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Estimates on Some General Classes of Holomorphic Function Spaces

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Abstract: In this current manuscript, some general classes of weighted analytic function spaces in a unit disc are defined and studied. Special functions significant in both analytic $T(p, q, m, s; \Psi)$ norms and analytic Ψ-Bloch norms serve as a framework for introducing new families of analytic classes. An application in operator theory is provided by establishing important properties of the composition-type operator C_ϕ such as the boundedness and compactness with the help of the defined new classes.

Keywords: composition operators; Bloch functions; $T(p, p-2, m, s; \Psi)$-classes; Carleson measure

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1. Introduction

The subjects of complex classes of Banach function spaces and composition operators have gained extremely considerable popularity and high importance during the last few decades. These important subjects have mainly demonstrated significant applications and interesting, widespread, different fields of mathematical analysis. In the last few years, regarding the above-mentioned evolved motivations, some extended complex classes of function spaces associated with certain weights have been introduced and studied actively; see the related references herein. The main goal of this article is to introduce some new relevant classes of related holomorphic function spaces with the help of some general relevant weights and then establish some numerous, potentially useful interesting properties and general formulas. Some important properties of the composition operator C_ϕ such as the boundedness and compactness are investigated using the defined general classes in the considered complex disc.

Mainly definitions, notations and basic concepts will be recalled in the next section.

The considered class that contains all the analytic functions in the unit disk $\mathbb{D} = \{w \in \mathbb{C} : |w| < 1\}$ is symbolized by $H(\mathbb{D})$.

Definition 1 ([1,2]). *Let $g \in H(\mathbb{D})$ and $\alpha \in (0, \infty)$. When*

$$\|g\|_{\mathcal{B}^\alpha} = \sup_{w \in \mathbb{D}} (1 - |w|^2)^\alpha |g'(w)| < \infty, \ w \in \mathbb{D}$$

Thus, g is said to belong to the known α-Bloch space $\mathcal{B}^\alpha$. Additionally, the specific space $\mathcal{B}^1$ is called the analytic Bloch space $\mathcal{B}$.

Definition 2 ([3]). *Let $g \in H(\mathbb{D})$ and suppose that $p \in (1, \infty)$. If*

$$\|g\|_{B_p}^p = \sup_{w \in \mathbb{D}} \int_{\mathbb{D}} |g'(w)|^p (1 - |w|^2)^{p-2} d\xi(w) < \infty,$$

then g *belongs to the Besov space* B_p.

Suppose that $\varphi : \mathbb{D} \longrightarrow \mathbb{D}$. Assume $d\xi(w) = dx\,dy$ defines the relevant Euclidean area element on $\mathbb{D}$. Using φ, the composition operator C_φ is given by

$$C_\varphi g = g \circ \varphi,$$

where $g \in H(\mathbb{D})$. This operator maps analytic functions g to analytic functions.

Composition operators are among the most interesting and widely studied of the different types of operators. The study of their properties (boundedness and compactness) is very important. For several studies of composition operators on some weighted classes of analytic function spaces, refer to [4–8] and others.

Let $F : X_1 \to X_2$ be a linear operator; then, F is called compact if it maps relevant bounded sets in X_1 to sets in X_2, where such sets have compact closure. For the spaces X_1 and X_2 of the Banach type, it is said that F is compact from X_1 to X_2 if each bounded sequence $\{a_n\} \in X_1$ and the sequence $\{Fa_n\} \in X_2$ contains a subsequence converging to some limits in X_2.

Let M_h and M_h^* be two quantities, both of them depending on the function h, for which $h \in H(\mathbb{D})$; thus, we can write $M_h \approx M_h^*$, when have a constant $\eta > 0$, for which

$$\frac{1}{\eta} M_h^* \leq L_g \leq \eta\, M_h^*.$$

Let $b \in \mathbb{D}$ and $0 < |w| = \rho < 1$. The symbol $\sigma(b, \rho)$ stands for the relevant pseudo-hyperbolic complex disc such that (see [9])

$$|\sigma(b, \rho)| = \frac{(1 - |b|^2)^2}{(1 - \rho^2 |b|^2)^2} \rho^2.$$

2. Analytic Bloch Characterizations

In the present article, some properties of analytic functions belonging to $T(p, q, m, s; \Psi)$ are studied. Additionally, certain Carleson measure characterizations of the relevant compact composition operator C_φ on $T(p, q, m, s; \Psi)$ spaces are given.
Interesting characterizations of C_φ on $T(p, q, m, s; \Psi)$ spaces are also presented. In this relevant study, the relevant function Ψ is supposed to be a nondecreasing bounded and continuous function, where $\Psi : (0, 1) \to (0, \infty)$.

