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Fourier Transforms of Some Finite Bivariate Orthogonal Polynomials

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Abstract: In this paper, we first obtain the Fourier transforms of some finite bivariate orthogonal polynomials and then by using the Parseval identity, we introduce some new families of bivariate orthogonal functions.

Keywords: bivariate orthogonal functions; Fourier transform; parseval identity; hypergeometric functions

Citation: Güldoğan Lekesiz, E.; Aktaş, R.; Masjed-Jamei, M. Fourier Transforms of Some Finite Bivariate Orthogonal Polynomials. *Symmetry* **2021**, *13*, 452. https://doi.org/10.3390/sym13030452

Academic Editor: Diego Caratelli

Received: 22 February 2021
Accepted: 8 March 2021
Published: 10 March 2021

1. Introduction

The integral transforms have wide applications in many branches of physics, engineering, mathematics and in other scientific disciplines. There are many applications of the integral transforms to differential, integral, and integro-differential equations, and in the theory of special functions. In particular, the integral transform technique can be applied to derive the solutions of integral equations of convolution type, integral equations, differential equations, or integro-differential equations. The literature in this subject is huge and includes many research papers and books. For more details regarding this subject, we refer the readers to [1–7]. Integral transforms are also used in the solutions of problems regarding mathematical modelling [8,9].

In this article, we focus on just Fourier transform which is an integral transform. The most important use of the Fourier transformation is to solve many of the partial differential equations of the mathematical physics, such as Laplace, Heat, and Wave equations. Some applications of the Fourier transform include vibration analysis, sound engineering, communication, data analysis, etc. [10–14]. The Fourier transform is also an important image processing tool, especially in transformation, representation, and encoding, smoothing and sharpening images [5]. By comparing with the signal process that uses one-dimensional Fourier transform in imaging analysis, two- or multi-dimensional Fourier transforms are being used. Fourier transform has been widely used in the fields of image analysis.

Consider the following differential equation

$$\left(ax^2+bx+c\right)y_n''+(dx+e)y_n'=n(d+(n-1)a)y_n, \tag{1}$$

where a, b, c, d, e are real parameters and n is a positive integer. According to [15], this equation has generally six sequences of orthogonal polynomial solutions. Three of them are Jacobi, Laguerre and Hermite infinitely orthogonal polynomials [16] and three other ones, which are denoted by $M_n^{(p,q)}(x)$, $N_n^{(p)}(x)$ and $I_n^{(p)}(x)$, are finitely orthogonal with respect to the F sampling, inverse Gamma and T sampling distributions, respectively (see [17,18]).

The study of orthogonal polynomials and their transformations have been the subject of many papers during the last several years. The families of orthogonal polynomials which are mapped onto each other can be introduced by using the well-known Fourier transform or other integral transforms [19]. For example, Hermite functions are eigenfunctions of a Fourier transform (see [20–23]). Likewise, the Jacobi polynomials are mapped onto the continuous Hahn polynomials [20] and by the Fourier-Jacobi transform, Jacobi polynomials are mapped onto the Wilson polynomials [23]. In [24] , new examples of orthogonal functions are obtained via Fourier transforms of the generalized Ultraspherical polynomials and the generalized Hermite polynomials. In [25], the Fourier transform of Routh-Romanovski polynomials is investigated. Furthermore, via the Fourier transforms of the finite classical orthogonal polynomials $M_n^{(p,q)}(x)$ and $N_n^{(p)}(x)$, and two symmetric sequences of finite orthogonal polynomials, new families of orthogonal functions are introduced in [21,26].

Recently, in [27] some new families of orthogonal functions in two variables were introduced by using Fourier transforms of specific functions derived from two-variable polynomials defined in [28,29] and then using the Parseval identity their orthogonality relations have been obtained. Also, in [30] the authors have defined finite bivariate orthogonal polynomials by using a Koornwinder's method [28].

Motivated by papers on Fourier transforms of univariate orthogonal polynomials mentioned above, a similar approach in those papers has been developed for two-dimensional Fourier transforms. This approach allows us to derive new families of bivariate orthogonal functions. Also, a similar approach can be applied for multivariate orthogonal polynomials and their properties can be investigated.

The aim of this paper is to obtain new families of bivariate orthogonal functions by two-dimensional Fourier transforms of bivariate finite orthogonal polynomials given in [30] by means of Koorwinder's method. The rest of the article is organized as follows: In Section 2, we first remind three classes of finite univariate orthogonal polynomials in [18] and then present fifteen classes of finite bivariate orthogonal polynomials which are introduced in [30]. In Section 3, via Fourier transforms of finite bivariate orthogonal polynomials, we obtain new families of bivariate orthogonal functions and then compute their orthogonality relations via Parseval identity.

2. Preliminaries

In this section, we recall the classes of finite univariate and bivariate orthogonal polynomials introduced in [18,30], respectively. We first start with three classes of finite univariate orthogonal polynomials.

2.1. The Classes of Finite Univariate Orthogonal Polynomials

2.1.1. The First Class of Finite Classical Orthogonal Polynomials

Consider the equation

$$x(x+1)y_n''(x)+((2-p)x+(1+q))y_n'(x)-n(n+1-p)y_n(x)=0, \tag{2}$$

as a special case of (1). By means of the Frobenius method, an explicit polynomial solution for Equation (2) is obtained as [18]

$$M_n^{(p,q)}(x)=(-1)^n n!\sum_{k=0}^{n}\binom{p-(n+1)}{k}\binom{q+n}{n-k}(-x)^k. \tag{3}$$

The first class of finite classical orthogonal polynomials denoted by $M_n^{(p,q)}$ is orthogonal on $[0,\infty)$ with respect to the weight function $W_1(x,p,q)=x^q(1+x)^{-(p+q)}$ if and only if $p>2\max\{m,n\}+1$ and $q>-1$. Indeed, if we rewrite Equation (2) in self-adjoint forms as

$$\begin{cases} \left(x^{1+q}(1+x)^{1-p-q}y_n'(x)\right)'=n(n+1-p)x^q(1+x)^{-(p+q)}y_n(x), \\ \left(x^{1+q}(1+x)^{1-p-q}y_m'(x)\right)'=m(m+1-p)x^q(1+x)^{-(p+q)}y_m(x), \end{cases} \tag{4}$$

where $y_n(x) = M_n^{(p,q)}(x)$, then if we multiply the equations in (4) by $y_m(x)$ and $y_n(x)$, respectively and subtract them, we arrive at

$$\begin{aligned}&\left[\frac{x^{q+1}}{(1+x)^{p+q-1}}\left(y_n'(x)y_m(x)-y_m'(x)y_n(x)\right)\right]_0^\infty\\&=(\lambda_n-\lambda_m)\int_0^\infty\frac{x^q}{(1+x)^{p+q}}M_n^{(p,q)}(x)M_m^{(p,q)}(x)dx,\end{aligned}\tag{5}$$

where $\lambda_n = n(n+1-p)$. Since

$$\max\deg\{y_n'(x)y_m(x)-y_m'(x)y_n(x)\}=m+n-1,$$

then if $q > -1$, $p > 2N+1$, $N = \max\{m,n\}$, the left hand side of (5) tends to zero. Thus, it follows

$$\int_0^\infty\frac{x^q}{(1+x)^{p+q}}M_n^{(p,q)}(x)M_m^{(p,q)}(x)dx=0\Leftrightarrow\left\{\begin{array}{c}m\neq n, p>2N+1, q>-1\\N=\max\{m,n\}\end{array}\right..$$

To calculate the norm square value of the polynomials $M_n^{(p,q)}(x)$, if we write the Rodrigues representation of the polynomials given by [18]

$$M_n^{(p,q)}(x)=(-1)^n\frac{(1+x)^{p+q}}{x^q}\frac{d^n\left(x^{n+q}(1+x)^{n-p-q}\right)}{dx^n},\quad n=0,1,\ldots,\tag{6}$$

in the norm square value, we have

$$\int_0^\infty\frac{x^q}{(1+x)^{p+q}}\left(M_n^{(p,q)}(x)\right)^2dx=(-1)^n\int_0^\infty M_n^{(p,q)}(x)\frac{d^n\left(x^{n+q}(1+x)^{n-p-q}\right)}{dx^n}dx,$$

then from integration by parts it follows

$$(-1)^n\int_0^\infty M_n^{(p,q)}(x)\frac{d^n\left(x^{n+q}(1+x)^{n-p-q}\right)}{dx^n}dx=\frac{n!(p-(n+1))!}{(p-(2n+1))!}\int_0^\infty x^{n+q}(1+x)^{n-p-q}dx.$$

Since

$$\int_0^\infty x^{n+q}(1+x)^{n-p-q}dx=\frac{(p-(2n+2))!(q+n)!}{(p+q-(n+1))!},$$

we find that

$$\int_0^\infty\frac{x^q}{(1+x)^{p+q}}\left(M_n^{(p,q)}(x)\right)^2dx=\frac{n!\Gamma(p-n)\Gamma(q+n+1)}{(p-(2n+1))\Gamma(p+q-n)},$$

where $\Gamma(z)$ is the well-known Gamma function defined by [31]

$$\Gamma(z)=\int_0^\infty x^{z-1}e^{-x}dx\ ,\qquad\Re(z)>0.$$

Thus, the following corollary holds.

Corollary 1 (Orthogonality relation)**.** *([18]) The following relation is satisfied*

$$\int_0^\infty \frac{x^q}{(1+x)^{p+q}} M_n^{(p,q)}(x) M_m^{(p,q)}(x) dx = \left(\frac{n!\Gamma(p-n)\Gamma(q+n+1)}{(p-(2n+1))\Gamma(p+q-n)} \right) \delta_{n,m}, \tag{7}$$

if and only if $m, n = 0, 1, 2, ..., N < \frac{p-1}{2}$, $q > -1$, $\delta_{n,m} = \begin{cases} 1 & \text{if} \quad m = n \\ 0 & \text{if} \quad m \neq n \end{cases}$.

