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Numerical Solutions Caused by DGJIM and ADM Methods for Multi-Term Fractional BVP Involving the Generalized ψ-RL-Operators

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Abstract: In this research study, we establish some necessary conditions to check the uniqueness-existence of solutions for a general multi-term ψ-fractional differential equation via generalized ψ-integral boundary conditions with respect to the generalized asymmetric operators. To arrive at such purpose, we utilize a procedure based on the fixed-point theory. We follow our study by suggesting two numerical algorithms called the Dafterdar-Gejji and Jafari method (DGJIM) and the Adomian decomposition method (ADM) techniques in which a series of approximate solutions converge to the exact ones of the given ψ-RLFBVP and the equivalent ψ-integral equation. To emphasize for the compatibility and the effectiveness of these numerical algorithms, we end this investigation by providing some examples showing the behavior of the exact solution of the existing ψ-RLFBVP compared with the approximate ones caused by DGJIM and ADM techniques graphically.

Keywords: ADM numerical method; DGJIM numerical method; boundary value problem; existence

MSC: (2020) 34A08; 65R20

1. Introduction

Fractional calculus is extending quickly and its attractive applications are completely used in various parts of the applied science [1–5]. It has appeared in financial structures [6], optimal control [7], epidemiological models [8,9], chaotic systems [10], engineering [11–14] and etc. Specifically, the fractional configurations of boundary problems (FBVPs) usually give a vast diversity of mathematical modelings for description of physical, chemical and biological processes which one can refer to them in the newly-published articles [15–19]. Besides these actual models caused by real phenomena, many researchers do research on the existence theory of solutions for general constructions of FBVPs furnished with boundary conditions involving multi-point nonlocal integral ones [20–29]. The most number of mathematicians have also worked on the numerical approaches to obtain the analytical and the approximate solutions of given FBVPs. In recent years, numerous numerical techniques are presented by researchers that have improved the convergence rate and errors resulting from the approximate solutions. If we aim to refer to some examples of these methods and how to apply them, we can consider some techniques based on Haar Wavelet Method [30,31], HATM and q-HATM [32,33], Bernstein Polynomials [34], Iterative Reproducing Kernel Hilbert Space Method [35] and etc.

Since fractional multi-term differential equations have appeared in vast applied domains, many mathematicians and researchers have begun to study the specifications and numerical solutions of this kind of FDEs. Additionally, because most of the time, the exact solution cannot be obtained, or it is very difficult to find it, so numerous numerical approaches have been utilized for such FBVPs to yield the approximate solutions. For example, in 2016, Bolandtalat, Babolian and Jafari [36] compared the convergence effects of exact and numerical solutions of multi-order FDEs by terms of Boubaker polynomials. In the same year, Hesameddini, Rahimi, Asadollahifard [37] gave a novel configuration of a reliable algorithm to solve multi-order FDEs and controlled its convergence. Firoozjaee, Yousefi, Jafari and Baleanu [38] followed a numerical procedure for a multi order FDE furnished with a combination of boundary-initial conditions. In 2017, Dabiri and Butcher [39] suggested a numerical algorithm relying on the spectral collocation method and generated the numerical solutions of given multi-order FDEs via multiple delays.

Besides these, in recent decades, many FBVPs involving abstract integral boundary conditions are formulated by many mathematicians, and they investigated the existence and the stability theory on their possible solutions. In 2018, Padhi, Graef and Pati [40] worked on the properties of the positive solutions for a fractional equation furnished with Riemann–Stieltjes integral conditions

$$\begin{cases} \mathfrak{D}^{\varrho}_{0^+} v(z) + \psi(z) h\big(z, v(z)\big) = 0, \quad z \in (0,1), \\ v(0) = v'(0) = \cdots = v^{(k-2)}(0) = 0, \qquad \mathfrak{D}^{\varpi}_{0^+} v(1) = \displaystyle\int_0^1 \varphi(r, v(r))\, \mathrm{d}A(r), \end{cases}$$

provided $\varrho \in (k-1, k]$ with $k > 2$ and $1 \le \varpi \le \varrho - 1$. In 2021, Thabet, Etemad and Rezapour [41] discussed the concept of the existence of solutions to a fractional system of the coupled Caputo conformable BVPs of the pantograph equations in the following format

$$\begin{cases} {}^{\mathcal{CC}}\mathfrak{D}^{\varrho,\sigma_1^*}_{z_0} v(z) = \tilde{\mathcal{P}}_1(z, m(z), m(\ell z)), \quad z \in [z_0, \tilde{K}],\ z_0 \ge 0, \\ {}^{\mathcal{CC}}\mathfrak{D}^{\varrho,\sigma_2^*}_{z_0} m(z) = \tilde{\mathcal{P}}_2(z, v(z), v(\ell z)), \end{cases}$$

via three-point-Riemann–Liouville (RL)-conformable integral conditions

$$\begin{cases} v(z_0) = 0, \qquad c_1 v(\tilde{K}) + c_2 {}^{\mathcal{RC}}\mathfrak{I}^{\varrho,\theta^*}_{z_0} v(\delta) = w_1^*, \\ m(z_0) = 0, \qquad c_1^* m(\tilde{K}) + c_2^* {}^{\mathcal{RC}}\mathfrak{I}^{\varrho,\theta^*}_{z_0} m(\nu) = w_2^*, \end{cases}$$

where $\varrho \in (0,1]$, $\sigma_1^*, \sigma_2^* \in (1,2)$, $\delta, \nu \in (z_0, \tilde{K})$, $c_1, c_2, c_1^*, c_2^*, w_1^*, w_2^* \in \mathbb{R}$, $\ell \in (0,1)$ and $\tilde{\mathcal{P}}_1, \tilde{\mathcal{P}}_2 \in C([z_0, \tilde{K}] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$. As you observed, in all above fractional FBVPs endowed with different combinations of integral boundary conditions, only the necessary criteria of solution's existence have been discussed and FBVPs have not been solved numerically. Hence, in order of the complexity of the structure of such FBVPs involving integral boundary conditions, and due to the difficulty of the computation process of the exact solution, several numerical algorithms and techniques were substituted to find approximate and analytical solutions.

In 2005, Dafterdar-Gejji and Jafari [42] used the Adomian decomposition method (ADM) to find solutions of a generalized structure of an initial system of multi-order FDEs. In 2006, these two mathematician [43] formulated an iterative algorithm to solve a functional equation approximately and called it as the Dafterdar-Gejji and Jafari method (DGJIM). Among other numerical techniques, these two methods DGJIM and ADM are known as two approximate tools with high accuracy and rapid convergence to the exact solution. For more information, we can refer to some works in the relevant field [44–46]. We shall apply these strong numerical techniques to approximate possible solutions of our generalized FBVP.

With the help of above ideas and mentioned techniques, in this research, we propose a generalized integral ψ-FBVP of the multi-term differential equation in the format of asymmetric ψ-RL-derivatives displayed as

$$\begin{cases} \mathfrak{D}_{0^+}^{\varrho;\psi}u(z) = \hat{\hbar}\big(z, u(z), \mathfrak{D}_{0^+}^{\sigma_1;\psi}u(z), \mathfrak{D}_{0^+}^{\sigma_2;\psi}u(z), \dots, \mathfrak{D}_{0^+}^{\sigma_n;\psi}u(z)\big), \\ u(0) = 0, \\ u(1) = p\mathfrak{I}_{0^+}^{\mu;\psi}k_1(\xi, u(\xi)) + q\mathfrak{I}_{0^+}^{\nu;\psi}k_2(\eta, u(\eta)), \end{cases} \tag{1}$$

where $0 \le z \le 1$, $1 < \varrho < 2$, $0 < \sigma_1 < \sigma_2 < \dots < \sigma_n < 1$, $\varrho > \sigma_n + 1$, $\hat{\hbar} : [0,1] \times \mathbb{R}^{n+1} \to \mathbb{R}$, $k_j : [0,1] \times \mathbb{R} \to \mathbb{R}$, $(j = 1,2)$ are continuous functions, $\mathfrak{D}_{0^+}^{\varrho;\psi}, \mathfrak{D}_{0^+}^{\sigma_1;\psi}, \dots, \mathfrak{D}_{0^+}^{\sigma_n;\psi}$ are the ψ-RL-derivatives depending on an increasing function ψ of orders $\varrho, \sigma_1, \dots, \sigma_n$, respectively, and $\mathfrak{I}_{0^+}^{\gamma;\psi}$ is the ψ-RL-integral depending on the special function ψ of order $\gamma \in \{\mu, \nu\}$ with $\mu, \nu, p, q > 0$ and $0 < \xi, \eta \le 1$. To begin, we first establish the corresponding ψ-integral equation of the given multi-term ψ-RLFBVP (1) by means of a theoretical proof and then verify the existence-uniqueness results with the aid of the fixed point approach. After that, we propose two numerical iterative algorithms named the DGJIM along with the ADM to find solutions approximately.

In fact, we must emphasize that the novelty and the main motivation of our research is that unlike other articles in which the ADM and the DGJIM techniques have been utilized to solve initial value problems, we here aim to explore approximate solutions for a complicated generalized multi-term ψ-RLFBVP subject to boundary conditions endowed with the generalized ψ-RL-fractional integrals. Also, notice that in the second boundary condition, the value of the unknown function at the endpoint $z = 1$ is proportional to a linear combination of ψ-RL-integrals via different orders $\mu, \nu > 0$ at the intermediate points $z = \xi, \eta \in (0,1)$, respectively. Besides this, we consider the nonlinear term $\hat{\hbar}$ as a multivariable function including the finite number of multi-order ψ-RL-derivatives. This research presents two numerical methods based on simple algorithms which help us to obtain approximate solutions of different generalized fractional mathematical models caused by real phenomena.

The organization of the next sections and subsections of this research is arranged by the following structure: Section 2 reviews fundamental notions on fractional calculus. Section 3 is devoted to establishing some criteria and conditions proving the solutions' existence. Section 4 proposes two numerical iterative ADM and DGJIM techniques in two distinct subsections. In Section 5, the mentioned approximate techniques are employed to show the compatibility and the accuracy of them in two separate stimulative examples. The conclusion summarizes our approach in Section 6.

2. Preliminaries

In the first place, for convenience of the readers, we need some fundamental properties and lemmas on the fractional calculus which are utilized further in this research.

Definition 1. *[2] Let $\varrho > 0$ and $\phi : (0, +\infty) \longrightarrow \mathbb{R}$. The following integral*

$$(\mathfrak{I}_{0^+}^{\varrho}\phi)(t) = \frac{1}{\Gamma(\varrho)} \int_0^t (t-s)^{\varrho-1}\phi(s)ds,$$

is called the Riemann–Liouville (RL-integral) fractional integral of order ϱ, if the integral in the right side exists.

Definition 2. *[2] The RL-derivative of order ϱ for a continuous function $\phi : (0, +\infty) \longrightarrow \mathbb{R}$ is presented as*

$$\mathfrak{D}^{\varrho}_{0^+}\phi(t) = \frac{1}{\Gamma(n-\varrho)}\left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\varrho-1}\phi(s)ds = \left(\frac{d}{dt}\right)^n \mathfrak{I}^{n-\varrho}_{0^+}\phi(t),$$

where $n = [\varrho] + 1$.