Definition 3. *For a given nondecreasing continuous function* $\Psi : (0, 1) \to (0, \infty)$, *the function* $g \in H(\mathbb{D})$ *is said to belong to the space* $\mathcal{B}_\Psi$ *if*

$$\|g\|_{\mathcal{B}_\Psi} = \sup_{w \in \mathbb{D}} \Psi(|w|)\, |g'(w)| < \infty.$$

Remark 1. *In the above definition, suppose that* $\Psi(t) = (1 - t)^\alpha$, $0 < t < \infty$; *then, we obtain the definition of an* α*-Bloch space where* $0 < \alpha < \infty$.

Let the Green's function $g(w, a) = \ln |\frac{1 - \bar{a}w}{a - w}| = \ln \frac{1}{|\varphi_a(w)|}$, where $\varphi_a(w) = \frac{1 - \bar{a}w}{a - w}$ stands for the Möbius transformation.
By the help of the function $g(w, a)$, the following definition can be given.

Definition 4. *Let* $0 < p < \infty$, $-2 < q < \infty$, $0 < m < \infty$ *and* $0 < s < \infty$. *Additionally, let* $q + s + m > -1$. *For a given nondecreasing continuous function* $\Psi : (0, 1) \to (0, \infty)$, *the function* $g \in H(\mathbb{D})$ *is said to belong to the space* $T(p, q, m, s; \Psi)$ *if*

$$\|g\|^p_{T(p,q,m,s;\Psi)} = \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |g'(w)|^p (1 - |w|^2)^q \Psi^p(|w|)\, g^s(w, a) (1 - |\varphi_a(w)|^2)^m d\xi(w) < \infty. \quad (1)$$

Remark 2. *The analytic $T(p,q,m,s;\Psi)$ function classes involve some various special analytic classes of complex-type function spaces. For example, when $m = 0$ and $\Psi \equiv 1$, the analytic $F(p,q,s)$ classes can be obtained (see [10]). Additionally, when $q = 0, m = 0, p = 2$ and $\Psi \equiv 1$, the analytic Q_s classes can be followed ([2]). Furthermore, upon setting $s = 0$ and $\Psi \equiv 1$, the known analytic Besov-type classes are also obtained (see [3,11]).*

In this current manuscript, the conditions of $q + s + m > -1$ and the boundedness of the weighted holomorphic function Ψ are necessary to guarantee nontrivial holomorphic spaces. Moreover, these classes of complex-type function spaces have interesting, clear importance and different applications in various research areas, for instance, in operator theory and in measure theory as well as in differential equations. In the present article, useful interesting discussions with the help of a class of composition operators on the defined analytic $T(p,q,m,s;\Psi)$-type classes are introduced.

Theorem 1. *Let $0 < |w| = \rho < 1$, $0 < p < \infty$, and $0 \le m < \infty$ with $q + s + m > -1$. Assume that $0 \le s < \infty$. Let $\Psi : (0,1) \to (0,\infty)$ be a given nondecreasing bounded and continuous function. Then, for $g \in H(\mathbb{D})$, the next fundamental quantities are equivalent:*

(A) $\|g\|^p_{\mathcal{B}_\Psi}$

(B)

$$\sup_{a\in\mathbb{D}} \frac{1}{|\sigma_{(a,\rho)}|^{1-\frac{p}{2}}} \int_{U(a,r)} |g'(w)|^p \Psi^p(|w|) d\xi(w) < \infty,$$

(C)

$$\sup_{a\in\mathbb{D}} \int_{\sigma_{(a,\rho)}} |g'(w)|^p (1-|w|^2)^{p-2} \Psi^p(|w|) d\xi(w) < \infty,$$

(D)

$$\sup_{a\in\mathbb{D}} \int_{\sigma_{(a,\rho)}} |g'(w)|^p (1-|w|^2)^{p-2} \Psi^p(|w|)(1-|\varphi_a(w)|^2)^{m+s} d\xi(w) < \infty,$$

(E)

$$\sup_{a\in\mathbb{D}} \int_{\sigma_{(a,\rho)}} |g'(w)|^p (1-|w|^2)^{p-2} \Psi^p(|w|) g^{m+s}(w,a) d\xi(w) < \infty,$$

(F)

$$\sup_{a\in\mathbb{D}} \int_{\sigma_{(a,\rho)}} |g'(w)|^p \left(\log \frac{1}{|w|}\right)^p \Psi^p(|w|) |\varphi'_a(w)|^2 d\xi(w) < \infty.$$

Proof. The proof can be obtained as the corresponding results in [1,12,13] with very simple modifications. □

3. Nevanlinna-Type Functions

Composition operators on weighted classes of function spaces are intensively researched by several authors (see [4,5,14–19] and others).
Nevanlinna-type counting functions are playing a significant role in such studies. Some types of Nevanlinna counting functions for different function spaces are defined. For instance, in [4], the known Nevanlinna-type counting holomorphic function is defined to help with studying some interesting useful properties of composition-type operators on holomorphic-type $F(p,q,s)$ classes.