2.1.2. The Second Class of Finite Classical Orthogonal Polynomials

Let consider the second order differential equation of the form

$$x^2 y_n''(x) + ((2-p)x+1) y_n'(x) - n(n+1-p) y_n(x) = 0, \tag{8}$$

as a special case of (1). By the Frobenius method, an explicit polynomial solution for this equation is obtained as [18]

$$N_n^{(p)}(x) = (-1)^n \sum_{k=0}^{n} k! \binom{p-(n+1)}{k} \binom{n}{n-k} (-x)^k. \tag{9}$$

By means of similar calculations applied for the first class of finite classical orthogonal polynomials $M_n^{(p,q)}(x)$ it is seen that these polynomials are orthogonal on $[0, \infty)$ with respect to the weight function $W_2(x, p) = x^{-p} e^{-1/x}$ if and only if $p > 2\max\{m, n\} + 1$ [18]. In other words

$$\int_0^\infty x^{-p} e^{-1/x} N_n^{(p)}(x) N_m^{(p)}(x) dx = \frac{n!\Gamma(p-n)}{(p-2n-1)} \delta_{n,m}.$$

2.1.3. The Third Class of Finite Classical Orthogonal Polynomials

The third class is defined by

$$I_n^{(p)}(x) = n! \sum_{k=0}^{[n/2]} (-1)^k \binom{p-1}{n-k} \binom{n-k}{k} (2x)^{n-2k}, \tag{10}$$

and they are solutions of the differential equation

$$\left(1+x^2\right) y_n''(x) + (3-2p) x y_n'(x) - n(n+2-2p) y_n(x) = 0. \tag{11}$$

They are orthogonal on $(-\infty, \infty)$ with respect to the weight function $W_3(x, p) = (1+x^2)^{-(p-1/2)}$ if and only if $p > \max\{m, n\} + 1$. Indeed, the orthogonality relation is as follows [18]

$$\int_{-\infty}^{\infty} \left(1+x^2\right)^{-(p-1/2)} I_n^{(p)}(x) I_m^{(p)}(x) dx = \frac{n! 2^{2n-1} \sqrt{\pi} \Gamma^2(p) \Gamma(2p-2n)}{(p-n-1)\Gamma(p-n)\Gamma(p-n+1/2)\Gamma(2p-n-1)} \delta_{n,m}.$$

2.2. The Classes of Finite Bivariate Orthogonal Polynomials

Recently, in [30], fifteen families of finite bivariate orthogonal polynomials have been introduced by using Koornwinder's method [28], which are now listed as follows:

2.2.1. The First Sequence

Finite orthogonal polynomials $\left\{ {}_1Q_{n,k}^{(p,q)}(x,y) \right\}_{k=0,n=0}^{k=n,n=N}$ defined as

$$ {}_1Q_{n,k}^{(p,q)}(x,y) = M_{n-k}^{(p-2k-1,q+2k+1)}(x)x^k M_k^{(p,q)}\left(\tfrac{y}{x}\right),\ k = 0,1,...,n, \tag{12} $$

are orthogonal with respect to the weight function

$$ w_1(x,y;p,q) = x^{p+q}y^q(1+x)^{-(p+q)}(x+y)^{-(p+q)}, $$

on the domain

$$ D_1 = \left\{ (x,y) \in \mathbb{R}^2 : 0 < x < \infty, 0 < y < \infty \right\}, $$

if and only if $p > 2N+2$, $q > -1$. In other words, we have

$$ \begin{aligned} &\iint\limits_{D_1} x^{p+q}y^q(1+x)^{-(p+q)}(x+y)^{-(p+q)}\ {}_1Q_{n,k}^{(p,q)}(x,y)\ {}_1Q_{r,s}^{(p,q)}(x,y)dxdy \\ &= \frac{(n-k)!k!\Gamma(p-k)\Gamma(p-n-k-1)\Gamma(q+k+1)\Gamma(q+n+k+2)}{(p-2n-2)(p-2k-1)\Gamma(p+q-(n-k))\Gamma(p+q-k)}\delta_{n,r}\delta_{k,s}, \end{aligned} \tag{13} $$

for $n,r = 0,1,2,...,N < \frac{p-2}{2}$, $q > -1$ and $N = \max\{n,r\}$.

2.2.2. The Second Sequence

Finite orthogonal polynomials $\left\{ {}_2Q_{n,k}^{(p,q)}(x,y) \right\}_{k=0,n=0}^{k=n,n=N}$ defined as

$$ {}_2Q_{n,k}^{(p,q)}(x,y) = M_{n-k}^{(p-2k-1,q)}(x)(1+x)^k M_k^{(p,q)}\left(\tfrac{y}{1+x}\right),\ \ k = 0,1,...,n, \tag{14} $$

are orthogonal with respect to the weight function

$$ w_2(x,y;p,q) = x^q y^q (1+x)^{-q}(1+x+y)^{-(p+q)}, $$

on the domain

$$ D_2 = \left\{ (x,y) \in \mathbb{R}^2 : 0 < x < \infty,\ 0 < y < \infty \right\}, $$

if and only if $p > 2N+2$, $q > -1$. In other words, we have

$$ \begin{aligned} &\iint\limits_{D_2} x^q y^q (1+x)^{-q}(1+x+y)^{-(p+q)}\ {}_2Q_{n,k}^{(p,q)}(x,y)\ {}_2Q_{r,s}^{(p,q)}(x,y)dxdy \\ &= \frac{(n-k)!k!\Gamma(p-n-k-1)\Gamma(p-k)\Gamma(q+n-k+1)\Gamma(q+k+1)}{(p-2n-2)(p-2k-1)\Gamma(p+q-n-k-1)\Gamma(p+q-k)}\delta_{n,r}\delta_{k,s}, \end{aligned} \tag{15} $$

for $n,r = 0,1,2,...,N < \frac{p-2}{2}$, $q > -1$ and $N = \max\{n,r\}$.

2.2.3. The Third Sequence

Finite orthogonal polynomials $\left\{ {}_3Q_{n,k}^{(p)}(x,y) \right\}_{k=0,n=0}^{k=n,n=N}$ defined as

$$ {}_3Q_{n,k}^{(p)}(x,y) = N_{n-k}^{(p-2k-1)}(x)x^k N_k^{(p)}\left(\tfrac{y}{x}\right), \quad k = 0,1,...,n, \tag{16} $$

are orthogonal with respect to the weight function

$$ w_3(x,y;p) = y^{-p}\exp\left(-\frac{1}{x} - \frac{x}{y}\right), $$

on the domain

$$D_3 = \{(x,y) : 0 < x < \infty, 0 < y < \infty\},$$

if and only if $p > 2N+2$. In other words, we have

$$\begin{aligned}&\iint\limits_{D_3} y^{-p} \exp\left(-\frac{1}{x} - \frac{x}{y}\right) {}_3Q_{n,k}^{(p)}(x,y)\, {}_3Q_{r,s}^{(p)}(x,y) dxdy \\ &= \frac{(n-k)!k!\Gamma(p-k)\Gamma(p-n-k-1)}{(p-2k-1)(p-2n-2)} \delta_{n,r}\delta_{k,s},\end{aligned} \tag{17}$$

for $n, r = 0, 1, ..., N < \frac{p-2}{2}$ and $N = \max\{n, r\}$.

2.2.4. The Fourth Sequence

Finite orthogonal polynomials $\left\{ {}_4Q_{n,k}^{(p,q)}(x,y) \right\}_{k=0,n=0}^{k=n,n=N}$ defined as

$$ {}_4Q_{n,k}^{(p,q)}(x,y) = M_{n-k}^{(p-2k-1,q+2k+1)}(x) x^k N_k^{(p)}\left(\tfrac{y}{x}\right), \quad k = 0, 1, ..., n, \tag{18}$$

are orthogonal with respect to the weight function

$$w_4(x,y;p,q) = x^{p+q} y^{-p} (x+1)^{-(p+q)} \exp(-x/y),$$

on the domain

$$D_4 = \left\{(x,y) \in \mathbb{R}^2 : 0 < x < \infty, 0 < y < \infty\right\},$$

if and only if $p > 2N+2$, $q > -2$. In other words, we have

$$\begin{aligned}&\iint\limits_{D_4} x^{p+q} y^{-p} (x+1)^{-(p+q)} \exp(-x/y)\, {}_4Q_{n,k}^{(p,q)}(x,y)\, {}_4Q_{r,s}^{(p,q)}(x,y) dxdy \\ &= \frac{(n-k)!k!\Gamma(p-k)\Gamma(p-n-k-1)\Gamma(q+n+k+2)}{(p-2k-1)(p-2n-2)\Gamma(p+q-n+k)} \delta_{n,r}\delta_{k,s},\end{aligned} \tag{19}$$

for $n, r = 0, 1, ..., N < \frac{p-2}{2}$, $q > -2$ and $N = \max\{n, r\}$.

