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Fractional Reverse Coposn's Inequalities via Conformable Calculus on Time Scales

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Abstract: This paper provides novel generalizations by considering the generalized conformable fractional integrals for reverse Copson's type inequalities on time scales. The main results will be proved using a general algebraic inequality, chain rule, Hölder's inequality, and integration by parts on fractional time scales. Our investigations unify and extend some continuous inequalities and their corresponding discrete analogues. In addition, when $\alpha = 1$, we obtain some well-known time scale inequalities due to Hardy, Copson, Bennett, and Leindler inequalities.

MSC: 26A15; 26D10; 39A13; 34A40; 34N05

Keywords: Copson's inequality; Hölder's inequality; time scales; conformable fractional calculus

Citation: Zakarya, M.; Altanji, M.; AlNemer, G.; Abd El-Hamid, H.A.; Cesarano, C.; Rezk, H.M. Fractional Reverse Coposn's Inequalities via Conformable Calculus on Time Scales. *Symmetry* **2021**, *13*, 542. https://doi.org/10.3390/sym13040542

Academic Editor: Ioan Rașa and Wei-Shih Du

Received: 2 March 2021
Accepted: 23 March 2021
Published: 25 March 2021

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1. Introduction

The Hardy discrete inequality is known as (see [1]):

$$\sum_{l=1}^{\infty}\left(\frac{1}{l}\sum_{j=1}^{l}w(j)\right)^{h} \leq \left(\frac{h}{h-1}\right)^{h}\sum_{l=1}^{\infty}w^{h}(l), \quad h>1, \tag{1}$$

where $w(l) > 0$ for all $l \geq 1$.

In [2], Hardy exemplified the continuous version of (1) by utilizing the calculus of variations, which has the form:

$$\int_{0}^{\infty}\left(\frac{1}{y}\int_{0}^{y}g(s)ds\right)^{h}dy \leq \left(\frac{h}{h-1}\right)^{h}\int_{0}^{\infty}g^{h}(y)dy, \qquad h>1, \tag{2}$$

where $g \geq 0$, which is integrable over $(0, y)$, g^{h} is a convergent and integrable function over $(0,\infty)$ and $(h/(h-1))^{h}$ is a sharp constant in (1) and (2).

In [3,4], Copson outreached the inequalities of Hardy (1) and (2). Particularly, he exemplified that if $h > 1, w(j) \geq 0, k(j) \geq 0, \forall j \geq 1, \vartheta(l) = \sum_{j=1}^{l} k(j)$ and $m > 1$, then:

$$\sum_{l=1}^{\infty}\frac{k(l)}{\vartheta^{m}(l)}\left(\sum_{j=1}^{l}w(j)k(j)\right)^{h}\le\left(\frac{h}{m-1}\right)^{h}\sum_{l=1}^{\infty}k(l)\vartheta^{h-m}(l)w^{h}(l),\tag{3}$$

and if $h>1$ *and* $0\le m<1$, then

$$\sum_{l=1}^{\infty}\frac{k(l)}{\vartheta^{m}(l)}\left(\sum_{j=1}^{\infty}w(j)k(j)\right)^{h}\le\left(\frac{h}{1-m}\right)^{h}\sum_{l=1}^{\infty}k(l)\vartheta^{h-m}(l)w^{h}(l).\tag{4}$$

The continuous transcriptions of (3) and (4) were exemplified by Copson in [4]. Particularly, he exemplified that if $h\ge 1, m>1$ *and* $\vartheta(s)=\int_0^s k(\zeta)d\zeta$, then:

$$\int_0^{\infty}\frac{k(s)}{\vartheta^{m}(s)}\Theta^{h}(s)ds\le\left(\frac{h}{m-1}\right)^{h}\int_0^{\infty}\frac{k(s)}{\vartheta^{m-h}(s)}g^{h}(s)ds,\tag{5}$$

where $\Theta(s)=\int_0^{\infty}k(\zeta)g(\zeta)d\zeta$ and if $h>1, 0\le m<1$, then

$$\int_0^{\infty}\frac{k(s)}{\vartheta^{m}(s)}\bar{\Theta}^{h}(s)ds\le\left(\frac{h}{1-m}\right)^{h}\int_0^{\infty}\frac{k(s)}{\vartheta^{m-h}(s)}g^{h}(s)ds,\tag{6}$$

where $\bar{\Theta}(s)=\int_s^{\infty}k(\zeta)g(\zeta)d\zeta$.

Leindler in [5] and Bennett in [6] obtains some generalizations of (3) and (4) by using new weighted function. Specially, Leindler exemplified that if $\vartheta^{*}(l)=\sum_{j=l}^{\infty}k(j)<\infty, h>1$ and $0\le m<1$, then:

$$\sum_{l=1}^{\infty}\frac{k(l)}{(\vartheta^{*}(l))^{m}}\left(\sum_{j=1}^{l}w(j)k(j)\right)^{h}\le\left(\frac{h}{1-m}\right)^{h}\sum_{l=1}^{\infty}k(l)\left(\vartheta^{*}(l)\right)^{h-m}w^{h}(l).\tag{7}$$

Bennett explicated that if $1<m\le h$, then:

$$\sum_{l=1}^{\infty}\frac{k(l)}{\left(\vartheta^{*}(l)\right)^{m}}\left(\sum_{j=1}^{\infty}w(j)k(j)\right)^{h}\le\left(\frac{h}{m-1}\right)^{h}\sum_{l=1}^{\infty}k(l)\left(\vartheta^{*}(l)\right)^{h-m}w^{h}(l).\tag{8}$$

Bennett in [7,8] established a converses of the inequalities (3) and (4). Particularly, he exemplified that if $m\le 0<h<1$, then:

$$\sum_{l=1}^{\infty}\frac{k(l)}{\vartheta^{m}(l)}\left(\sum_{j=1}^{\infty}w(j)k(j)\right)^{h}\ge\left(\frac{h}{1-m}\right)^{h}\sum_{l=1}^{\infty}k(l)\left(\sum_{j=1}^{l}k(j)\right)^{h-m}w^{h}(l),\tag{9}$$

and if $m>1>h>0, \vartheta(l)\to\infty$, then

$$\sum_{l=1}^{\infty}\frac{k(l)}{\vartheta^{m}(l)}\left(\sum_{j=1}^{l}w(j)k(j)\right)^{h}\ge\left(\frac{hL}{m-1}\right)^{h}\sum_{l=1}^{\infty}k(l)\,\vartheta^{h-m}(l)w^{h}(l),\tag{10}$$

where

$$L:=\inf_{l\in\mathbb{N}}\frac{k(l)}{k(l+1)}.$$

In the last decades, the study of the dynamic equations and inequalities on time scales became a main field in applied and pure mathematics. We refer to the papers ([9–15]). In fact, Refs. [16–24] mentions forms of the above inequalities on a time-scale and their extensions.

For example, in [25], Saker et al. exemplified the time scale version of a converse of the inequalities (7) and (8), respectively, as follows:

Assume that $\mathbb{T}$ be a time scale with $w \in (0,\infty)_{\mathbb{T}}$. If $m \leq 0 < h < 1, \vartheta(\zeta) = \int_{\zeta}^{\infty} k(s)\Delta s$ and $\Omega(\zeta) = \int_{w}^{\zeta} k(s)\eta(s)\Delta s$, then

$$\int_{w}^{\infty} \frac{k(\zeta)}{\vartheta^{m}(\zeta)} (\Omega^{\sigma}(\zeta))^{h} \Delta\zeta \geq \left(\frac{h}{1-m}\right)^{h} \int_{w}^{\infty} k(\zeta)\eta^{h}(\zeta)\vartheta^{h-m}(\zeta)\Delta\zeta. \tag{11}$$

If $0 < h < 1 < m, \vartheta(\zeta) = \int_{\zeta}^{\infty} k(s)\Delta s$ and $\overline{\Omega}(\zeta) = \int_{\zeta}^{\infty} k(s)\eta(s)\Delta s$, then

$$\int_{w}^{\infty} \frac{k(\zeta)}{\vartheta^{m}(\zeta)} (\overline{\Omega}(\zeta))^{h} \Delta\zeta \geq \left(\frac{hM^{m}}{m-1}\right)^{h} \int_{w}^{\infty} k(\zeta)\eta^{h}(\zeta)\vartheta^{h-m}(\zeta)\Delta\zeta, \tag{12}$$

where

$$M := \inf_{\zeta \in \mathbb{T}} \frac{\vartheta^{\sigma}(\zeta)}{\vartheta(\zeta)} > 0.$$

