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Asymptotic Stability of Nonlinear Discrete Fractional Pantograph Equations with Non-Local Initial Conditions

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Abstract: Pantograph, the technological successor of trolley poles, is an overhead current collector of electric bus, electric trains, and trams. In this work, we consider the discrete fractional pantograph equation of the form $\Delta_*^\beta[k](t) = w(t + \beta, k(t + \beta), k(\lambda(t + \beta)))$, with condition $k(0) = p[k]$ for $t \in \mathbb{N}_{1-\beta}, 0 < \beta \leq 1, \lambda \in (0,1)$ and investigate the properties of asymptotic stability of solutions. We will prove the main results by the aid of Krasnoselskii's and generalized Banach fixed point theorems. Examples involving algorithms and illustrated graphs are presented to demonstrate the validity of our theoretical findings.

Keywords: fractional pantograph equations; fractional difference equation; asymptotic stability; caputo difference operator

MSC: 26A33; 47H10; 93D20

1. Introduction

Graphical methods in engineering are very much useful to present clear results, develop reasoning, and spatial thinking. Dependency on computer-based simulations has led to the demise of graphical methods [1]. Though computer simulations with the correct programming convey invariably accurate results, they fail to provide ingenuity, understanding, and conceptual thinking. Graphical methods provide practical knowledge which is more efficient than just going through texts. The science of the mechanisms can be extended beyond its classical limits to include pneumatic, hydraulic, electrical, and electronic links.

A special type of differential equation with delay was discovered when J.R. Ockendon and A.B. Tayler studied motion of pantograph head on an electric locomotive [2]. The equation is of the form

$$x'(t) = ax(t) + bx(\lambda t) \tag{1}$$

where $x(t)$ represents the motion of the locomotive and a, b are real constants with $0 < \lambda < 1$. The pantograph is used in locomotion and trams to transfer power from the

wire to the traction unit by maintaining electrical contact. They are also used to increase or reduce motion in some definite proportion, as in the indicator rig on an engine where the motion of the crosshead is reduced proportionally to the desired length of the indicator diagram [3]. The pantograph is a four-bar mechanism used to enlarge or reduce drawings for it is evident that similar curves may be traced as well as straight lines. It was originally used in drafting for copying and scale line drawings. Three-dimensional pantograph is used in sculpting to enlarge sculptures by interchanging the positions. Windscreen wipers on pantograph in some vehicles are used to allow blade to cover more windscreen on each wipe. In 1890, the US census made use of keyboard punch which is a pantograph design [4]. Some heavy-duty applications of pantograph include scissor lifts, material handling equipment, stage lifts, etc. During the past few decades, there was a gradual development of the modeling of nonlinear phenomena that occurs in various science and engineering fields [5].

Fractional calculus, which is a generalization of classical integer order calculus, has become popular among the scientists and engineers as it renders new dimension and flexibility in dealing with real-world problems [6]. Increasing interest towards this field is due to non-local behavior and ultimate convergence to the integer order systems. Potential of fractional derivatives has already been widely explored by researchers from different parts of the world by studying its applications in a range of problems in biology, physics, electronics (circuit theory), chemistry, etc. Non-standard Lagrangians have wide range of applications in nonlinear differential equations, dynamical systems, etc. [7–13]. Fractional action-like variational approach is very useful in giving better description of dissipative system. The fractional non-standard Lagrangians have been effective in various areas of physics like astrophysics, cosmology, quantum and classical dynamical systems. Recent works can be seen in [14–18]. Discrete fractional calculus is gaining its importance in recent years. Recently, Atici and Eloe [19–22], and Miller and Ross [23], have studied discrete delta fractional calculus. The study of stability is a venerable branch in the qualitative theory of differential equations. Asymptotic stability results for fractional difference equations have been developed by Chen et al. [24–26] for both Caputo- and Riemann Liouville-type operators. Other authors studied stability results of nabla fractional equations [27–29]. In 2019, the authors investigated the k-dimensional system of Langevin Hadamard-type fractional differential inclusions with $2k$ different fractional orders and they established existence results for a fraction hybrid differential inclusion with Caputo–Hadamard-type fractional derivative [30,31]. In 2020, Zhou et al. studied a nonlinear non-autonomous model which is composed of two species in a rocky intertidal community and occupy each other by individual organisms, in a rocky intertidal community [32]. One can see some significant applications of fractional differential equations in [33–41].

Though a standard pantograph equation is available in literature, the varying design of the pantograph in accordance with its application has inspired us to consider the generalized version of the equation. Motivated by the works in [42–46], we consider the nonlinear discrete fractional pantograph equation

$$
\begin{cases}
\Delta_*^\beta[k](t) = w\big(t + \beta, k(t + \beta), k(\lambda(t + \beta))\big), \\
k(0) = p[k],
\end{cases}
\tag{2}
$$

for $t \in \mathbb{N}_{1-\beta}$, where $0 < \beta \leq 1$, $0 < \lambda < 1$, Δ_*^β is a Caputo like difference operator, k represents the motion of the pantograph, $w : \mathcal{E} \to \mathbb{R}$ is continuous with respect to k, and t. Here, $\mathcal{E} = [0, \infty) \times \mathcal{C} \times \mathcal{C}$, $\mathbb{N}_t = \{t, t+1, t+2, \ldots\}$, and $p : \mathcal{C} \to \mathbb{R}$ is Lipschitz continuous in k where $\mathcal{C} = C([0, \infty), \mathbb{C})$. That is, there is a positive constant $M \in (0, 1)$ such that

$$
\|p[k](t) - p[l](t)\| \leq M\|k(t) - l(t)\|,
\tag{3}
$$

for each $t \in \mathbb{N}_t$ and almost all $k, l \in \mathcal{C}$. The discretized form of standard pantograph equation can be obtained from Equation (2) when $\beta = 1$ and $w(t + \beta, k(t + \beta), k(\lambda(t +$

$\beta))) = ak(t + \beta) + bk(\lambda(t + \beta))$ for $t \in \mathbb{N}_{1-\beta}$. By employing fixed point hypotheses based on Krasnoselskii's and generalized Banach fixed point theorems, we investigate the asymptotic stability of solutions of Equation (2). Particular examples are presented to demonstrate the validity of our theoretical findings. Some interesting observations are presented at the end of the paper.

This paper is organized as follows. In Section 2, some notations, definitions, and lemmas that are essential in our further analysis are presented. In Section 3, we analyze the asymptotic stability of the problem expressed by (2). Section 4 contains some illustrative examples to show the validity and applicability of our results.

2. Essential Preliminaries

This section is committed to state some notations and essential preliminaries that are acting as necessary prerequisites for the subsequent sections. First, we recall $\sigma-$th fractional sum of function $k \in \mathcal{C}$.

Definition 1 ([19,20]). *Let $\sigma > 0$. The $\sigma-$th fractional sum of k is defined by*

$$\Delta^{-\sigma}[k](t) = \frac{1}{\Gamma(\sigma)} \sum_{s=a}^{t-\sigma} (t - s - 1)^{(\sigma-1)} k(s), \tag{4}$$

where $k(s)$ and $\Delta^{-\sigma}[k](t)$ are defined for $s \equiv a \mod (1)$, for $t \equiv (a + \sigma) \mod (1)$, respectively,

$$t^{(-\sigma)} = \frac{\Gamma(t+1)}{\Gamma(t+\sigma+1)}, \tag{5}$$

for $t \in \mathbb{N}_a$ and $\Delta^{-\sigma}$ maps functions defined on $\mathbb{N}_a$ to functions defined on $\mathbb{N}_{a+\sigma}$, which, upon substitution in Equation (4), leads to

$$\Delta^{-\sigma}[k](t) = \frac{1}{\Gamma(\sigma)} \sum_{s=a}^{t-\sigma} \frac{\Gamma(t-s)}{\Gamma(t-s-\sigma+1)} k(s). \tag{6}$$

Figure 1 presents the convergence of $t^{(-\sigma)}$ in 3-dimensional view, and it is clear that the greater the value of σ, the lesser the time taken for $t^{(-\sigma)}$ to approach zero. Figure 2, indeed, illustrates the behavior of $t^{(-\sigma)}$ in (5) whenever a and σ are changed, respectively. These results are presented in Table 1, and they show that the operator $t^{(-\sigma)}$ is decreasing with respect to both σ and t. Thus, $t^{(-\sigma)} \to 0$ as $t \to \infty$.
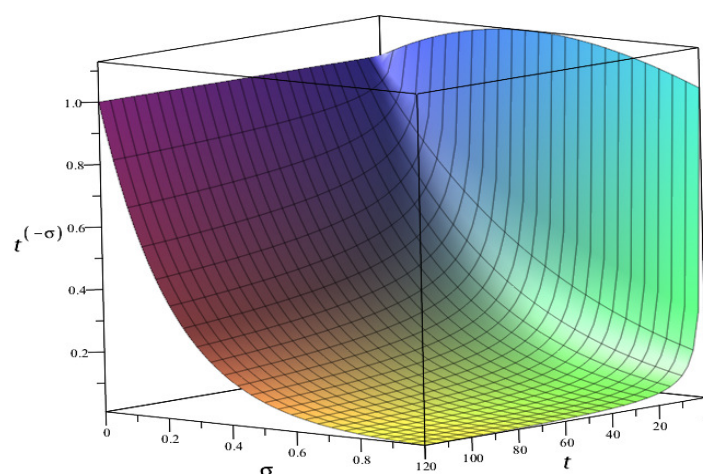