2.2.5. The Fifth Sequence

Finite orthogonal polynomials $\left\{ {}_5Q_{n,k}^{(p,q)}(x,y) \right\}_{k=0,n=0}^{k=n,n=N}$ defined as

$$ {}_5Q_{n,k}^{(p,q)}(x,y) = M_{n-k}^{(p-2k-1,q)}(x) (1+x)^k N_k^{(p)}\left(\tfrac{y}{1+x}\right), \quad k = 0, 1, ..., n, \tag{20}$$

are orthogonal with respect to the weight function

$$w_5(x,y;p,q) = x^q y^{-p} (1+x)^{-q} \exp\left(-\frac{1+x}{y}\right),$$

on the domain

$$D_5 = \left\{(x,y) \in \mathbb{R}^2 : 0 < x < \infty,\ 0 < y < \infty\right\},$$

if and only if $p > 2N+2$, $q > -1$. In other words, we have

$$\begin{aligned}&\iint\limits_{D_5} x^q y^{-p} (1+x)^{-q} \exp\left(-\frac{1+x}{y}\right) {}_5Q_{n,k}^{(p,q)}(x,y)\, {}_5Q_{r,s}^{(p,q)}(x,y) dxdy \\ &= \frac{(n-k)!k!\Gamma(p-n-k-1)\Gamma(p-k)\Gamma(q+n-k+1)}{(p-2n-2)(p-2k-1)\Gamma(p+q-n-k-1)} \delta_{n,r}\delta_{k,s},\end{aligned} \tag{21}$$

for $n, r = 0, 1, ..., N < \frac{p-2}{2}$, $q > -1$ and $N = \max\{n, r\}$.

2.2.6. The Sixth Sequence

Finite orthogonal polynomials $\left\{ {}_6Q_{n,k}^{(p,q)}(x,y) \right\}_{k=0,n=0}^{k=n,n=N}$ defined as

$$ {}_6Q_{n,k}^{(p,q)}(x,y) = N_{n-k}^{(p-2k-1)}(x) x^k M_k^{(p,q)}\left(\tfrac{y}{x}\right), \quad k = 0, 1, ..., n, \tag{22} $$

are orthogonal with respect to the weight function

$$ w_6(x, y; p, q) = y^q (x+y)^{-(p+q)} \exp(-1/x), $$

on the domain

$$ D_6 = \{(x, y) : 0 < x < \infty,\ 0 < y < \infty\}, $$

if and only if $p > 2N + 2$, $q > -1$. In other words, we have

$$ \begin{aligned} &\iint\limits_{D_6} y^q (x+y)^{-(p+q)} \exp(-1/x)\, {}_6Q_{n,k}^{(p,q)}(x,y)\, {}_6Q_{r,s}^{(p,q)}(x,y) dxdy \\ &= \frac{(n-k)!k!\Gamma(p-k)\Gamma(p-n-k-1)\Gamma(q+k+1)}{(p-2n-2)(p-2k-1)\Gamma(p+q-k)} \delta_{n,r}\delta_{k,s}, \end{aligned} \tag{23} $$

for $n, r = 0, 1, ..., N < \frac{p-2}{2}$, $q > -1$ and $N = \max\{n, r\}$.

2.2.7. The Seventh Sequence

Finite orthogonal polynomials $\left\{ {}_7Q_{n,k}^{(p,q,u,v)}(x,y) \right\}_{k=0,n=0}^{k=n,n=N}$ defined as

$$ {}_7Q_{n,k}^{(p,q,u,v)}(x,y) = M_{n-k}^{(p,q)}(x) M_k^{(u,v)}(y), \quad k = 0, 1, ..., n, \tag{24} $$

are orthogonal with respect to the weight function

$$ w_7(x, y; p, q, u, v) = x^q y^v (1+x)^{-(p+q)} (1+y)^{-(u+v)}, $$

on the domain

$$ D_7 = \{(x, y) : 0 < x < \infty, 0 < y < \infty\}, $$

if and only if $p, u > 2N + 1$, $q, v > -1$. In other words, we have

$$ \begin{aligned} &\iint\limits_{D_7} x^q y^v (1+x)^{-(p+q)} (1+y)^{-(u+v)}\, {}_7Q_{n,k}^{(p,q,u,v)}(x,y)\, {}_7Q_{r,s}^{(p,q,u,v)}(x,y) dxdy \\ &= \int_0^\infty x^q (1+x)^{-(p+q)} M_{n-k}^{(p,q)}(x) M_{r-s}^{(p,q)}(x) dx \\ &\times \int_0^\infty y^v (1+y)^{-(u+v)} M_k^{(u,v)}(y) M_s^{(u,v)}(y) dy. \end{aligned} $$

Here, using the orthogonality relation (7) for polynomial $M_n^{(p,q)}(x)$, the following orthogonality relation

$$ \begin{aligned} &\iint\limits_{D_7} x^q y^v (1+x)^{-(p+q)} (1+y)^{-(u+v)}\, {}_7Q_{n,k}^{(p,q,u,v)}(x,y)\, {}_7Q_{r,s}^{(p,q,u,v)}(x,y) dxdy \\ &= \frac{(n-k)!k!\Gamma(p-(n-k))\Gamma(q+n-k+1)\Gamma(u-k)\Gamma(v+k+1)}{(p-2(n-k)-1)(u-2k-1)\Gamma(p+q-n+k)\Gamma(u+v-k)} \delta_{n,r}\delta_{k,s} \end{aligned} \tag{25} $$

is satisfied for $n, r = 0, 1, ..., N < \min\left\{\frac{p-1}{2}, \frac{u-1}{2}\right\}, q, v > -1$ and $N = \max\{n, r\}$.

2.2.8. The Eight Sequence

Finite orthogonal polynomials $\left\{ {}_8Q_{n,k}^{(p,q)}(x,y) \right\}_{k=0,n=0}^{k=n,n=N}$ defined as

$$ {}_8Q_{n,k}^{(p,q)}(x,y) = N_{n-k}^{(p)}(x)N_k^{(q)}(y), \quad k = 0, 1, ..., n, \tag{26}$$

are orthogonal with respect to the weight function

$$ w_8(x, y; p, q) = x^{-p}y^{-q} \exp\left(-\frac{1}{x} - \frac{1}{y}\right), $$

on the domain

$$ D_8 = \{(x, y) : 0 < x < \infty, 0 < y < \infty\}, $$

if and only if $p, q > 2N + 1$. In other words, we have

$$ \begin{aligned} &\iint_{D_8} e^{\left(-\frac{1}{x}-\frac{1}{y}\right)} x^{-p}y^{-q} \, {}_8Q_{n,k}^{(p,q)}(x,y) \, {}_8Q_{r,s}^{(p,q)}(x,y) dxdy \\ &= \frac{(n-k)!k!\Gamma(p-(n-k))\Gamma(q-k)}{(p-2(n-k)-1)(q-2k-1)} \delta_{n,r}\delta_{k,s}, \end{aligned} \tag{27}$$

for $n, r = 0, 1, ..., N < \min\left\{\frac{p-1}{2}, \frac{q-1}{2}\right\}$ and $N = \max\{n, r\}$.

2.2.9. The Ninth Sequence

Finite orthogonal polynomials $\left\{ {}_9Q_{n,k}^{(p,q,u)}(x,y) \right\}_{k=0,n=0}^{k=n,n=N}$ defined as

$$ {}_9Q_{n,k}^{(p,q,u)}(x,y) = M_{n-k}^{(p,q)}(x)N_k^{(u)}(y), \quad k = 0, 1, ..., n, \tag{28}$$

are orthogonal with respect to the weight function

$$ w_9(x, y; p, q, u) = x^q(1+x)^{-(p+q)}y^{-u} \exp(-1/y), $$

on the domain

$$ D_9 = \{(x, y) : 0 < x < \infty, 0 < y < \infty\}, $$

if and only if $p, u > 2N + 1, q > -1$. In other words, we have

$$ \begin{aligned} &\iint_{D_9} x^q(1+x)^{-(p+q)}y^{-u} \exp(-1/y) \, {}_9Q_{n,k}^{(p,q,u)}(x,y) \, {}_9Q_{r,s}^{(p,q,u)}(x,y) dxdy \\ &= \frac{(n-k)!k!\Gamma(p-(n-k))\Gamma(q+n-k+1)\Gamma(u-k)}{(p-2(n-k)-1)(u-2k-1)\Gamma(p+q-(n-k))} \delta_{n,r}\delta_{k,s}, \end{aligned} \tag{29}$$

for $n, r = 0, 1, ..., N < \min\left\{\frac{p-1}{2}, \frac{u-1}{2}\right\}, q > -1$ and $N = \max\{n, r\}$.

2.2.10. The Tenth Sequence

Finite orthogonal polynomials $\left\{ {}_{10}Q_{n,k}^{(p)}(x,y) \right\}_{k=0,n=0}^{k=n,n=N}$ defined as

$$ {}_{10}Q_{n,k}^{(p)}(x,y) = I_{n-k}^{(p-k-1/2)}(x)\left(1+x^2\right)^{k/2} I_k^{(p)}\left(\frac{y}{\sqrt{1+x^2}}\right), \quad k = 0, 1, ..., n, \tag{30}$$

are orthogonal with respect to the weight function

$$w_{10}(x,y;p) = \left(1+x^2+y^2\right)^{-\left(p-\frac{1}{2}\right)},$$

on the domain

$$D_{10} = \{(x,y) : -\infty < x < \infty, -\infty < y < \infty\},$$

if and only if $p > N + \frac{3}{2}$. In other words, we have

$$\iint_{D_{10}} \left(1+x^2+y^2\right)^{-\left(p-\frac{1}{2}\right)} {}_{10}Q_{n,k}^{(p)}(x,y)\, {}_{10}Q_{r,s}^{(p)}(x,y)dxdy$$
$$= \frac{(n-k)!k!2^{2(n-1)}\pi\Gamma^2(p-k-1/2)\Gamma^2(p)\Gamma(2p-2n-1)\Gamma(2p-2k)}{(p-n-3/2)(p-k-1)\Gamma(p-n-1/2)\Gamma(p-k)\Gamma(p-n)\Gamma(p-k+1/2)\Gamma(2p-n-k-2)\Gamma(2p-k-1)}\delta_{n,r}\delta_{k,s},$$

for $n,r = 0,1,...,N < p - \frac{3}{2}$ and $N = \max\{n,r\}$.