In the same paper [25], Saker et al. proved the time scale transcript of the Bennet-Leindler inequalities (9) and (10), respectively, as follows: Assume that $\mathbb{T}$ is a time scale with $w \in (0,\infty)_{\mathbb{T}}$. If $m \leq 0 < h < 1, \Gamma(\zeta) = \int_{w}^{\zeta} k(s)\Delta s$ and $\overline{\Omega}(\zeta) = \int_{\zeta}^{\infty} k(s)\eta(s)\Delta s$, then:

$$\int_{w}^{\infty} \frac{k(\zeta)}{\left(\Gamma^{\sigma}(\zeta)\right)^{m}} (\overline{\Omega}(\zeta))^{h} \Delta\zeta \geq \left(\frac{h}{1-m}\right)^{h} \int_{w}^{\infty} k(\zeta)\eta^{h}(\zeta) \left(\Gamma^{\sigma}(\zeta)\right)^{h-m}\Delta\zeta. \tag{13}$$

If $0 < h \leq m, \Gamma(\zeta) = \int_{w}^{\zeta} k(s)\Delta s$, such that

$$L := \inf_{\zeta \in \mathbb{T}} \frac{\Gamma(\zeta)}{\Gamma^{\sigma}(\zeta)} > 0,$$

and $\overline{\Phi}(\zeta) = \int_{w}^{\zeta} k(s)\eta(s)\Delta s$, then

$$\int_{w}^{\infty} \frac{k(\zeta)}{\left(\Gamma^{\sigma}(\zeta)\right)^{m}} (\overline{\Phi}^{\sigma}(\zeta))^{h} \Delta\zeta \geq \left(\frac{hL^{1-m}}{1-m}\right)^{h} \int_{w}^{\infty} k(\zeta)\eta^{h}(\zeta) \left(\Gamma^{\sigma}(\zeta)\right)^{h-m}\Delta\zeta. \tag{14}$$

In recent years, a lot of work has been published on fractional inequalities and the subject has become an active field of research with several authors interested in proving the inequalities of fractional type by using the Riemann-Liouville and Caputo derivative (see [26–28]).

On the other hand, the authors in [29,30] introduced a new fractional calculus called the conformable calculus and gave a new definition of the derivative with the base properties of the calculus based on the new definition of derivative and integrals.

The main question that arises now is: Is it possible to prove new fractional inequalities on timescales and give a unified approach of such studies? This in fact needs a new fractional calculus on timescales. Very recently Torres and others, in [31,32], combined a time scale calculus and conformable calculus and obtained the new fractional calculus on timescales. Thus, it is natural to look on new fractional inequalities on timescales and give an affirmative answer to the above question.

In particular, in this paper, we will prove the fractional forms of the classical Hardy, Copson type and its reversed and Leindler inequalities with employing conformable calculus on time scales. The article is structured as follows: Section 2 is an introduction of the basics of fractional calculus on timescales and Section 3 contains the main results.

2. Basic Concepts

In this part, we introduce the essentials of conformable fractional integral and derivative of order $\alpha \in [0,1]$ on time scales that will be used in this article (see [33–35]). A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. We define the operator $\sigma: \mathbb{T} \to \mathbb{T}$, as $\sigma(\theta) := \inf\{s \in \mathbb{T}: s > \theta\}$. In addition, we define the function $\mu: \mathbb{T} \to [0,\infty)$ by $\mu(\theta) := \sigma(\theta) - \theta$. Finally, for any $\theta \in \mathbb{T}$, we refer to the notation $\zeta^{\sigma}(\theta)$ by $\zeta(\sigma(\theta))$, i.e., $\zeta^{\sigma} = \zeta \circ \sigma$.

In the following, we define conformable α-fractional derivative and α-fractional integral on $\mathbb{T}$.

Definition 1 (**Definition 1,** [31])**.** *Suppose that* $\zeta: \mathbb{T} \to \mathbb{R}$ *and* $\alpha \in (0,1]$. *Then for* $\theta > 0$, *we define* $D_\alpha^\Delta(\zeta)(\theta)$ *to be the number with the property that, for any* $\varepsilon > 0$, *there is a neighborhood* V *of* θ *s.t.* $\forall \theta \in V$, *we have:*

$$\left|[\zeta^\sigma(\theta) - \zeta(s)]\sigma^{1-\alpha(\theta)} - D_\alpha^\Delta(\zeta)(\theta)[\sigma(\theta) - s]\right| \le \varepsilon|\sigma(\theta) - s|.$$

The conformable α-fractional derivative on $\mathbb{T}$ at 0 as:

$$D_\alpha^\Delta\big(\zeta(0)\big) = \lim_{\theta \to 0} D_\alpha^\Delta\big(\zeta(\theta)\big).$$

Theorem 1 (**Theorem 51,** [31])**.** *Assume* $\alpha \in (0,1]$ *and* $v, \zeta: \mathbb{T} \to \mathbb{R}$ *be conformable* α*-fractional derivative on* $\mathbb{T}$, *then*

(i) The $v + \zeta: \mathbb{T} \to \mathbb{R}$ is conformable α-fractional derivative and

$$D_\alpha^\Delta(v + \zeta) = D_\alpha^\Delta(v) + D_\alpha^\Delta(\zeta).$$

(ii) For $\lambda \in \mathbb{R}$, then $\lambda v: \mathbb{T} \to \mathbb{R}$ α-fractional differentiable and

$$D_\alpha^\Delta(\lambda v) = \lambda D_\alpha^\Delta(v).$$

(iii) If v and ζ are α-fractional differentiable, then $v\zeta: \mathbb{T} \to \mathbb{R}$ is a α-fractionald differentiable and

$$D_\alpha^\Delta(v\zeta) = D_\alpha^\Delta(v)\zeta + (v \circ \sigma)D_\alpha^\Delta(\zeta) = D_\alpha^\Delta(v)(\zeta \circ \sigma) + vD_\alpha^\Delta(\zeta). \tag{15}$$

(iv) If v is α-fractional differentiable, then $1/v$ is α-fractional differentiable with:

$$D_\alpha^\Delta\left(\frac{1}{v}\right) = -\frac{D_\alpha^\Delta(v)}{v(v \circ \sigma)}.$$

(v) If v and ζ are α-fractional differentiable, then v/ζ is α-fractional differentiable with:

$$D_\alpha^\Delta\left(\frac{v}{\zeta}\right) = \frac{\zeta D_\alpha^\Delta(v) - vD_\alpha^\Delta(\zeta)}{\zeta(\zeta \circ \sigma)}, \tag{16}$$

valid at all points $\theta \in \mathbb{T}^k$ for which $\zeta(\theta)(\zeta(\sigma(\theta)) \neq 0$.

Lemma 1 (**Chain rule** [32])**.** *Suppose that* $\zeta: \mathbb{T} \to \mathbb{R}$ *is continuous and* α*-fractional differentiable at* $\theta \in \mathbb{T}$, *for* $\alpha \in (0,1]$ *and* $v: \mathbb{R} \to \mathbb{R}$ *is continuously differentiable. Then* $(v \circ \zeta): \mathbb{T} \to \mathbb{R}$ *is* α*-fractional differentiable and*

$$D_\alpha^\Delta(v \circ \zeta)(\theta) = v'\big(\zeta(d)\big)D_\alpha^\Delta\big(\zeta(\theta)\big), \;\; where \;\; d \in [\theta, \sigma(\theta)]. \tag{17}$$

Definition 2 (**Definition 26**, [31])**.** *For* $0 < \alpha \le 1$, *then the* α*-fractional integral of* ζ, *is defined as*

$$I_\alpha^\Delta(\zeta(s)) = \int \zeta(s)\Delta_\alpha s = \int \zeta(s)s^{\alpha-1}\Delta s.$$

Theorem 2 (**Theorem 31**, [31])**.** *Suppose that* $l, m, n \in \mathbb{T}$ *and* $\lambda \in \mathbb{R}$. *If* $v, \zeta: \mathbb{T} \to \mathbb{R}$, *then*

(vi) $\int_l^m [v(s) + \zeta(s)]\Delta_\alpha s = \int_l^m v(s)\Delta_\alpha s + \int_l^m \zeta(s)\Delta_\alpha s.$

(vii) $\int_l^m \lambda v(s)\Delta_\alpha s = \lambda \int_l^m v(s)\Delta_\alpha s.$

(viii) $\int_l^m v(s)\Delta_\alpha s = -\int_l^m v(s)\Delta_\alpha s.$

(ix) $\int_l^m v(s)\Delta_\alpha s = \int_l^n v(s)\Delta_\alpha s + \int_n^m v(s)\Delta_\alpha s.$

(x) $\int_l^l v(s)\Delta_\alpha s = 0$

Lemma 2 (**Integration by parts formula** [31])**.** *Suppose that $l, m \in \mathbb{T}$ where $m > l$. If v, ζ are conformable α-fractional differentiable and $\alpha \in (0,1]$, then:*

$$\int_l^m v(s)\ D_\alpha^\Delta(\zeta(s))\Delta_\alpha s = [v(s)\zeta(s)]_l^m - \int_l^m \zeta^\sigma(s) D_\alpha^\Delta\big(v(s)\big)\Delta_\alpha s. \tag{18}$$

Lemma 3 (**Hölder's inequality** [32])**.** *Let $l, m \in \mathbb{T}$ where $m > l$. If $\alpha \in (0,1]$ and $F, G\colon \mathbb{T} \to \mathbb{R}$, then*

$$\int_l^m |F(s)G(s)|\ \Delta_\alpha s \leq \left(\int_l^m |F(s)|^\lambda \Delta_\alpha s\right)^{\frac{1}{\lambda}} \left(\int_l^m |G(s)|^\mu \Delta_\alpha s\right)^{\frac{1}{\mu}}, \tag{19}$$

where $\lambda > 1$ and $1/\lambda + /\mu = 1$.