Next, an alternative definition for the Nevanlinna-type counting holomorphic function will be introduced to characterize certain properties of C_φ on holomorphic $T(p,q,m,s;\Psi)$-type classes.

Definition 5. *The alternative definition of the Nevanlinna-type counting holomorphic function for the holomorphic $T(p,q,m,s;\Psi)$-type classes is given by:*

$$L_{p,q,m,s;\omega}(w) = \sum_{\varphi(z)=w} \{|\varphi'(z)|^{p-2}(1-|z|^2)^q \Psi^p(|z|) g^s(z,a)(1-|\varphi_a(z)|^2)^m\}, \tag{2}$$

for $w \in \varphi(\mathbb{D})$, $p \in (2,\infty)$, $q \in (-2,\infty)$, $s \in (0,\infty)$ *and* $0 < m < \infty$.

Remark 3. *In Definition 5, let* $\Psi(|z|) \equiv 1$ *and* $m = 0$*; then, we obtain the Nevanlinna-type function as introduced in [4]. Additionally, if we let* $\Psi(|z|) \equiv 1$ *and* $s = m = 0$*, then we obtain the counting function that is given in [20].*

The alternative known Nevanlinna function is introduced by using the change of variables formula in $T(p,q,m,s;\Psi)$ classes as follows:

For $g \in T(p,q,m,s;\Psi)$, $2 \le p < \infty$, $-2 < q < \infty$, $0 < s < \infty$ and $0 < m < \infty$,

$$\begin{aligned}\|C_\varphi g\|^p_{T(p,q,m,s;\Psi)} &= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |(g\circ\varphi)'(z)|^p (1-|z|^2)^q \Psi^p(|z|) g^s(z,a)(1-|\varphi_a(z)|^2)^m \, d\xi(z) \\ &= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |g'(\varphi(z))|^p |\varphi'(z)|^p (1-|z|^2)^q \Psi^p(|z|) g^s(z,a)(1-|\varphi_a(z)|^2)^m \, d\xi(z).\end{aligned}$$

By using the known change in variables as obtained in [4,20], we yield

$$\|C_\varphi g\|^p_{T(p,q,m,s;\Psi)} = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |g'(w)|^p L_{p,q,m,s;\Psi}(w) d\xi(w). \tag{3}$$

Next, there are some restrictions of C_φ to $T(p,q,,m,s;\Psi)$ that shall be clarified.

Then, C_φ is a bounded operator if and only if there is a positive constant K such that

$$\|C_\varphi g\|^p_{T(p,q,m,s;\Psi)} \le K \|g\|^p_{T(p,q,m,s;\Psi)} \tag{4}$$

for all $g \in T(p,q,m,s;\Psi)$.

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |g'(w)|^p L_{p,q,m,s;\Psi}(w) d\xi(w) \le C \|g\|^p_{T(p,q,m,s;\Psi)}$$

for all $g \in T(p,q,m,s;\Psi)$.

Suppose that $h \in (0,1)$, $\theta \in [0,\, 2\pi]$; also let

$$\Omega(h,\theta) = \{re^{it} \,:\, r \in (1-h,1) \text{ with } h > |\theta - t|\},$$

$$S(h,\theta) = \{re^{it} \,:\, h > |re^{it} - re^{i\theta}|\}.$$

It is known that any positive-type measure μ on $\mathbb{D}$ is a Carleson-type measure when we find a positive M_1 such that

$$\mu(S(h,\theta)) \le M_1 h, \text{ for } h \in (0,1) \text{ and } \theta \in [0 \le 2\pi].$$

Now, we clarify things by using the considered measures, which induce certain types of Carleson conditions, playing an important role in understanding how the analytic function φ mapping $\mathbb{D}$ onto itself provides some types of bounded composition operators on the classes $W = (T(p,q,m,s;\Psi)$ or $\mathcal{B}_\Psi$.

For various intensive and essential results on Carleson measures, see [4,5,20,21] and others.