2.2.11. The Eleventh Sequence

Finite orthogonal polynomials $\left\{ {}_{11}Q_{n,k}^{(p,q)}(x,y) \right\}_{k=0,n=0}^{k=n,n=N}$ defined as

$${}_{11}Q_{n,k}^{(p,q)}(x,y) = M_{n-k}^{(p-2k-1,q+2k+1)}(x)x^k I_k^{(p)}\left(\tfrac{y}{x}\right), \quad k = 0,1,...,n, \tag{31}$$

are orthogonal with respect to the weight function

$$w_{11}(x,y;p,q) = x^{2p+q-1}(1+x)^{-(p+q)}\left(x^2+y^2\right)^{-\left(p-\frac{1}{2}\right)},$$

on the domain

$$D_{11} = \{(x,y) : 0 < x < \infty,\ -\infty < y < \infty\},$$

if and only if $p > 2N+2$ and $q > -2$. In other words, we have

$$\iint_{D_{11}} x^{2p+q-1}(1+x)^{-(p+q)}\left(x^2+y^2\right)^{-\left(p-\frac{1}{2}\right)} {}_{11}Q_{n,k}^{(p,q)}(x,y)\, {}_{11}Q_{r,s}^{(p,q)}(x,y)dydx$$
$$= \frac{(n-k)!k!2^{2k-1}\sqrt{\pi}\Gamma^2(p)\Gamma(p-n-k-1)\Gamma(2p-2k)\Gamma(q+n+k+2)}{(p-2n-2)(p-k-1)\Gamma(p-k)\Gamma\left(p-k+\frac{1}{2}\right)\Gamma(2p-k-1)\Gamma(p+q-n+k)}\delta_{n,r}\delta_{k,s},$$

for $n,r = 0,1,...,N < \frac{p-2}{2}$, $q > -2$ and $N = \max\{n,r\}$.

2.2.12. The Twelfth Sequence

Finite orthogonal polynomials $\left\{ {}_{12}Q_{n,k}^{(p)}(x,y) \right\}_{k=0,n=0}^{k=n,n=N}$ defined as

$${}_{12}Q_{n,k}^{(p)}(x,y) = N_{n-k}^{(p-2k-1)}(x)x^k I_k^{(p)}\left(\tfrac{y}{x}\right), \quad k = 0,1,...,n, \tag{32}$$

are orthogonal with respect to the weight function

$$w_{12}(x,y;p) = x^{p-1}\left(x^2+y^2\right)^{-\left(p-\frac{1}{2}\right)}\exp(-1/x),$$

on the domain

$$D_{12} = \{(x,y) : 0 < x < \infty,\ -\infty < y < \infty\},$$

if and only if $p > 2N+2$. In other words, we have

$$\iint\limits_{D_{12}} x^{p-1}\left(x^2+y^2\right)^{-\left(p-\frac{1}{2}\right)} \exp(-1/x)\ {}_{12}Q_{n,k}^{(p)}(x,y)\ {}_{12}Q_{r,s}^{(p)}(x,y)dydx$$
$$= \frac{(n-k)!k!2^{2k-1}\sqrt{\pi}\Gamma^2(p)\Gamma(2p-2k)\Gamma(p-n-k-1)}{(p-2n-2)(p-k-1)\Gamma(p-k)\Gamma\left(p-k+\frac{1}{2}\right)\Gamma(2p-k-1)}\delta_{n,r}\delta_{k,s},$$

for $n, r = 0, 1, ..., N < \frac{p-2}{2}$ and $N = \max\{n, r\}$.

2.2.13. The Thirteenth Sequence

Finite orthogonal polynomials $\left\{ {}_{13}Q_{n,k}^{(p,q)}(x,y)\right\}_{k=0,n=0}^{k=n,n=N}$ defined as

$$ {}_{13}Q_{n,k}^{(p,q)}(x,y) = I_{n-k}^{(p)}(x)I_k^{(q)}(y), \quad k = 0, 1, ..., n, \tag{33}$$

are orthogonal with respect to the weight function

$$w_{13}(x,y;p,q) = \left(1+x^2\right)^{-\left(p-\frac{1}{2}\right)}\left(1+y^2\right)^{-\left(q-\frac{1}{2}\right)},$$

on the domain

$$D_{13} = \{(x,y) : -\infty < x < \infty, -\infty < y < \infty\},$$

if and only if $p, q > N+1$. In other words, we have

$$\begin{aligned}
&\iint\limits_{D_{13}} \left(1+x^2\right)^{-\left(p-\frac{1}{2}\right)}\left(1+y^2\right)^{-\left(q-\frac{1}{2}\right)}\ {}_{13}Q_{n,k}^{(p,q)}(x,y)\ {}_{13}Q_{r,s}^{(p,q)}(x,y)dydx \\
&= \frac{(n-k)!k!2^{2(n-1)}\pi\Gamma^2(p)\Gamma^2(q)}{(p-n+k-1)(q-k-1)\Gamma(p-n+k)\Gamma(q-k)\Gamma\left(p-n+k+\frac{1}{2}\right)} \\
&\times \frac{\Gamma(2p-2n+2k)\Gamma(2q-2k)}{\Gamma\left(q-k+\frac{1}{2}\right)\Gamma(2p-n+k-1)\Gamma(2q-k-1)}\delta_{n,r}\delta_{k,s},
\end{aligned} \tag{34}$$

for $n, r = 0, 1, ..., N < \min\{p-1, q-1\}$ and $N = \max\{n, r\}$.

2.2.14. The Fourteenth Sequence

Finite orthogonal polynomials $\left\{ {}_{14}Q_{n,k}^{(p,q,u)}(x,y)\right\}_{k=0,n=0}^{k=n,n=N}$ defined as

$$ {}_{14}Q_{n,k}^{(p,q,u)}(x,y) = M_{n-k}^{(p,q)}(x)I_k^{(u)}(y), \quad k = 0, 1, ..., n, \tag{35}$$

are orthogonal with respect to the weight function

$$w_{14}(x,y;p,q,u) = x^q(1+x)^{-(p+q)}\left(1+y^2\right)^{-\left(u-\frac{1}{2}\right)},$$

on the domain

$$D_{14} = \{(x,y) : 0 < x < \infty, -\infty < y < \infty\},$$

if and only if $p > 2N+1$, $u > N+1$ and $q > -1$. In other words, we have

$$\iint_{D_{14}} x^q(1+x)^{-(p+q)}\left(1+y^2\right)^{-\left(u-\frac{1}{2}\right)} {}_{14}Q_{n,k}^{(p,q,u)}(x,y)\, {}_{14}Q_{r,s}^{(p,q,u)}(x,y)dydx \tag{36}$$
$$= \frac{(n-k)!k!2^{2k-1}\sqrt{\pi}\Gamma(p-n+k)\Gamma(q+n-k+1)}{(p-2(n-k)-1)(u-k-1)\Gamma(p+q-n+k)}$$
$$\times \frac{\Gamma^2(u)\Gamma(2u-2k)}{\Gamma(u-k)\Gamma\left(u-k+\frac{1}{2}\right)\Gamma(2u-k-1)}\delta_{n,r}\delta_{k,s},$$

for $n,r=0,1,...,N<\min\left\{\frac{p-1}{2},u-1\right\}$, $q>-1$ and $N=\max\{n,r\}$.

2.2.15. The Fifteenth Sequence

Finite orthogonal polynomials $\left\{ {}_{15}Q_{n,k}^{(p,q)}(x,y)\right\}_{k=0,n=0}^{k=n,n=N}$ defined as

$${}_{15}Q_{n,k}^{(p,q)}(x,y) = N_{n-k}^{(p)}(x)I_k^{(q)}(y), \quad k=0,1,...,n, \tag{37}$$

are orthogonal with respect to the weight function

$$w_{15}(x,y;p,q) = x^{-p}\left(1+y^2\right)^{-\left(q-\frac{1}{2}\right)}\exp(-1/x),$$

on the domain

$$D_{15} = \{(x,y) : 0<x<\infty, -\infty<y<\infty\},$$

if and only if $p>2N+1$ and $q>N+1$. In other words, we have

$$\iint_{D_{15}} x^{-p}\left(1+y^2\right)^{-\left(q-\frac{1}{2}\right)}\exp(-1/x)\, {}_{15}Q_{n,k}^{(p,q)}(x,y)\, {}_{15}Q_{r,s}^{(p,q)}(x,y)dydx$$
$$= \frac{(n-k)!k!2^{2k-1}\sqrt{\pi}\Gamma(p-n+k)\Gamma^2(q)\Gamma(2q-2k)}{(p-2(n-k)-1)(q-k-1)\Gamma(q-k)\Gamma\left(q-k+\frac{1}{2}\right)\Gamma(2q-k-1)}\delta_{n,r}\delta_{k,s}, \tag{38}$$

for $n,r=0,1,...,N<\min\left\{\frac{p-1}{2},q-1\right\}$ and $N=\max\{n,r\}$.

In the present paper, we first consider Fourier transforms of some specific functions in terms of finite bivariate orthogonal polynomials listed above except for tenth, eleventh and twelft polynomial sequences and then we introduce new families of bivariate orthogonal functions via Parseval identity.

3. Fourier Transforms for the Set of the Polynomials $Q_{n,k}$

The Fourier transform for a function of one variable is defined as [32]

$$\mathcal{F}(f(x)) = \int_{-\infty}^{\infty} e^{-i\xi x} f(x)dx,$$

and the corresponding Parseval identity is given by

$$\int_{-\infty}^{\infty} f(x)\overline{p(x)}dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{F}(f(x))\overline{\mathcal{F}(p(x))}d\xi,$$

for $p,f\in L^2(\mathbb{R})$.