Through our paper, we will consider the integrals are given exist (are finite i.e., convergent).

3. Results

Here, we will exemplify our main results in this article by utilizing Hölder's inequality, chain rule, and integration by parts for fractional on time scale.

Theorem 3. *Suppose that $\mathbb{T}$ is a time scale with $w \in (0,\infty)_\mathbb{T}, k \leq 0 < h < 1$ and $\alpha \in (0,1]$. Define*

$$\vartheta(y) = \int_y^\infty x(s)\Delta_\alpha s \text{ and } \Omega(y) = \int_w^y x(s)\eta(s)\Delta_\alpha s.$$

Then

$$\int_w^\infty \frac{x(y)}{\vartheta^{k-\alpha+1}(y)} \big(\Omega^\sigma(y)\big)^h \Delta_\alpha y \geq \left(\frac{h}{\alpha - k}\right)^h \int_w^\infty x(y)\eta^h(y)\vartheta^{h-k+\alpha-1}(y)\Delta_\alpha y. \tag{20}$$

Proof. By utilizing the formula of integration by parts (18) on

$$\int_w^\infty \frac{x(y)}{\vartheta^{k-\alpha+1}(y)} \big(\Omega^\sigma(y)\big)^h \Delta_\alpha y,$$

with $\zeta^\sigma(y) = \big(\Omega^\sigma(y)\big)^h$ and $D_\alpha^\Delta\ (v(y)) = \frac{x(y)}{\vartheta^{k-\alpha+1}(y)}$, we have

$$\int_w^\infty \frac{x(y)}{\vartheta^{k-\alpha+1}(y)} \big(\Omega^\sigma(y)\big)^h \Delta_\alpha y = v(y)\Omega^h(y)\Big|_w^\infty + \int_w^\infty \big(-v(y)\big) D_\alpha^\Delta\big(\Omega^h(y)\big)\Delta_\alpha y, \tag{21}$$

where

$$v(y) = -\int_y^\infty \frac{x(s)}{\vartheta^{k-\alpha+1}(s)} \Delta_\alpha s.$$

Using $\Omega(w) = 0$ and $v(\infty) = 0$ in (21), we see that

$$\int_w^\infty \frac{x(y)}{\vartheta^{k-\alpha+1}(y)} \big(\Omega^\sigma(y)\big)^h \Delta_\alpha y = -\int_w^\infty v(y) D_\alpha^\Delta\big(\Omega^h(y)\big)\Delta_\alpha y. \tag{22}$$

By utilizing chain rule, we get:

$$D_\alpha^\Delta\big(\Omega^h(y)\big) = h\Omega^{h-1}(d) D_\alpha^\Delta\big(\Omega(y)\big) = \frac{h D_\alpha^\Delta\big(\Omega(y)\big)}{\big(\Omega(d)\big)^{1-\mathrm{h}}} \geq \frac{h D_\alpha^\Delta\big(\Omega(y)\big)}{\big(\Omega^\sigma(y)\big)^{1-h}}.$$

Since $D_\alpha^\Delta\big(\Omega(y)\big) = x(y)\ \eta(y)$, we have

$$D_{\alpha}^{\Delta}\left(\Omega^{h}(y)\right) \geq \frac{hx(y)\,\eta(y)}{\left(\Omega^{\sigma}(y)\right)^{1-h}}. \tag{23}$$

Next note $D_{\alpha}^{\Delta}\,\vartheta(y) = -x(y) \leq 0$. By the chain rule, we have (note $k \leq 0$)

$$\begin{aligned} D_{\alpha}^{\Delta}\left(\vartheta^{\alpha-k}(y)\right) &= (\alpha-k)\vartheta^{\alpha-k-1}(d)D_{\alpha}^{\Delta}\vartheta(y) \\ &= \frac{\alpha-k}{\vartheta^{k+1-\alpha}(d)}D_{\alpha}^{\Delta}\vartheta(y) \geq \frac{\alpha-k}{\vartheta^{k+1-\alpha}(y)}D_{\alpha}^{\Delta}\vartheta(y) \\ &= \frac{\alpha-k}{\vartheta^{k+1-\alpha}(y)}(-x(y)) \geq \frac{-(\alpha-k)x(y)}{\vartheta^{k+1-\alpha}(y)}. \end{aligned}$$

This leads to

$$\vartheta^{-k+\alpha-1}(y)x(y) \geq \frac{-1}{\alpha-k}D_{\alpha}^{\Delta}\left(\vartheta^{\alpha-k}(y)\right),$$

and then, we have

$$\begin{aligned} -v(y) = \int_{y}^{\infty}\frac{x(s)}{\vartheta^{k-\alpha+1}(y)}\Delta_{\alpha}s &\geq \frac{-1}{\alpha-k}\int_{y}^{\infty}D_{\alpha}^{\Delta}\left(\vartheta^{\alpha-k}(s)\right)\Delta_{\alpha}s \\ &= \frac{1}{(\alpha-k)\vartheta^{k-\alpha}(y)} \end{aligned} \tag{24}$$

Substituting (23), (24) into (22) yields:

$$\int_{w}^{\infty}\frac{x(y)}{\vartheta^{k-\alpha+1}(y)}\left(\Omega^{\sigma}(y)\right)^{h}\Delta_{\alpha}y \geq \left(\frac{h}{\alpha-k}\right)\int_{w}^{\infty}\frac{\eta(y)x(y)}{\vartheta^{k-\alpha}(y)\left(\Omega^{\sigma}(y)\right)^{1-h}}\Delta_{\alpha}y. \tag{25}$$

Raises (25) to the factor h, we have:

$$\left(\int_{y}^{\infty}\frac{x(y)}{\vartheta^{k-\alpha+1}(y)}\left(\Omega^{\sigma}(y)\right)^{h}\Delta_{\alpha}y\right)^{h} \geq \left(\frac{h}{\alpha-k}\right)^{h}\left(\int_{w}^{\infty}\left(\frac{\eta^{h}(y)x^{h}(y)}{\vartheta^{h(k-\alpha)}(y)\left(\Omega^{\sigma}(y)\right)^{h(1-h)}}\right)^{\frac{1}{h}}\Delta_{\alpha}y\right)^{h}. \tag{26}$$

By applying Hölder's inequality (19) on the term

$$\left(\int_{w}^{\infty}\left(\frac{\eta^{h}(y)x^{h}(y)}{\vartheta^{h(k-\alpha)}(y)\left(\Omega^{\sigma}(y)\right)^{h(1-h)}}\right)^{\frac{1}{h}}\Delta_{\alpha}y\right)^{h},$$

with indices $\lambda = 1/h > 1, \mu = 1/(1-h)$ (note that $\frac{1}{\lambda}+\frac{1}{\mu} = 1$, where $\lambda > 1$) and

$$F(y) = \frac{\eta^{h}(y)x^{h}(y)}{\vartheta^{h(k-\alpha)}(y)\left(\Omega^{\sigma}(y)\right)^{h(1-h)}} \quad \text{and} \quad G(y) = \left(\frac{x(y)}{\vartheta^{k-\alpha+1}(y)}\right)^{1-h}\left(\Omega^{\sigma}(y)\right)^{h(1-h)},$$