Definition 3. *[1] Let $\phi : [a,b] \to \mathbb{R}$ be integrable and $\psi \in C^n[a,b]$ be an increasing mapping which satisfies $\psi'(z) \neq 0$, for any $z \in [a,b]$. The ψ-RL-integral of order $\varrho > 0$ of the function ϕ is represented as*

$$\mathfrak{I}^{\varrho;\psi}_{a^+}\phi(z) = \frac{1}{\Gamma(\varrho)}\int_a^z \psi'(s)\big(\psi(z) - \psi(s)\big)^{\varrho-1}\phi(s)ds,$$

and the ψ-RL-derivative of order $\varrho > 0$ for the same function is defined by

$$\begin{aligned}\mathfrak{D}^{\varrho;\psi}_{a^+}\phi(z) &= \left(\frac{1}{\psi'(z)}\frac{d}{dz}\right)^n \mathfrak{I}^{n-\varrho;\psi}_{a^+}\phi(z)\\ &= \frac{1}{\Gamma(n-\varrho)}\left(\frac{1}{\psi'(z)}\frac{d}{dz}\right)^n \int_a^z \psi'(s)\big(\psi(z)-\psi(s)\big)^{n-\varrho-1}\phi(s)ds,\end{aligned}$$

where $n = [\varrho] + 1$.

The following semigroup specification holds. For all $\mu, \nu > 0$

$$\mathfrak{I}^{\mu;\psi}_{a^+}\mathfrak{I}^{\nu;\psi}_{a^+}u(z) = \mathfrak{I}^{\mu+\nu;\psi}_{a^+}u(z).$$

If $\psi(z) = z$, then ψ-operators displayed in Definition 3 coincide respectively with the classical RL-integral and RL-derivative given in Definitions 1 and 2. When $\psi(z) = \ln z$, then the same ψ-operators reduce repectively to the Hadamard fractional integral and Hadamard fractional derivative [1–3].

Definition 4. *[1] Let $\psi \in C^n[a,b]$ such that $\psi'(z) \neq 0$ for all $z \in [a,b]$. Then we define*

$$AC^{n;\psi}[a,b] = \left\{\delta : [a,b] \to \mathbb{R}, \quad \delta^{[n-1]} = \left(\frac{1}{\psi'(z)}\frac{d}{dz}\right)^{n-1}, \ \delta \in AC[a,b]\right\}.$$

Lemma 1. *[1] Let $\mu > 0$ and $\nu > 0$. If $u(r) = \big[\psi(r) - \psi(a)\big]^{\nu-1}$, then*

$$\left(\mathfrak{D}^{\mu;\psi}_{a^+}u(r)\right)(z) = \frac{\Gamma(\nu)}{\Gamma(\nu-\mu)}\big[\psi(z)-\psi(a)\big]^{\nu-\mu-1}, \tag{2}$$

and

$$\left(\mathfrak{I}^{\mu;\psi}_{a^+}u(r)\right)(z) = \frac{\Gamma(\nu)}{\Gamma(\nu+\mu)}\big[\psi(z)-\psi(a)\big]^{\mu+\nu-1}. \tag{3}$$

As a particular case of (2) and (3), we have respectively the following expressions

$$\left(\mathfrak{D}^{\mu;z^\sigma}_{0^+}r^{\sigma(\nu-1)}\right)(z) = \frac{\Gamma(\nu)}{\Gamma(\nu-\mu)}z^{\sigma(\nu-\mu-1)},$$

and

$$\left(\mathfrak{I}_{0^+}^{\mu;z^\sigma} r^{\sigma(\nu-1)}\right)(z) = \frac{\Gamma(\nu)}{\Gamma(\nu+\mu)} z^{\sigma(\mu+\nu-1)}.$$

Lemma 2. *[47] Let $\varrho > n$ with $n \in \mathbb{N}$. Then*

$$\left(\frac{1}{\psi'(z)}.\frac{d}{dz}\right)^n \mathfrak{I}_{a^+}^{\varrho;\psi}\phi(z) = \mathfrak{I}_{a^+}^{\varrho-n;\psi}\phi(z).$$

Lemma 3. *[47] Let $\mu > \nu$, $n-1 < \nu < n$, $n \in \mathbb{N}$. Then*

$$\mathfrak{D}_{a^+}^{\nu;\psi}\mathfrak{I}_{a^+}^{\mu;\psi}\phi(z) = \mathfrak{I}_{a^+}^{\mu-\nu;\psi}\phi(z);$$

In particular

$$\mathfrak{D}_{a^+}^{\mu;\psi}\mathfrak{I}_{a^+}^{\mu;\psi}\phi(z) = \phi(z).$$

Lemma 4. *[47] Let $\mu > 0$, $n = [\mu]+1$, $\phi \in L[a,b]$ and $\mathfrak{I}_{a^+}^{\mu;\psi}\phi \in AC^{n;\psi}[a,b]$. Then*

$$\left(\mathfrak{I}_{a^+}^{\mu;\psi}\mathfrak{D}_{a^+}^{\mu;\psi}\right)\phi(z) = \phi(z) - \sum_{j=1}^{n} \frac{\mathfrak{I}_{a^+}^{j-\mu;\psi}\phi(a)}{\Gamma(\mu-j+1)}(\psi(z)-\psi(a))^{\mu-j}.$$

In special case if $0 < \mu < 1$, we have

$$\left(\mathfrak{I}_{a^+}^{\mu;\psi}\mathfrak{D}_{a^+}^{\mu;\psi}\right)\phi(z) = \phi(z) - \frac{\mathfrak{I}_{a^+}^{1-\mu;\psi}\phi(a)}{\Gamma(\mu)}(\psi(z)-\psi(a))^{\mu-1}.$$

Lemma 5. *[47] Let $\varrho > 0$ and $\mathfrak{D}_{a^+}^{\varrho;\psi}\phi \in AC^{n;\psi}[a.b] \cap L^1[a,b]$, then*

$$\mathfrak{I}_{a^+}^{\varrho;\psi}\mathfrak{D}_{a^+}^{\varrho;\psi}\phi(z) = \phi(z) + k_1(\psi(z)-\psi(a))^{\varrho-1} + k_2(\psi(z)-\psi(a))^{\varrho-2} + \cdots + k_n(\psi(z)-\psi(a))^{\varrho-n},$$

where $k_1, \ldots, k_n \in \mathbb{R}$ and $n = [\varrho]+1$.

3. Results on the Existence Criteria

In this part, we first derive a ψ-integral equation corresponding to the suggested multi-term ψ-RLFBVP (1) and then establish some conditions to admit the existence of solutions for (1).

Definition 5. *We call $u(z)$ as a solution for the supposed multi-term ψ-RLFBVP (1) if u satisfies (1) and $\mathfrak{D}_{0^+}^{\varrho;\psi}u(z) \in AC^{n;\psi}[0,1]$ and $u(z) \in AC^{n;\psi}[0,1]$.*

Theorem 1. *Let $1 < \varrho < 2$, $0 < \sigma_1 < \sigma_2 < \cdots < \sigma_n < 1$, $\varrho > \sigma_n + 1$, $\mu, \nu, p, q > 0$ and $0 < \xi, \eta < 1$. Then a function $u(z)$ is a solution of the multi-term ψ-RLFBVP*

$$\begin{cases} \mathfrak{D}_{0^+}^{\varrho;\psi}u(z) = \hbar\left(z, u(z), \mathfrak{D}_{0^+}^{\sigma_1;\psi}u(z), \mathfrak{D}_{0^+}^{\sigma_2;\psi}u(z), \ldots, \mathfrak{D}_{0^+}^{\sigma_n;\psi}u(z)\right), \\ u(0) = 0, \\ u(1) = p\mathfrak{I}_{0^+}^{\mu;\psi}k_1(\xi, u(\xi)) + q\mathfrak{I}_{0^+}^{\nu;\psi}k_2(\eta, u(\eta)), \end{cases} \tag{4}$$

if and only if $v(z) = \mathfrak{D}_{0^+}^{\sigma_n;\psi}u(z)$ satisfies the following ψ-integral equation

$$
\begin{aligned}
v(z) \;=\;& \mathfrak{I}_{0^+}^{\varrho-\sigma_n;\psi}\hat{h}\big(z,\mathfrak{I}_{0^+}^{\sigma_n;\psi}v(z),\mathfrak{I}_{0^+}^{\sigma_n-\sigma_1;\psi}v(z),\ldots,v(z)\big) \\
&+\frac{\Gamma(\varrho)}{\Gamma(\varrho-\sigma_n)\big(\psi(1)-\psi(0)\big)^{\varrho-1}}\Bigg[\frac{p}{\Gamma(\mu)}\int_0^{\xi}\psi'(s)\big(\psi(\xi)-\psi(s)\big)^{\mu-1}k_1(s,\mathfrak{I}_{0^+}^{\sigma_n;\psi}v(s))ds \\
&+\frac{q}{\Gamma(\nu)}\int_0^{\eta}\psi'(s)\big(\psi(\eta)-\psi(s)\big)^{\nu-1}k_2(s,\mathfrak{I}_{0^+}^{\sigma_n;\psi}v(s))ds \\
&-\mathfrak{I}_{0^+}^{\varrho;\psi}\hat{h}\big(z,\mathfrak{I}_{0^+}^{\sigma_n;\psi}v(z),\mathfrak{I}_{0^+}^{\sigma_n-\sigma_1;\psi}v(z),\ldots,v(z)\big)\Big|_{z=1}\Bigg]\big(\psi(z)-\psi(0)\big)^{\varrho-\sigma_n-1}.
\end{aligned} \tag{5}
$$

Proof. Let $u(z)\in AC^{n;\psi}[0,1]$ be a solution of the multi-term ψ-RLFBVP (4), then we get $v(z)=\mathfrak{D}_{0^+}^{\sigma_n;\psi}u(z)\in AC^{n;\psi}[0,1]$. Taking the operator $\mathfrak{I}_{0^+}^{\sigma_n;\psi}$ to both sides of equation $v(z)=\mathfrak{D}_{0^+}^{\sigma_n;\psi}u(z)$, we get

$$
\mathfrak{I}_{0^+}^{\sigma_n;\psi}v(z)=\mathfrak{I}_{0^+}^{\sigma_n;\psi}\mathfrak{D}_{0^+}^{\sigma_n;\psi}u(z)=u(z)-\frac{(\mathfrak{I}_{0^+}^{1-\sigma_n;\psi}u)(0)}{\Gamma(\sigma_n)}\big(\psi(z)-\psi(0)\big)^{\sigma_n-1}. \tag{6}
$$