The Fourier transform for a function of two variables is in the form [1]

$$\mathcal{F}(g(x,y)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(\xi_1 x+\xi_2 y)} g(x,y)\,dxdy,$$

and the corresponding Parseval identity is given by

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\overline{h(x,y)}\,dxdy = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathcal{F}(g(x,y))\overline{\mathcal{F}(h(x,y))}\,d\xi_1 d\xi_2. \tag{39}$$

Now, let us obtain the Fourier transforms of finite bivariate orthogonal polynomials given in the previous section in order to define some new families of bivariate orthogonal functions using the Parseval identity.

3.1. Fourier Transform of the Polynomials ${}_1Q_{n,k}^{(p,q)}(x,y)$

Let us define

$$\begin{aligned} h_{n,k}(x,y;\kappa_1,\kappa_2,\lambda,\mu) &= e^{\kappa_2 y}(e^x+1)^{-(\kappa_1+\kappa_2)}(e^x+e^y)^{-(\kappa_1+\kappa_2)} \\ &\quad\times e^{\left(\kappa_1+\kappa_2+\frac{1}{2}\right)x}\, {}_1Q_{n,k}^{(\lambda,\mu)}(e^x,e^y), \end{aligned} \tag{40}$$

where $\kappa_1,\kappa_2,\lambda$ and μ are real parameters, and the polynomials ${}_1Q_{n,k}^{(p,q)}(x,y)$ are defined in (12). By using appropriate substitutions $e^x = u$, $e^y = v$ and $\frac{v}{u} = t$, we get

$$\begin{aligned} &\mathcal{F}(h_{n,k}(x,y;\kappa_1,\kappa_2,\lambda,\mu)) \\ &= \left(\int_0^{\infty} u^{k+\kappa_2-\frac{1}{2}-i(\xi_1+\xi_2)}(u+1)^{-(\kappa_1+\kappa_2)} M_{n-k}^{(\lambda-2k-1,\mu+2k+1)}(u)\,du\right) \\ &\times \left(\int_0^{\infty} t^{\kappa_2-1-i\xi_2}(1+t)^{-(\kappa_1+\kappa_2)} M_k^{(\lambda,\mu)}(t)\,dt\right). \end{aligned}$$

If we apply (3), then

$$\begin{aligned} \mathcal{F}(h_{n,k}(x,y;\kappa_1,\kappa_2,\lambda,\mu)) &= (-1)^n\frac{\Gamma(\mu+n+k+2)\Gamma(\mu+k+1)}{\Gamma(\mu+2k+2)\Gamma(\mu+1)} \\ \times\Bigg[\sum_{l_1=0}^{n-k}\frac{(-(n-k))_{l_1}(2-\lambda+n+k)_{l_1}}{l_1!(-1)^{l_1}(\mu+2k+2)_{l_1}}&\left(\int_0^{\infty} u^{k+\kappa_2-\frac{1}{2}-i(\xi_1+\xi_2)+l_1}(u+1)^{-(\kappa_1+\kappa_2)}\,du\right)\Bigg] \\ \times\Bigg[\sum_{l_2=0}^{k}\frac{(-k)_{l_2}(1-\lambda+k)_{l_2}}{l_2!(-1)^{l_2}(\mu+1)_{l_2}}&\left(\int_0^{\infty} t^{\kappa_2-1-i\xi_2+l_2}(1+t)^{-(\kappa_1+\kappa_2)}\,dt\right)\Bigg] \\ = (-1)^n\frac{\Gamma(\mu+n+k+2)\Gamma(\mu+k+1)}{\Gamma(\mu+2k+2)\Gamma(\mu+1)}&\Bigg[\sum_{l_1=0}^{n-k}\frac{(-(n-k))_{l_1}(2-\lambda+n+k)_{l_1}}{l_1!(-1)^{l_1}(\mu+2k+2)_{l_1}} \\ \times\frac{\Gamma\left(\kappa_2+k+\frac{1}{2}-i(\xi_1+\xi_2)+l_1\right)\Gamma\left(\kappa_1-k-\frac{1}{2}+i(\xi_1+\xi_2)-l_1\right)}{\Gamma(\kappa_1+\kappa_2)}&\Bigg] \\ \times\Bigg[\sum_{l_2=0}^{k}\frac{(-k)_{l_2}(1-\lambda+k)_{l_2}}{l_2!(-1)^{l_2}(\mu+1)_{l_2}}&\frac{\Gamma(\kappa_2-i\xi_2+l_2)\Gamma(\kappa_1+i\xi_2-l_2)}{\Gamma(\kappa_1+\kappa_2)}\Bigg], \end{aligned}$$

and we can conclude that

$$\mathcal{F}(h_{n,k}(x,y;\kappa_1,\kappa_2,\lambda,\mu)) = \frac{(-1)^n\Gamma(\mu+n+k+2)\Gamma(\mu+k+1)}{\Gamma(\mu+2k+2)\Gamma(\mu+1)\Gamma^2(\kappa_1+\kappa_2)} \times C_1(k,\kappa_1,\kappa_2,\xi_1,\xi_2)\Theta_1(n,k,\kappa_1,\kappa_2,\lambda,\mu,\xi_1,\xi_2),$$

where

$$C_1(k,\kappa_1,\kappa_2,\xi_1,\xi_2) = \Gamma(\kappa_1+i\xi_2)\Gamma(\kappa_2-i\xi_2)\Gamma(\kappa_1-k-1/2+i(\xi_1+\xi_2)) \times \Gamma(\kappa_2+k+1/2-i(\xi_1+\xi_2)),$$

and

$$\Theta_1(n,k,\kappa_1,\kappa_2,\lambda,\mu,\xi_1,\xi_2) = {}_3F_2\left(\begin{matrix} -k,\ k+1-\lambda,\ \kappa_2-i\xi_2 \\ \mu+1,\ 1-\kappa_1-i\xi_2 \end{matrix} \mid 1\right) \times {}_3F_2\left(\begin{matrix} -(n-k),\ n+k+2-\lambda,\ k+\kappa_2+1/2-i(\xi_1+\xi_2) \\ \mu+2k+2,\ k-\kappa_1+3/2-i(\xi_1+\xi_2) \end{matrix} \mid 1\right),$$

such that ${}_3F_2$ is a special case of the hypergeometric function given by [31]

$${}_pF_q\left(\begin{matrix} a_1,a_2,\ldots,a_p \\ b_1,b_2,\ldots,b_q \end{matrix} \mid x\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k(a_2)_k\ldots(a_p)_k}{(b_1)_k(b_2)_k\ldots(b_q)_k}\frac{x^k}{k!},$$

in which $(\lambda)_k = \lambda(\lambda+1)\ldots(\lambda+k-1), k=1,2,\ldots\,; (\lambda)_0 = 1$ is the Pochhammer symbol. Also, from the definition of Gamma function, it can be verified that

$$(\lambda)_k = \frac{\Gamma(\lambda+k)}{\Gamma(\lambda)} \ \mathit{and}\ \Gamma(a-k) = \frac{(-1)^k\Gamma(a)}{(1-a)_k}. \tag{41}$$

Hence, from the Parseval identity (39) we obtain

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_{n,k}(x,y;\kappa_1,\kappa_2,\lambda,\mu)h_{r,s}(x,y;\varrho_1,\varrho_2,\alpha,\beta)dxdy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{(\kappa_1+\kappa_2+\varrho_1+\varrho_2+1)x}e^{(\kappa_2+\varrho_2)y}(e^x+1)^{-(\kappa_1+\kappa_2+\varrho_1+\varrho_2)}(e^x+e^y)^{-(\kappa_1+\kappa_2+\varrho_1+\varrho_2)} \\
&\qquad \times\ {}_1Q_{n,k}^{(\lambda,\mu)}(e^x,e^y)\ {}_1Q_{r,s}^{(\alpha,\beta)}(e^x,e^y)dxdy \\
&= \int_0^{\infty}\int_0^{\infty} u^{\kappa_1+\kappa_2+\varrho_1+\varrho_2}v^{\kappa_2+\varrho_2-1}(u+1)^{-(\kappa_1+\kappa_2+\varrho_1+\varrho_2)}(u+v)^{-(\kappa_1+\kappa_2+\varrho_1+\varrho_2)} \\
&\times\ {}_1Q_{n,k}^{(\lambda,\mu)}(u,v)\ {}_1Q_{r,s}^{(\alpha,\beta)}(u,v)dudv \\
&= \frac{(-1)^{n+r}}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\Gamma(\mu+n+k+2)\Gamma(\mu+k+1)\Gamma(\beta+r+s+2)\Gamma(\beta+s+1)}{\Gamma(\mu+2k+2)\Gamma(\mu+1)\Gamma(\beta+2s+2)\Gamma(\beta+1)} \\
&\times \frac{C_1(k,\kappa_1,\kappa_2,\xi_1,\xi_2)\overline{C_1(s,\varrho_1,\varrho_2,\xi_1,\xi_2)}}{\Gamma^2(\kappa_1+\kappa_2)\Gamma^2(\varrho_1+\varrho_2)} \\
&\times \Theta_1(n,k,\kappa_1,\kappa_2,\lambda,\mu,\xi_1,\xi_2)\overline{\Theta_1(r,s,\varrho_1,\varrho_2,\alpha,\beta,\xi_1,\xi_2)}d\xi_1 d\xi_2.
\end{aligned} \tag{42}$$