Figure 1. Results of $t^{(-\sigma)}$ with respect to t where $\sigma \in [0, 1]$ in (5).
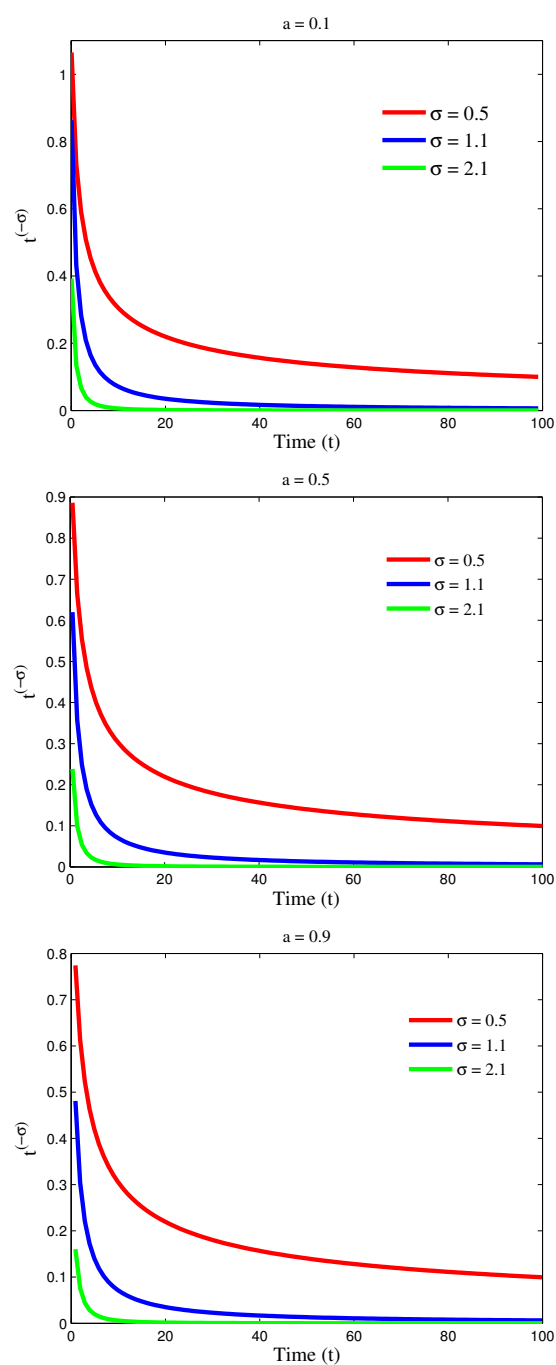
Figure 2. Results of $t^{(-\sigma)}$ with respect to t where $a = 0.1, 0.5, 0.9$ in (5) for $t \in \mathbb{N}_a$, $\sigma = 0.5, 1.1, 2.1$, respectively, according to Table 1.

Further, the authors of [20] proved that

$$\Delta^{-\mu} t^{(\sigma)} = \frac{\Gamma(\sigma + 1)}{\Gamma(\sigma + \mu + 1)} t^{(\sigma + \mu)}, \tag{7}$$

for $\sigma \in \mathbb{R} \setminus \{\cdots, -2, -1\}$. At present, suppose that $\mu > 0$ and $\ell - 1 < \mu < \ell$, where ℓ denotes a positive integer, $\ell = \lceil \mu \rceil$, here $\lceil . \rceil$ denotes the ceiling of number [5]. Set $\sigma = \ell - \mu$. The $\mu-$th fractional Caputo-like difference is defined as

$$\Delta_*^{\mu}[k](t) = \Delta^{-\sigma}\left[\Delta^{\ell}[k]\right](t) = \frac{1}{\Gamma(\sigma)} \sum_{s=a}^{t-\sigma} (t - s - 1)^{(\sigma-1)} \Delta^{\ell}[k](s), \tag{8}$$

where Δ^ℓ is the $\ell-$th order forward difference operator and Δ_*^μ maps functions defined on $\mathbb{N}_a$ to functions defined on $\mathbb{N}_{a-\mu}$.

Table 1. Numerical results of $t^{(-\sigma)}$ where $\sigma = 0.5, 1.1, 2.1$ in (5) for $t \in \mathbb{N}_a$, $a = 0.1, 0.5, 0.9$ and $n = 1, 2, \cdots, 100$ (Algorithm 1).

	t	$t^{(-\sigma)}$	t	$t^{(-\sigma)}$	t	$t^{(-\sigma)}$
$(\sigma = 0.5)\, n$	$a = 0.1$		$a = 0.5$		$a = 0.9$	
1	0.1	1.06472	0.5	0.88622	0.9	0.77426
2	1.1	0.73200	1.5	0.66467	1.9	0.61295
3	2.1	0.59123	2.5	0.55389	2.9	0.52281
4	3.1	0.50911	3.5	0.48465	3.9	0.46340
5	4.1	0.45377	4.5	0.43618	4.9	0.42049
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
97	96.1	0.10161	96.5	0.10140	96.9	0.10119
98	97.1	0.10109	97.5	0.10088	97.9	0.10068
99	98.1	0.10057	98.5	0.10037	98.9	0.10017
100	99.1	0.10007	99.5	0.09987	99.9	0.09967
$(\sigma = 1.1)\, n$	$a = 0.1$		$a = 0.5$		$a = 0.9$	
1	0.1	0.86344	0.5	0.61990	0.9	0.48088
2	1.1	0.43172	1.5	0.35763	1.9	0.30455
3	2.1	0.28331	2.5	0.24835	2.9	0.22080
4	3.1	0.20911	3.5	0.18896	3.9	0.17222
5	4.1	0.16488	4.5	0.15184	4.9	0.14065
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
97	96.1	0.00651	96.5	0.00648	96.9	0.00645
98	97.1	0.00644	97.5	0.00641	97.9	0.00638
99	98.1	0.00636	98.5	0.00634	98.9	0.00631
100	99.1	0.00629	99.5	0.00627	99.9	0.00624
$(\sigma = 2.1)\, n$	$a = 0.1$		$a = 0.5$		$a = 0.9$	
1	0.1	0.392477	0.5	0.238423	0.9	0.160294
2	1.1	0.134913	1.5	0.099343	1.9	0.076139
3	2.1	0.067456	2.5	0.053990	2.9	0.044161
4	3.1	0.040214	3.5	0.033744	3.9	0.028704
5	4.1	0.026593	4.5	0.023007	4.9	0.020093
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
97	96.1	0.000091	96.5	0.000065	96.9	0.000065
98	97.1	0.000089	97.5	0.000064	97.9	0.000063
99	98.1	0.000087	98.5	0.000063	98.9	0.000062
100	99.1	0.000085	99.5	0.000061	99.9	0.000061

Lemma 1 ([5]). *For $\mu > 0$, μ is non-integer, $\ell = \lceil \mu \rceil$, $\sigma = \ell - \mu$, it holds*

$$k(t) = \sum_{m=0}^{\ell-1} \frac{(t-a)^{(m)}}{m!} \Delta^m[k](a)$$

$$+ \frac{1}{\Gamma(\mu)} \sum_{s=a+\sigma}^{t-\sigma} (t-s-1)^{(\mu-1)} \Delta_*^\mu[k](s),$$

where k is defined on $\mathbb{N}_a$ with $a \in \mathbb{Z}^+$. In particular, when $0 < \mu < 1$ and $a = 0$, we have

$$k(t) = k(0) + \frac{1}{\Gamma(\mu)} \sum_{s=1-\mu}^{t-\mu} (t-s-1)^{(\mu-1)} \Delta_*^\mu[k](s), \tag{9}$$

where k is defined on $\mathbb{N}_1$ and Δ_*^μ is defined on $\mathbb{N}_{1-\mu}$.

Algorithm 1: The MATLAB lines of $t^{(-\sigma)}$ where $\sigma = 0.5$, 1.1, 2.1 in (5) for $t \in \mathbb{N}_a$, $a = 0.1$, 0.5, 0.9 and $n = 1, 2, \cdots, N$.

```
format long
sigma = [0.5 1.1 2.1];
[xsigma ysigma] = size(sigma);
a = [0.1 0.5 0.9];
N = 100;
column = 1;
for   i = 1:ysigma
       for   j = 1:ya
             n = 1;
             t = a(j);
             while t ≤ N
                   parammatrix(n, column) = n;
                   parammatrix(n, column+1) = t;
                   parammatrix(n, column+2) = round(gamma(t + 1)/gamma(t + 1 + sigma(i)), 6);
                   t = t + 1;
                   n = n + 1;
             end;
             column = column + 3;
       end;
end;
```

Remark 1. *According to Lemma 1, k in (9) should be defined on $\mathbb{N}_0$. The sum*

$$\sum_{s=1-\mu}^{t-\mu} (t - s - 1)^{(\mu-1)} \Delta_*^\mu [k](s),$$

has no sense when $t = 0$, then we define k on $\mathbb{N}_1$.