we see that

$$\begin{aligned} \left(\int_{w}^{\infty}F^{\frac{1}{h}}(y)\Delta_{\alpha}y\right)^{h} &= \left(\int_{w}^{\infty}\left(\frac{\eta^{h}(y)x^{h}(y)}{\vartheta^{h(k-\alpha)}(y)\left(\Omega^{\sigma}(y)\right)^{h(1-h)}}\right)^{\frac{1}{h}}\Delta_{\alpha}y\right)^{h} \\ &\geq \frac{\int_{w}^{\infty}F(y)G(y)\Delta_{\alpha}y}{\left(\int_{w}^{\infty}G^{\frac{1}{1-h}}(y)\Delta_{\alpha}y\right)^{1-h}} \\ &= \left(\int_{w}^{\infty}\left(\frac{\eta^{h}(y)\left(x(y)\vartheta^{-k+\alpha-1}(y)\right)^{1-h}x^{h}(y)\Omega^{\sigma}(y))^{h(1-h)}}{\vartheta^{h(k-\alpha)}(y)\left(\Omega^{\sigma}(y)\right)^{h(1-h)}}\right)\Delta_{\alpha}y\right) \\ &\times\left(\int_{w}^{\infty}\frac{x(y)\left(\Omega^{\sigma}(y)\right)^{h}}{\vartheta^{k-\alpha+1}(y)}\Delta_{\alpha}y\right)^{h-1} \end{aligned}$$

$$= \left(\int_w^\infty \left(\frac{\eta^h(y)x(y)}{\vartheta^{h(k-\alpha)}(y)\left(\vartheta^{k-\alpha+1}(y)\right)^{h-1}} \right) \Delta_\alpha y \right)$$
$$\times \left(\int_w^\infty \frac{x(y)\left(\Omega^\sigma(y)\right)^h}{\vartheta^{k-\alpha+1}(y)} \Delta_\alpha y \right)^{h-1}.$$

This means that

$$\int_w^\infty \left(\left(\frac{\eta^h(y)x^h(y)}{\vartheta^{h(k-\alpha)}(y)\left(\Omega^\sigma(y)\right)^{h(1-h)}} \right)^{\frac{1}{h}} \Delta_\alpha y \right)^h \geq \frac{\int_w^\infty \eta^h(y)x(y)\vartheta^{h-k+\alpha-1}(y)\Delta_\alpha y}{\left(\int_w^\infty \frac{x(y)\left(\Omega^\sigma(y)\right)^h}{\vartheta^{k-\alpha+1}(y)} \Delta_\alpha y \right)^{1-h}}, \tag{27}$$

by substitution (27) into (26), we get

$$\left(\int_w^\infty \frac{x(y)}{\vartheta^{k-\alpha+1}(y)} \left(\Omega^\sigma(y)\right)^h \Delta_\alpha y \right)^h \geq \left(\frac{h}{\alpha-k} \right)^h \frac{\int_w^\infty \eta^h(y)x(y)\vartheta^{h-k+\alpha-1}(y)\Delta_\alpha y}{\left(\int_w^\infty \frac{x(y)\left(\Omega^\sigma(y)\right)^h}{\vartheta^{k-\alpha+1}(y)} \Delta_\alpha y \right)^{1-h}}.$$

This means that

$$\int_w^\infty \frac{x(y)}{\vartheta^{k-\alpha+1}(y)} \left(\Omega^\sigma(y)\right)^h \Delta_\alpha y \geq \left(\frac{h}{\alpha-k} \right)^h \int_w^\infty x(y)\eta^h(y)\vartheta^{h-k+\alpha-1}(y)\Delta_\alpha y,$$

which the wanted inequality (20). □

Corollary 1. *If we put* $\alpha = 1$ *in Theorem 6, then we get*

$$\int_w^\infty \frac{x(y)}{\vartheta^{k}(y)} \left(\Omega^\sigma(y)\right)^h \Delta y \geq \left(\frac{h}{1-k} \right)^h \int_w^\infty x(y)\eta^h(y)\vartheta^{h-k}(y)\Delta y, \tag{28}$$

where $w \in (0,\infty)_{\mathbb{T}}, k \leq 0 < h < 1$,

$$\vartheta(y) = \int_y^\infty x(s)\Delta s \quad \text{and} \quad \Omega(y) = \int_w^y x(s)\eta(s)\Delta s,$$

which is (11) in the Introduction.

Remark 1. *If we take* $\mathbb{T} = \mathbb{R}$ *in Theorem 6, then:*

$$\int_w^\infty \frac{x(y)}{\vartheta^{k-\alpha+1}(y)} \left(\Omega(y)\right)^h y^{\alpha-1} dy \geq \left(\frac{h}{\alpha-k} \right)^h \int_w^\infty x(y)\eta^h(y)\vartheta^{h-k+\alpha-1}(y)\, y^{\alpha-1} dy, \tag{29}$$

where $w \in (0,\infty), k \leq 0 < h < 1, \vartheta(y) = \int_y^\infty x(s)s^{\alpha-1}ds$ and $\Omega(y) = \int_w^y x(s)\eta(s)s^{\alpha-1}ds$.

Remark 2. *Clearly, for* $\alpha = 1$ *and* $w = 1$, *Remark 1 coincides with Remark 1 in* [25].

Remark 3. *As a result, if* $\mathbb{T} = \mathbb{Z}$ *in (20), and* $k \leq 0 < h < 1$, *then*:

$$\sum_{y=w}^\infty \frac{x(w)}{\vartheta^{k-\alpha+1}(w)} \left(\Omega^\sigma(y)\right)^h y^{\alpha-1}\,(y+1)^{\alpha-1} \geq \left(\frac{h}{\alpha-k} \right)^h \sum_{y=w}^\infty x(w)\,\vartheta^{h-k+\alpha-1}(y)\,\eta^h(w)\,(y+1)^{\alpha-1}, \tag{30}$$

where, $\Omega^\sigma(y) = \Omega(y+1) = \sum_{s=w}^y x(s)\,\eta(s)(s+1)^{\alpha-1}$ *and* $\vartheta(y) = \sum_{s=y}^\infty x(s)\,(s+1)^{\alpha-1}$.

If $\alpha = 1$, then (30) becomes

$$\sum_{y=w}^\infty \frac{x(w)}{\vartheta^{k}(w)} \left(\Omega^\sigma(y)\right)^h \geq \left(\frac{h}{1-k} \right)^h \sum_{y=w}^\infty x(w)\,\vartheta^{h-k}(y)\,\eta^h(w), \tag{31}$$

where, $\Omega^\sigma(y) = \Omega(y+1) = \sum_{s=w}^y x(s)\,\eta(s)$ *and* $\vartheta(y) = \sum_{s=y}^\infty x(s)$. which is Remark 2 in [25].

Theorem 4. *Suppose that* $\mathbb{T}$ *be a time scale with* $w \in (0,\infty)_{\mathbb{T}}, 0 < h < 1 < k$ *and* $\alpha \in (0,1]$. *Assume that* $\vartheta(y)$ *is defined as in Theorem 6 such that:*

$$M := \inf_{y\in\mathbb{T}} \frac{\vartheta^{\sigma}(y)}{\vartheta(y)} > 0, \tag{32}$$

and define $\overline{\Omega}(y) = \int_y^{\infty} x(s)\eta(s)\Delta_\alpha s$. Then

$$\int_w^{\infty} \frac{x(y)}{\vartheta^{k-\alpha+1}(y)} \left(\overline{\Omega}(y)\right)^h \Delta_\alpha y \geq \left(\frac{hM^{k-\alpha+1}}{k-\alpha}\right)^h \int_w^{\infty} x(y)\eta^h(y)\vartheta^{h-k+\alpha-1}(y)\Delta_\alpha y. \tag{33}$$

Proof. Utilizing the formula of integration by parts (18) on

$$\int_w^{\infty} \frac{x(y)}{\vartheta^{k-\alpha+1}(y)} \left(\overline{\Omega}(y)\right)^h \Delta_\alpha y,$$

with $(y) = \left(\overline{\Omega}(y)\right)^h$ and $D_\alpha^\Delta \zeta(y) = \frac{x(y)}{\vartheta^{k-\alpha+1}(y)}$, we have

$$\int_w^{\infty} \frac{x(y)}{\vartheta^{k-\alpha+1}(y)} \left(\overline{\Omega}(y)\right)^h \Delta_\alpha y = \zeta(y)\left(\overline{\Omega}(y)\right)^h\Big|_w^{\infty} + \int_w^{\infty} \zeta^{\sigma}(y)\left(-D_\alpha^\Delta\left(\overline{\Omega}(y)\right)^h\right)\Delta_\alpha y,$$

where $\zeta(y) = \int_w^y \frac{x(s)}{\vartheta^{k-\alpha+1}(y)}\Delta_\alpha s$. This with $\overline{\Omega}(\infty) = 0$ *and* $\zeta(w) = 0$ implies that

$$\int_w^{\infty} \frac{x(y)}{\vartheta^{k-\alpha+1}(y)} \left(\overline{\Omega}(y)\right)^h \Delta_\alpha y = \int_w^{\infty} \zeta^{\sigma}(y)\left(-D_\alpha^\Delta\left(\overline{\Omega}(y)\right)^h\right)\Delta_\alpha y. \tag{34}$$