Now by taking $\kappa_1+\varrho_1+1=\lambda=\alpha$ and $\kappa_2+\varrho_2-1=\mu=\beta$ in (42), if we use the orthogonality relation (13) in the left-hand side of (42), we obtain

$$\begin{aligned}&\frac{4\pi^2(n-k)!k!\Gamma(\kappa_1+\varrho_1-n-k)\Gamma(\kappa_1+\varrho_1-k+1)\Gamma^2(\kappa_2+\varrho_2+2k+1)}{(\kappa_1+\varrho_1-2n-1)(\kappa_1+\varrho_1-2k)\Gamma(\kappa_1+\kappa_2+\varrho_1+\varrho_2-(n-k))}\\&\quad\times\frac{\Gamma^2(\kappa_1+\kappa_2)\Gamma^2(\kappa_2+\varrho_2)\Gamma^2(\varrho_1+\varrho_2)}{\Gamma(\kappa_1+\kappa_2+\varrho_1+\varrho_2-k)\Gamma(\kappa_2+\varrho_2+n+k+1)\Gamma(\kappa_2+\varrho_2+k)}\delta_{n,r}\delta_{k,s}\\&=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}C_1(k,\kappa_1,\kappa_2,\xi_1,\xi_2)\overline{C_1(s,\varrho_1,\varrho_2,\xi_1,\xi_2)}\\&\quad\times\Theta_1(n,k,\kappa_1,\kappa_2,\kappa_1+\varrho_1+1,\kappa_2+\varrho_2-1,\xi_1,\xi_2)\\&\quad\times\overline{\Theta_1(r,s,\varrho_1,\varrho_2,\varrho_1+\kappa_1+1,\varrho_2+\kappa_2-1,\xi_1,\xi_2)}d\xi_1d\xi_2.\end{aligned}$$

Theorem 1. *The special function*

$$\begin{aligned}{}_1E_{n,k}(x,y;\kappa_1,\kappa_2,\varrho_2,\varrho_1)&=\frac{(\kappa_2+1/2-(x+y))_k}{(3/2-\kappa_1-(x+y))_k}\\&\quad\times\Theta_1(n,k,\kappa_1,\kappa_2,\kappa_1+\varrho_1+1,\kappa_2+\varrho_2-1,-ix,-iy),\end{aligned}$$

has an orthogonality relation of form

$$\begin{aligned}&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Gamma\left(\kappa_1-\frac{1}{2}+i(x+y)\right)\Gamma\left(\kappa_2+\frac{1}{2}-i(x+y)\right)\\&\times\Gamma\left(\varrho_2+\frac{1}{2}+i(x+y)\right)\Gamma\left(\varrho_1-\frac{1}{2}-i(x+y)\right)\\&\times\Gamma(\kappa_1+iy)\Gamma(\kappa_2-iy)\Gamma(\varrho_2+iy)\Gamma(\varrho_1-iy)\\&\times\ {}_1E_{n,k}(ix,iy;\kappa_1,\kappa_2,\varrho_2,\varrho_1)\ {}_1E_{r,s}(-ix,-iy;\varrho_1,\varrho_2,\kappa_2,\kappa_1)dxdy\\&=\frac{4\pi^2(n-k)!k!\Gamma(\kappa_1+\varrho_1-n-k)\Gamma(\kappa_1+\varrho_1-k+1)}{(\kappa_1+\varrho_1-2n-1)(\kappa_1+\varrho_1-2k)\Gamma(\kappa_1+\kappa_2+\varrho_1+\varrho_2-(n-k))}\\&\times\frac{\Gamma^2(\kappa_2+\varrho_2+2k+1)\Gamma^2(\kappa_1+\kappa_2)\Gamma^2(\kappa_2+\varrho_2)\Gamma^2(\varrho_2+\varrho_1)}{\Gamma(\kappa_1+\kappa_2+\varrho_1+\varrho_2-k)\Gamma(\kappa_2+\varrho_2+n+k+1)\Gamma(\kappa_2+\varrho_2+k)}\delta_{n,r}\delta_{k,s},\end{aligned}$$

for $\kappa_1,\varrho_1>1/2$, $\kappa_2,\varrho_2>0$ and $\kappa_1+\varrho_1>2n+1$. Please note that the weight function of this orthogonality relation is positive for $\kappa_1=\varrho_1$ and $\kappa_2=\varrho_2$.

3.2. Fourier Transform of the Polynomials ${}_2Q_{n,k}^{(p,q)}(x,y)$

Let us define

$$h_{n,k}(x,y;\kappa_1,\kappa_2,\lambda,\mu)=e^{\left(\kappa_2+\frac{1}{2}\right)(x+y)}(1+e^x)^{-\kappa_2}(1+e^x+e^y)^{-(\kappa_1+\kappa_2)}\ {}_2Q_{n,k}^{(\lambda,\mu)}(e^x,e^y),\quad(43)$$

where $\kappa_1, \kappa_2, \lambda$ and μ are real parameters, and the polynomials ${}_2Q_{n,k}^{(p,q)}(x,y)$ are defined in (14). If we apply the Fourier transform to the function $h_{n,k}(x,y;\kappa_1,\kappa_2,\lambda,\mu)$, under the subsitutions $e^x = u$, $e^y = v$ and $\frac{v}{1+u} = t$, respectively, we get

$$\begin{aligned}
&\mathcal{F}(h_{n,k}(x,y;\kappa_1,\kappa_2,\lambda,\mu)) \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(\xi_1 x+\xi_2 y)} e^{\left(\kappa_2+\frac{1}{2}\right)(x+y)}(1+e^x)^{-\kappa_2}(1+e^x+e^y)^{-(\kappa_1+\kappa_2)}\,{}_2Q_{n,k}^{(\lambda,\mu)}(e^x,e^y)dxdy \\
&= \left(\int_0^\infty u^{\kappa_2-i\xi_1-\frac{1}{2}}(1+u)^{k-\kappa_1-\kappa_2-i\xi_2+\frac{1}{2}} M_{n-k}^{(\lambda-2k-1,\mu)}(u)du\right) \\
&\times \left(\int_0^\infty t^{\kappa_2-i\xi_2-\frac{1}{2}}(1+t)^{-(\kappa_1+\kappa_2)} M_k^{(\lambda,\mu)}(t)dt\right)
\end{aligned}$$

$$\begin{aligned}
&= (-1)^n \frac{\Gamma(\mu+n-k+1)\Gamma(\mu+k+1)}{\Gamma^2(\mu+1)} \left[\sum_{l_1=0}^{n-k} \frac{(-(n-k))_{l_1}(n+k+2-\lambda)_{l_1}}{l_1!(-1)^{l_1}(\mu+1)_{l_1}}\right. \\
&\times \left.\left(\int_0^\infty u^{\kappa_2-i\xi_1-\frac{1}{2}+l_1}(1+u)^{k-\kappa_1-\kappa_2-i\xi_2+\frac{1}{2}}du\right)\right] \\
&\times \left[\sum_{l_2=0}^{k} \frac{(-k)_{l_2}(k+1-\lambda)_{l_2}}{l_2!(-1)^{l_2}(\mu+1)_{l_2}} \left(\int_0^\infty t^{\kappa_2-i\xi_2-\frac{1}{2}+l_2}(1+t)^{-(\kappa_1+\kappa_2)}dt\right)\right],
\end{aligned}$$

from the relations (41) and definition of Beta function

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^\infty \frac{u^{x-1}}{(1+u)^{x+y}}du, \qquad \Re(x), \Re(y) > 0,$$

the latter expression can be also expressed as

$$\begin{aligned}
\mathcal{F}(h_{n,k}(x,y;\kappa_1,\kappa_2,\lambda,\mu)) &= \frac{(-1)^n\Gamma(\mu+n-k+1)\Gamma(\mu+k+1)}{\Gamma^2(\mu+1)\Gamma(\kappa_1+\kappa_2)} \\
&\times C_2(k,\kappa_1,\kappa_2,\xi_1,\xi_2)\Theta_2(n,k,\kappa_1,\kappa_2,\lambda,\mu,\xi_1,\xi_2),
\end{aligned}$$

where

$$\begin{aligned}
C_2(k,\kappa_1,\kappa_2,\xi_1,\xi_2) &= \Gamma(\kappa_2 - i\xi_1 + 1/2)\Gamma(\kappa_1 + i\xi_2 - 1/2)\Gamma(\kappa_2 - i\xi_2 + 1/2) \\
&\times \frac{\Gamma(\kappa_1 - k - 1 - i(\xi_1+\xi_2))}{\Gamma(\kappa_1+\kappa_2-k-1/2+i\xi_2)},
\end{aligned}$$

and

$$\begin{aligned}
\Theta_2(n,k,\kappa_1,\kappa_2,\lambda,\mu,\xi_1,\xi_2) &= {}_3F_2\left(\begin{matrix}-(n-k),\ n+k+2-\lambda,\ \kappa_2+1/2-i\xi_1 \\ \mu+1,\ k-\kappa_1+2-i(\xi_1+\xi_2)\end{matrix} \,\Big|\, 1\right) \\
&\times {}_3F_2\left(\begin{matrix}-k,\ k+1-\lambda,\ \kappa_2-i\xi_2+1/2 \\ \mu+1,\ 3/2-\kappa_1-i\xi_2\end{matrix} \,\Big|\, 1\right).
\end{aligned}$$