Lemma 2 ([20])**.** *Assume that the following factorial functions are well defined.*

(i)   If $0 < \nu < 1$, then $t^{(c\nu)} \geq \left(t^{(\nu)} \right)^c$.

(ii) $t^{(\nu+c)} = (t - c)^{(\nu)} t^{(c)}$.

Lemma 3 ([47])**.** *The quotient expansion of two Gamma functions at infinity is*

$$\frac{\Gamma(y + a)}{\Gamma(y + b)} = y^{a-b} \left[1 + O\left(\frac{1}{y} \right) \right], \qquad (\|\arg(y + a)\| < \pi, |y| \to \infty). \tag{10}$$

Definition 2 ([25])**.** *Let $k = \varphi(t)$ be a solution of Equation (2).*

(1)   The solution k is said to be stable, whenever for any $\epsilon > 0$ and $t_0 \in \mathbb{R}^+$, there exists $\delta = \delta(t_0, \epsilon) > 0$ such that

$$\|k(t, k_0, t_0) - \varphi(t)\| < \epsilon,$$

for $\|k_0 - \varphi(t_0)\| \leq \delta(t_0, \epsilon)$ and each $t \geq t_0$.

(2)   The solution k is said to be attractive, if there exists $\eta(t_0) > 0$ such that $\|k_0\| \leq \eta$ implies $\lim_{t \to \infty} k(t, k_0, t_0) = 0$.

(3)   The solution k is said to be asymptotically stable, whenever it is stable and attractive.

Definition 3 ([48])**.** *Let $k = \varphi(t)$ be a solution of Equation (2). A set Ψ of sequences in $l_{n_0}^\infty$ is uniformly Cauchy or equi-Cauchy, if for every $\epsilon > 0$, there exists an integer N such that $\|k(i) - k(j)\| < \epsilon$ whenever $i, j > N$ for every $k = \{k(n)\}$ in Ψ.*

Theorem 1 ([48] Discrete Arzelà–Ascoli theorem). *A bounded, uniformly Cauchy subset Ω of $l_{n_0}^\infty$ is relatively compact.*

Theorem 2 ([49] Krasnoselskii fixed point theorem). *Let Ψ be a nonempty, closed, convex, and bounded subset of the Banach space $\mathcal{X}$ and let $G : \mathcal{X} \to \mathcal{X}$ and $H : \Psi \to \mathcal{X}$ be two operators such that*

(a)   *G is a contraction with constant $M < 1$.*
(b)   *H is continuous, $G(\Psi)$ resides in a compact subset of $\mathcal{X}$.*
(c)   *For all $l \in \Psi$, if $k = G(k) + H(l)$ then $k \in \Psi$.*
      Then the operator equation $G[k] + H[k] = k$ has a solution in Ψ.

Lemma 4 ([50] Generalized Banach Fixed Point Theorem). *Let Ψ be a nonempty, closed subset of a Banach space $(\mathcal{X}, \|\cdot\|)$ and $\rho^n \geq 0$ for every $n \in \mathbb{N}_0$ such that $\sum_{n=0}^{\infty} \rho^n$ converges. Moreover, let the mapping $Q : \Psi \to \Psi$ satisfy the inequality*

$$\|Q^n[k] - Q^n[l]\| \leq \rho^n \|k - l\|,$$

for all $n \in \mathbb{N}_1$ and any $k, l \in \Psi$. Then, Q has a uniquely defined fixed point k^*. Furthermore, for any $k_0 \in \Psi$, the sequence $\{Q^n[k_0]\}_{n=1}^{\infty}$ converges to this fixed point k^*.

3. Main Results

For the purpose of convenience, we set

$$\mathcal{W}_\lambda^\beta[k](t) = w(t + \beta, k(t + \beta), k(\lambda(t + \beta))). \tag{11}$$

Let l_1^∞ be the set of all real sequences $k = \{k(t)\}_{t=1}^{\infty}$ with norm

$$\|k\| = \sup_{t \in \mathbb{N}_1} \|k(t)\|,$$

then l_1^∞ is a Banach space. Define the operators $G[k](t) = p[k](t)$ and

$$
H[k](t) = \frac{1}{\Gamma(\beta)} \sum_{s=1-\beta}^{t-\beta} (t - s - 1)^{(\beta-1)} \mathcal{W}_\lambda^\beta[k](s),
$$

$$
Q[k](t) = p[k](t) + \frac{1}{\Gamma(\beta)} \sum_{s=1-\beta}^{t-\beta} (t - s - 1)^{(\beta-1)} \mathcal{W}_\lambda^\beta[k](s),
\tag{12}
$$

where $p : \mathcal{C} \to \mathbb{R}$ is Lipschitz continuous map and $\mathcal{W}_\lambda^\beta[k](s)$ is defined in Equation (11). Clearly, $Q[k] = G[k] + H[k]$. Let $k, l \in l_\beta^\infty$. Then, we have

$$\|G[k](t) - G[l](t)\| = \|p[k](t) - p[l](t)\| \leq M \|k - l\|.$$

Thus, the operator G is contraction with $M < 1$. Condition (a) of the Lemma 2 holds and $k(t)$ is a solution of (2) if it is a fixed point of Q. Now, we proof our key lemmas.

Lemma 5. *The map $k : \mathbb{N}_1 \to \mathbb{R}$ is a solution of (2) if and only if $k(t)$ is a solution of the fractional Taylor's difference formula given by*

$$
k(t) = k(0) + \frac{1}{\Gamma(\beta)} \sum_{s=1-\beta}^{t-\beta} (t - s - 1)^{(\beta-1)} \mathcal{W}_\lambda^\beta[k](s),
\tag{13}
$$

for each $t \in \mathbb{N}_1$, where $\mathcal{W}_\lambda^\beta[k](s)$ is defined in (11).

Proof. Suppose that $k(t)$ is a solution of (2), we have from (9)

$$k(t) = k(0) + \frac{1}{\Gamma(\beta)} \sum_{s=1-\beta}^{t-\beta} (t-s-1)^{(\beta-1)} \Delta_*^{\beta}[k](s)$$

$$= k(0) + \frac{1}{\Gamma(\beta)} \sum_{s=1-\beta}^{t-\beta} (t-s-1)^{(\beta-1)} \mathcal{W}_{\lambda}^{\beta}[k](s).$$

This implies that (13) holds. Conversely, if $k(t)$ is solution of (13), comparing (9) and (13) yields,

$$\sum_{s=1-\beta}^{t-\beta} (t-s-1)^{(\beta-1)} \Delta_*^{\beta}[k](s) = \sum_{s=1-\beta}^{t-\beta} (t-s-1)^{(\beta-1)} \mathcal{W}_{\lambda}^{\beta}[k](s),$$

which takes the form

$$\sum_{s=1-\beta}^{t-\beta} (t-s-1)^{(\beta-1)} \left[\Delta_*^{\beta}[k](s) - \mathcal{W}_{\lambda}^{\beta}[k](s) \right] = 0, \tag{14}$$

for each $t \in \mathbb{N}_1$. If $t = 1$ then (14) becomes

$$(\beta - 1)^{(\beta-1)} \left[\Delta_*^{\beta}[k](1-\beta) - \mathcal{W}_{\lambda}^{\beta}[k](1-\beta) \right] = 0,$$

which implies

$$\Delta_*^{\beta}[k](1-\beta) = \mathcal{W}_{\lambda}^{\beta}[k](1-\beta). \tag{15}$$

If $t = 2$ then from (14) it follows that

$$(\beta)^{(\beta-1)} \left[\Delta_*^{\beta}[k](1-\beta) - \mathcal{W}_{\lambda}^{\beta}[k](1-\beta) \right]$$
$$+ (\beta - 1)^{(\beta-1)} \left[\Delta_*^{\beta}[k](2-\beta) - \mathcal{W}_{\lambda}^{\beta}[k](2-\beta) \right] = 0.$$

By using (15), the above equation becomes

$$\Delta_*^{\beta}[k](2-\beta) = \mathcal{W}_{\lambda}^{\beta}[k](2-\beta).$$

Thus, by induction, we have that $\Delta_*^{\beta}[k](t) = \mathcal{W}_{\lambda}^{\beta}[k](t)$ for all $t \in \mathbb{N}_{1-\beta}$ and so $k(t)$ is a solution of (2). This completes the proof. □

In order to prove the main results, we make the following assumption.
(W_1) There exists constants $\sigma_1 \in (\beta, 1)$ and $B_1, B_2 > 0$ such that

$$\left\| w(t, k(t), k(\lambda t)) \right\| \leq (B_1 + B_2) t^{(-\sigma_1)}, \tag{16}$$

for $t \in \mathbb{N}_1$.