Definition 6. *Assume that μ is a positive-type measure on $\mathbb{D}$, and t $W = \mathcal{B}_\Psi$ or $T(p,q,m,s;\Psi)$ for $0 < p < \infty$, $q \in (-2,\infty)$, $s \in (0,\infty)$ and $m \in (0,\infty)$. Thus, the measure μ is a (W,p)-Carleson-type measure when we find a constant $C > 0$; hence*

$$\int_{\mathbb{D}} |g'(w)|^p d\mu(w) \leq C\|g\|_W^p,$$

for all $g \in W$.

Definition 7. *For $p \in (1,\infty)$, the specific measure μ is called a vanishing-type p-Carleson measure when*

$$\lim_{h\to 0} \sup_{\theta\in[0,2\pi)} F(\mu,h) = 0,$$

where $F(\mu,h) = \frac{\mu(S(h,\theta))}{h^p}$.

With the help of (2) and (3), we can deduce that the considered operator C_φ is bounded on $T(p,q,m,s;\Psi)$ if the measure $L_{p,q,m,s;\Psi}(w)\,d\xi(w)$ is a $(T(p,q,m,s;\Psi),p)-$Carleson measure.

Next, some global and interesting characterizations of compact composition operators on $T(p,q,m,s;\Psi)$ classes in terms of p_1-Carleson-type measures are given.

Theorem 2. *Let $2 \leq p < \infty$, $0 < m < \infty$ and $0 < s < \infty$. For a given nondecreasing continuous function $\Psi : (0,1) \to (0,\infty)$, the next fundamental quantities can be equivalent:*

(a) *The measure μ is a $(T(p,p-2,m,s;\Psi),p)-$Carleson-type measure,*

(b) *We can find a positive constant C_1, for which $\mu(S(h,\theta)) \leq C_1 h^p$, $\forall\ 0 < h < 1$ and $0 \leq \theta < 2\pi$,*

(c) *We can find a specific positive constant C_2 for which*

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |\varphi_a'(z)|^p d\mu(z) \leq C_2, \quad \forall\ a \in \mathbb{D}.$$

Proof. Now, assume that (a) holds. Thus, by using Theorem 1 and Definition 7, we infer that

$$\int_{\mathbb{D}} |g'(z)|^p d\mu(z) \leq C \int_{\mathbb{D}} |g'(z)|^p (1-|z|^2)^{p-2}\Psi^p(|z|) g^s(z,a)(1-|\varphi_a(z)|^2)^m d\xi(z),$$

for all $g \in T(p,p-2,m,s;\Psi)$. This can be obtained for

$$g(z) = \varphi_a(z) = \frac{a-z}{1-\bar{a}z}.$$

Thus,

$$\begin{aligned}\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |\varphi_a'(z)|^p d\mu(z) &\leq C \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |\varphi_a'(z)|^p (1-|z|^2)^{p-2}\Psi^p(|z|) g^s(z,a)(1-|\varphi_a(z)|^2)^m d\xi(z) \\ &\leq C\,\|\varphi_a\|^p_{T(p,p-2,m,s;\Psi)} \leq C \ \ const.,\end{aligned}$$

for all $a \in \mathbb{D}$. This gives (c). It is not hard to prove the equivalence between (b) and (c). Now, assume that (b) holds; we shall prove that (a) is true. For $z = re^{i\theta}$, let

$$\sigma_1(z) = \left\{ t : |t-z| < \frac{1-|z|}{2} \right\},$$

$$\sigma_2(z) = \left\{ t : |t-z| < 1-|z| \right\}.$$

Then,

$$\sigma_1(z) \subseteq \sigma_2(z) \subseteq S(2(1-|z|), \theta).$$

Furthermore, if $a \in \sigma_1(z)$, then

$$\frac{1}{2} \leq \frac{1-|t|}{1-|z|} \leq \frac{3}{2}.$$

Let $g \in T(p, q, m, s; \Psi)$; because the function g is analytic,

$$g'(z) = \frac{4}{\pi(1-|z|)^2} \int_{\sigma_1(z)} g'(t) d\xi(t).$$

By Jensen's inequality (see [22]), we obtain

$$|g'(z)|^p \leq \frac{4}{\pi(1-|z|)^2} \int_{\sigma_1(z)} |g'(t)|^p d\xi(t).$$

Thus,

$$\begin{aligned}
\int_{\mathbb{D}} |g'(z)|^p d\mu(z) &\leq \int_{\mathbb{D}} \frac{4}{\pi(1-|z|)^2} \left(\int_{\sigma_1(z)} |g'(t)|^p d\xi(t) \right) d\mu(z) \\
&\leq \frac{4}{\pi} \int_{\mathbb{D}} \left(\int_{\sigma_1(z)} |g'(t)|^p \left(\frac{3}{2(1-|t|)} \right)^2 d\xi(t) \right) d\mu(z) \\
&\leq \frac{9}{\pi} \int_{\mathbb{D}} \int_{\mathbb{D}} |g'(t)|^p \Lambda_{\sigma_1(z)}(t) (1-|t|)^{-2} d\xi(t) d\mu(z) \\
&\leq \frac{9}{\pi} \int_{\mathbb{D}} |g'(t)|^p (1-|t|)^{-2} \int_{\mathbb{D}} \Lambda_{\sigma_1(z)}(t) d\mu(z) d\xi(t),
\end{aligned}$$