Since $\big(\mathfrak{I}_{0^+}^{1-\sigma_n;\psi}u\big)(0)=0$, then it follows that

$$
u(z)=\mathfrak{I}_{0^+}^{\sigma_n;\psi}v(z). \tag{7}
$$

From Lemma 3, we get

$$
\begin{aligned}
\mathfrak{D}_{0^+}^{\sigma_n;\psi}u(z) &= \mathfrak{D}_{0^+}^{\sigma_n;\psi}\mathfrak{I}_{0^+}^{\sigma_n;\psi}v(z)=v(z), \\
\mathfrak{D}_{0^+}^{\sigma_n-1;\psi}u(z) &= \mathfrak{D}_{0^+}^{\sigma_n-1;\psi}\mathfrak{I}_{0^+}^{\sigma_n;\psi}v(z)=\mathfrak{I}_{0^+}^{\sigma_n-\sigma_{n-1};\psi}v(z), \\
\vdots &= \vdots \\
\mathfrak{D}_{0^+}^{\sigma_1;\psi}u(z) &= \mathfrak{D}_{0^+}^{\sigma_1;\psi}\mathfrak{I}_{0^+}^{\sigma_n;\psi}v(z)=\mathfrak{I}_{0^+}^{\sigma_n-\sigma_1;\psi}v(z).
\end{aligned}
$$

Since $1<\varrho<2$, so by utilizing a property of the ψ-RL-derivative and by (7) and by exploiting the semigroup specification, we can write

$$
\begin{aligned}
\mathfrak{D}_{0^+}^{\varrho;\psi}u(z) &= \mathfrak{D}_{0^+}^{2;\psi}\mathfrak{I}_{0^+}^{2-\varrho;\psi}u(z)=\mathfrak{D}_{0^+}^{2;\psi}\mathfrak{I}_{0^+}^{2-\varrho;\psi}\mathfrak{I}_{0^+}^{\sigma_n;\psi}v(z) \\
&= \mathfrak{D}_{0^+}^{2;\psi}\mathfrak{I}_{0^+}^{2-\varrho+\sigma_n;\psi}v(z) \\
&= \mathfrak{D}_{0^+}^{\varrho-\sigma_n;\psi}v(z).
\end{aligned}
$$

Consequently, the multi-term equation illustrated by (4) can be written as

$$
\mathfrak{D}_{0^+}^{\varrho-\sigma_n;\psi}v(z)=\hat{h}\big(z,\mathfrak{I}_{0^+}^{\sigma_n;\psi}v(z),\mathfrak{I}_{0^+}^{\sigma_n-\sigma_1;\psi}v(z),\ldots,v(z)\big),\quad 0\le z\le 1. \tag{8}
$$

Putting $\lambda=\varrho-\sigma_n>1$, $\lambda_j=\sigma_n-\sigma_j$, $\sigma_0=0$ $(j=0,1,\ldots n)$, then (8) can be represented as

$$
\mathfrak{D}_{0^+}^{\lambda;\psi}v(z)=\hat{h}\big(z,\mathfrak{I}_{0^+}^{\lambda_0;\psi}v(z),\mathfrak{I}_{0^+}^{\lambda_1;\psi}v(z),\ldots,v(z)\big),\quad 0\le z\le 1. \tag{9}
$$

Hence by (7), as $u(0) = 0$, thus we can find the value of the initial condition $v(0)$. Therefore, because of $v(z) \in AC^{n;\psi}[0,1]$ and

$$\mathfrak{I}_{0^+}^{\sigma_n;\psi} v(z) = \frac{1}{\Gamma(\sigma_n)} \int_0^z \psi'(s)\big(\psi(z) - \psi(s)\big)^{\sigma_n - 1} v(\theta) ds,$$

we can derive the initial value of $v(z)$ arbitrarily in which $u(0) = \mathfrak{I}_{0^+}^{\sigma_n;\psi} v(z)\Big|_{z=0}$ and we have

$$v(0) = 0. \tag{10}$$

Applying the ψ-Riemann–Liouville fractional integral $\mathfrak{I}_{0^+}^{\lambda;\psi}$ to both sides of (9), we find

$$\mathfrak{I}_{0^+}^{\lambda;\psi} \mathfrak{D}_{0^+}^{\lambda;\psi} v(z) = \mathfrak{I}_{0^+}^{\lambda;\psi} \hat{h}\big(z, \mathfrak{I}_{0^+}^{\lambda_0;\psi} v(z), \mathfrak{I}_{0^+}^{\lambda_1;\psi} v(z), \dots, v(z)\big), \quad 0 \le z \le 1. \tag{11}$$

Since $\lambda = \varrho - \sigma_n > 1$, then from Lemma 5, the left-side of (11) becomes

$$\mathfrak{I}_{0^+}^{\lambda;\psi} \mathfrak{D}_{0^+}^{\lambda;\psi} v(z) = v(z) + c_1\big(\psi(z) - \psi(0)\big)^{\lambda - 1} + c_2\big(\psi(z) - \psi(0)\big)^{\lambda - 2}, \tag{12}$$

and (11) gives

$$\begin{aligned} v(z) &= \mathfrak{I}_{0^+}^{\lambda;\psi} \hat{h}\big(z, \mathfrak{I}_{0^+}^{\lambda_0;\psi} v(z), \mathfrak{I}_{0^+}^{\lambda_1;\psi} v(z), \dots, v(z)\big) \\ &\quad - c_1\big(\psi(z) - \psi(0)\big)^{\lambda - 1} - c_2\big(\psi(z) - \psi(0)\big)^{\lambda - 2}. \end{aligned} \tag{13}$$

In this place, we focus to find constants c_1 and c_2. From $v(0) = 0$ and $\lambda > 1$ and by (10), it follows that $c_2 = 0$; consequently, Equation (13) can be written as

$$v(z) = \mathfrak{I}_{0^+}^{\lambda;\psi} \hat{h}\big(z, \mathfrak{I}_{0^+}^{\lambda_0;\psi} v(z), \mathfrak{I}_{0^+}^{\lambda_1;\psi} v(z), \dots, v(z)\big) - c_1\big(\psi(z) - \psi(0)\big)^{\lambda - 1}. \tag{14}$$

Using the second boundary condition in (4) and by (7), we get

$$\begin{aligned} u(1) &= \mathfrak{I}_{0^+}^{\sigma_n;\psi} v(z)\Big|_{z=1} \\ &= p\mathfrak{I}_{0^+}^{\mu;\psi} k_1(\xi, \mathfrak{I}_{0^+}^{\sigma_n;\psi} v(\xi)) + q\mathfrak{I}_{0^+}^{\nu;\psi} k_2(\eta, \mathfrak{I}_{0^+}^{\sigma_n;\psi} v(\eta)). \end{aligned} \tag{15}$$

By Lemma 1 and from (14) and (15), we get

$$\begin{aligned} u(1) &= \mathfrak{I}_{0^+}^{\sigma_n;\psi} v(z)\Big|_{z=1} \\ &= \mathfrak{I}_{0^+}^{\sigma_n+\lambda;\psi} \hat{h}\big(z, \mathfrak{I}_{0^+}^{\lambda_0;\psi} v(z), \mathfrak{I}_{0^+}^{\lambda_1;\psi} v(z), \dots, v(z)\big)\Big|_{z=1} - c_1 \mathfrak{I}_{0^+}^{\sigma_n;\psi}\big(\psi(z) - \psi(0)\big)^{\lambda - 1}\Big|_{z=1} \\ &= \mathfrak{I}_{0^+}^{\sigma_n+\lambda;\psi} \hat{h}\big(z, \mathfrak{I}_{0^+}^{\lambda_0;\psi} v(z), \mathfrak{I}_{0^+}^{\lambda_1;\psi} v(z), \dots, v(z)\big)\Big|_{z=1} \\ &- c_1 \tfrac{\Gamma(\lambda)}{\Gamma(\lambda+\sigma_n)}\big(\psi(z) - \psi(0)\big)^{\lambda+\sigma_n-1}\Big|_{z=1} \\ &= \tfrac{p}{\Gamma(\mu)} \int_0^\xi \psi'(s)\big(\psi(\xi) - \psi(s)\big)^{\mu-1} k_1(s, \mathfrak{I}_{0^+}^{\sigma_n;\psi} v(s)) ds \\ &+ \tfrac{q}{\Gamma(\nu)} \int_0^\eta \psi'(s)\big(\psi(\eta) - \psi(s)\big)^{\nu-1} k_2(s, \mathfrak{I}_{0^+}^{\sigma_n;\psi} v(s)) ds. \end{aligned} \tag{16}$$

We know that $\lambda + \sigma_n - 1 = \varrho - \sigma_n + \sigma_n - 1 = \varrho - 1 > 0$, then we have

$$\begin{aligned} &\tfrac{p}{\Gamma(\mu)} \int_0^\xi \psi'(s)\big(\psi(\xi) - \psi(s)\big)^{\mu-1} k_1(s, \mathfrak{I}_{0^+}^{\sigma_n;\psi} v(s)) ds \\ &+ \tfrac{q}{\Gamma(\nu)} \int_0^\eta \psi'(s)\big(\psi(\eta) - \psi(s)\big)^{\nu-1} k_2(s, \mathfrak{I}_{0^+}^{\sigma_n;\psi} v(s)) ds \\ = \;& \mathfrak{I}_{0^+}^{\sigma_n+\lambda;\psi} \hat{h}\big(z, \mathfrak{I}_{0^+}^{\lambda_0;\psi} v(z), \mathfrak{I}_{0^+}^{\lambda_1;\psi} v(z), \dots, v(z)\big)\Big|_{z=1} - c_1 \tfrac{\Gamma(\lambda)}{\Gamma(\lambda+\sigma_n)}\big(\psi(1) - \psi(0)\big)^{\lambda+\sigma_n-1} \\ = \;& \mathfrak{I}_{0^+}^{\varrho;\psi} \hat{h}\big(z, \mathfrak{I}_{0^+}^{\sigma_n;\psi} v(z), \mathfrak{I}_{0^+}^{\sigma_n-\sigma_1;\psi} v(z), \dots, v(z)\big)\Big|_{z=1} - c_1 \tfrac{\Gamma(\varrho-\sigma_n)}{\Gamma(\varrho)}\big(\psi(1) - \psi(0)\big)^{\varrho-1}. \end{aligned}$$

Then we conclude that

$$\begin{aligned} c_1 &= \frac{\Gamma(\varrho)}{\Gamma(\varrho-\sigma_n)\big(\psi(1)-\psi(0)\big)^{\varrho-1}}\Bigg[\mathfrak{I}_{0^+}^{\varrho;\psi}\hat{\hbar}\big(z,\mathfrak{I}_{0^+}^{\sigma_n;\psi}v(z),\mathfrak{I}_{0^+}^{\sigma_n-\sigma_1;\psi}v(z),\dots,v(z)\big)\Big|_{z=1} \\ &- \frac{p}{\Gamma(\mu)}\int_0^{\xi}\psi'(s)\big(\psi(\xi)-\psi(s)\big)^{\mu-1}k_1(s,\mathfrak{I}_{0^+}^{\sigma_n;\psi}v(s))ds \\ &- \frac{q}{\Gamma(\nu)}\int_0^{\eta}\psi'(s)\big(\psi(\eta)-\psi(s)\big)^{\nu-1}k_2(s,\mathfrak{I}_{0^+}^{\sigma_n;\psi}v(s))ds\Bigg]. \end{aligned} \tag{17}$$