Lemma 6. *Assume that* (3) *and* (W_1) *hold. Then, H is continuous and $H[\Psi]$ is a compact subset of $\mathbb{R}$ for $t \in \mathbb{N}_1$, where*

$$\Psi = \left\{ k(t) : \|k(t)\| \leq t^{(-\xi_1)}, \, t \in \mathbb{N}_1 \right\}, \tag{17}$$

$\xi_1 = -\frac{1}{2}(\beta - \sigma_1)$ *satisfies*

$$\frac{(B_1 + B_2)\Gamma(1 - \sigma_1)}{(1 - M)\Gamma(1 + \beta - \sigma_1)} (1 + \xi_1)^{(-\xi_1)} \leq 1. \tag{18}$$

Proof. For $t \in \mathbb{N}_1, \xi_1 > 0$

$$t^{(-\xi_1)} = \frac{\Gamma(t+1)}{\Gamma(t+\xi_1+1)}. \tag{19}$$

Clearly the set Ψ defined in (17) is closed, bounded, and convex subset of $\mathbb{R}$. First, we prove the continuity of the operator H. Using Equations (7) and (12) and the condition (W_1), we have

$$\|H[k](t)\| = \frac{1}{\Gamma(\beta)} \left\| \sum_{s=1-\beta}^{t-\beta} (t-s-1)^{(\beta-1)} \mathcal{W}_\lambda^\beta[k](s) \right\|$$

$$\leq \frac{1}{\Gamma(\beta)} \sum_{s=1-\beta}^{t-\beta} (t-s-1)^{(\beta-1)} \left\| \mathcal{W}_\lambda^\beta[k](s) \right\|$$

$$\leq \frac{1}{\Gamma(\beta)} \sum_{s=1-\beta}^{t-\beta} (t-s-1)^{(\beta-1)} (B_1+B_2)(s+\beta)^{(-\sigma_1)}$$

$$= (B_1+B_2)\Delta^{-\beta}(t+\beta)^{(-\sigma_1)}$$

$$= (B_1+B_2)\frac{\Gamma(1-\sigma_1)}{\Gamma(1+\beta-\sigma_1)}(t+\beta)^{(\beta-\sigma_1)}$$

$$= (B_1+B_2)\frac{\Gamma(1-\sigma_1)}{\Gamma(1+\beta-\sigma_1)}(t+\beta)^{(-2\xi_1)}.$$

For $t \in \mathbb{N}_1$, by using Lemma 2 we obtain

$$\|H[k](t)\| \leq (B_1+B_2)\frac{\Gamma(1-\sigma_1)}{\Gamma(1+\beta-\sigma_1)}(t+\beta)^{(-2\xi_1)}$$

$$= (B_1+B_2)\frac{\Gamma(1-\sigma_1)}{\Gamma(1+\beta-\sigma_1)}(t+\beta+\xi_1)^{(-\xi_1)}(t+\beta)^{(-\xi_1)}$$

$$\leq (B_1+B_2)\frac{\Gamma(1-\sigma_1)}{\Gamma(1+\beta-\sigma_1)}(1+\xi_1)^{(-\xi_1)}(t)^{(-\xi_1)}.$$

Using (18), it is clear that $\|H[k](t)\| \leq t^{(-\xi_1)}$. Thus, $H[\Psi] \subset \Psi$ for $t \in \mathbb{N}_1$. Let $\epsilon > 0$ be given. Then, there exists $S_1 \in \mathbb{N}_1$, such that $t > S_1$ implies

$$(B_1+B_2)\frac{\Gamma(1-\sigma_1)}{\Gamma(1+\beta-\sigma_1)}t^{(\beta-\sigma_1)} < \frac{\epsilon}{2}. \tag{20}$$

Consider the sequence $\{k_n\}$ such that $k_n \to k$. By the continuity of the function f and Lemma 5 for $t \in \{1, 2, \ldots S_1\}$, we obtain

$$\|H[k_n](t) - H[k](t)\|$$

$$\leq \frac{1}{\Gamma(\beta)} \sum_{s=1-\beta}^{t-\beta} (t-s-1)^{(\beta-1)} \left\| \mathcal{W}_\lambda^\beta[k_n](s) - \mathcal{W}_\lambda^\beta[k](s) \right\|$$

$$\leq \frac{t^{(\beta)}}{\Gamma(\beta+1)} \max_{s \in \{1-\beta,\ldots,S_1-\beta\}} \left\| \mathcal{W}_\lambda^\beta[k_n](s) - \mathcal{W}_\lambda^\beta[k](s) \right\|$$

$$\leq \frac{\Gamma(\beta+S_1)}{\Gamma(S_1)\Gamma(\beta+1)} \max_{s \in \{1-\beta,\ldots,S_1-\beta\}} \left\| \mathcal{W}_\lambda^\beta[k_n](s) - \mathcal{W}_\lambda^\beta[k](s) \right\| \to 0,$$

as $n \to \infty$, For $t \in \mathbb{N}_{S_1+1}$,

$$\|H[k_n](t) - H[k](t)\|$$

$$\leq \frac{1}{\Gamma(\beta)} \sum_{s=1-\beta}^{t-\beta} (t-s-1)^{(\beta-1)} \left\| \mathcal{W}_\lambda^\beta[k_n](s) - \mathcal{W}_\lambda^\beta[k](s) \right\|$$

$$\leq \frac{2(B_1+B_2)}{\Gamma(\beta)} \sum_{s=1-\beta}^{t-\beta} (t-s-1)^{(\beta-1)}(s+\beta)^{(-\sigma_1)}$$

$$\leq \frac{2(B_1+B_2)\Gamma(1-\sigma_1)}{\Gamma(1+\beta-\sigma_1)}(t)^{(\beta-\sigma_1)}$$

$$< \epsilon.$$

Thus,

$$\|H[k_n](t) - H[k](t)\| \to 0,$$

as $n \to \infty$ for all $t \in \mathbb{N}_1$. Therefore, the operator H is continuous. Let $\delta_1, \delta_2 \in \mathbb{N}_1$ and $\delta_1 < \delta_2$. Then, we get

$$\|H[k](\delta_2) - H[k](\delta_1)\| \leq \frac{1}{\Gamma(\beta)} \sum_{s=1-\beta}^{\delta_2-\beta} (\delta_2-s-1)^{(\beta-1)} \left\| \mathcal{W}_\lambda^\beta[k](s) \right\|$$

$$+ \frac{1}{\Gamma(\beta)} \sum_{s=1-\beta}^{\delta_1-\beta} (\delta_2-s-1)^{(\beta-1)} \left\| \mathcal{W}_\lambda^\beta[k](s) \right\|$$

$$\leq \frac{(B_1+B_2)\Gamma(1-\sigma_1)}{\Gamma(1+\beta-\sigma_1)} \delta_2^{(\beta-\sigma_1)}$$

$$+ \frac{(B_1+B_2)\Gamma(1-\sigma_1)}{\Gamma(1+\beta-\sigma_1)} \delta_1^{(\beta-\sigma_1)} < \epsilon.$$

It is clear from the definition of uniformly Cauchy that $\{H[k], k \in \Psi\}$ is bounded and uniformly Cauchy subset and from Discrete Arzelà–Ascoli's Theorem stated in Lemma 1, $H[\Psi]$ is relatively compact. This completes the proof. $\square$

Lemma 7. *Assume that* (3) *and condition* (W_1) *hold, then for* $t \in \mathbb{N}_1$ *a solution of* (2) *is in* Ψ.

Proof. Condition (c) of Lemma 2 is yet to be proved. If $k = G[k] + H[l]$, $l \in \Psi$ for $t \in \mathbb{N}_1$, we have

$$\|k(t)\| \leq \|G[k](t)\| + \|H[l](t)\|$$

$$\leq \|p[k](t)\| + (B_1+B_2)\frac{\Gamma(1-\sigma_1)}{\Gamma(1+\beta-\sigma_1)}(t+\beta)^{(\beta-\sigma_1)}$$

$$\leq M\|k(t)\| + (B_1+B_2)\frac{\Gamma(1-\sigma_1)}{\Gamma(1+\beta-\sigma_1)}(t+\beta)^{(\beta-\sigma_1)}.$$

Therefore,

$$\|k(t)\| \leq (B_1+B_2)\frac{\Gamma(1-\sigma_1)}{\Gamma(1+\beta-\sigma_1)(1-M)}(t)^{(\beta-\sigma_1)}$$

$$\leq \left[\frac{(B_1+B_2)\Gamma(1-\sigma_1)}{(1-M)\Gamma(1+\beta-\sigma_1)}(1+\xi_1)^{(-\tilde{\xi}_1)} \right] t^{(-\tilde{\xi}_1)}.$$

Indeed $\|k(t)\| \leq t^{(-\tilde{\xi}_1)}$. Thus, $k(t) \in \Psi$ for $t \in \mathbb{N}_1$. By Theorem 2, Q has a fixed point in Ψ which is solution of (2). $\square$

Theorem 3. *Assume that* (3) *and condition* (W_1) *hold, then the solutions of* (2) *are attractive.*

Proof. By Lemma 7, the solutions of (2) exist and are in Ψ. Further, the function $k(t)$ in Ψ tends to zero as $t \to \infty$. Then, clearly the solutions of (2) tend to zero with t approaching infinity. The proof is complete. $\square$

Before establishing the theorems, we make the following assumption.