where $\Lambda_{\sigma_1(z)}(t)$ stands for the characteristic function with $\Lambda_{\sigma_1(z)}(t) \leq \Lambda_{S(2(1-|z|),\theta)}(t)$, $z = re^{i\theta}$, because $a \in \sigma_1(z)$, which means that

$$|t - e^{i\theta}| < 2(1-|t|).$$

Now, by applying (b), we infer that

$$\int_{\mathbb{D}} \Lambda_{\sigma_1(z)} d\mu(z) \leq \mu(S(2(1-|t|), \theta)) \leq A 2^p (1-|t|)^p.$$

Therefore,

$$\begin{aligned}
\int_{\mathbb{D}} |g'(z)|^p d\mu(z) &\leq \frac{9}{\pi} A 2^p \int_{\mathbb{D}} |g'(t)|^p (1-|t|)^{p-2} d\xi(t) \\
&\leq C \int_{\mathbb{D}} |g'(t)|^p (1-|t|)^{p-2} d\xi(t),
\end{aligned}$$

where C_1 is a positive constant. From Theorem 1, the statements (C) and (E) are equivalent; hence,

$$\begin{aligned}
\int_{\mathbb{D}} |g'(z)|^p d\mu(z) &\leq C \int_{\mathbb{D}} |g'(t)|^p (1-|t|)^{p-2} \Psi^p(|z|) g^s(z,a) (1-|\varphi_a(z)|^2)^m \, d\xi(t) \\
&\leq C \, \|g\|_{T(p,p-2,m,s;\Psi)},
\end{aligned}$$

which is (a). The proof is thereby finished. □

Using Theorem 2, the following result can be proved.

Theorem 3. *Let φ be an analytic function on $\mathbb{D}$, $2 \leq p < \infty$, $0 < m < \infty$ and $0 < s < \infty$. Let $\Psi : (0,1) \to (0,\infty)$ be a given nondecreasing continuous function. Then, the operator C_φ is bounded on $T(p, p-2, m, s; \Psi) \Longleftrightarrow$*

$$\sup_{a \in \mathbb{D}} \|C_\varphi \ \varphi_a\|_{T(p,p-2,m,s;\Psi)} < \infty.$$

Next, some interesting results on $T(p, q, m, s; \Psi)$ spaces are proved.

Lemma 1. *Let $\Psi : (0,1) \to (0,\infty)$ be a given nondecreasing bounded and continuous function. Let $0 < p < \infty$, $0 < q < \infty, 0 < s < \infty$ $0 < m < \infty$. Suppose that*

$$\int_0^1 \frac{r\,dr}{\Psi(r)} < \infty$$

Then:

(i) Every bounded sequence $(h_n) \subset T(p, q, s, m; \Psi)$ is converging, uniformly bounded in compact specific sets.

(ii) For any specific sequence (h_n) on $T(p, q, s, m; \Psi)$, $\|h_n\|_{T(p,q,s,m;\Psi)} \to 0$ by the uniform convergence $h_n - h_n(0) \to 0$ on compact sets.

Proof. From Theorem 1, we have

$$\|g\|_{\mathcal{B}_\Psi} \leq \lambda_1 \|g\|_{T(p,q,m,s;\Psi)},$$

λ_1 is a positive constant. If $z \in U(0,r)$, $0 < r < 1$, then we get

$$|h_n - h_n(0)| = \left| \int_0^1 h_n'(zt) z dt \right| \leq \lambda_1 \|h_n\|_{\mathcal{B}_\Psi} \leq \lambda \|h_n\|_{T(p,q,m,s;\Psi)}.$$

where λ and λ_1 are positive constants. The lemma is therefore completely established. □

Lemma 2. *Let $\Psi : (0,1) \to (0,\infty)$ be a given nondecreasing continuous function. Suppose that $2 \leq p < \infty, 0 < q < \infty$, $0 < m < \infty$, and $0 < s < \infty$. Let $X_1, X_2 = T(p, q, m, s; \Psi)$ or $\mathcal{B}_\Psi$. Then,*

$$C_\varphi : X_1 \to X_2 \quad \text{is a compact operator} \Longleftrightarrow \|C_\varphi g_n\|_{X_2} \to 0$$

where $(g_n) \subset X_1$ is a bounded sequence converging uniformly to zero on compact sets as $n \to \infty$, $g_n \to 0$.