Hence, from the Parseval identity (39), we obtain

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_{n,k}(x,y;\kappa_1,\kappa_2,\lambda,\mu)h_{r,s}(x,y;\varrho_1,\varrho_2,\alpha,\beta)dxdy \\
&= \int_0^{\infty}\int_0^{\infty} u^{\kappa_2+\varrho_2}v^{\kappa_2+\varrho_2}(1+u)^{-(\kappa_2+\varrho_2)}(1+u+v)^{-(\kappa_1+\kappa_2+\varrho_1+\varrho_2)} \\
&\times\ {}_2Q_{n,k}^{(\lambda,\mu)}(u,v)\ {}_2Q_{r,s}^{(\alpha,\beta)}(u,v)dudv \\
&= \frac{(-1)^{n+r}}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\Gamma(\mu+n-k+1)\Gamma(\mu+k+1)\Gamma(\beta+r-s+1)\Gamma(\beta+s+1)}{\Gamma^2(\mu+1)\Gamma^2(\beta+1)\Gamma(\kappa_1+\kappa_2)\Gamma(\varrho_1+\varrho_2)} \\
&\times C_2(k,\kappa_1,\kappa_2,\xi_1,\xi_2)\overline{C_2(s,\varrho_1,\varrho_2,\xi_1,\xi_2)}\Theta_2(n,k,\kappa_1,\kappa_2,\lambda,\mu,\xi_1,\xi_2) \\
&\times \overline{\Theta_2(r,s,\varrho_1,\varrho_2,\alpha,\beta,\xi_1,\xi_2)}d\xi_1 d\xi_2.
\end{aligned} \tag{44}
$$

Now by taking $\kappa_2+\varrho_2=\mu=\beta$ and $\kappa_1+\varrho_1=\lambda=\alpha$ in (44), and we use the orthogonality relation (15) in the left-hand side of (44), we obtain

$$
\begin{aligned}
&\frac{4\pi^2(n-k)!k!\Gamma(\kappa_1+\varrho_1-n-k-1)\Gamma(\kappa_1+\varrho_1-k)}{(\kappa_1+\varrho_1-2n-2)(\kappa_1+\varrho_1-2k-1)\Gamma(\kappa_1+\kappa_2+\varrho_1+\varrho_2-n-k-1)} \\
&\times \frac{\Gamma^4(\kappa_2+\varrho_2+1)\Gamma(\kappa_1+\kappa_2)\Gamma(\varrho_1+\varrho_2)}{\Gamma(\kappa_1+\kappa_2+\varrho_1+\varrho_2-k)\Gamma(\kappa_2+\varrho_2+n-k+1)\Gamma(\kappa_2+\varrho_2+k+1)}\delta_{n,r}\delta_{k,s} \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} C_2(k,\kappa_1,\kappa_2,\xi_1,\xi_2)\overline{C_2(s,\varrho_1,\varrho_2,\xi_1,\xi_2)} \\
&\times \Theta_2(n,k,\kappa_1,\kappa_2,\kappa_1+\varrho_1,\kappa_2+\varrho_2,\xi_1,\xi_2) \\
&\times \overline{\Theta_2(r,s,\varrho_1,\varrho_2,\varrho_1+\kappa_1,\varrho_2+\kappa_2,\xi_1,\xi_2)}d\xi_1 d\xi_2.
\end{aligned}
$$

Theorem 2. *The special function*

$$
\begin{aligned}
{}_2E_{n,k}(x,y;\kappa_1,\kappa_2,\varrho_2,\varrho_1) = &\frac{(3/2-\kappa_1-\kappa_2-y)_k}{(2-\kappa_1+x+y)_k} \\
&\times\Theta_2(n,k,\kappa_1,\kappa_2,\kappa_1+\varrho_1,\kappa_2+\varrho_2,-ix,-iy)
\end{aligned}
$$

has an orthogonality relation of form

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\Gamma(\kappa_2-ix+1/2)\Gamma(\kappa_1+iy-1/2)\Gamma(\kappa_2-iy+1/2)\Gamma(\kappa_1-i(x+y)-1)}{\Gamma(\kappa_1+\kappa_2+iy-1/2)} \\
&\times\frac{\Gamma(\varrho_2+ix+1/2)\Gamma(\varrho_2+iy+1/2)\Gamma(\varrho_1-iy-1/2)\Gamma(\varrho_1+i(x+y)-1)}{\Gamma(\varrho_2+\varrho_1-iy-1/2)} \\
&\times\ {}_2E_{n,k}(ix,iy;\kappa_1,\kappa_2,\varrho_2,\varrho_1)\ {}_2E_{r,s}(-ix,-iy;\varrho_1,\varrho_2,\kappa_2,\kappa_1)dxdy \\
&= \frac{4\pi^2(n-k)!k!\Gamma(\kappa_1+\varrho_1-n-k-1)\Gamma(\kappa_1+\varrho_1-k)}{(\kappa_1+\varrho_1-2n-2)(\kappa_1+\varrho_1-2k-1)\Gamma(\kappa_1+\kappa_2+\varrho_1+\varrho_2-n-k-1)} \\
&\times\frac{\Gamma^4(\kappa_2+\varrho_2+1)\Gamma(\kappa_1+\kappa_2)\Gamma(\varrho_2+\varrho_1)}{\Gamma(\kappa_1+\kappa_2+\varrho_1+\varrho_2-k)\Gamma(\kappa_2+\varrho_2+n-k+1)\Gamma(\kappa_2+\varrho_2+k+1)}\delta_{n,r}\delta_{k,s},
\end{aligned}
$$

for $\kappa_1,\varrho_1>n+1$ and $\kappa_2,\varrho_2>-1/2$. Please note that the weight function of this orthogonality relation is positive for $\kappa_1=\varrho_1$ and $\kappa_2=\varrho_2$.

3.3. Fourier Transform of the Polynomials ${}_3Q^{(p)}_{n,k}(x,y)$

Let us define

$$h_{n,k}(x,y;\kappa_1,\lambda) = \exp\left[\frac{x}{2} - \left(\kappa_1 - \frac{1}{2}\right)y - \frac{e^{-x}+e^{x-y}}{2}\right] {}_3Q^{(\lambda)}_{n,k}(e^x,e^y), \tag{45}$$

where κ_1 and λ are real parameters, and the polynomials ${}_3Q^{(p)}_{n,k}(x,y)$ are defined in (16). By using appropriate substitutions, we derive the Fourier transform of the function given above by taking into account the relations (41) as

$$\begin{aligned}
&\mathcal{F}(h_{n,k}(x,y;\kappa_1,\lambda)) \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(\xi_1 x+\xi_2 y)} e^{\frac{x}{2}-\left(\kappa_1-\frac{1}{2}\right)y-\frac{e^{-x}+e^{x-y}}{2}}\, {}_3Q^{(\lambda)}_{n,k}(e^x,e^y)dxdy \\
&= \left(\int_0^{\infty} u^{k-\kappa_1-i(\xi_1+\xi_2)} e^{-\frac{1}{2u}} N^{(\lambda-2k-1)}_{n-k}(u)du\right)\left(\int_0^{\infty} t^{-\left(\kappa_1+i\xi_2+\frac{1}{2}\right)} e^{-\frac{1}{2t}} N^{(\lambda)}_{k}(t)dt\right) \\
&= (-1)^n\left[\sum_{l_1=0}^{n-k}\frac{(-(n-k))_{l_1}(n+k+2-\lambda)_{l_1}}{(-1)^{l_1}l_1!}\left(\int_0^{\infty} u^{k-\kappa_1-i(\xi_1+\xi_2)+l_1} e^{-\frac{1}{2u}}du\right)\right] \\
&\times\left[\sum_{l_2=0}^{k}\frac{(-k)_{l_2}(k+1-\lambda)_{l_2}}{(-1)^{l_2}l_2!}\left(\int_0^{\infty} t^{-\left(\kappa_1+i\xi_2+\frac{1}{2}-l_2\right)} e^{-\frac{1}{2t}}dt\right)\right],
\end{aligned}$$

since

$$\int_0^{\infty} t^{-(p+is-k+1)} e^{-\frac{1}{2t}}dt = 2^{p+is-k}\Gamma(p+is-k), \tag{46}$$

we can conclude that

$$\mathcal{F}(h_{n,k}(x,y;\kappa_1,\lambda)) = (-1)^n C_3(k,\kappa_1,\xi_1,\xi_2)\Theta_3(n,k,\kappa_1,\lambda,\xi_1,\xi_2),$$

where

$$C_3(k,\kappa_1,\xi_1,\xi_2) = 2^{2\kappa_1-k+i(\xi_1+2\xi_2)-3/2}\Gamma(\kappa_1-k-1+i(\xi_1+\xi_2))\Gamma(\kappa_1-1/2+i\xi_2),$$

and

$$\Theta_3(n,k,\kappa_1,\lambda,\xi_1,\xi_2) = {}_2F_1\left(\begin{matrix}-(n-k),\ n+k+2-\lambda\\ k-\kappa_1+2-i(\xi_1+\xi_2)\end{matrix}\,\Big|\,\frac{1}{2}\right){}_2F_1\left(\begin{matrix}-k,\ k+1-\lambda\\ 3/2-\kappa_1-i\xi_2\end{matrix}\,\Big|\,\frac{1}{2}\right),$$

where ${}_2F_1$ is a special case of the hypergeometric function. Hence, from the Parseval identity (39), we obtain

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_{n,k}(x,y;\kappa_1,\lambda)h_{r,s}(x,y;\varrho_1,\alpha)dxdy \\
&= \int_0^{\infty}\int_0^{\infty} v^{-(\kappa_1+\varrho_1)} e^{-\left(\frac{1}{u}+\frac{u}{v}\right)}\, {}_3Q^{(\lambda)}_{n,k}(u,v)\, {}_3Q^{(\alpha)}_{r,s}(u,v)dudv \\
&= \frac{(-1)^{n+r}}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} C_3(k,\kappa_1,\xi_1,\xi_2)\overline{C_3(s,\varrho_1,\xi_1,\xi_2)} \\
&\times\Theta_3(n,k,\kappa_1,\lambda,\xi_1,\xi_2)\overline{\Theta_3(r,s,\varrho_1,\alpha,\xi_1,\xi_2)}d\xi_1 d\xi_2.
\end{aligned} \tag{47}$$