By inserting c_1 into Equation (14), we get

$$\begin{aligned} v(z) &= \mathfrak{I}_{0^+}^{\varrho-\sigma_n;\psi}\hat{\hbar}\big(z,\mathfrak{I}_{0^+}^{\sigma_n;\psi}v(z),\mathfrak{I}_{0^+}^{\sigma_n-\sigma_1;\psi}v(z),\dots,v(z)\big) \\ &+ \frac{\Gamma(\varrho)}{\Gamma(\varrho-\sigma_n)\big(\psi(1)-\psi(0)\big)^{\varrho-1}}\Bigg[\frac{p}{\Gamma(\mu)}\int_0^{\xi}\psi'(s)\big(\psi(\xi)-\psi(s)\big)^{\mu-1}k_1(s,\mathfrak{I}_{0^+}^{\sigma_n;\psi}v(s))ds \\ &+ \frac{q}{\Gamma(\nu)}\int_0^{\eta}\psi'(s)\big(\psi(\eta)-\psi(s)\big)^{\nu-1}k_2(s,\mathfrak{I}_{0^+}^{\sigma_n;\psi}v(s))ds \\ &- \mathfrak{I}_{0^+}^{\varrho;\psi}\hat{\hbar}\big(z,\mathfrak{I}_{0^+}^{\sigma_n;\psi}v(z),\mathfrak{I}_{0^+}^{\sigma_n-\sigma_1;\psi}v(z),\dots,v(z)\big)\Big|_{z=1}\Bigg]\big(\psi(z)-\psi(0)\big)^{\varrho-\sigma_n-1}, \end{aligned}$$

which yields that $v(z)=\mathfrak{D}_{0^+}^{\sigma_n;\psi}u(z)\in AC^{n;\psi}[0,1]$ is a solution of the ψ-integral Equation (5). For the opposite direction, suppose that $v(z)=\mathfrak{D}_{0^+}^{\sigma_n;\psi}u(z)\in AC^{n;\psi}[0,1]$ is a solution of the ψ-integral Equation (5). Then in view of Lemma 3, we have

$$\begin{aligned} u(z) &= \mathfrak{I}_{0^+}^{\sigma_n;\psi}v(z), \\ \mathfrak{D}_{0^+}^{\sigma_{n-1};\psi}u(z) &= \mathfrak{D}_{0^+}^{\sigma_{n-1};\psi}\mathfrak{I}_{0^+}^{\sigma_n;\psi}v(z)=\mathfrak{I}_{0^+}^{\sigma_n-\sigma_{n-1};\psi}v(z), \\ \dots &= \dots \\ \mathfrak{D}_{0^+}^{\sigma_1;\psi}u(z) &= \mathfrak{D}_{0^+}^{\sigma_1;\psi}\mathfrak{I}_{0^+}^{\sigma_n;\psi}v(z)=\mathfrak{I}_{0^+}^{\sigma_n-\sigma_1;\psi}v(z). \end{aligned} \tag{18}$$

By applying the ψ-RL-integral $\mathfrak{I}_{0^+}^{\sigma_n;\psi}$ on both sides of the ψ-integral Equation (5), we get

$$\begin{aligned} \mathfrak{I}_{0^+}^{\sigma_n;\psi}v(z) &= \mathfrak{I}_{0^+}^{\varrho;\psi}\hat{\hbar}\big(z,\mathfrak{I}_{0^+}^{\sigma_n;\psi}v(z),\mathfrak{I}_{0^+}^{\sigma_n-\sigma_1;\psi}v(z),\dots,v(z)\big) \\ &+ \frac{\Gamma(\varrho)}{\Gamma(\varrho-\sigma_n)\big(\psi(1)-\psi(0)\big)^{\varrho-1}}\Bigg[\frac{p}{\Gamma(\mu)}\int_0^{\xi}\psi'(s)\big(\psi(\xi)-\psi(s)\big)^{\mu-1}k_1(s,\mathfrak{I}_{0^+}^{\sigma_n;\psi}v(s))ds \\ &+ \frac{q}{\Gamma(\nu)}\int_0^{\eta}\psi'(s)\big(\psi(\eta)-\psi(s)\big)^{\nu-1}k_2(s,\mathfrak{I}_{0^+}^{\sigma_n;\psi}v(s))ds \\ &- \mathfrak{I}_{0^+}^{\varrho;\psi}\hat{\hbar}\big(z,\mathfrak{I}_{0^+}^{\sigma_n;\psi}v(z),\mathfrak{I}_{0^+}^{\sigma_n-\sigma_1;\psi}v(z),\dots,v(z)\big)\Big|_{z=1}\Bigg]\mathfrak{I}_{0^+}^{\sigma_n;\psi}\big(\psi(z)-\psi(0)\big)^{\varrho-\sigma_n-1} \end{aligned}$$

and thus

$$
\begin{aligned}
u(z) \;=\;& \mathfrak{I}_{0^+}^{\varrho;\psi}\hat{\hbar}\big(z,\mathfrak{I}_{0^+}^{\sigma_n;\psi}v(z),\mathfrak{I}_{0^+}^{\sigma_n-\sigma_1;\psi}v(z),\ldots,v(z)\big)\\
&+\frac{\Gamma(\varrho)}{\Gamma(\varrho-\sigma_n)\big(\psi(1)-\psi(0)\big)^{\varrho-1}}\Bigg[\frac{p}{\Gamma(\mu)}\int_0^{\xi}\psi'(s)\big(\psi(\xi)-\psi(s)\big)^{\mu-1}k_1(s,\mathfrak{I}_{0^+}^{\sigma_n;\psi}v(s))ds\\
&+\frac{q}{\Gamma(\nu)}\int_0^{\eta}\psi'(s)\big(\psi(\eta)-\psi(s)\big)^{\nu-1}k_2(s,\mathfrak{I}_{0^+}^{\sigma_n;\psi}v(s))ds\\
&-\mathfrak{I}_{0^+}^{\varrho;\psi}\hat{\hbar}\big(z,\mathfrak{I}_{0^+}^{\sigma_n;\psi}v(z),\mathfrak{I}_{0^+}^{\sigma_n-\sigma_1;\psi}v(z),\ldots,v(z)\big)\Big|_{z=1}\Bigg]\mathfrak{I}_{0^+}^{\sigma_n;\psi}\big(\psi(z)-\psi(0)\big)^{\varrho-\sigma_n-1}.
\end{aligned}
\tag{19}
$$

Taking now the ψ-Riemann–Liouville derivative $\mathfrak{D}_{0^+}^{\varrho;\psi}$ to both sides of (19) we find

$$
\begin{aligned}
\mathfrak{D}_{0^+}^{\varrho;\psi}u(z) \;=\;& \mathfrak{D}_{0^+}^{\varrho;\psi}\mathfrak{I}_{0^+}^{\varrho;\psi}\hat{\hbar}\big(z,\mathfrak{I}_{0^+}^{\sigma_n;\psi}v(z),\mathfrak{I}_{0^+}^{\sigma_n-\sigma_1;\psi}v(z),\ldots,v(z)\big)\\
&+\frac{\Gamma(\varrho)}{\Gamma(\varrho-\sigma_n)\big(\psi(1)-\psi(0)\big)^{\varrho-1}}\Bigg[\frac{p}{\Gamma(\mu)}\int_0^{\xi}\psi'(s)\big(\psi(\xi)-\psi(s)\big)^{\mu-1}k_1(s,\mathfrak{I}_{0^+}^{\sigma_n;\psi}v(s))ds\\
&+\frac{q}{\Gamma(\nu)}\int_0^{\eta}\psi'(s)\big(\psi(\eta)-\psi(s)\big)^{\nu-1}k_2(s,\mathfrak{I}_{0^+}^{\sigma_n;\psi}v(s))ds\\
&-\mathfrak{I}_{0^+}^{\varrho;\psi}\hat{\hbar}\big(z,\mathfrak{I}_{0^+}^{\sigma_n;\psi}v(z),\mathfrak{I}_{0^+}^{\sigma_n-\sigma_1;\psi}v(z),\ldots,v(z)\big)\Big|_{z=1}\Bigg]\mathfrak{D}_{0^+}^{\varrho;\psi}\mathfrak{I}_{0^+}^{\sigma_n;\psi}\big(\psi(z)-\psi(0)\big)^{\varrho-\sigma_n-1}.
\end{aligned}
$$

By Lemma 1, $\mathfrak{I}_{0^+}^{\sigma_n;\psi}(\psi(z)-\psi(0))^{\varrho-\sigma_n-1} = \dfrac{\Gamma(\varrho-\sigma_n)}{\Gamma(\varrho)}(\psi(z)-\psi(0))^{\varrho-1}$ and due to the equality $\mathfrak{D}_{0^+}^{\varrho;\psi}(\psi(z)-\psi(0))^{\varrho-1}=0$, we get

$$
\begin{aligned}
\mathfrak{D}_{0^+}^{\varrho;\psi}u(z) \;=\;& \hat{\hbar}\big(z,\mathfrak{I}_{0^+}^{\sigma_n;\psi}v(z),\mathfrak{I}_{0^+}^{\sigma_n-\sigma_1;\psi}v(z),\ldots,v(z)\big)\\
&+\frac{1}{\big(\psi(1)-\psi(0)\big)^{\varrho-1}}\Bigg[\frac{p}{\Gamma(\mu)}\int_0^{\xi}\psi'(s)\big(\psi(\xi)-\psi(s)\big)^{\mu-1}k_1(s,\mathfrak{I}_{0^+}^{\sigma_n;\psi}v(s))ds\\
&+\frac{q}{\Gamma(\nu)}\int_0^{\eta}\psi'(s)\big(\psi(\eta)-\psi(s)\big)^{\nu-1}k_2(s,\mathfrak{I}_{0^+}^{\sigma_n;\psi}v(s))ds\\
&-\mathfrak{I}_{0^+}^{\varrho;\psi}\hat{\hbar}\big(z,\mathfrak{I}_{0^+}^{\sigma_n;\psi}v(z),\mathfrak{I}_{0^+}^{\sigma_n-\sigma_1;\psi}v(z),\ldots,v(z)\big)\Big|_{z=1}\Bigg]\mathfrak{D}_{0^+}^{\varrho;\psi}\big(\psi(z)-\psi(0)\big)^{\varrho-1}\\
=\;& \hat{\hbar}\big(z,\mathfrak{I}_{0^+}^{\sigma_n;\psi}v(z),\mathfrak{I}_{0^+}^{\sigma_n-\sigma_1;\psi}v(z),\ldots,v(z)\big).
\end{aligned}
\tag{20}
$$

Based on (18), since $\mathfrak{I}_{0^+}^{\sigma_n;\psi}v(z) = u(z)$, $\mathfrak{I}_{0^+}^{\sigma_n-\sigma_1;\psi}v(z) = \mathfrak{D}_{0^+}^{\sigma_1;\psi}u(z)$, …and $v(z) = \mathfrak{D}_{0^+}^{\sigma_n;\psi}u(z)$, so the fractional differential Equation (20) is transformed into

$$
\mathfrak{D}_{0^+}^{\varrho;\psi}u(z)=\hat{\hbar}\big(z,u(z),\mathfrak{D}_{0^+}^{\sigma_1;\psi}u(z),\mathfrak{D}_{0^+}^{\sigma_2;\psi}u(z),\ldots,\mathfrak{D}_{0^+}^{\sigma_n;\psi}u(z)\big).
$$