Proof. For the spaces $T(p, q, m, s; \Psi)$ or $\mathcal{B}_\Psi$, the quantities (i) and (ii) can be proved in view of Lemma 2.10 [20]. From Lemma 1, it is not hard to clarify that (i) holds. To clarify that the quantity (ii) holds, assume that (g_n) is a sequence in the closed unit ball of X_1. Then, by Lemma 1, the sequence (g_n) is uniformly bounded on compact sets. Applying Montel's theorem (see [23]), we can find a subsequence $(g_{n_k}), (n_1 < n_2 < \ldots)$ such that $g_{n_k} \to f$ uniformly on compact sets, for some $f \in H(\mathbb{D})$. Now, we clarify that $f \in X_1$.

(i) If $X_1 = T(p, q, m, s; \Psi), (2 \leq p < \infty, -2 < q < \infty,\ 0 < m < \infty,\ 0 < s < \infty)$, then by applying Fatou's theorem, we deduce that

$$\begin{aligned} \int_{\mathbb{D}} |g'(z)|^p \, S(\Psi, g, \varphi_a)\, d\xi(z) &= \int_{\mathbb{D}} \lim_{k\to\infty} |g_{n_k}'(z)|^p S(\Psi, g, \varphi_a)\, d\xi(z) \\ &\leq \liminf_{k\to\infty} \int_{\mathbb{D}} |g_{n_k}'(z)|^p S(\Psi, g, \varphi_a)\, d\xi(z), \\ &= \liminf_{k\to\infty} \|g_{n_k}\|^p_{T(p,q,m,s;\Psi)} < \infty, \end{aligned}$$

where

$$S(\Psi, g, \varphi_a) = (1-|z|^2)^q \Psi^p(|z|) g^s(z,a)(1-|\varphi_a(z)|^2)^m.$$

(ii) If $X_1 = \mathcal{B}_\Psi$, we get

$$|g'(z)|(1-|z|^2)\Psi(|z|) \quad = \quad \lim_{k\to\infty} |g'_{n_k}(z)|\Psi(|z|)(1-|z|^2) \leq \lim_{k\to\infty} \|g'_{n_k}\|_{\mathcal{B}_\Psi} < \infty,$$

Lemma 1 gives

$$C_\varphi : X_1 \to X_2 \quad \text{is a compact operator} \Longleftrightarrow \|C_\varphi g_n\|_{X_2} \to 0$$

where the sequence $(g_n) \subset X_1$ is a bounded specific sequence that converges uniformly to zero on compact sets as $n \to \infty$. Hence, $|g_n(\varphi(0))| + \|C_\varphi g_n\|_{X_2} \to 0$, as $n \to \infty$. The lemma is therefore proved.

For the compactness of the operator C_φ on $T(p,q,m,s;\Psi)$ spaces, the next results can be proved. □

Theorem 4. *Let $\Psi : (0,1) \to (0,\infty)$ be a given nondecreasing continuous function. Let $2 \leq p < p_1 < \infty$, $0 < q < \infty$, and $0 < s < \infty$. Then, the next essential quantities are equivalent:*

(a) *The operator $C_\varphi : T(p,q,m,s;\Psi) \to T(p_1,q,m,s;\Psi)$ is compact.*

(b) *The measure $L_{p_1,q,m,s;\Psi}(w)d\xi(w)$ is a vanishing p_1-Carleson measure.*

(c) *The norm $\|C_\varphi\ \varphi_a\|_{T(p_1,q,m,s;\Psi)} \to 0$ as $|a| \to 1$.*

Proof. Using (3), we infer that

$$\|C_\varphi\ \varphi_a\|^{p_1}_{T(p_1,q,m,s;\Psi)} = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |\varphi'_a(w)|^{p_1} L_{p_1,q,m,s;\Psi}(w)d\xi(w)$$

