} \Big\}, \end{split}$$

where we used the identity

$$\begin{split} \int_{\varrho^{-1}(\mathbf{c}_{1})}^{\varrho^{-1}\left(\frac{\mathbf{c}_{1}+\mathbf{c}_{2}}{2}\right)} \left| \int_{\varrho^{-1}(\mathbf{c}_{1})}^{\kappa} \varrho'(x)(\varrho(x)-\mathbf{c}_{1})^{\nu-1}dx \right|^{p} \varrho'(\kappa)d\kappa \\ &= \int_{\varrho^{-1}\left(\frac{\mathbf{c}_{1}+\mathbf{c}_{2}}{2}\right)}^{\varrho^{-1}(\mathbf{c}_{2})} \left| \int_{\kappa}^{\varrho^{-1}(\mathbf{c}_{2})} \varrho'(x)(\mathbf{c}_{2}-\varrho(x))^{\nu-1}dx \right|^{p} \varrho'(\kappa)d\kappa = \frac{(\mathbf{c}_{2}-\mathbf{c}_{1})^{p\nu+1}}{2^{p\nu+1}(p\nu+1)\nu^{p}}. \end{split}$$

This ends our proof. \Box

Remark 7. From Theorem 4, we can obtain some special cases as follows:

(i) If $\varrho(x) = x$, then inequality (40) becomes

$$\begin{split} \bigg| \mathsf{u} \bigg(\frac{c_1 + c_2}{2} \bigg) \bigg[\Big(\frac{c_1 + c_2}{2} \Big)^{RL} \mathcal{I}^{\nu} w(c_2) + \frac{RL}{2} \mathcal{I}^{\nu}_{\left(\frac{c_1 + c_2}{2}\right) -} w(c_1) \bigg] \\ & - \bigg[w(c_2) \bigg(\Big(\frac{c_1 + c_2}{2} \Big)^{RL} \mathcal{I}^{\nu}_{w} \mathsf{u} \bigg)(c_2) + w(c_1) \bigg(\frac{RL}{w} \mathcal{I}^{\nu}_{\left(\frac{c_1 + c_2}{2}\right) -} \mathsf{u} \bigg)(c_1) \bigg] \bigg| \\ & \leq \frac{(c_2 - c_1)^{\nu + 1}}{2^{\nu + 1 + \frac{2}{q}} (p\nu + 1)^{\frac{1}{p}} \Gamma(\nu + 1)} \bigg\{ \|w\|_{\left[c_1, \frac{c_1 + c_2}{2}\right], \infty} \big[3 |\mathsf{u}'(c_1)|^q + |\mathsf{u}'(c_2)|^q \big]^{\frac{1}{q}} \\ & + \|w\|_{\left[\frac{c_1 + c_2}{2}, c_2\right], \infty} \big[|\mathsf{u}'(c_1)|^q + 3 |\mathsf{u}'(c_2)|^q \big]^{\frac{1}{q}} \bigg\} \leq \frac{(c_2 - c_1)^{\nu + 1} \|w\|_{\left[c_1, c_2\right], \infty}}{2^{\nu + 1 + \frac{2}{q}} (p\nu + 1)^{\frac{1}{p}} \Gamma(\nu + 1)} \\ & \times \bigg\{ \big[3 |\mathsf{u}'(c_1)|^q + |\mathsf{u}'(c_2)|^q \big]^{\frac{1}{q}} + \big[|\mathsf{u}'(c_1)|^q + 3 |\mathsf{u}'(c_2)|^q \big]^{\frac{1}{q}} \bigg\}. \end{split}$$

(ii) If $\varrho(x) = x$ and w(x) = 1, then inequality (40) becomes

$$\begin{split} \left| \frac{2^{\nu-1}\Gamma(\nu+1)}{(c_2-c_1)^{\nu}} \left[\left(\frac{c_1+c_2}{2} \right)_+^{RL} \mathcal{I}^{\nu} \mathsf{u}(c_2) + \frac{RL}{2} \mathcal{I}^{\nu}_{\left(\frac{c_1+c_2}{2} \right)_-} \mathsf{u}(c_1) \right] - \mathsf{u}\left(\frac{c_1+c_2}{2}\right) \right| \\ & \leq \frac{(c_2-c_1)^{\nu+1} \|w\|_{[c_1,c_2],\infty}}{2^{\nu+1+\frac{2}{q}} (p\nu+1)^{\frac{1}{p}} \Gamma(\nu+1)} \\ & \times \left\{ \left[3|\mathsf{u}'(c_1)|^q + |\mathsf{u}'(c_2)|^q \right]^{\frac{1}{q}} + \left[|\mathsf{u}'(c_1)|^q + 3|\mathsf{u}'(c_2)|^q \right]^{\frac{1}{q}} \right\}, \end{split}$$

which is already obtained in ([39] [Theorem 6]).

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(iii) If $\varrho(x) = x$, w(x) = 1 and $\nu = 1$, then inequality (40) becomes

$$\left| \frac{1}{c_2 - c_1} \int_{c_1}^{c_2} \mathsf{u}(x) dx - \mathsf{u}\left(\frac{c_1 + c_2}{2}\right) \right| \le \frac{c_2 - c_1}{16} \left(\frac{4}{p+1}\right)^{\frac{1}{p}} \\ \times \left\{ \left[3|\mathsf{u}'(c_1)|^q + |\mathsf{u}'(c_2)|^q \right]^{\frac{1}{q}} + \left[|\mathsf{u}'(c_1)|^q + 3|\mathsf{u}'(c_2)|^q \right]^{\frac{1}{q}} \right\},$$

which is already obtained in ([45] [Theorem 2.3]).

4. Concluding Remarks

In the present article, we have investigated a midpoint fractional HHF integral inequality by using the weighted fractional integrals with positive weighted symmetric function kernels, which is also the midpoint version of (9). Moreover, we have investigated some related results.

The existing versions of HHF integral inequalities (7) and (8) have been successfully applied to other classes of convex functions, see [46–48]. Therefore, our present results can be applied to those classes of convex functions as well.

Furthermore, one can observe that our results in this article are very generic and can be extended to give further potentially useful and interesting HHF integral inequalities of end-midpoint version, like the following one

$$\begin{split} \mathsf{u}\bigg(\frac{\mathbf{c}_1+\mathbf{c}_2}{2}\bigg) &\leq \frac{2^{\nu-1}\Gamma(\nu+1)}{(\mathbf{c}_2-\mathbf{c}_1)^{\nu}} \bigg[{}^{RL}\mathcal{I}^{\nu}_{\mathbf{c}_1+} \mathsf{u}\bigg(\frac{\mathbf{c}_1+\mathbf{c}_2}{2}\bigg) + {}^{RL}\mathcal{I}^{\nu}_{\mathbf{c}_2-} \mathsf{u}\bigg(\frac{\mathbf{c}_1+\mathbf{c}_2}{2}\bigg)\bigg] \\ &\leq \frac{\mathsf{u}(\mathbf{c}_1) + \mathsf{u}(\mathbf{c}_2)}{2} \text{,} \end{split}$$

which was already established by Mohammed and Brevik in [49].

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Abbreviations

The following abbreviations are used in our manuscript:

HH Hermite-Hadamard

HHF Hermite-Hadamard-Fejér

RL Riemann-Liouville

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