(W_2) There exists $\sigma_2 \in (\beta, 1)$ and $B_3, B_4 \geq 0$ such that

$$\|w(t, k_1(t), l_1(t)) - w(t, k_2(t), l_2(t))\| \leq \left[B_3 \|k_1 - k_2\| + B_4 \|l_1 - l_2\| \right] t^{(-\sigma_2)}, \quad (21)$$

for any $k_i, l_i \in l^\infty, i = 1, 2$.

Theorem 4. *Assume that* (3) *together with the condition* (W_2) *is satisfied, then the solution of* (2) *is unique bounded solution in l^∞ provided that*

$$\rho = M + \frac{(B_3 + B_4)\Gamma(1 - \sigma_2)}{\Gamma(1 + \beta - \sigma_2)\Gamma(2 - \beta + \sigma_2)} < 1. \quad (22)$$

Proof. Let the iterates of operator Q be defined as $Q^1 = Q$ and $Q^n = Q(Q^{n-1})$ for each $n \in \mathbb{N}_1$. Now, we shall prove that Q is a contraction operator for sufficiently large n. We have that

$$\|Q^n[k] - Q^n[l]\| \leq \rho^n \|k - l\| \quad (23)$$

and

$$\|Q[k](t) - Q[l](t)\| \leq \|p[k] - p[l]\|$$
$$+ \frac{1}{\Gamma(\beta)} \sum_{s=1-\beta}^{t-\beta} (t - s - 1)^{(\beta-1)} \left\| \mathcal{W}_\lambda^\beta[k_n](s) - \mathcal{W}_\lambda^\beta[k](s) \right\|$$
$$\leq M\|k - l\| + (B_3 + B_4) \frac{\Gamma(1 - \sigma_2)}{\Gamma(1 + \beta - \sigma_2)} (t)^{(\beta - \sigma_2)} \|k - l\|$$
$$\leq M\|k - l\| + (B_3 + B_4) \frac{\Gamma(1 - \sigma_2)}{\Gamma(1 + \beta - \sigma_2)} (1)^{(\beta - \sigma_2)} \|k - l\|$$
$$\leq \left[M + \frac{(B_3 + B_4)\Gamma(1 - \sigma_2)}{\Gamma(1 + \beta - \sigma_2)\Gamma(2 - \beta + \sigma_2)} \right] \|k - l\|,$$

which implies

$$\|Q[k] - Q[l]\| \leq \rho \|k - l\|.$$

Therefore, the (23) is true for $n = 1$. Assuming (23) is true for n, we obtain

$$\left\| Q^{n+1}[k](t) - Q^{n+1}[l](t) \right\| = \|Q(Q^n)[k] - Q(Q^n)[l]\|$$
$$\leq \|p[Q^n[k]] - p[Q^n[l]]\|$$
$$+ \frac{1}{\Gamma(\beta)} \sum_{s=1-\beta}^{t-\beta} (t - s - 1)^{(\beta-1)} \left\| \mathcal{W}_\lambda^\beta[Q^n[k]](s) - \mathcal{W}_\lambda^\beta[Q^n[l]](s) \right\|$$
$$\leq \rho^n M\|k - l\| + (B_3 + B_4)\rho^n \frac{\Gamma(1 - \sigma_2)}{\Gamma(1 + \beta - \sigma_2)} (t)^{(\beta - \sigma_2)} \|k - l\|$$
$$\leq M\|k - l\| + (B_3 + B_4) \frac{\Gamma(1 - \sigma_2)}{\Gamma(1 + \beta - \sigma_2)} (1)^{(\beta - \sigma_2)} \|k - l\|$$
$$\leq \rho^n \left[M + \frac{(B_3 + B_4)\Gamma(1 - \sigma_2)}{\Gamma(1 + \beta - \sigma_2)\Gamma(2 - \beta + \sigma_2)} \right] \|k - l\|,$$

which implies

$$\left\| Q^{n+1}[k] - Q^{n+1}[l] \right\| \leq \rho^{n+1} \|k - l\|.$$

By the principle of mathematical induction on n, the statement (23) is true for all $n \in \mathbb{N}_1$. The geometric series $\sum_{n=0}^{\infty} \rho^n$ converges, as $\rho < 1$ and so Q has a unique bounded fixed point in Ψ. □

Theorem 5. *Assume that* (3) *and condition* (W_2) *hold, then the solutions of* (2) *are stable provided that* (22) *holds.*

The proof follows from Theorem 4.

Theorem 6. *Assume that* (3) *and conditions* (W_1) *and* (W_2) *hold. Then, the solutions of* (2) *are asymptotically stable provided that* (22) *holds.*

The proof is the consequence of Theorems 3 and 5.

4. Numerical Examples

Example 1. *Consider the following discrete fractional pantograph equation,*

$$\begin{cases} \Delta^{0.8}[k](t) = \dfrac{1}{10}\left[0.04\, k(t+0.8) + 0.01 \sin\left(k\left(\dfrac{t+0.8}{2} \right) \right) \right], \\ k(0) = p[k], \end{cases} \tag{24}$$

for each $t \in \mathbb{N}_{0.2}$, where

$$\|p[k]\| \leq 0.2\|k\|.$$

Clearly, $\beta = 0.8 \in (0,1]$, $M = 0.2 \in (0,1)$, $\lambda = \frac{1}{2} \in (0,1)$. Take

$$\mathcal{W}_\lambda^\beta[k](t) = \frac{1}{10}\left[0.04\, k(t+0.8) + 0.01 \sin\left(k\left(\frac{t+0.8}{2} \right) \right) \right]. \tag{25}$$

Let $k_1, k_2, l_1, l_2 \in \mathcal{C}$. Then, we have

$$\| w(t, k_1(t), l_1(\lambda t)) - w(t, k_2(t), l_2(\lambda t)) \|$$
$$= \left\| \frac{1}{10}\left[0.04\, k_1(t+0.8) + 0.01 \sin\left(l_1\left(\frac{t+0.8}{2} \right) \right) \right] \right.$$
$$\left. - \frac{1}{10}\left[0.04\, k_2(t+0.8) + 0.1 \sin\left(l_2\left(\frac{t+0.8}{2} \right) \right) \right] \right\|$$
$$\leq 0.004\| k_1(t+0.8) - k_2(t+0.8) \|$$
$$+ 0.001\left\| \sin\left(l_1\left(\frac{t+0.8}{2} \right) \right) - \sin\left(l_2\left(\frac{t+0.8}{2} \right) \right) \right\|$$
$$\leq 0.004\| k_1(t+0.8) - k_2(t+0.8) \|$$
$$+ 0.001\left\| l_1\left(\frac{t+0.8}{2} \right) - l_2\left(\frac{t+0.8}{2} \right) \right\|.$$

Then, from (21)*, for $\sigma_2 = 0.85 \in (\beta, 1)$, we get*

$$\| w(t, k_1(t), l_1(\lambda t)) - w(t, k_2(t), l_2(\lambda t)) \|$$
$$\leq \left[0.004\| k_1(t+0.8) - k_2(t+0.8) \| + 0.001\| l_1(t+0.8) - l_2(t+0.8) \| \right] t^{(-0.85)},$$

for all $t \in \mathbb{N}_{1-\beta} = \mathbb{N}_{0.2} = \{0.2, 1.2, 2.2, \cdots \}$.

By using (5) *and consider three sample values for* σ, *we will have*

$$t^{(-\sigma_2)} = \frac{\Gamma(t+1)}{\Gamma(t+\sigma_2+1)} = \begin{cases} \dfrac{\Gamma(t+1)}{\Gamma(t+0.81+1)} = 0.9143, 0.5458, \cdots, 0.1109, \\[3mm] \dfrac{\Gamma(t+1)}{\Gamma(t+0.90+1)} = 0.8774, 0.5014, \cdots, 0.0866, \\[3mm] \dfrac{\Gamma(t+1)}{\Gamma(t+0.99+1)} = 0.8379, 0.4591, \cdots, 0.0676, \end{cases}$$

for $t = 0.2, 1.2, \cdots, 14.2$ *respectively. Table* 2 *shows these numerical results. Therefore,*

$$B_3 = 0.004, \qquad B_4 = 0.001.$$

Figure 3 *illustrates* $t^{(-\sigma_2)}$ *for* $\sigma_2 = 0.81, 0.90, 0.99$ *and* $t \in \mathbb{N}_{0.2}$. *These results are shown in Table* 2.
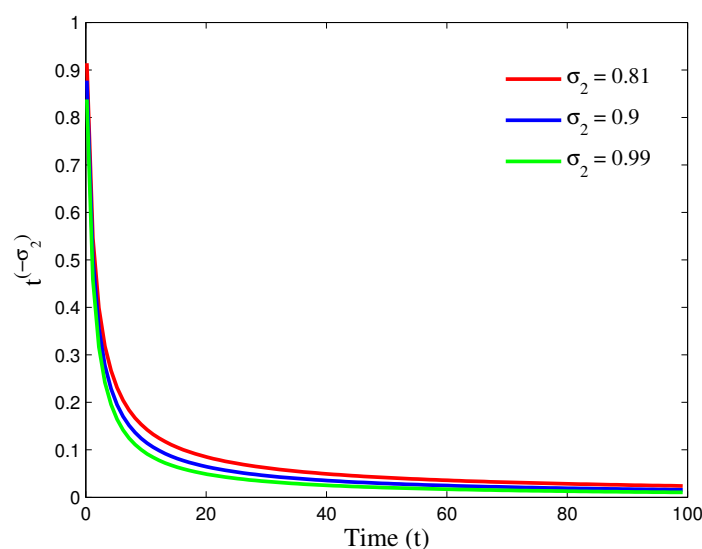

Figure 3. The behavior of $t^{(-\sigma_2)}$ with respect to t where $\sigma_2 = 0.81, 0.9, 0.99$ in Example 1, respectively, for $t \in \mathbb{N}_{0.2}$ and $n = 1, 2, \cdots, 100$, according to Table 2.