But utilizing chain rule, we obtain:

$$-D_\alpha^\Delta\left(\overline{\Omega}(y)\right)^h = -h\overline{\Omega}^{\mathrm{h}-1}(d) D_\alpha^\Delta \overline{\Omega}(y) = \frac{hx(y)\eta(y)}{\left(\overline{\Omega}(d)\right)^{1-\mathrm{h}}} \geq \frac{hx(y)\eta(y)}{\left(\overline{\Omega}(y)\right)^{1-\mathrm{h}}}. \tag{35}$$

Since $D_\alpha^\Delta \overline{\Omega}(y) = -x(y)\eta(y) \leq 0$ and $d \geq y$, we find that $\overline{\Omega}(y) \geq \overline{\Omega}(d)$. By substituting (35) into (34) and using that $D_\alpha^\Delta \zeta(y) \geq 0$, we get

$$\int_w^{\infty} \frac{x(y)}{\vartheta^{k-\alpha+1}(y)} (\overline{\Omega}(y))^h \Delta_\alpha y \geq h\int_w^{\infty} \zeta(y) \frac{x(y)\eta(y)}{(\overline{\Omega}(y))^{1-h}} \Delta_\alpha y. \tag{36}$$

Next note $D_\alpha^\Delta\,\vartheta(y) = -x(y) \leq 0$. By the chain rule, we have (note $k \leq 0$)

$$\begin{aligned}
&D_\alpha^\Delta\left(\vartheta^{\alpha-k}(y)\right) = (\alpha-k)\vartheta^{\alpha-k-1}(d) D_\alpha^\Delta\,\vartheta(y) \\
&= \frac{\alpha-k}{\vartheta^{k+1-\alpha}(d)} D_\alpha^\Delta\,\vartheta(y) \leq \frac{\alpha-k}{\vartheta^{k+1-\alpha}(y)} D_\alpha^\Delta\,\vartheta(y) \\
&\leq \frac{k-\alpha}{\vartheta^{k+1-\alpha}(d)} x(y) \leq \frac{k-\alpha}{\left(\vartheta^{\sigma}(y)\right)^{k+1-\alpha}} x(y) \\
&= (k-\alpha)\frac{x(y)}{\left(\vartheta^{\sigma}(y)\right)^{k+1-\alpha}} \frac{\vartheta^{k+1-\alpha}(y)}{\vartheta^{k+1-\alpha}(y)} \\
&\qquad \leq \frac{(k-\alpha)x(y)}{M^{k+1-\alpha}\left(\vartheta(y)\right)^{k+1-\alpha}}.
\end{aligned}$$

This implies that

$$\zeta(y) = \int_w^y \frac{x(x)}{\vartheta^{k+1-\alpha}(s)}\Delta_\alpha s \geq \left(\frac{M^{k+1-\alpha}}{k-\alpha}\right)\int_w^y D_\alpha^\Delta\left(\vartheta^{\alpha-k}(s)\right)\Delta_\alpha s = \left(\frac{M^{k+1-\alpha}}{k-\alpha}\right)\vartheta^{\alpha-k}(y). \tag{37}$$

By substituting (37) into (36) yields

$$\int_w^{\infty} \frac{x(y)}{\vartheta^{k+1-\alpha}(y)} (\overline{\Omega}(y))^{\mathrm{h}} \Delta_\alpha s \geq \left(\frac{hM^{k+1-\alpha}}{k-\alpha}\right)\int_w^{\infty} \frac{x\,(y)\eta(y)}{\vartheta^{\alpha-k}(y)\overline{\Omega}^{(1-\mathrm{h})}(y)}\Delta_\alpha y. \tag{38}$$

Raising (38) to the factor h, we get:

$$\left(\int_w^\infty \frac{x(y)}{\vartheta^{k+1-\alpha}(y)}(\bar{\Omega}(y))^{\mathrm{h}}\Delta_\alpha y\right)^h \geq \left(\frac{hM^{k+1-\alpha}}{k-\alpha}\right)^h \left(\int_w^\infty \left(\frac{x^{\,h}(y)\eta^h(y)}{\vartheta^{h(k-\alpha)}(y)\bar{\Omega}^{h(1-\mathrm{h})}(y)}\right)^{\frac{1}{h}} \Delta_\alpha y\right)^h. \quad (39)$$

The rest of the proof is identical to the proof of Theorem 6 and hence is deleted. □

Corollary 2. *If we put* $\alpha = 1$ *in Theorem* 8*, then:*

$$\int_w^\infty \frac{x(y)}{\vartheta^k(y)}(\bar{\Omega}(y))^{\mathrm{h}}\Delta y \geq \left(\frac{hM^k}{k-1}\right)^h \int_w^\infty x\,(y)\eta^h(y)\vartheta^{h-k}(y)\Delta(y), \quad (40)$$

where $w \in (0,\infty)_{\mathbb{T}}, 0 < h < 1 < k, \bar{\Omega}(y) = \int_y^\infty x(s)\eta\,(s)\Delta s$ *and* $\vartheta(y) = \int_y^\infty x(s)\Delta s$ such that

$$M := \inf_{y\in\mathbb{T}} \frac{\vartheta^\sigma(y)}{\vartheta(y)} > 0,$$

which is (12) in the Introduction

Remark 4. *If we take* $\mathbb{T} = \mathbb{R}\ (i,e,\sigma(y) = y)$ *in Theorem* 8*, then*

$$\int_w^\infty \frac{x(y)}{\vartheta^{k-\alpha+1}(y)}\left(\bar{\Omega}(y)\right)^{\mathrm{h}} y^{\alpha-1}dy \geq \left(\frac{h}{k-\alpha}\right)^h \int_w^\infty x\,(y)\eta^h(y)\vartheta^{h-k+\alpha-1}(y)y^{\alpha-1}dy, \quad (41)$$

where $0 < h < 1 < k, \alpha \in (0{,}1], \vartheta(y) = \int_y^\infty x(s)s^{\alpha-1}ds, \bar{\Omega}(y) = \int_y^\infty x(s)\eta(s)s^{\alpha-1}ds$ *and*

$$M := \inf_{y\in\mathbb{T}} \frac{\vartheta^\sigma(y)}{\vartheta(y)} = \inf_{y\in\mathbb{R}} \frac{\vartheta(y)}{\vartheta(y)} = 1.$$

If $\alpha = 1$ *and* $w = 1$, then (41) becomes:

$$\int_1^\infty \frac{x(y)}{\vartheta^k(y)}\left(\bar{\Omega}(y)\right)^h dy \geq \left(\frac{h}{k-1}\right)^h \int_1^\infty x\,(y)\eta^h(y)\vartheta^{h-k}(y)dy, \quad (42)$$

which is Remark 3 in [25].

Remark 5. *As a special case of* (33)*, when* $\mathbb{T} = \mathbb{Z}\ (i.e.\,\sigma(y) = y+1)$ *and* $0 < h < 1 < k$*, we get:*

$$\sum_{y=w}^\infty \frac{x(y)}{\vartheta^{k-\alpha+1}(y)}\left(\bar{\Omega}(y)\right)^h \geq \left(\frac{hM^k}{k-\alpha}\right)^h \sum_{y=w}^\infty x(y)\eta^h(y)\vartheta^{h-k+\alpha-1}(y), \quad (43)$$

where $\vartheta(y) = \sum_{s=y}^\infty x(s)\,(s+1)^{\alpha-1}$, and $\bar{\Omega}(y) = \sum_{s=w}^\infty x(s)\eta(s)\,(s+1)^{\alpha-1}$.

$$M := \inf_{y\in\mathbb{T}} \frac{\vartheta^\sigma(y)}{\vartheta(y)} = \inf_{y\in\mathbb{Z}} \frac{\vartheta(l+1)}{\vartheta(l)} > 0,$$

which is Remark 4 in [25], when $\alpha = 1$.