Now, it remains to review the boundary conditions of our multi-term ψ-RLFBVP (4). From ψ-integral Equation (5), we can write

$$\begin{aligned} v(0) &= \mathfrak{I}_{0^+}^{\varrho-\sigma_n;\psi}\hat{h}\big(z, \mathfrak{I}_{0^+}^{\sigma_n;\psi}v(z), \mathfrak{I}_{0^+}^{\sigma_n-\sigma_1;\psi}v(z), \dots, v(z)\big)\Big|_{z=0} \\ &+ \frac{\Gamma(\varrho)}{\Gamma(\varrho-\sigma_n)\big(\psi(1)-\psi(0)\big)^{\varrho-1}}\Bigg[\frac{p}{\Gamma(\mu)}\int_0^{\xi}\psi'(s)\big(\psi(\xi)-\psi(s)\big)^{\mu-1}k_1(s, \mathfrak{I}_{0^+}^{\sigma_n;\psi}v(s))ds \\ &+ \frac{q}{\Gamma(\nu)}\int_0^{\eta}\psi'(s)\big(\psi(\eta)-\psi(s)\big)^{\nu-1}k_2(s, \mathfrak{I}_{0^+}^{\sigma_n;\psi}v(s))ds \\ &- \mathfrak{I}_{0^+}^{\varrho;\psi}\hat{h}\big(z, \mathfrak{I}_{0^+}^{\sigma_n;\psi}v(z), \mathfrak{I}_{0^+}^{\sigma_n-\sigma_1;\psi}v(z), \dots, v(z)\big)\Big|_{z=1}\Bigg]\big(\psi(z)-\psi(0)\big)^{\varrho-\sigma_n-1}\Bigg|_{z=0}. \\ &= 0. \end{aligned}$$

So $v(0) = 0$. On the other side, we also have $u(z) = \mathfrak{I}_{0^+}^{\sigma_n;\psi}v(z)$, then

$$u(0) = \mathfrak{I}_{0^+}^{\sigma_n;\psi}v(z)\big|_{z=0} = 0$$

and so $u(0) = 0$. This states that the first boundary condition holds. At this moment, to investigate the second boundary condition, by inserting $z = 1$ into (19), we arrive at the following

$$\begin{aligned} u(1) &= \mathfrak{I}_{0^+}^{\varrho;\psi}\hat{h}\big(z, \mathfrak{I}_{0^+}^{\sigma_n;\psi}v(z), \mathfrak{I}_{0^+}^{\sigma_n-\sigma_1;\psi}v(z), \dots, v(z)\big)\big|_{z=1} \\ &+ \frac{\Gamma(\varrho)}{\Gamma(\varrho-\sigma_n)\big(\psi(1)-\psi(0)\big)^{\varrho-1}}\Bigg[\frac{p}{\Gamma(\mu)}\int_0^{\xi}\psi'(s)\big(\psi(\xi)-\psi(s)\big)^{\mu-1}k_1(s, \mathfrak{I}_{0^+}^{\sigma_n;\psi}v(s))ds \\ &+ \frac{q}{\Gamma(\nu)}\int_0^{\eta}\psi'(s)\big(\psi(\eta)-\psi(s)\big)^{\nu-1}k_2(s, \mathfrak{I}_{0^+}^{\sigma_n;\psi}v(s))ds \\ &- \mathfrak{I}_{0^+}^{\varrho;\psi}\hat{h}\big(z, \mathfrak{I}_{0^+}^{\sigma_n;\psi}v(z), \mathfrak{I}_{0^+}^{\sigma_n-\sigma_1;\psi}v(z), \dots, v(z)\big)\big|_{z=1}\Bigg]\mathfrak{I}_{0^+}^{\sigma_n;\psi}\big(\psi(z)-\psi(0)\big)^{\varrho-\sigma_n-1}\big|_{z=1} \\ &= \mathfrak{I}_{0^+}^{\varrho;\psi}\hat{h}\big(z, \mathfrak{I}_{0^+}^{\sigma_n;\psi}v(z), \mathfrak{I}_{0^+}^{\sigma_n-\sigma_1;\psi}v(z), \dots, v(z)\big)\big|_{z=1} \\ &+ \frac{\Gamma(\varrho)}{\Gamma(\varrho-\sigma_n)\big(\psi(1)-\psi(0)\big)^{\varrho-1}}\Bigg[\frac{p}{\Gamma(\mu)}\int_0^{\xi}\psi'(s)\big(\psi(\xi)-\psi(s)\big)^{\mu-1}k_1(s, \mathfrak{I}_{0^+}^{\sigma_n;\psi}v(s))ds \\ &+ \frac{q}{\Gamma(\nu)}\int_0^{\eta}\psi'(s)\big(\psi(\eta)-\psi(s)\big)^{\nu-1}k_2(s, \mathfrak{I}_{0^+}^{\sigma_n;\psi}v(s))ds \\ &- \mathfrak{I}_{0^+}^{\varrho;\psi}\hat{h}\big(z, \mathfrak{I}_{0^+}^{\sigma_n;\psi}v(z), \mathfrak{I}_{0^+}^{\sigma_n-\sigma_1;\psi}v(z), \dots, v(z)\big)\Big|_{z=1}\Bigg]\frac{\Gamma(\varrho-\sigma_n)}{\Gamma(\varrho)}\big(\psi(z)-\psi(0)\big)^{\varrho-1}\big|_{z=1} \\ &= p\mathfrak{I}_{0^+}^{\mu;\psi}k_1(\xi, u(\xi)) + q\mathfrak{I}_{0^+}^{\nu;\psi}k_2(\eta, u(\eta)). \end{aligned}$$

In consequence, we understand that $u(z)$ satisfies the multi-term ψ-RLFBVP (4) and accordingly u is a solution of the multi-term ψ-RLFBVP (4). □

Here, we introduce some new notations based on above theorem. Consider the Banach space $\mathbf{B} = AC^{n;\psi}[0,1]$ via $\|v\| = \max_{z\in[0,1]}|v(z)|$ and besides this, by Theorem 1, we define the operator $K : \mathbf{B} \to \mathbf{B}$ by

$$\begin{aligned}(Kv)(z) &= \mathfrak{I}_{0^+}^{\varrho-\sigma_n;\psi}\hat{h}\big(z,\mathfrak{I}_{0^+}^{\sigma_n;\psi}v(z),\mathfrak{I}_{0^+}^{\sigma_n-\sigma_1;\psi}v(z),\ldots,v(z)\big)\\ &+ \frac{\Gamma(\varrho)}{\Gamma(\varrho-\sigma_n)\big(\psi(1)-\psi(0)\big)^{\varrho-1}}\Bigg[\frac{p}{\Gamma(\mu)}\int_0^{\xi}\psi'(s)\big(\psi(\xi)-\psi(s)\big)^{\mu-1}k_1(s,\mathfrak{I}_{0^+}^{\sigma_n;\psi}v(s))ds\\ &+ \frac{q}{\Gamma(\nu)}\int_0^{\eta}\psi'(s)\big(\psi(\eta)-\psi(s)\big)^{\nu-1}k_2(s,\mathfrak{I}_{0^+}^{\sigma_n;\psi}v(s))ds\\ &- \mathfrak{I}_{0^+}^{\varrho;\psi}\hat{h}\big(z,\mathfrak{I}_{0^+}^{\sigma_n;\psi}v(z),\mathfrak{I}_{0^+}^{\sigma_n-\sigma_1;\psi}v(z),\ldots,v(z)\big)\Big|_{z=1}\Bigg]\big(\psi(z)-\psi(0)\big)^{\varrho-\sigma_n-1}.\end{aligned} \tag{21}$$

Clearly, the equation

$$Kv = v, \quad v \in \mathbf{B}, \tag{22}$$

and the ψ-integral Equation (5) are equivalent. When K involves a fixed point, in that case, it will be the solution of the multi-term ψ-RLFBVP (1). On the other direction, note that the continuity of $\hat{h}$, k_1 and k_2 implies the one of the operator K. In this position, we would like to express the existence theorem.

Theorem 2. *Assume that the following assertions occur:*
$(H1)$ *There exist* $M_j \in \mathbb{R}$, $j = 0,1,\ldots,n$ *such that*

$$\big|\hat{h}(z,u_0,u_1,\ldots,u_n) - \hat{h}(z,U_0,U_1,\ldots,U_n)\big| \le \sum_{j=0}^{n} M_j|u_j - U_j|,$$

for any $z \in [0,1]$ *and* $(u_0,u_1,\ldots,u_n),(U_0,U_1,\ldots,U_n) \in \mathbb{R}^{n+1}$.
$(H2)$ *There exist two integrable functions* $l_1 : [0,1] \to \mathbb{R}^+$ *and* $l_2 : [0,1] \to \mathbb{R}^+$ *such that*

$$\begin{aligned}\big|k_1(z,v) - k_1(z,u)\big| &\le l_1(z)|v-u|, \quad v,u \in \mathbb{R},\\ \big|k_2(z,v) - k_2(z,u)\big| &\le l_2(z)|v-u|, \quad v,u \in \mathbb{R}.\end{aligned}$$

$(H3)$ *We have* $0 < \Phi < 1$, *where*

$$\begin{aligned}\Phi &= \frac{p\Gamma(\varrho)}{\Gamma(\varrho-\sigma_n)\Gamma(\sigma_n+1)}\mathfrak{I}_{0^+}^{\mu;\psi}l_1(\xi) + \frac{q\Gamma(\varrho)}{\Gamma(\varrho-\sigma_n)\Gamma(\sigma_n+1)}\mathfrak{I}_{0^+}^{\nu;\psi}l_2(\eta)\\ &+\sum_{j=0}^{n}\left(\frac{M_j\big(\psi(1)-\psi(0)\big)^{\varrho-\sigma_j}}{\Gamma(\varrho-\sigma_j+1)} + \frac{M_j\Gamma(\varrho)\big(\psi(1)-\psi(0)\big)^{\varrho-\sigma_j}}{\Gamma(\varrho-\sigma_n)\Gamma(\varrho+\sigma_n-\sigma_j+1)}\right) < 1.\end{aligned}$$

Then the multi-term ψ*-RLFBVP* (1) *includes a unique solution.*

Proof. From Theorem 1 and as it said above, it follows that the existence of solutions to the multi-term ψ-RLFBVP (1) is equivalent to the existence of the solutions for Equation (21). Therefore it is sufficient to establish that the Equation (21) involves a fixed point uniquely. Setting $\lambda = \varrho - \sigma_n$, $\sigma_0 = 0$, $\lambda_j = \sigma_n - \sigma_j$, $(j = 0,1,\ldots,n)$, then from $(H1)$, we have for any $v_1, v_2 \in \mathbf{B}$

$$\begin{aligned}&\Big|\hat{h}\big(z,\mathfrak{I}_{0^+}^{\lambda_0;\psi}v_1(z),\ldots,v_1(z)\big) - \hat{h}\big(z,\mathfrak{I}_{0^+}^{\lambda_0;\psi}v_2(z),\ldots,v_2(z)\big)\Big|\\ &\le \sum_{j=0}^{n} M_j\Big|\mathfrak{I}_{0^+}^{\lambda_j;\psi}v_1(z) - \mathfrak{I}_{0^+}^{\lambda_j;\psi}v_2(z)\Big|.\end{aligned} \tag{23}$$