Table 2. Numerical results of $t^{(-\sigma_2)}$ where $\sigma_2 = 0.81, 0.90, 0.99$ in Example 1 for $t \in \mathbb{N}_{1-\beta} = \mathbb{N}_{0.2}$ and $n = 1, 2, \cdots, 15$ (Algorithm 2).

		$t^{(-\sigma_2)}$		
n	t	0.81	0.9	0.99
1	0.2000	0.9143	0.8774	0.8379
2	1.2000	0.5458	0.5014	0.4591
3	2.2000	0.3989	0.3558	0.3166
4	3.2000	0.3184	0.2777	0.2418
5	4.2000	0.2669	0.2287	0.1957
6	5.2000	0.2309	0.1950	0.1644
7	6.2000	0.2042	0.1702	0.1418
8	7.2000	0.1836	0.1513	0.1246
9	8.2000	0.1671	0.1364	0.1112
10	9.2000	0.1536	0.1242	0.1004
11	10.2000	0.1423	0.1141	0.0915
12	11.2000	0.1327	0.1056	0.0841
13	12.2000	0.1244	0.0984	0.0778
14	13.2000	0.1172	0.0921	0.0723
15	14.2000	0.1109	0.0866	0.0676

Now, by employing Equation (22), the ρ obtained for different fractional order β and for different values of σ_2. For this purpose, let $\sigma_2 = 0.81, 0.90$, and 0.99, and because σ_2 should be in $(\beta, 1)$, then we have

$$\rho = M + \frac{(B_3 + B_4)\Gamma(1 - \sigma_2)}{\Gamma(1 + \beta - \sigma_2)\Gamma(2 - \beta + \sigma_2)}$$

$$= \begin{cases} 0.2 + \dfrac{0.005\,\Gamma(1 - 0.81)}{\Gamma(1 + \beta - 0.81)\Gamma(2 - \beta + 0.81)}, \\[2ex] 0.2 + \dfrac{0.005\,\Gamma(1 - 0.90)}{\Gamma(1 + \beta - 0.90)\Gamma(2 - \beta + 0.90)}, \\[2ex] 0.2 + \dfrac{0.005\,\Gamma(1 - 0.99)}{\Gamma(1 + \beta - 0.99)\Gamma(2 - \beta + 0.99)}, \end{cases}$$

$$= \begin{cases} 0.2 + \dfrac{0.005\,\Gamma(0.19)}{\Gamma(\beta + 0.19)\Gamma(2.81 - \beta)} = 0.2030, 0.2039, \cdots, 0.2240, \\[2ex] 0.2 + \dfrac{0.005\,\Gamma(0.10)}{\Gamma(\beta - 0.10)\Gamma(2.90 - \beta)} = 0.2027, 0.2044, \cdots, 0.2451, \\[2ex] 0.2 + \dfrac{0.005\,\Gamma(0.01)}{\Gamma(\beta - 0.01)\Gamma(2.99 - \beta)} = 0.2025, 0.2163, \cdots, 0.6768, \end{cases}$$

for

$$\beta = 0.05, 0.10, 0.15, \cdots, 0.8,$$

whenever $\sigma_2 = 0.81$, for

$$\beta = 0.05, 0.10, 0.15, \cdots, 0.9,$$

whenever $\sigma_2 = 0.90$, for

$$\beta = 0.05, 0.10, 0.15, \cdots, 0.95,$$

whenever $\sigma_2 = 0.99$, respectively.

The solution is given by

$$k(t) = p(k) + \frac{1}{\Gamma(0.8)} \sum_{s=0.2}^{t-0.8} (t - s - 1)^{(\beta - 1)}$$

$$\times \frac{1}{10}\left[0.04\,k(s + 0.8) + 0.01\sin\left(k\left(\frac{s + 0.8}{2}\right)\right)\right],$$

for almost all $t \in \mathbb{N}_1$. Simple calculations yield,

$$B_3 = 0.004, \qquad B_4 = 0.001,$$

$M = 0.2, \sigma_2 = 0.85$ and $\rho = 0.2295$. The conditions in Theorem 4 hold and thus the solutions of (24) are asymptotically stable.

Remark 2. *It is clear from Figure 4 that the increase in fractional order (β) results in gradual increase in the value of ρ in (22) and all the values of ρ are less than one. The values of ρ are tabulated in Table 3 which ensures the stability of (24). The value of ρ is plotted against $\beta \in (0, 0.8)$ and σ_2 in the range $(0.8, 1)$ in Figure 5. Thus, it is evident that the solutions of discrete fractional equations of order $\beta = 0.8$ remains stable (i.e., $\rho < 1$) for any values of $\sigma_2 \in (0.8, 1)$.*

Table 3. Numerical results of ρ versus $\sigma_2 \in \{0.15, 0.30, 0.45, 0.60, 0.75, 0.9\}$ for $\beta \in (0,1)$ in Example 1 (Algorithm 3).

		$\rho(\sigma_2)$					
n	β	0.15	0.30	0.45	0.60	0.75	0.9
1	0.0400	0.2049	0.2046	0.2043	0.2040	0.2038	0.2040
2	0.0800	0.2052	0.2049	0.2047	0.2045	0.2044	0.2054
3	0.1200	0.2054	0.2052	0.2050	0.2050	0.2052	0.2070
4	0.1600		0.2055	0.2054	0.2055	0.2059	0.2086
5	0.2000		0.2058	0.2058	0.2060	0.2067	0.2103
6	0.2400		0.2061	0.2062	0.2065	0.2075	0.2121
7	0.2800		0.2064	0.2066	0.2071	0.2083	0.2140
8	0.3200			0.2070	0.2076	0.2092	0.2160
9	0.3600			0.2073	0.2081	0.2100	0.2181
10	0.4000			0.2077	0.2086	0.2109	0.2202
11	0.4400			0.2080	0.2092	0.2118	0.2224
12	0.4800				0.2097	0.2126	0.2246
13	0.5200				0.2102	0.2135	0.2268
14	0.5600				0.2106	0.2143	0.2291
15	0.6000					0.2152	0.2314
16	0.6400					0.2160	0.2337
17	0.6800					0.2168	0.2360
18	0.7200					0.2176	0.2382
19	0.7600						0.2404
20	0.8000						0.2425
21	0.8400						0.2446
22	0.8800						0.2466

Algorithm 2: The MATLAB lines of $t^{(-\sigma_2)}$ where $\sigma = 0.81, 0.9, 0.99$ in Example 1 for $t \in \mathbb{N}_{1-\beta}$, $t = 0.2, 1.2, 2.2, \cdots, 14.2$ and $n = 1, 2, \cdots, N$.

```
format long
sigma = [0.81 0.9 0.99];
[xsigma ysigma] = size(sigma);
beta = 0.2;
N = 100;
column = 1;
for   i = 1:ysigma
      n = 1;
      t = beta;
      while t ≤ N
              parammatrix(n, column) = n;
              parammatrix(n, column+1) = t;
              parammatrix(n, column+2) = round(gamma(t + 1)/gamma(t + 1 + sigma(i)), 6);
              t = t + 1;
              n = n + 1;
      end;
      column = column + 3;
end;
```

Example 2. *Consider the following discrete fractional pantograph equation,*

$$
\begin{cases}
\Delta^{0.5}[k](t) = \dfrac{1}{25}\left[0.09\, k(t+0.5) + 0.3\cos^2\left(k\left(\dfrac{t+0.5}{5}\right)\right)\right], \\
k(0) = p[k],
\end{cases}
\tag{26}
$$

for each $t \in \mathbb{N}_{0.5}$, where

$$
\|p[k]\| \le 0.1\|k\|.
$$

Clearly, $\beta = 0.5 \in (0,1]$, $M = 0.1 \in (0,1)$, $\lambda = \frac{1}{5} \in (0,1)$. Take

$$
\mathcal{W}_\lambda^\beta[k](t) = \dfrac{1}{25}\left[0.09\, k(t+0.5) + 0.3\cos^2\left(k\left(\dfrac{t+0.5}{5}\right)\right)\right].
\tag{27}
$$