Theorem 5. *Suppose that* $\mathbb{T}$ *is a time scale with* $w \in (0,\infty)_{\mathbb{T}}, k \leq 0 < h < 1$ *and* $\alpha \in (0{,}1]$*. Assume that* $\Gamma(y) = \int_w^y x(s)\Delta_\alpha s$ *and* $\bar{\Omega}(y) = \int_y^\infty x(s)\eta(s)\,\Delta_\alpha s$*. Then*

$$\int_w^\infty \frac{x(y)}{\left(\Gamma^\sigma(y)\right)^{k-\alpha+1}}\left(\bar{\Omega}(y)\right)^h \Delta_\alpha y \geq \left(\frac{h}{\alpha-k}\right)^h \int_w^\infty x(y)\eta^h(y)\left(\Gamma^\sigma(y)\right)^{h-k+\alpha-1}\Delta_\alpha y. \quad (44)$$

Proof. Utilizing the formula of integration by parts (18) on

$$\int_w^\infty \frac{x(y)}{\left(\Gamma^\sigma(y)\right)^{k-\alpha+1}}\left(\bar{\Omega}(y)\right)^h \Delta_\alpha y,$$

with $v(y) = \left(\bar{\Omega}(y)\right)^h$ and $D_\alpha^\Delta \zeta(y) = \frac{x(y)}{\left(\Gamma^\sigma(y)\right)^{k-\alpha+1}}$, we get

$$\int_w^\infty \frac{x(y)}{\left(\Gamma^\sigma(y)\right)^{k-\alpha+1}} \left(\bar{\Omega}(y)\right)^h \Delta_\alpha y = \zeta(y)\left(\bar{\Omega}(y)\right)^h \Big|_w^\infty + \int_w^\infty \zeta^\sigma(y) \left(-D_\alpha^\Delta \left(\bar{\Omega}(y)\right)^h\right) \Delta_\alpha y,$$

where $\zeta(y) = \int_w^y \frac{x(s)}{\left(\Gamma^\sigma(s)\right)^{k-\alpha+1}} \Delta_\alpha s$. This with $\bar{\Omega}(\infty) = 0$ *and* $\Gamma(w) = 0$ imply that

$$\int_w^\infty \frac{x(y)}{\left(\Gamma^\sigma(y)\right)^{k-\alpha+1}} \left(\bar{\Omega}(y)\right)^h \Delta_\alpha y = \int_w^\infty \zeta^\sigma(y) \left(-D_\alpha^\Delta \left(\bar{\Omega}(y)\right)^h \Delta_\alpha y. \tag{45}$$

By utilizing chain rule, we get:

$$-D_\alpha^\Delta \left(\bar{\Omega}(y)\right)^h = -h\bar{\Omega}^{\mathrm{h}-1}(d) D_\alpha^\Delta \bar{\Omega}(\mathrm{y}) = \frac{hx(y)\eta(y)}{\bar{\Omega}^{1-h}(d)} \geq \frac{hx(y)\eta(y)}{\bar{\Omega}^{1-h}(y)}. \tag{46}$$

Since $D_\alpha^\Delta \bar{\Omega}(\mathrm{y}) = -x(y)\eta(y) \leq 0$. By substituting (46) into (45) and using that $D_\alpha^\Delta \zeta(\mathrm{y}) \geq 0$, we have:

$$\int_w^\infty \frac{x(y)}{\left(\Gamma^\sigma(y)\right)^{k-\alpha+1}} \left(\bar{\Omega}(y)\right)^h \Delta_\alpha y \geq h \int_w^\infty \zeta^\sigma(y) \frac{x(y)\eta(y)}{\bar{\Omega}^{1-\mathrm{h}}(y)} \Delta_\alpha y. \tag{47}$$

Next note $D_\alpha^\Delta \Gamma(\mathrm{y}) = x(y) \geq 0$. By the chain rule, we have (note $k \leq 0$)

$$D_\alpha^\Delta \left(\Gamma^{\alpha-k}(y)\right) = (\alpha - k)\Gamma^{\alpha-k-1}(d) D_\alpha^\Delta \Gamma(y)$$
$$= \frac{\alpha - k}{\Gamma^{k+1-\alpha}(d)} D_\alpha^\Delta \Gamma(y) \leq \frac{\alpha - k}{\Gamma^{k+1-\alpha}(y)} D_\alpha^\Delta \Gamma(y)$$
$$\leq \frac{\alpha-k}{\Gamma^{k+1-\alpha}(y)} x(y) \leq \frac{\alpha-k}{\left(\Gamma^\sigma(y)\right)^{k+1-\alpha}} x(y).$$

This implies that

$$\zeta^\sigma(y) = \int_w^{\sigma(y)} \frac{x(s)}{\left(\Gamma^\sigma(s)\right)^{k-\alpha+1}} \Delta_\alpha s \geq \frac{1}{\alpha - k} \int_w^{\sigma(y)} D_\alpha^\Delta \left(\Gamma(s)\right)^{\alpha-k} \Delta_\alpha s = \frac{1}{\alpha - k} \left(\Gamma^\sigma(y)\right)^{\alpha-k}. \tag{48}$$

By substituting (48) into (47) yields

$$\int_w^\infty \frac{x(y)}{\left(\Gamma^\sigma(y)\right)^{k-\alpha+1}} \left(\bar{\Omega}(y)\right)^h \Delta_\alpha y \geq \frac{h}{\alpha - k} \int_w^\infty \left(\Gamma^\sigma(y)\right)^{\alpha-k} \frac{x(y)\eta\,(y)}{\bar{\Omega}^{1-\mathrm{h}}(y)} \Delta_\alpha y. \tag{49}$$

Raises (49) to the factor h, we get:

$$\left(\int_w^\infty \frac{x(y)}{\left(\Gamma^\sigma(y)\right)^{k-\alpha+1}} \left(\bar{\Omega}(y)\right)^h \Delta_\alpha y\right)^h \geq \left(\frac{h}{\alpha-k}\right)^h \left(\int_w^\infty \left(\left(\Gamma^\sigma(y)\right)^{h(\alpha-k)} \frac{x^{\,h}(y)\eta^h(y)}{\bar{\Omega}^{h(1-\mathrm{h})}(y)}\right)^{\frac{1}{h}} \Delta_\alpha y\right)^h. \tag{50}$$

The rest of the proof is identical to the proof of Theorem 6 and hence is deleted. □

Corollary 3. *If we put* $\alpha = 1$ *in Theorem* 10, *then:*

$$\int_w^\infty \frac{x(y)}{\left(\Gamma^\sigma(y)\right)^k} \left(\bar{\Omega}(y)\right)^h \Delta\, y \geq \left(\frac{h}{1-k}\right)^h \int_w^\infty x(y)\eta^h(y)\left(\Gamma^\sigma(y)\right)^{h-k} \Delta\, y, \tag{51}$$

where $w \in (0, \infty)_{\mathbb{T}}, k \leq 0 < h < 1$,

$$\Gamma(y) = \int_w^y x(s)\Delta s \;\; and \;\; \bar{\Omega}(y) = \int_y^\infty x(s)\eta(s)\Delta s,$$

which is (13) in the Introduction.

Remark 6. *In Theorem 10, if we take* $\mathbb{T} = \mathbb{R}$ $(i.e. \sigma(y) = y)$, *then:*

$$\int_w^\infty \frac{x(y)}{\left(\Gamma(y)\right)^{k-\alpha+1}} \left(\bar{\Omega}(y)\right)^h y^{\alpha-1} dy \geq \left(\frac{h}{\alpha - k}\right)^h \int_w^\infty x(y)\eta^h(y) \left(\Gamma(y)\right)^{h-k+\alpha-1} y^{\alpha-1} dy, \tag{52}$$

where $w \in (0,\infty), k \le 0 < h < 1, \alpha \in (0,1]$

$$\Gamma(y) = \int_w^y x(s)s^{\alpha-1}ds \text{ and } \overline{\Omega}(y) = \int_y^\infty x(s)\eta(s)s^{\alpha-1}ds.$$

If $\alpha = 1$ and $w = 1$, then (52) becomes

$$\int_w^\infty \frac{x(y)}{(\Gamma(y))^k} \left(\overline{\Omega}(y)\right)^h dy \ge \left(\frac{h}{1-k}\right)^h \int_w^\infty x(y)\eta^h(y) \left(\Gamma(y)\right)^{h-k} dy, \tag{53}$$

where

$$\Gamma(y) = \int_1^y x(s)ds \text{ and } \overline{\Omega}(y) = \int_y^\infty x(s)\eta(s)ds,$$

which is Remark 5 in [25]

Remark 7. *As a special case of* (44) *when* $\mathbb{T} = \mathbb{Z}$ $(i.e. \sigma(y) = y+1)$, *we get:*

$$\sum_{y=w}^\infty \frac{x(y)}{(\Gamma^\sigma(y))^{k-\alpha+1}} \left(\overline{\Omega}(y)\right)^h (y+1)^{\alpha-1} \ge \left(\frac{h}{\alpha-k}\right)^h \sum_{y=w}^\infty x(y)\eta^h(y)\left(\Gamma^\sigma(y)\right)^{h-k+\alpha-1} (y+1)^{\alpha-1}, \tag{54}$$

where $\Gamma^\sigma(y) = \Gamma(y+1) = \sum_{s=w}^y x(s)\,(s+1)^{\alpha-1}$, $\overline{\Omega}(y) = \sum_{s=y}^\infty x(s)\eta(s)\,(s+1)^{\alpha-1}$, $k \le 0 < h < 1$.