Applying the ψ-Riemann–Liouville integral $\Im_{0^+}^{\lambda;\psi}$ to both sides of Inequality (23), we obtain

$$\begin{aligned}
&\Im_{0^+}^{\lambda;\psi}\Big|\hat{h}\big(z,\Im_{0^+}^{\lambda_0;\psi}v_1(z),\ldots,v_1(z)\big)-\hat{h}\big(z,\Im_{0^+}^{\lambda_0;\psi}v_2(z),\ldots,v_2(z)\big)\Big|\\
&\le \Im_{0^+}^{\lambda;\psi}\sum_{j=0}^{n}M_j\Big|\Im_{0^+}^{\lambda_j;\psi}v_1(z)-\Im_{0^+}^{\lambda_j;\psi}v_2(z)\Big|\\
&\le \sum_{j=0}^{n}M_j\Im_{0^+}^{\lambda+\lambda_j;\psi}\Big|v_1(z)-v_2(z)\Big|\\
&\le \|v_1-v_2\|\sum_{j=0}^{n}\frac{M_j\big(\psi(1)-\psi(0)\big)^{\lambda+\lambda_j}}{\Gamma(\varrho-\sigma_j+1)}.
\end{aligned}\tag{24}$$

Also, from $(H2)$, it follows that

$$\begin{aligned}
&\Bigg|\frac{\Gamma(\varrho)}{\Gamma(\lambda)\big(\psi(1)-\psi(0)\big)^{\varrho-1}}\Bigg[\frac{p}{\Gamma(\mu)}\int_0^{\xi}\psi'(s)\big(\psi(\xi)-\psi(s)\big)^{\mu-1}k_1(s,\Im_{0^+}^{\sigma_n;\psi}v_1(s))ds\\
+&\ \frac{q}{\Gamma(\nu)}\int_0^{\eta}\psi'(s)\big(\psi(\eta)-\psi(s)\big)^{\nu-1}k_2(s,\Im_{0^+}^{\sigma_n;\psi}v_1(s))ds\\
-&\ \Im_{0^+}^{\sigma_n+\lambda;\psi}\hat{h}\big(z,\Im_{0^+}^{\lambda_0;\psi}v_1(z),\Im_{0^+}^{\lambda_1;\psi}v_1(z),\ldots,v_1(z)\big)\Big|_{z=1}\Bigg]\big(\psi(z)-\psi(0)\big)^{\lambda-1}\\
-&\ \frac{\Gamma(\varrho)}{\Gamma(\lambda)\big(\psi(1)-\psi(0)\big)^{\varrho-1}}\Bigg[\frac{p}{\Gamma(\mu)}\int_0^{\xi}\psi'(s)\big(\psi(\xi)-\psi(s)\big)^{\mu-1}k_1(s,\Im_{0^+}^{\sigma_n;\psi}v_2(s))ds\\
+&\ \frac{q}{\Gamma(\nu)}\int_0^{\eta}\psi'(s)\big(\psi(\eta)-\psi(s)\big)^{\nu-1}k_2(s,\Im_{0^+}^{\sigma_n;\psi}v_2(s))ds\\
-&\ \Im_{0^+}^{\sigma_n+\lambda;\psi}\hat{h}\big(z,\Im_{0^+}^{\lambda_0;\psi}v_2(z),\Im_{0^+}^{\lambda_1;\psi}v_2(z),\ldots,v_2(z)\big)\Big|_{z=1}\Bigg]\big(\psi(z)-\psi(0)\big)^{\lambda-1}\Bigg|\\
\le&\ \frac{\Gamma(\varrho)}{\Gamma(\lambda)\big(\psi(1)-\psi(0)\big)^{\sigma_n}}\Bigg[\frac{p}{\Gamma(\mu)}\int_0^{\xi}\psi'(s)\big(\psi(\xi)-\psi(s)\big)^{\mu-1}\Big|k_1\big(s,\Im_{0^+}^{\sigma_n;\psi}v_1(s)\big)-k_1\big(s,\Im_{0^+}^{\sigma_n;\psi}v_2(s)\Big|ds\\
+&\ \frac{q}{\Gamma(\nu)}\int_0^{\eta}\psi'(s)\big(\psi(\eta)-\psi(s)\big)^{\nu-1}\Big|k_2\big(s,\Im_{0^+}^{\sigma_n;\psi}v_1(s)\big)-k_2\big(s,\Im_{0^+}^{\sigma_n;\psi}v_2(s)\Big|ds\\
+&\ \Im_{0^+}^{\sigma_n+\lambda;\psi}\Big|\hat{h}\big(z,\Im_{0^+}^{\lambda_0;\psi}v_1(z),\Im_{0^+}^{\lambda_1;\psi}v_1(z),\ldots,v_1(z)\big)-\hat{h}\big(z,\Im_{0^+}^{\lambda_0;\psi}v_2(z),\Im_{0^+}^{\lambda_1;\psi}v_2(z),\ldots,v_2(z)\big)\Big|\Big|_{z=1}\Bigg]\\
\le&\ \frac{\Gamma(\varrho)}{\Gamma(\lambda)\big(\psi(1)-\psi(0)\big)^{\sigma_n}}\Bigg[\frac{p}{\Gamma(\mu)}\int_0^{\xi}\psi'(s)\big(\psi(\xi)-\psi(s)\big)^{\mu-1}l_1(s)\Big|\Im_{0^+}^{\sigma_n;\psi}v_1(s)-\Im_{0^+}^{\sigma_n;\psi}v_2(s)\Big|ds\\
+&\ \frac{q}{\Gamma(\nu)}\int_0^{\eta}\psi'(s)\big(\psi(\eta)-\psi(s)\big)^{\nu-1}l_2(s)\Big|\Im_{0^+}^{\sigma_n;\psi}v_1(s)-\Im_{0^+}^{\sigma_n;\psi}v_2(s)\Big|ds\\
+&\ \Im_{0^+}^{\sigma_n+\lambda;\psi}\sum_{j=0}^{n}M_j\Im_{0^+}^{\lambda_j;\psi}\big|v_1(z)-v_2(z)\big|\Big|_{z=1}\Bigg]\\
\le&\ \frac{\Gamma(\varrho)}{\Gamma(\lambda)\big(\psi(1)-\psi(0)\big)^{\sigma_n}}\Bigg[p\frac{\big(\psi(1)-\psi(0)\big)^{\sigma_n}}{\Gamma(\sigma_n+1)}\Im_{0^+}^{\mu;\psi}l_1(\xi)\\
+&\ q\frac{\big(\psi(1)-\psi(0)\big)^{\sigma_n}}{\Gamma(\sigma_n+1)}\Im_{0^+}^{\nu;\psi}l_2(\eta)+\sum_{j=0}^{n}\frac{M_j\big(\psi(1)-\psi(0)\big)^{\sigma_n+\lambda+\lambda_j}}{\Gamma(\sigma_n+\lambda+\lambda_j+1)}\Bigg]\|v_1-v_2\|\\
=&\ \frac{\Gamma(\varrho)}{\Gamma(\lambda)}\Bigg[\frac{p}{\Gamma(\sigma_n+1)}\Im_{0^+}^{\mu;\psi}l_1(\xi)+\frac{q}{\Gamma(\sigma_n+1)}\Im_{0^+}^{\nu;\psi}l_2(\eta)+\sum_{j=0}^{n}\frac{M_j\big(\psi(1)-\psi(0)\big)^{\lambda+\lambda_j}}{\Gamma(\sigma_n+\lambda+\lambda_j+1)}\Bigg]\|v_1-v_2\|.
\end{aligned}\tag{25}$$

Therefore, by (24) and (25) we conclude that

$$\begin{aligned}\|Kv_1(z)-Kv_2(z)\| \leq & \Big[\tfrac{p\Gamma(\varrho)}{\Gamma(\varrho-\sigma_n)\Gamma(\sigma_n+1)}\Im_{0^+}^{\mu;\psi}l_1(\xi)+\tfrac{q\Gamma(\varrho)}{\Gamma(\varrho-\sigma_n)\Gamma(\sigma_n+1)}\Im_{0^+}^{\nu;\psi}l_2(\eta) \\ & +\textstyle\sum_{j=0}^{n}\Big(\tfrac{M_j\left(\psi(1)-\psi(0)\right)^{\varrho-\sigma_j}}{\Gamma(\varrho-\sigma_j+1)}+\tfrac{M_j\Gamma(\varrho)\left(\psi(1)-\psi(0)\right)^{\varrho-\sigma_j}}{\Gamma(\varrho-\sigma_n)\Gamma(\varrho+\sigma_n-\sigma_j+1)}\Big)\Big]\|v_1-v_2\|.\end{aligned}$$

By using $(H3)$, we figure out that

$$\|Kv_1-Kv_2\| \leq \Phi\|v_1-v_2\|,$$

where $\Phi \in (0,1)$. We deduce that K will be a contraction. Thus, By Banach's principle, we conclude that K admits a fixed point uniquely which it interprets the existence of a solution uniqyely for the multi-term ψ-RLFBVP (1). □

4. Numerical Solutions via DGJIM and ADM Methods

This section is assigned for the formulation and the analysis of algorithms of two numerical techniques called DGJIM and ADM. In fact, we here express how we can implement these techniques to our supposed multi-term ψ-RLFBVP. In each algorithm, appropriate recursion relations are obtained to approximate the solutions of (4) as well as their convergence. Our methods are motivated by [42,43].

4.1. The Numerical Technique DGJIM

As you saw before, we showed that the solutions of Equations (1) and (5) are equivalent. Let's write the right-hand side of (22) under the following decomposition (this decomposition is not unique)

$$(Kv)(z)=\mathbb{L}v(z))+\mathbb{N}(v(z))+\Theta(z),$$

where $\mathbb{L}$ is a linear operator, the operator $\mathbb{N}$ represents the nonlinear terms and Θ is a known function. Then, Equation (5) can be written in the decomposed format

$$v(z)=\mathbb{L}(v(z))+\mathbb{N}(v(z))+\Theta(z). \tag{26}$$

Suppose that the solution of (26) is expanded in the form of a series as follows

$$v(z)=\sum_{n=0}^{+\infty}v_n(z). \tag{27}$$

By replacing (27) in (26), we get

$$\sum_{n=0}^{+\infty}v_n(z)=\mathbb{L}\Big(\sum_{n=0}^{+\infty}v_n(z)\Big)+\mathbb{N}\Big(\sum_{n=0}^{+\infty}v_n(z)\Big)+\Theta(z). \tag{28}$$

Because of the linearity of $\mathbb{L}$ and by a simple computation we get the following numerical algorithm named DGJIM method:

$$\begin{cases} v_0(z) = \Theta(z), \\ v_1(z) = \mathbb{L}\big(v_0(z)\big) + \mathbb{N}\big(v_0(z)\big), \\ v_2(z) = \mathbb{L}\big(v_1(z)\big) + \mathbb{N}\big(v_0(z) + v_1(z)\big) - \mathbb{N}\big(v_0(z)\big), \\ v_3(z) = \mathbb{L}\big(v_2(z)\big) + \mathbb{N}\big(v_0(z) + v_1(z) + v_2(z)\big) - \mathbb{N}\big(v_0(z) + v_1(z)\big), \\ \ \vdots \ = \quad \vdots \\ v_n(z) = \mathbb{L}\big(v_{n-1}(z)\big) + \mathbb{N}\Big(\sum_{i=0}^{n-1} v_i(z)\Big) - \mathbb{N}\Big(\sum_{i=0}^{n-2} v_i(z)\Big), \\ \ \vdots \ = \quad \vdots \end{cases} \tag{29}$$