Let $k_1, k_2, l_1, l_2 \in \mathcal{C}$. Then, we have

$$
\begin{aligned}
\big\|w\big(t, & k_1(t), l_1(\lambda t)\big) - w\big(t, k_2(t), l_2(\lambda t)\big)\big\| \\
&= \left\|\dfrac{1}{25}\left[0.09\, k_1(t+0.5) + 0.3\cos^2\left(l_1\left(\dfrac{t+0.5}{5}\right)\right)\right]\right. \\
&\quad \left. - \dfrac{1}{25}\left[0.09\, k_2(t+0.5) + 0.3\cos^2\left(l_2\left(\dfrac{t+0.5}{5}\right)\right)\right]\right\| \\
&\le \dfrac{9}{2500}\|k_1(t+0.5) - k_2(t+0.5)\| \\
&\quad + \dfrac{3}{250}\left\|\cos^2\left(l_1\left(\dfrac{t+0.5}{5}\right)\right) - \cos^2\left(l_2\left(\dfrac{t+0.8}{5}\right)\right)\right\| \\
&\le \dfrac{9}{2500}\|k_1(t+0.5) - k_2(t+0.5)\| \\
&\quad + \dfrac{3}{250}\left\|l_1\left(\dfrac{t+0.5}{5}\right) - l_2\left(\dfrac{t+0.5}{5}\right)\right\|.
\end{aligned}
$$

Then, from (21), for $\sigma_2 = 0.75 \in (\beta, 1)$, we get

$$
\begin{aligned}
\big\|w\big(t, & k_1(t), l_1(\lambda t)\big) - w\big(t, k_2(t), l_2(\lambda t)\big)\big\| \\
&\le \big[0.0036\|k_1(t+0.5) - k_2(t+0.5)\| + 0.012\|l_1(t+0.5) - l_2(t+0.5)\|\big] t^{(-0.75)},
\end{aligned}
$$

for all $t \in \mathbb{N}_{1-\beta} = \mathbb{N}_{0.5} = \{0.5, 1.5, 2.5, \cdots\}$.

By using (5) and consider three sample values for σ, we will have

$$
t^{(-\sigma_2)} = \dfrac{\Gamma(t+1)}{\Gamma(t+\sigma_2+1)} =
\begin{cases}
\dfrac{\Gamma(t+1)}{\Gamma(t+0.55+1)} = 0.8669, 0.6343, \cdots, 0.2232, \\[2ex]
\dfrac{\Gamma(t+1)}{\Gamma(t+0.75+1)} = 0.7821, 0.5214, \cdots, 0.1287, \\[2ex]
\dfrac{\Gamma(t+1)}{\Gamma(t+0.9+1)} = 0.7134, 0.4459, \cdots, 0.0851,
\end{cases}
$$

for $t = 0.5, 1.5, \cdots, 14.5$, respectively. Table 4 contains values of $t^{(-\sigma_2)}$ for $\sigma_2 = 0.55, 0.75, 0.9$ and $t \in \mathbb{N}_{0.5}$. Therefore, $B_3 = 0.0036$ and $B_4 = 0.012$. Now, by employing Equation (22), the ρ obtained for different fractional order β and for different values of σ_2. For this purpose, let $\sigma_2 = 0.55$, 0.75, and 0.9, and because σ_2 should be in $(\beta, 1)$, then we have

$$\rho = M + \frac{(B_3 + B_4)\Gamma(1 - \sigma_2)}{\Gamma(1 + \beta - \sigma_2)\Gamma(2 - \beta + \sigma_2)}$$

$$= \begin{cases} 0.1 + \dfrac{0.0156\,\Gamma(1 - 0.55)}{\Gamma(1 + \beta - 0.55)\Gamma(2 - \beta + 0.55)}, \\[2mm] 0.1 + \dfrac{0.0156\,\Gamma(1 - 0.75)}{\Gamma(1 + \beta - 0.75)\Gamma(2 - \beta + 0.75)}, \\[2mm] 0.1 + \dfrac{0.0156\,\Gamma(1 - 0.9)}{\Gamma(1 + \beta - 0.9)\Gamma(2 - \beta + 0.9)}. \end{cases}$$
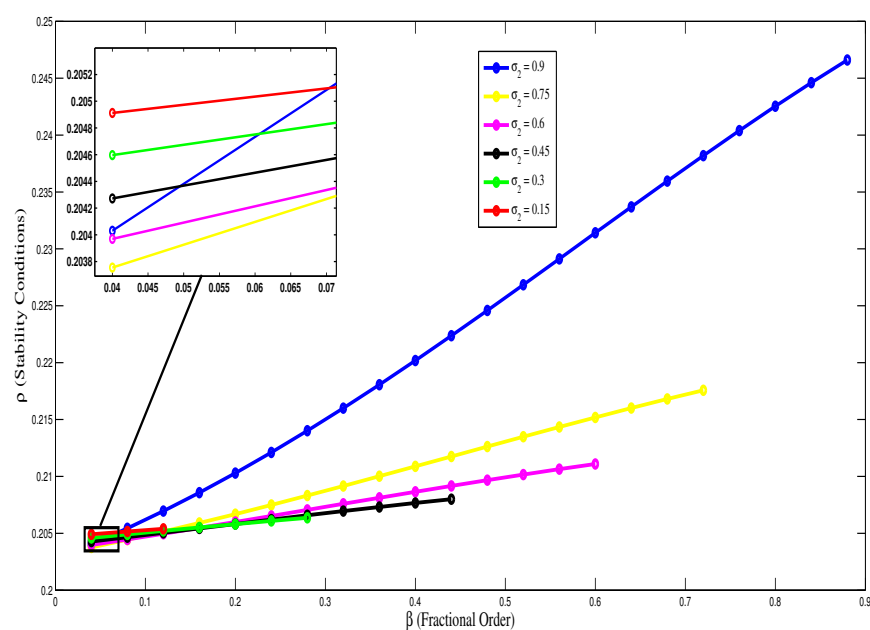

Figure 4. ρ versus β where $\sigma_2 \in \{0.15, 0.30, 0.45, 0.60, 0.75, 0.9\}$ in Example 1, respectively, for $\beta \in (0,1)$, according to Table 3.

Table 4. Numerical results of $t^{(-\sigma_2)}$ where $\sigma_2 = 0.55, 0.75, 0.9$ in Example 2 for $t \in \mathbb{N}_{1-\beta} = \mathbb{N}_{0.5}$ and $n = 1, 2, \cdots, 15$ (Algorithm 2).

|		$t^{(-\sigma_2)}$		
n	t	0.55	0.75	0.9
1	0.5000	0.8669	0.7821	0.7134
2	1.5000	0.6343	0.5214	0.4459
3	2.5000	0.5199	0.4011	0.3278
4	3.5000	0.4493	0.3303	0.2608
5	4.5000	0.4004	0.2831	0.2173
6	5.5000	0.3640	0.2491	0.1867
7	6.5000	0.3356	0.2233	0.1640
8	7.5000	0.3126	0.2030	0.1464
9	8.5000	0.2936	0.1866	0.1322
10	9.5000	0.2776	0.1729	0.1209
11	10.5000	0.2638	0.1614	0.1114
12	11.5000	0.2517	0.1515	0.1033
13	12.5000	0.2411	0.1429	0.0964
14	13.5000	0.2317	0.1354	0.0903
15	14.5000	0.2232	0.1287	0.0851
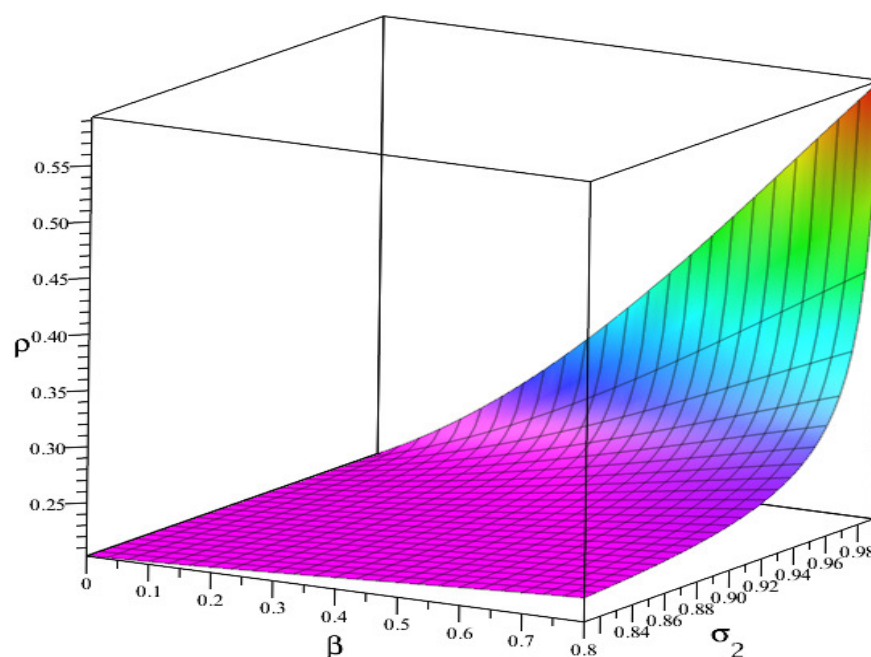
Figure 5. ρ versus β and σ_2 where $\beta = 0.8$ and $\sigma_2 \in (0.8, 1)$ in Example 1.