For $\alpha = 1$ in (54), then we get the inequality in Remark 6 in [21].

Theorem 6. *Suppose that* $\mathbb{T}$ *is a time scale with* $w \in (0,\infty)_\mathbb{T}, 0 < h \le 1 < k$ *and* $\alpha \in (0,1]$. *Assume that* $\Gamma(y) = \int_w^y x(s)\Delta_\alpha s$ *such that*

$$L := \inf_{y\in\mathbb{T}} \frac{\Gamma(y)}{\Gamma^\sigma(y)} > 0, \tag{55}$$

and define $\overline{\Phi}(y) = \int_w^y x(s)\eta(s)\Delta_\alpha s$

$$\int_w^\infty \frac{x(y)}{(\Gamma^\sigma(y))^{k-\alpha+1}} \left(\overline{\Phi}(\sigma(y))\right)^h \Delta_\alpha y \ge \left(\frac{hL^{\alpha-k}}{k-\alpha}\right)^h \int_w^\infty x(y)\eta^h(y) \left(\Gamma^\sigma(y)\right)^{h-k+\alpha-1}(y)\Delta_\alpha y. \tag{56}$$

Proof. Utilizing the formula of integration by parts (18) on

$$\int_w^\infty \frac{x(y)}{(\Gamma^\sigma(y))^{k-\alpha+1}} \left(\overline{\Phi}(\sigma(y))\right)^h \Delta_\alpha y,$$

with $\zeta^\sigma(y) = \left(\overline{\Phi}(\sigma(y))\right)^h$ and $D_\alpha^\Delta v(y) = \frac{x(y)}{(\Gamma^\sigma(y))^{k-\alpha+1}}$, we have

$$\int_w^\infty \frac{x(y)}{(\Gamma^\sigma(y))^{k-\alpha+1}} \left(\overline{\Phi}(\sigma(y))\right)^h \Delta_\alpha y = v(y)\left(\overline{\Phi}(y)\right)^h\Big|_w^\infty + \int_w^\infty \left(-v(y)\right)\left(D_\alpha^\Delta(\overline{\Phi}(y))^h\right)\Delta_\alpha y,$$

where $v(y) = \int_w^y \frac{x(y)}{(\Gamma^\sigma(y))^{k-\alpha+1}}\Delta_\alpha s$. This with $\overline{\Phi}(\infty) = 0$ and $\Gamma(w) = 0$ imply that

$$\int_w^\infty \frac{x(y)}{(\Gamma^\sigma(y))^{k-\alpha+1}} \left(\overline{\Phi}(\sigma(y))\right)^h \Delta_\alpha y = \int_w^\infty \left(-v(y)\right)\left(D_\alpha^\Delta(\overline{\Phi}(y))^h\right)\Delta_\alpha y. \tag{57}$$

By utilizing chain rule, we obtain:

$$D_\alpha^\Delta(\overline{\Phi}(y))^h = h\left(\overline{\Phi}(y)\right)^{h-1}(d) D_\alpha^\Delta \overline{\Phi}(y) = \frac{hx(y)\eta(y)}{\overline{\Phi}^{1-h}(d)} \ge \frac{hx(y)\eta(y)}{\left(\overline{\Phi}(\sigma(y))\right)^{1-h}}. \tag{58}$$

Since $D_\alpha^\Delta \overline{\Phi}(y) = x(y)\eta(y) \ge 0$. By substituting (58) into (57) and using that $D_\alpha^\Delta \overline{\Phi}(y) \ge 0$, we have:

$$\int_w^\infty \frac{x(y)}{\left(\Gamma^\sigma(y)\right)^{k-\alpha+1}} \left(\overline{\Phi}(\sigma(y))\right)^h \Delta_\alpha y \geq h \int_w^\infty (-v(y)) \frac{x(y)\eta(y)}{\left(\overline{\Phi}(\sigma(y))\right)^{1-h}} \Delta_\alpha y. \tag{59}$$

Next note $D_\alpha^\Delta \Gamma(y) = x(y) \geq 0$. By the chain rule, we have (note $k \leq 0$)

$$\begin{aligned} D_\alpha^\Delta\left(\Gamma^{\alpha-k}(y)\right) &= (\alpha-k)\Gamma^{\alpha-k-1}(d) D_\alpha^\Delta \Gamma(y) \\ &= \frac{\alpha-k}{\Gamma^{k+1-\alpha}(d)} D_\alpha^\Delta \Gamma(y) \leq \frac{\alpha-k}{\Gamma^{k+1-\alpha}(y)} D_\alpha^\Delta \Gamma(y) \\ &\leq \frac{\alpha-k}{\Gamma^{k+1-\alpha}(y)} x(y) \leq \frac{\alpha-k}{(\Gamma^\sigma(y))^{k+1-\alpha}} x(y), \end{aligned}$$

And

$$\begin{aligned} -v(y) &= -\int_w^y \frac{x(s)}{(\Gamma^\sigma(s))^{k-\alpha+1}} \Delta_\alpha s \\ &\geq \frac{1}{k-\alpha} \int_w^y D_\alpha^\Delta (\Gamma^{\alpha-k}(s)) \Delta_\alpha s \\ &= \left(\frac{1}{k-\alpha}\right) - \left(\frac{\Gamma(y)}{\Gamma^\sigma(y)}\right)^{\alpha-k} \left(\Gamma^\sigma(y)\right)^{\alpha-k} \\ &\geq \left(\frac{L^{\alpha-k}}{k-\alpha}\right) \left(\Gamma^\sigma(y)\right)^{\alpha-k}, \end{aligned} \tag{60}$$

by substituting (60) into (59), we get

$$\int_w^\infty \frac{x(y)}{\left(\Gamma^\sigma(y)\right)^{k-\alpha+1}} \left(\overline{\Phi}(\sigma(y))\right)^h \Delta_\alpha y \geq \left(\frac{hL^{\alpha-k}}{k-\alpha}\right) \int_w^\infty \left(\Gamma^\sigma(y)\right)^{\alpha-k} \frac{x(y)\eta(y)}{\left(\overline{\Phi}(\sigma(y))\right)^{1-h}} \Delta_\alpha y. \tag{61}$$

Raises (61) to the factor h, we get

$$\left(\int_w^\infty \frac{x(y)}{\left(\Gamma^\sigma(y)\right)^{k-\alpha+1}} \left(\overline{\Phi}(\sigma(y))\right)^h \Delta_\alpha y\right)^h \geq \left(\frac{hL^{\alpha-k}}{k-\alpha}\right)^h \left(\int_w^\infty \left(\left(\Gamma^\sigma(y)\right)^{h(\alpha-k)} \frac{x^h(y)\eta^h(y)}{\left(\overline{\Phi}^\sigma(y)\right)^{h(1-h)}}\right)^{\frac{1}{h}} \Delta_\alpha y\right)^h.$$

The rest of the proof is identical to the proof Theorem 6 and hence is deleted. □

Corollary 4. *If we put* $\alpha = 1$ *in Theorem* 12, *then:*

$$\int_w^\infty \frac{x(y)}{\left(\Gamma^\sigma(y)\right)^k} \left(\overline{\Phi}(\sigma(y))\right)^h \Delta y \geq \left(\frac{hL^{\alpha-k}}{k-\alpha}\right)^h \int_w^\infty x(y)\eta^h(y)\left(\Gamma^\sigma(y)\right)^{h-k} \Delta y, \tag{62}$$

where $w \in (0,\infty)_\mathbb{T}, 0 < h \leq 1 < k, \overline{\Phi}(y) = \int_w^y x(s)\eta(s)\Delta s$ and $\Gamma(y) = \int_w^y x(s)\Delta_\alpha s$ such that

$$L := \inf_{y \in \mathbb{T}} \frac{\Gamma(y)}{\Gamma^\sigma(y)} > 0,$$

which is (14) in the Introduction

Remark 8. *If we take* $\mathbb{T} = \mathbb{R}$ $(i.e., \sigma(y) = y)$ *in Theorem* 12, *then:*

$$\int_w^\infty \frac{x(y)}{\left(\Gamma\,(y)\right)^{k-\alpha+1}} \left(\overline{\Phi}(y)\right)^h y^{\alpha-1} dy \geq \left(\frac{h}{k-\alpha}\right)^h \int_w^\infty x(y)\eta^h(y)\left(\Gamma(y)\right)^{h-k+\alpha-1}(y) y^{\alpha-1} dy, \tag{63}$$

where $w \in (0,\infty)_\mathbb{T}, 0 < h \leq 1 < k, \overline{\Phi}(y) = \int_w^y x(s)\eta(s)s^{\alpha-1}ds$ and $\Gamma(y) = \int_w^y x(s)s^{\alpha-1}ds$ such that

$$L := \inf_{y \in \mathbb{T}} \frac{\Gamma(y)}{\Gamma^\sigma(y)} = \inf_{y \in \mathbb{R}} \frac{\Gamma(y)}{\Gamma(y)} = 1.$$