Using Lemma 2.1 in [24], we have (b)$\Longleftrightarrow$(c). Next, we show that (a)$\Longrightarrow$(c). Let us assume that

$$C_\varphi : T(p,q,m,s;\Psi) \to T(p_1,q,m,s;\Psi)$$

is a specific compact operator. Because the set $\{\varphi_a : a \in \mathbb{D}\}$ is bounded in classes $T(p,q,m,s;\Psi)$,

$$\|\varphi_a\|_{T(p,q,m,s;\Psi)} = \|z \circ \varphi_a\|_{T(p,q,m,s;\Psi)},$$

with the norm of φ_a in $T(p,q,m,s;\Psi)$ such that

$$|\varphi_a(0)| + \|\varphi_a\|_{T(p,q,m,s;\Psi)} < 1 + \|\varphi_a\|_{T(p,q,m,s;\Psi)} < \infty.$$

Additionally, the convergence $(\varphi_a - a) \longrightarrow 0$ as $|a| \to 1$ is uniform on specific compact sets, since

$$|\varphi_a - a| = |z|\frac{1-|a|^2}{|1-\bar{a}z|}, \text{ for } r = |z|;\ z \in \mathbb{D}.$$

Hence, by Lemma 2, we obtain

$$\|C_\varphi(\varphi_a - a)\|_{T(p_1,q,m,s;\Psi)} \to 0, \text{ as } |a| \to 1.$$

Now, we clarify that (b)$\Rightarrow$(a). Let (g_n) be a certain bounded sequence in $T(p,q,m,s;\Psi)$ that converges to 0 uniformly on compact sets. Using the mean-value property for the g'_n, shows that

$$g'_n(\zeta) = \frac{4}{\pi(1-|\zeta|)^2} \int_{|\zeta-z|<\frac{1-|\zeta|}{2}} |g'_n(z)|d\xi(z). \tag{5}$$

Jensen's inequality gives

$$\sigma_1(\zeta) = \left\{ z : |\zeta - z| < \frac{1-|\zeta|}{2} \right\},$$

$$|g_n'(\zeta)|^{p_1} \leq \frac{4}{\pi(1-|\zeta|)^2} \int_{\sigma_1(\zeta)} |g_n'(z)|^{p_1} d\xi(z). \tag{6}$$

Thus, by using (6) and applying the known Fubini's theorem, the following inequality can be deduced

$$\begin{aligned} \|C_\varphi g_n\|^{p_1}_{T(p_1,q,m,s;\Psi)} &= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |g_n'(\zeta)|^{p_1} L_{p_1,q,m,s;\Psi}(\zeta) d\xi(\zeta) \\ &\leq \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{4}{\pi(1-|\zeta|)^2} \left(\int_{\sigma_1(\zeta)} |g_n'(z)|^{p_1} d\xi(z) \right) L_{p_1,q,m,s;\Psi}(\zeta) d\xi(\zeta). \end{aligned}$$

Then,

$$\|C_\varphi g_n\|^{p_1}_{T(p_1,q,m,s;\Psi)} \leq \frac{4}{\pi} \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |g_n'(z)|^{p_1} \left(\int_{\mathbb{D}} \frac{\Lambda_{\sigma_1(\zeta)}(z)}{(1-|\zeta|)^2} L_{p_1,q,m,s;\Psi}(\zeta) d\xi(\zeta) \right) d\xi(z), \tag{7}$$

When $|\zeta - z| < \frac{1-|\zeta|}{2}$, $\zeta \in S(2(1-|z|), \theta)$, where $z = |z|e^{i\theta}$, since

$$|\zeta - e^{i\theta}| \leq |z-\zeta| + |e^{i\theta} - z| < \frac{1-|\zeta|}{2} + |\frac{z}{|z|} - z| < 2(1-|z|).$$

Hence, when $|\zeta - z| < \frac{1-|w|}{2}$,

$$\frac{1}{(1-|\zeta|)^2} < c_1 \frac{1}{(1-|z|)^2},$$

where c_1 is a positive specific constant. Then,

$$\begin{aligned} &\|C_\varphi g_n\|^{p_1}_{T(p_1,q,m,s;\Psi)} \leq c_1 \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{|g_n'(z)|^{p_1}}{(1-|z|)^2} \left(\int_{S(2(1-|z|),\theta)} L_{p_1,q,m,s;\Psi}(\zeta)\, d\xi(\zeta) \right) d\xi(z) \\ &= c_2 \sup_{a\in\mathbb{D}} \left(\int_{|z|>1-\frac{\delta}{2}} + \int_{|z|\leq 1-\frac{\delta}{2}} \frac{|g_n'(z)|^{p_1}}{(1-|z|)^2} \left(\int_{S(2(1-|z|),\theta)} L_{p_1,q,m,s;\Psi}(b) d\xi(\zeta) \right) d\xi(z) \right) \\ &= c_3 \sup_{a\in\mathbb{D}} (J + J^*) \end{aligned}$$

for any $0 < \delta < 1$, where c_2 and c_3 are specific positive constants. By fixing $\varepsilon > 0$ with $\delta > 0$ for which $\theta \in [0, 2\pi]$ with $h_1 < \delta$,

$$\sup_{a\in\mathbb{D}} \int_{S(h,\theta)} L_{p_1,q,m,s,\varphi;\Psi}(w) d\xi(w) \quad \leq \quad \varepsilon\, h_1^{p_1}. \tag{8}$$