Accordingly, we obtain the n-term approximate solution of the ψ-integral Equation (5) by

$$\Upsilon_n(z) = \sum_{i=0}^{n} v_i(z). \tag{30}$$

From expression (30), we easily reach

$$v_n(z) = \Upsilon_n(z) - \Upsilon_{n-1}(z). \tag{31}$$

By combining (29) and (31),

$$\Upsilon_n(z) = \Upsilon_{n-1}(z) + \mathbb{L}\big(\Upsilon_{n-1}(z) - \Upsilon_{n-2}(z)\big) + \mathbb{N}\big(\Upsilon_{n-1}(z)\big) - \mathbb{N}\big(\Upsilon_{n-2}(z)\big). \tag{32}$$

Now, let us write

$$\begin{aligned} \big\|\mathbb{L}v - \mathbb{L}u\big\| &\leq \mu_1\|v-u\|, \quad 0<\mu_1<1, \\ \big\|\mathbb{N}v - \mathbb{N}u\big\| &\leq \mu_2\|v-u\|, \quad 0<\mu_2<1, \end{aligned}$$

where $\mu_1 + \mu_2 < 1$. Therefore, the Banach's fixed point theorem ensures the existence of a unique solution $\widetilde{\Upsilon}(z)$ for Equation (26) and so for the ψ-integral Equation (5). Based on (32), we can write the following iterative inequalities

$$\begin{aligned} \big\|\Upsilon_n - \Upsilon_{n-1}\big\| &\leq \mu_1\big\|\Upsilon_{n-1} - \Upsilon_{n-2}\big\| + \mu_2\big\|\Upsilon_{n-1} - \Upsilon_{n-2}\big\| \\ &= (\mu_1+\mu_2)\big\|\Upsilon_{n-1} - \Upsilon_{n-2}\big\| \\ &\leq (\mu_1+\mu_2)^2\big\|\Upsilon_{n-2} - \Upsilon_{n-3}\big\| \\ &\leq \ \vdots \quad \vdots \quad \vdots \\ &\leq (\mu_1+\mu_2)^{n-1}\big\|\Upsilon_1 - \Upsilon_0\big\|, \end{aligned}$$

which points out the absolute convergence and the uniform one of the sequence $\{\Upsilon_n\}$ to the exact solution $\widetilde{\Upsilon}(z)$.

4.2. The Numerical Technique ADM

To establish the numerical technique ADM, the nonlinear term $\mathbb{N}\Big(\sum_{n=0}^{+\infty} v_n(z)\Big)$ in (28) can be decomposed into a series of Adomian polynomials displayed by

$$\mathbb{N}\Big(\sum_{n=0}^{+\infty} v_n(z)\Big) = \sum_{n=0}^{+\infty} \mathbf{A}_n(v_0, v_1, \ldots, v_n),$$

in which $\mathbf{A}_n(v_0, v_1, \ldots, v_n)$ is generated by

$$\mathbf{A}_n(v_0, v_1, \ldots, v_n) = \frac{1}{n!}\frac{\partial^n}{\partial z^n}\Big[\mathbb{N}\Big(\sum_{k=0}^{+\infty} v_k z^k\Big)\Big]_{z=0}, \quad n = 0, 1, \ldots \tag{33}$$

Therefore, Equation (28) is transformed into

$$\sum_{n=0}^{+\infty} v_n(z) = \mathbb{L}\Big(\sum_{n=0}^{+\infty} v_n(z)\Big) + \sum_{n=0}^{+\infty} \mathbf{A}_n\Big(v_0(z), v_1(z), \ldots, v_n(z)\Big) + \Theta(z),$$

which yields the following iterative structures named as ADM technique:

$$\begin{cases} v_0(z) = \Theta(z), \\ v_1(z) = \mathbb{L}(v_0(z)) + \mathbf{A}_0(v_0(z), v_1(z), \ldots, v_n(z)), \\ v_2(z) = \mathbb{L}(v_1(z)) + \mathbf{A}_1(v_0(z), v_1(z), \ldots, v_n(z)), \\ v_3(z) = \mathbb{L}(v_2(z)) + \mathbf{A}_2(v_0(z), v_1(z), \ldots, v_n(z)), \\ \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ v_n(z) = \mathbb{L}(v_{n-1}(z)) + \mathbf{A}_{n-1}(v_0(z), v_1(z), \ldots, v_n(z)), \\ \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \end{cases} \tag{34}$$

Eventually, by representing K-term approximate solution of the ψ-integral Equation (5) as

$$\Upsilon_K(z) = \sum_{n=0}^{K} v_n(z), \tag{35}$$

the exact solution of (5) is given by

$$v(z) = \lim_{K \to +\infty} \Upsilon_K(z). \tag{36}$$

Finally, we figure out that the approximate solutions and the exact solution of the multi-term ψ-RLFBVP (1) are explored by $u_n(z) = \mathfrak{I}_{0^+}^{\sigma_n} \Upsilon_n(z)$ and $u(z) = \mathfrak{I}_{0^+}^{\sigma_n} v(z)$, respectively.

5. Examples

At this moment, we give two illustrative and distinct applied examples. In the first one, the theoretical existence results are checked and in the second one, the numerical solutions of the supposed ψ-RLFBVP are extracted caused by the numerical techniques DGJIM and ADM formulated above.

Example 1. *Consider the following ψ-RLFBVP which has a structure as*

$$\begin{cases} \mathfrak{D}_{0^+}^{1.8;z^2} u(z) = z + \dfrac{1}{16}\sin(2u(z)) + \dfrac{1}{8}\mathfrak{D}_{0^+}^{0.1}u(z) + \dfrac{1}{10}\sin\left(\mathfrak{D}_{0^+}^{0.2}u(z)\right), \quad z \in [0,1], \\ u(0) = 0, \\ u(1) = 2\displaystyle\int_0^{\frac{1}{2}} s\Big(\frac{1}{4} - s^2\Big)^3 \big(1 + 1000u(s)e^s\big)ds + 2\int_0^{\frac{1}{4}} s\Big(\frac{1}{16} - s^2\Big)^4 \big(1 + 2e^s\sin\big(u(s)\big)\big)ds. \end{cases}$$

In this example, we have $\psi(z) = z^2$, $\varrho = 1.8$, $\sigma_0 = 0$, $\sigma_1 = 0.1$, $\sigma_2 = 0.2$, $\xi = \frac{1}{2}$, $\eta = \frac{1}{4}$, $p = \Gamma(4)$, $q = \Gamma(5)$, $\mu = 4$, $\nu = 5$ and $k_1\big(z, u(z)\big) = 1 + 1000u(z)e^z$, $k_2\big(z, u(z)\big) = 1 + 2e^z\sin\big(u(z)\big)$.

Then $M_0 = 0.125$, $M_1 = 0.125$, $M_2 = 0.1$, $l_1(z) = 1000e^z$, $l_2(z) = 2e^z$. A simple computation gives us

$$\begin{aligned} \Phi &= \frac{p\Gamma(\varrho)}{\Gamma(\varrho - \sigma_n)\Gamma(\sigma_n + 1)}\mathfrak{I}_{0^+}^{\mu;\psi} l_1(\xi) + \frac{q\Gamma(\varrho)}{\Gamma(\varrho - \sigma_n)\Gamma(\sigma_n + 1)}\mathfrak{I}_{0^+}^{\nu;\psi} l_2(\eta) \\ &+ \sum_{j=0}^{n}\left(\frac{M_j\big(\psi(1) - \psi(0)\big)^{\varrho - \sigma_j}}{\Gamma(\varrho - \sigma_j + 1)} + \frac{M_j\Gamma(\varrho)\big(\psi(1) - \psi(0)\big)^{\varrho - \sigma_j}}{\Gamma(\varrho - \sigma_n)\Gamma(\varrho + \sigma_n - \sigma_j + 1)}\right) \approx 0.9003 < 1. \end{aligned}$$

Hence, by Theorem 2 it follows that our ψ-RLFBVP designed in the current example involves a solution uniquely.

In the next example, we consider a case of the suggested ψ-RLFBVP and compare approximate numerical results with exact outcomes which states the effectiveness and the compatibility of both numerical techniques DGJIM and ADM.

Example 2. *Let us consider the following ψ-RLFBVP which has a format as*

$$\begin{cases} \mathfrak{D}_{0^+}^{1.7;z^2} u(z) = 1 + \mathfrak{D}_{0^+}^{0.5;z^2} u(z) + \chi(z), \quad z \in [0,1], \\ u(0) = 0, \\ u(1) = \displaystyle\int_0^{\frac{1}{2}} s\Big(\frac{1}{4} - s^2\Big)\Big(\frac{1}{4} - u(s)\Big)ds + 2\int_0^{\frac{1}{3}} s\Big(\frac{1}{9} - s^2\Big)^2\Big(\frac{1}{9} - u(s)\Big)ds, \end{cases} \tag{37}$$

where

$$\chi(z) = \frac{1}{\Gamma(0.3)}z^{-1.4} - \frac{1}{\Gamma(1.5)}z - 1.$$

In the above structure, we have chosen parameters $\psi(z) = z^2$, $\varrho = 1.7$, $\xi = 1/2$, $\eta = 1/3$, $\sigma_n = 0.5$, $\mu = 2$, $\nu = 3$, $p = 1$ and $q = 2$. It is evident that $\varrho - \sigma_n = 1.2 > 1$. Also

$k_1(z,u(z)) = \frac{1}{4} - u(z)$ *and* $k_2(z,u(z)) = \frac{1}{9} - u(z)$ *for* $z \in [0,1]$. *By assuming* $v(z) = \mathfrak{D}_{0^+}^{0.5;z^2} u(z)$, *the corresponding* ψ*-integral equation of the* ψ*-RLFBVP* (37) *is displayed as*

$$\begin{aligned}
v(z) &= \mathfrak{I}_{0^+}^{1.2;z^2}[1 + v(z) + \chi(z)] + \frac{\Gamma(1.7)}{\Gamma(1.2)}\Bigg[\int_0^{\frac{1}{2}} 2s\left(\frac{1}{4} - s^2\right)\left(\frac{1}{4} - \mathfrak{I}_{0^+}^{0.5;z^2} v(s)\right)ds \\
&\quad + \int_0^{\frac{1}{3}} 2s\left(\frac{1}{9} - s^2\right)^2\left(\frac{1}{9} - \mathfrak{I}_{0^+}^{0.5;z^2} v(s)\right)ds - \mathfrak{I}_{0^+}^{1.7;z^2}[1 + v(z) + \chi(z)]\Big|_{z=1}\Bigg](z^2)^{0.2} \\
&= \mathfrak{I}_{0^+}^{1.2;z^2}1 + \mathfrak{I}_{0^+}^{1.2;z^2} v(z) + \mathfrak{I}_{0^+}^{1.2;z^2}\chi(z) + \frac{2\Gamma(1.7)z^{0.4}}{\Gamma(1.2)}\int_0^{\frac{1}{2}} s\left(\frac{1}{4} - s^2\right)\left(\frac{1}{4} - \mathfrak{I}_{0^+}^{0.5;z^2} v(s)\right)ds \\
&\quad + \frac{2\Gamma(1.7)z^{0.4}}{\Gamma(1.2)}\int_0^{\frac{1}{3}} s\left(\frac{1}{9} - s^2\right)^2\left(\frac{1}{9} - \mathfrak{I}_{0^+}^{0.5;z^2} v(s)\right)ds - \frac{\Gamma(1.7)}{\Gamma(1.2)}\mathfrak{I}_{0^+}^{1.7;z^2}1\Big|_{z=1} \times z^{0.4} \\
&\quad - \frac{\Gamma(1.7)}{\Gamma(1.2)}\mathfrak{I}_{0^+}^{1.7;z^2} v(z)\Big|_{z=1} \times z^{0.4} - \frac{\Gamma(1.7)}{\Gamma(1.2)}\mathfrak{I}_{0^+}^{1.7;z^2}\chi(z)\Big|_{z=1} \times z^{0.4}.
\end{aligned} \tag{38}$$