The conditions in Theorem 4 hold and are plotted against the fractional order (β) and σ_2 in Figure 6. Figure 7 presents the corresponding 2-dimensional plot of ρ against the fractional order (β) for fixed values of $\sigma_2 = \{0.25, 0.45, 0.65, 0.75, 0.85, 0.95\}$ and numerical values are tabulated in Table 5. Thus, the solutions of (26) are asymptotically stable.

Table 5. Numerical results of ρ versus $\sigma_2 \in \{0.25, 0.45, 0.65, 0.75, 0.85, 0.95\}$ for $\beta \in (0, 1)$ in (26) (Algorithm 4).

|			$\rho(\sigma_2)$				
n		β	0.25	0.45	0.65	0.75	0.85
1	0.0400	0.1147	0.1133	0.1121	0.1117	0.1118	0.1155
2	0.0800	0.1156	0.1145	0.1138	0.1138	0.1150	0.1236
3	0.1200	0.1165	0.1157	0.1155	0.1161	0.1183	0.1324
4	0.1600	0.1173	0.1170	0.1173	0.1184	0.1219	0.1419
5	0.2000	0.1181	0.1182	0.1191	0.1209	0.1257	0.1521
6	0.2400	0.1189	0.1194	0.1210	0.1234	0.1296	0.1629
7	0.2800		0.1205	0.1229	0.1259	0.1337	0.1744
8	0.3200		0.1217	0.1248	0.1286	0.1379	0.1864
9	0.3600		0.1228	0.1267	0.1312	0.1423	0.1990
10	0.4000		0.1239	0.1286	0.1339	0.1468	0.2120
11	0.4400		0.1250	0.1305	0.1367	0.1513	0.2255
12	0.4800			0.1324	0.1394	0.1559	0.2393
13	0.5200			0.1342	0.1421	0.1606	0.2534
14	0.5600			0.1360	0.1448	0.1652	0.2678
15	0.6000			0.1377	0.1474	0.1699	0.2823
16	0.6400			0.1393	0.1499	0.1745	0.2969
17	0.6800				0.1524	0.1791	0.3115
18	0.7200				0.1248	0.1835	0.3260
19	0.7600					0.1878	0.3403
20	0.8000					0.1920	0.3544
21	0.8400					0.1961	0.3682
22	0.8800						0.3816
23	0.9200						0.3945
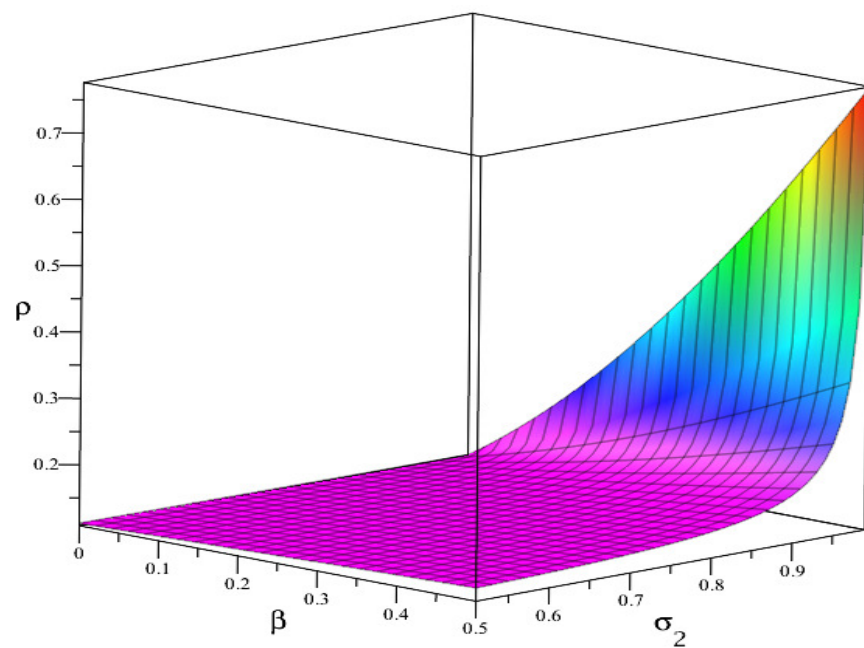
Figure 6. ρ versus β and σ_2 where $\beta = 0.5$ and $\sigma_2 \in (0.5, 1)$ in Example 2.
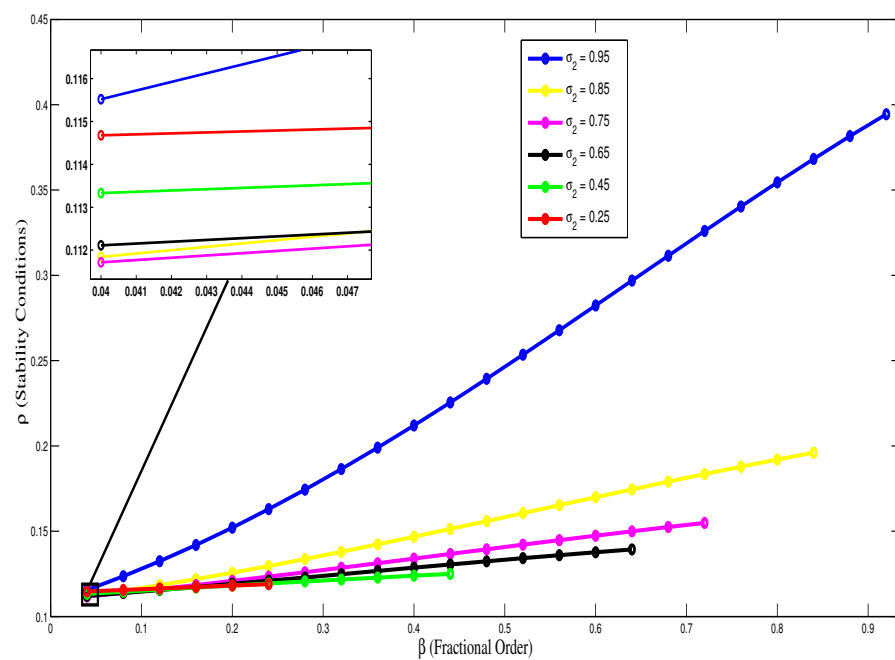

Figure 7. ρ versus β where $\sigma_2 \in \{0.25, 0.45, 0.65, 0.75, 0.85, 0.95\}$ in (26), respectively, for $\beta \in (0, 1)$, according to Table 5.

Algorithm 3: The MATLAB lines of ρ where $\sigma_2 \in \{0.15, 0.30, 0.45, 0.60, 0.75, 0.9\}$ for $\beta \in (0, 1)$ in Example 1.

```
format long
M = 0.2;
B3 = 0.004; B4 = 0.001;
beta = 0.04;
sigma = [0.15 0.3 0.45 0.6 0.75 0.9];
[xsigma ysigma] = size(sigma);
n = 1;
while beta < 1
    column = 1;
    for i = 1:ysigma
        if sigma(i)>beta
            parammatrix(n, column) = n;
            parammatrix(n, column + 1) = beta;
            parammatrix(n, column + 2) = sigma(i);
            parammatrix(n, column + 3) = round(M + (B3 + B4) · · ·
            *gamma(1-sigma(i))/(gamma(1 + beta-sigma(i))· · ·
            *gamma(2-beta + sigma(i))), 6);
        end;
        column = column + 4;
    end;
beta = beta + 0.04;
n = n + 1;
end;
```

Algorithm 4: The MATLAB lines of ρ where $\beta \in (0, 1)$ where $\sigma_2 = 0.25, 0.45, 0.65, 0.75, 0.85, 0.95$ in (26).

```
format long
M = 0.1;
B3 = 0.0036; B4 = 0.012;
sigma = [0.25 0.45 0.65 0.75 0.85 0.95];
[xsigma ysigma] = size(sigma);
for   i = 1:ysigma
    column = 1;;
    beta = 0.04;
    while beta < sigma(i)
        parammatrix(i, column) = i;
        parammatrix(i, column + 1) = sigma(i);
        parammatrix(i, column + 2) = beta;
        parammatrix(i, column + 3) = round(M + (B3 + B4)· · ·
        *gamma(1-sigma(i))/(gamma(1 + beta-sigma(i))· · ·
        *gamma(2-beta + sigma(i))), 6);
        beta = beta + 0.04;
        column = column + 4;
    end;
end;
```

5. Conclusions

Asymptotic stability of the initial value discrete fractional pantograph equation is established using Krasnoselskii theorem, generalized Banach fixed point theorem, and discrete Arzelà-Ascoli theorem. Numerical simulations are carried out for the stability results illustrating the effects of the fractional order on the stability conditions. The values are tabulated and plotted. The 3-dimensional images are presented to analyze the stability of the equation with simultaneous variation of the fractional order and $\sigma_2 \in (\beta, 1)$.

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