Therefore, we deduce that

$$\begin{aligned} J \quad &\leq \quad \varepsilon\, 2^{p_1} \int_{|z|>1-\frac{\delta}{2}} \frac{|g_n'(z)|^{p_1}}{(1-|z|)^2} (1-|z|)^{p_1} d\xi(z) \\ &\leq \quad \varepsilon\, 2^{p_1} \int_{|z|>1-\frac{\delta}{2}} |g_n'(z)|^{p_1} (1-|z|)^{p_1-2} d\xi(z) \\ &\leq \quad \varepsilon\ c_4 \quad \|g_n\|_{\mathcal{B}^{p_1}} \quad < \quad \varepsilon\ c_5, \end{aligned}$$

where c_4 and c_5 are positive constants.

Additionally, we have

$$J^* \leq k \sup_{a\in\mathbb{D}} \int_{|z|\leq 1-\frac{\delta}{2}} \frac{|g_n'(z)|^{p_1}}{(1-|z|)^2} \left(\int_{S(2(1-|z|),\theta)} L_{p_1,q,s,m;\Psi}(w) d\xi(w) \right) d\xi(z)$$
$$= k_1 \sup_{a\in\mathbb{D}} \left(\int_{\mathbb{D}} L_{p_1,q,s,m,\Psi}(w) d\xi(w) \right) \int_{|z|\leq 1-\frac{\delta}{2}} |g_n'(z)|^{p_1} d\xi(z) \; < \; const.,$$

where k, k_1 and k_2 are positive constants. For the case when n is large enough, because the convergence type of $g_n' \to 0$ is uniform on considered compact sets, the following inequality can be deduced

$$\|C_\varphi g_n'\|^{p_1}_{T(p_1,q,m,s;\Psi)} < c^* \sup_{a\in\mathbb{D}} (J + J^*) \; < \; \varepsilon \quad const.,$$

for a larger n, where c^* is a specific positive constant. Then, $\|C_\varphi g_n'\|^{p_1}_{T(p_1,q,m,s;\Psi)} \to 0$, when $n \to \infty$, so Lemma 2 implies that

$$C_\varphi : T(p,q,m,s;\Psi) \to T(p_1,q,m,s;\Psi)$$

is an actually compact operator. The proof is thereby clearly established. □

Remark 4. *For our current study, when $\varphi(w) = w$, the composition operator C_φ is not a compact operator. The condition is $\varphi(e^{i\theta}) < 1$ (almost everywhere). Thus, the operator C_φ is compact whenever*

$$\int_0^{2\pi} \frac{d\theta}{1-\varphi(e^{i\theta})} < \infty.$$

4. Conclusions

This manuscript covers a global study of a class of composition operators acting between a new general family of analytic function spaces on a disk $\mathbb{D}$. The elementary properties of the new class of functions are characterized with a class of specific univalent functions.

Motivated by a nondecreasing and continuous function as well as the Green's function, some certain weighted holomorphic classes of functions are defined and carefully discussed in the present manuscript. Important and global characterizations for the weighted classes $T(p,q,m,s;\Psi)$ with the general weighted Bloch-type class $\mathcal{B}_\Psi$ are introduced. Concerning the defined classes, the values of all the parameters have significant and large effects in the obtained results. Some interesting specific properties for the class of composition operators are discussed. The essential width for the operator C_φ is clarified between the defined weighted classes. The compactness and boundedness properties of the operator C_φ are discussed with the help of Carleson-type measure roles as well as the alternative considered Nevanlinna function.

For future research on the classes $T(p,q,m,s;\Psi)$, we make the following remarks.

Remark 5. *How can we establish some properties for other classes of operators, such as superposition and integral operators on the analytic $T(p,q,m,s;\Psi)$ classes?*

Remark 6. *There are some interesting recent studies on hyperbolic-type spaces (see [25,26]). Thus, it is possible to discuss some problems by using hyperbolic $T(p,q,m,s;\Psi)$-type classes.*

Remark 7. *Generalizing the analytic $T(p,q,m,s;\Psi)$-type spaces to the known quaternion analysis can be considered a future open problem, too. For some numerous recent studies with the help of quaternion techniques, one may refer to [27–30] and others.*

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