Now, we decompose the right side of (38) *as*

$$v(z) = \mathbb{L}(v(z)) + \mathbb{N}(v(z)) + \Theta(z),$$

where

$$\begin{aligned}
\mathbb{L}(v(z)) &= \mathfrak{I}_{0^+}^{1.2;z^2} v(z) - \frac{\Gamma(1.7)}{\Gamma(1.2)}\mathfrak{I}_{0^+}^{1.7;z^2} v(z)\Big|_{z=1} \times z^{0.4}, \\
\mathbb{N}(v(z)) &= \mathfrak{I}_{0^+}^{1.2;z^2}1 + \frac{2\Gamma(1.7)z^{0.4}}{\Gamma(1.2)}\int_0^{\frac{1}{2}} s\left(\frac{1}{4} - s^2\right)\left(\frac{1}{4} - \mathfrak{I}_{0^+}^{0.5;z^2} v(s)\right)ds \\
&\quad + \frac{2\Gamma(1.7)z^{0.4}}{\Gamma(1.2)}\int_0^{\frac{1}{3}} s\left(\frac{1}{9} - s^2\right)^2\left(\frac{1}{9} - \mathfrak{I}_{0^+}^{0.5;z^2} v(s)\right)ds - \frac{\Gamma(1.7)}{\Gamma(1.2)}\mathfrak{I}_{0^+}^{1.7;z^2}1\Big|_{z=1} \times z^{0.4}, \\
\Theta(z) &= \mathfrak{I}_{0^+}^{1.2;z^2}\chi(z) - \frac{\Gamma(1.7)}{\Gamma(1.2)}\mathfrak{I}_{0^+}^{1.7;z^2}1\Big|_{z=1} \times z^{0.4}.
\end{aligned}$$

In this case, the sequence of approximate solutions of (37) *and* (38) *are obtained based on algorithms of numerical techniques DGJIM and ADM as follows:*

- *Numerical solutions via DGJIM method:*

By using the algorithms of DGJIM and after some computations, we obtain four approximate solutions as

$$\mathrm{Y}_0(z) = 1.1284z - 0.6474z^{3.4} - 0.9076z^{2.4} - 0.6407z^{0.4}$$

$$\mathrm{Y}_1(z) = 1.1284z - 1.8152z^{2.4} - 1.2306z^{0.4} - 0.1887z^{5.8} - 0.3354z^{4.8} - 0.4736z^{2.8}$$

$$\begin{aligned}
\mathrm{Y}_2(z) &= 1.1284z - 1.8152z^{2.4} - 3.4315z^{0.4} - 0.6708z^{4.8} - 0.0472z^{2.8} - 0.0747z^{7.2} \\
&\quad - 0.1583z^{5.2}
\end{aligned}$$

$$\begin{aligned}
\mathrm{Y}_3(z) &= 1.1284z - 1.8152z^{2.4} - 0.0039z^{0.4} - 0.6708z^{4.8} + 2.0.6992z^{2.8} - 0.1494z^{7.2} \\
&\quad - 3.3022z^{5.2} - 0.0050z^{10.6} - 0.0117z^{9.6}
\end{aligned}$$

and

$$u_0(z) = z^2 - 0.4126z^{4.4} - 0.6474z^{3.4} - 0.6474z^{1.4}$$

$$u_1(z) = z^2 - 1.2948z^{3.4} - 1.2435z^{1.4} - 0.0340z^{6.8} - 0.1887z^{5.8} - 0.2300z^{3.8}$$

$$u_2(z) = z^2 - 1.2948z^{3.4} - 3.4675z^{1.4} - 0.3774z^{5.8} - 0.0321z^{3.8} - 0.0358z^{8.2} - 0.0864z^{6.2}$$

$$u_3(z) = z^2 - 1.2948z^{3.4} - 0.0039z^{1.4} - 0.3774z^{5.8} + 1.8348z^{3.8} - 0.0716z^{8.2} - 0.1649z^{6.2} - 0.0020z^{11.6} - 0.0050z^{10.6}.$$

- *Numerical solutions via ADM method:*

By the algorithms of ADM along with some computations, we obtain four approximate solutions as

$$Y_0(z) = 1.1284z - 0.6474z^{3.4} - 0.9076z^{2.4} - 0.6407z^{0.4}$$

$$Y_1(z) = 1.1284z - 1.8152z^{2.4} - 1.4106z^{0.4} - 0.1887z^{5.8} - 0.3354z^{4.8} - 0.4736z^{2.8}$$

$$Y_2(z) = 1.1284z - 1.8152z^{2.4} - 0.8212z^{0.4} - 0.6718z^{4.8} - 0.9096z^{2.8} - 0.0358z^{8.2} - 0.0747z^{7.2} - 0.1583z^{5.2}$$

$$Y_3(z) = 1.1284z - 1.8152z^{2.4} - 0.0138z^{0.4} - 0.6718z^{4.8} + 2.6764z^{2.8} - 0.1494z^{7.2} - 0.3022z^{5.2} - 0.0050z^{10.6} - 0.0117z^{9.6} - 0.0295$$

and

$$u_0(z) = z^2 - 0.4126z^{4.4} - 0.6474z^{3.4} - 0.6474z^{1.4}$$

$$u_1(z) = z^2 - 1.2948z^{3.4} - 1.4254z^{1.4} - 0.6215z^{6.8} - 0.1887z^{5.8} - 0.3219z^{3.8}$$

$$u_2(z) = z^2 - 1.2948z^{3.4} - 0.8298z^{1.4} - 0.3774z^{5.8} - 0.6183z^{3.8} - 0.0162z^{9.2} - 0.0358z^{8.2} - 0.0864z^{6.2}$$

$$u_3(z) = z^2 - 1.2948z^{3.4} + 0.1394z^{1.4} - 0.3774z^{5.8} + 1.8193z^{3.8} - 0.0716z^{8.2} - 0.1649z^{6.2} - 0.0020z^{11.6} - 0.0050z^{10.6} - 0.0138z^{8.6}.$$

In the final step, you can see the graphs of the fourth approximate solution caused by the DGJIM and the ADM algorithms for the supposed ψ-RLFBVP (37) and the ψ-integral Equation (38) plotted in Figures 1 and 2 in which these approximate solutions are compared with the exact solution of the ψ-RLBVP (37) given by $u(z) = z^2$ and the exact solution of the ψ-integral equation given by $v(z) = \frac{1}{\Gamma(1.5)}z$, respectively.

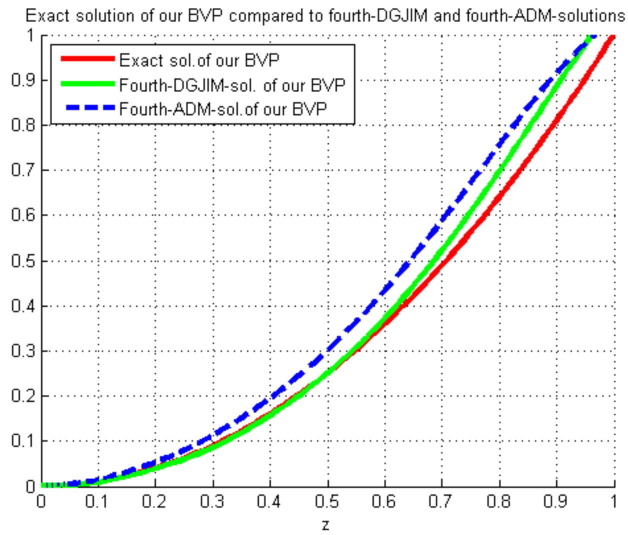

Figure 1. The graph of the exact solution of the ψ-RLFBVP (37) compared with the graphs of the fourth Dafterdar-Gejji and Jafari method (DGJIM)-approximate solution and the fourth Adomian decomposition method (ADM)-approximate one.

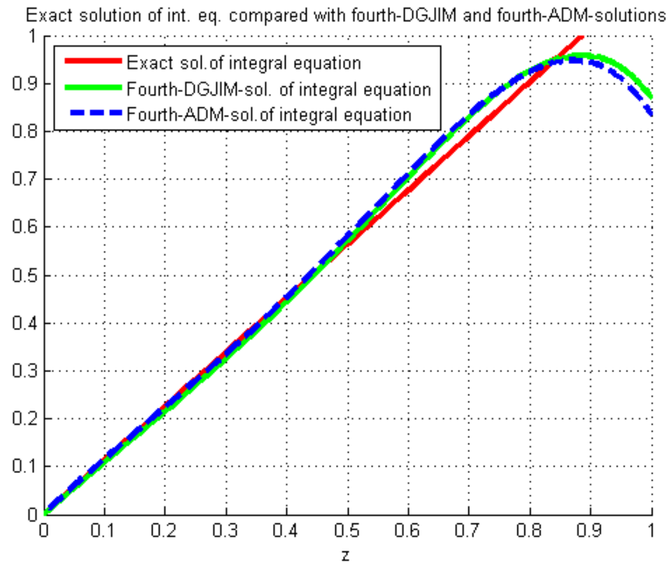

Figure 2. The graph of the exact solution of the ψ-integral Equation (38) compared with the graphs of the fourth DGJIM-approximate solution and the fourth ADM-approximate one.

6. Conclusions

In this research, in the first step, we reviewed the existence of solutions for a generalized multi-term ψ-RLFBVP subject to the generalized ψ-integral boundary conditions. Then, we implemented two numerical techniques called DGJIM and ADM algorithms for solving the supposed multi-term ψ-RLBVP via a decomposition approach. In the sequel, we have illustrated by a numerical example that the approximate solutions caused by these numerical techniques are in excellent compatibility with the exact solutions. These algorithms give the approximate solution as a series that converges to the exact one quickly whenever it exists. Consequently, the current research study emphasizes that these numerical techniques can be implemented in many other various fractional multi-term FBVPs subject to different boundary conditions by means of some symmetric and asymmetric derivation operators.

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