If $\alpha = 1$ *and* $w = 1$, then (63) becomes

$$\int_1^{\infty} \frac{x(y)}{\left(\Gamma(y)\right)^k}\left(\bar{\Phi}(y)\right)^h dy \geq \left(\frac{h}{k-1}\right)^h \int_1^{\infty} x(y)\eta^h(y)\left(\Gamma(y)\right)^{h-k} dy, \tag{64}$$

where

$$\Gamma(y) = \int_1^y x(s)ds \quad and \quad \bar{\Phi}(y) = \int_1^y x(s)\,\eta(s)ds,$$

which is Remark 7 in [25]

Remark 9. *As a special case of* (56)*, when* $\mathbb{T} = \mathbb{Z}\ (i.e. \sigma(y) = y+1)$ *and* $0 < h \leq 1 < k$*, we get:*

$$\sum_{w=y}^{\infty} \frac{x(y)}{\left(\Gamma^{\sigma}(y)\right)^{k-\alpha+1}}\left(\bar{\Phi}(\sigma(y))\right)^h (y+1)^{\alpha-1} \geq \left(\frac{hL^{\alpha-k}}{k-\alpha}\right)^h \sum_{w=y}^{\infty} x(y)\eta^h(y)\left(\Gamma^{\sigma}(y)\right)^{h-k+\alpha-1}(y)\ y+1^{\alpha-1}, \tag{65}$$

where, $\Gamma^{\sigma}(y) = \Gamma(y+1) = \sum_{s=w}^{y} x(s)\,(s+1)^{\alpha-1}$, $\bar{\Phi}(\sigma(y)) = \bar{\Phi}(y+1) = \sum_{s=w}^{y} x(s)\eta(s)\,(s+1)^{\alpha-1}$.

$$L \coloneqq \inf_{y\in\mathbb{T}} \frac{\Gamma(y)}{\Gamma^{\sigma}(y)} = \inf_{y\in\mathbb{Z}} \frac{\Gamma(y)}{\Gamma(y+1)},$$

which is Remark 8 in [25], when $\alpha = 1$.

Applications

The applications of quantum calculus play an important role in mathematics and the field of natural sciences, such as physics and chemistry. It has many applications in orthogonal polynomials, number theory, quantum theory, etc. In this section, some example for Reverse Coposn's Inequalities in fractional quantum calculus are selected to fulfil the applicability of the obtained results.

Now, we give an example using the time scale $\mathbb{T} = q^{\mathbb{N}_0}, q > 1$ which is a time scale with interesting applications in quantum calculus.

Example 1. *(Quantum calculus case 1.): Let* $\mathbb{T} = q^{\mathbb{N}_0} = \{t: t = q^n, n \in \mathbb{N}_0, q > 1\}$*. Then for all* $t \in q^{\mathbb{N}_0}$*, we have*

$$\sigma(t) = qt, \mu(t) = (q-1)t \quad and \quad \int_a^b f(t)\Delta_{\alpha} t = \sum_{n=\log_q a}^{\log_q b-1} f(q^n)\mu(q^n)(q^n)^{\alpha-1},\ \forall\, \mathrm{a,b} \in q^{\mathbb{N}_0} \tag{66}$$

Now, with the help of Theorem 3 and the above identities in (66), we can deduce

$$\sum_{n=\log_q w}^{\infty} \frac{x(q^n)}{\vartheta^{k-\alpha+1}(q^n)} (\Omega^{\sigma}(q^n))^h \mu(q^n)(q^n)^{\alpha-1} \geq \left(\frac{h}{\alpha-k}\right)^h \sum_{n=\log_q w}^{\infty} x(q^n)\eta^h(q^n)\left(\vartheta(q^n)\right)^{h-k+\alpha-1} \mu(q^n)(q^n)^{\alpha-1}.$$

where,

$$\Omega^{\sigma}(q^n) = \Omega\big(\sigma(q^n)\big) = \int_w^{\sigma(q^n)} x(s)\eta(s)\Delta_{\alpha} s = \sum_{m=\log_q w}^{\log_q \sigma(q^n)-1} x(q^m)\eta(q^m)\mu(q^m)\,(q^m)^{\alpha-1},$$

and

$$\vartheta(q^n) = \int_{q^n}^{\infty} x(s)\Delta_{\alpha} s = \sum_{m=\log_q q^n}^{\infty} x(q^m)\mu(q^m)\,(q^m)^{\alpha-1}$$

For an application of Theorem 4, we give the following example**.**

Example 2. *(Quantum calculus case 2.): Let* $\mathbb{T} = q^{\mathbb{N}_0}, q > 1$*, then the relation* (66) *is satisfy. Hence, we have:*

$$\sigma(t) = qt, \mu(t) = (q-1)t \quad and \quad \int_a^b f(t)\Delta_{\alpha} t = \sum_{n=\log_q a}^{\log_q b-1} f(q^n)\mu(q^n)(q^n)^{\alpha-1},\ \forall\, \mathrm{a,b} \in q^{\mathbb{N}_0}$$

Now, with the help of Theorem 4 and the above identities in (66), we can deduce:

$$\sum_{n=\log_q w}^{\infty} \frac{x(q^n)}{\vartheta^{k-\alpha+1}(q^n)} (\bar{\Omega}(q^n))^h \mu(q^n)(q^n)^{\alpha-1} \geq \left(\frac{hM^{k-\alpha+1}}{k-\alpha}\right)^h \sum_{n=\log_q w}^{\infty} x(q^n)\eta^h(q^n)\left(\vartheta(q^n)\right)^{h-k+\alpha-1}\mu(q^n)(q^n)^{\alpha-1}.$$

where, $\bar{\Omega}(q^n) = \int_{q^n}^{\infty} x(s)\eta(s)\Delta_\alpha s = \sum_{m=\log_q q^n}^{\infty} x(q^m)\eta(q^m)\mu(q^m)\,(q^m)^{\alpha-1}$. $M := \inf_{q^n \in \mathbb{T}} \frac{\vartheta^\sigma(q^n)}{\vartheta(q^n)} > 0$, $\vartheta(q^n)$ is defined in the above example.

Note that. By using theorems 10 and 12, we can apply the technique used in the above examples to obtain different applications. In addition, the above result is important not only for arbitrary time scales, but also for quantum calculus.

4. Conclusions and Future Work

The new fractional calculus on timescales is presented with applications in new fractional inequalities on timescales like Hardy, Bennett, Copson, and Leindler types. Inequalities are considered in rather general forms and contain several special integral and discrete inequalities. The technique is based on the applications of well-known inequalities and new tools from fractional calculus. In future research, we will continue to generalize more dynamic inequalities by using Specht's ratio, Kantorovich's ratio, functional generalization, and n-tuple fractional diamond- α integral. It will be interesting to find the inequalities in α,β-symmetric quantum and stochastic calculus.

Author Contributions: M.Z. and H.M.R. contributed by preparing the introduction, preliminaries, and formulating (theorem 6, theorem 8) its proof and corollaries (cor.7, cor. 9 and cor.10). M.A. contributed by preparing the introduction, preliminaries, and formulating (theorem 10) its proof and corollaries (cor.11). G.A. contributed by preparing the introduction, preliminaries, and formulating theorem 12, its proof and corollaries (cor.7). C.C. and H.A.A.E.-H. contributed by preparing the introduction, preliminaries, and formulating theorem 6, its proof and corollaries (cor. 13). All authors contributed equally to the writing of the manuscript. All authors have read and agreed to the published version of the manuscript.

Funding: The authors extend their appreciation to the Deanship of Scientific Research at King Khalid University for funding this work through the General Research Project, grant number GRP/56/42.

Data Availability Statement: No data were used to support this study.

Acknowledgments: The authors extend their appreciation to the Deanship of Scientific Research at King Khalid University for funding this work through the General Research Project, grant number GRP/56/42.

Conflicts of Interest: The authors declare no conflict of interest.

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