Now by taking $\kappa_1+\varrho_1=\lambda=\alpha$ in (47) and using the orthogonality relation (17) in the left-hand side of (47), we obtain

$$\begin{aligned}&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} C_3(k,\kappa_1,\xi_1,\xi_2)\overline{C_3(s,\varrho_1,\xi_1,\xi_2)}\\&\times\Theta_3(n,k,\kappa_1,\kappa_1+\varrho_1,\xi_1,\xi_2)\overline{\Theta_3(r,s,\varrho_1,\varrho_1+\kappa_1,\xi_1,\xi_2)}d\xi_1 d\xi_2\\&=\frac{4\pi^2(n-k)!k!\Gamma(\kappa_1+\varrho_1-n-k-1)\Gamma(\kappa_1+\varrho_1-k)}{(\kappa_1+\varrho_1-2n-2)(\kappa_1+\varrho_1-2k-1)}\delta_{n,r}\delta_{k,s}.\end{aligned}$$

Theorem 3. *The special function*

$$ {}_3E_{n,k}(x,y;\kappa_1,\varrho_1)=\frac{1}{(2-\kappa_1-(x+y))_k}\Theta_3(n,k,\kappa_1,\kappa_1+\varrho_1,-ix,-iy),$$

has an orthogonality relation of form

$$\begin{aligned}&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Gamma(\kappa_1-1+i(x+y))\Gamma(\varrho_1-1-i(x+y))\Gamma\left(\kappa_1-\frac{1}{2}+iy\right)\Gamma\left(\varrho_1-\frac{1}{2}-iy\right)\\&\times\ {}_3E_{n,k}(ix,iy;\kappa_1,\varrho_1)\ {}_3E_{r,s}(-ix,-iy;\varrho_1,\kappa_1)dxdy\\&=\frac{2^{1-2(\kappa_1+\varrho_1-k-2)}\pi^2(n-k)!k!\Gamma(\kappa_1+\varrho_1-n-k-1)\Gamma(\kappa_1+\varrho_1-k)}{(\kappa_1+\varrho_1-2n-2)(\kappa_1+\varrho_1-2k-1)}\delta_{n,r}\delta_{k,s},\end{aligned}$$

for $\kappa_1,\varrho_1>1$ and $\kappa_1+\varrho_1>2n+2$. Please note that the weight function is positive for $\kappa_1=\varrho_1$.

3.4. Fourier Transform of the Polynomials ${}_4Q_{n,k}^{(p,q)}(x,y)$

Let us define

$$\begin{aligned}h_{n,k}(x,y;\kappa_1,\kappa_2,\lambda,\mu)=\exp&\left[\left(\kappa_1+\kappa_2+\frac{1}{2}\right)x-\kappa_1 y-\frac{e^{x-y}}{2}\right]\\&\times(e^x+1)^{-(\kappa_1+\kappa_2)}\ {}_4Q_{n,k}^{(\lambda,\mu)}(e^x,e^y),\end{aligned}\tag{48}$$

where $\kappa_1,\kappa_2,\lambda$ and μ are real parameters, and the polynomials ${}_4Q_{n,k}^{(p,q)}(x,y)$ are defined by (18). If we apply similar calculations as in the first function family for finding the Fourier transform of the function given by (48), by considering the relations (41) and (46), we have

$$\begin{aligned}\mathcal{F}(h_{n,k}(x,y;\kappa_1,\kappa_2,\lambda,\mu))&=(-1)^n\frac{\Gamma(\mu+n+k+2)}{\Gamma(\mu+2k+2)}\left[\sum_{l_1=0}^{n-k}\frac{(-(n-k))_{l_1}}{(-1)^{l_1}l_1!}\right.\\&\left.\times\frac{(n+k+2-\lambda)_{l_1}}{(\mu+2k+2)_{l_1}}\frac{\Gamma\left(k+\kappa_2+\frac{1}{2}-i(\xi_1+\xi_2)+l_1\right)\Gamma\left(\kappa_1-k-\frac{1}{2}+i(\xi_1+\xi_2)-l_1\right)}{\Gamma(\kappa_1+\kappa_2)}\right]\\&\times\left[\sum_{l_2=0}^{k}\frac{(-k)_{l_2}(k+1-\lambda)_{l_2}}{(-1)^{l_2}l_2!}2^{\kappa_1+i\xi_2-l_2}\Gamma(\kappa_1+i\xi_2-l_2)\right],\end{aligned}$$

and we can conclude that

$$\begin{aligned}\mathcal{F}(h_{n,k}(x,y;\kappa_1,\kappa_2,\lambda,\mu))=&\frac{(-1)^n\Gamma(\mu+n+k+2)}{\Gamma(\mu+2k+2)\Gamma(\kappa_1+\kappa_2)}\\&\times C_4(k,\kappa_1,\kappa_2,\xi_1,\xi_2)\Theta_4(n,k,\kappa_1,\kappa_2,\lambda,\mu,\xi_1,\xi_2),\end{aligned}\tag{49}$$

where

$$C_4(k,\kappa_1,\kappa_2,\xi_1,\xi_2) = 2^{\kappa_1+i\xi_2}\Gamma(\kappa_1+i\xi_2) \\ \times\Gamma(\kappa_1-k-1/2+i(\xi_1+\xi_2))\Gamma(\kappa_2+k+1/2-i(\xi_1+\xi_2)),$$

and

$$\Theta_4(n,k,\kappa_1,\kappa_2,\lambda,\mu,\xi_1,\xi_2) = {}_2F_1\left(\begin{matrix}-k,\ k+1-\lambda \\ 1-\kappa_1-i\xi_2\end{matrix}\ \Big|\ \frac{1}{2}\right) \\ \times {}_3F_2\left(\begin{matrix}-(n-k),\ n+k+2-\lambda,\ \kappa_2+k+\frac{1}{2}-i(\xi_1+\xi_2) \\ \mu+2k+2,\ k-\kappa_1+3/2-i(\xi_1+\xi_2)\end{matrix}\ \Big|\ 1\right).$$

Hence, from the Parseval identity (39), we write

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_{n,k}(x,y;\kappa_1,\kappa_2,\lambda,\mu)h_{r,s}(x,y;\varrho_1,\varrho_2,\alpha,\beta)dxdy \\ = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathcal{F}(h_{n,k}(x,y;\kappa_1,\kappa_2,\lambda,\mu))\overline{\mathcal{F}(h_{r,s}(x,y;\varrho_1,\varrho_2,\alpha,\beta))}d\xi_1 d\xi_2$$

and thus, by using result (49) and definition (48), and then applying substitutions $e^x = u, e^y = v$, we obtain

$$\int_0^{\infty}\int_0^{\infty} u^{\kappa_1+\kappa_2+\varrho_1+\varrho_2}v^{-(\kappa_1+\varrho_1+1)}e^{-\frac{u}{v}}(u+1)^{-(\kappa_1+\kappa_2+\varrho_1+\varrho_2)} \\ \times\ {}_4Q_{n,k}^{(\lambda,\mu)}(u,v)\ {}_4Q_{r,s}^{(\alpha,\beta)}(u,v)dudv \\ = \frac{(-1)^{n+r}}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\Gamma(\mu+n+k+2)\Gamma(\beta+r+s+2)C_4(k,\kappa_1,\kappa_2,\xi_1,\xi_2)}{\Gamma(\mu+2k+2)\Gamma(\beta+2s+2)\Gamma(\kappa_1+\kappa_2)\Gamma(\varrho_1+\varrho_2)} \\ \times\overline{C_4(s,\varrho_1,\varrho_2,\xi_1,\xi_2)}\Theta_4(n,k,\kappa_1,\kappa_2,\lambda,\mu,\xi_1,\xi_2) \\ \times\overline{\Theta_4(r,s,\varrho_1,\varrho_2,\alpha,\beta,\xi_1,\xi_2)}d\xi_1 d\xi_2. \tag{50}$$

Now by taking $\kappa_1+\varrho_1+1=\lambda=\alpha$ and $\kappa_2+\varrho_2-1=\mu=\beta$ in the left hand side of (50), then according to the orthogonality relation (19), we obtain

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\overline{\Theta_4(r,s,\varrho_1,\varrho_2,\varrho_1+\kappa_1+1,\varrho_2+\kappa_2-1,\xi_1,\xi_2)} \\ \times\Theta_4(n,k,\kappa_1,\kappa_2,\kappa_1+\varrho_1+1,\kappa_2+\varrho_2-1,\xi_1,\xi_2) \\ \times C_4(k,\kappa_1,\kappa_2,\xi_1,\xi_2)\overline{C_4(s,\varrho_1,\varrho_2,\xi_1,\xi_2)}d\xi_1 d\xi_2 \\ = \frac{4\pi^2(n-k)!k!\Gamma(\kappa_1+\varrho_1-n-k)\Gamma(\kappa_1+\varrho_1-k+1)}{(\kappa_1+\varrho_1-2n-1)(\kappa_1+\varrho_1-2k)\Gamma(\kappa_1+\kappa_2+\varrho_1+\varrho_2-(n-k))} \\ \times\frac{\Gamma^2(\kappa_2+\varrho_2+2k+1)\Gamma(\kappa_1+\kappa_2)\Gamma(\varrho_1+\varrho_2)}{\Gamma(\kappa_2+\varrho_2+n+k+1)}\delta_{n,r}\delta_{k,s}.$$

Theorem 4. *The special function*

$${}_4E_{n,k}(x,y;\kappa_1,\kappa_2,\varrho_2,\varrho_1) = \frac{(\kappa_2+1/2-(x+y))_k}{(3/2-\kappa_1-(x+y))_k} \\ \times\Theta_4(n,k,\kappa_1,\kappa_2,\kappa_1+\varrho_1+1,\kappa_2+\varrho_2-1,-ix,-iy),$$

has an orthogonality relation in form

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Gamma(\kappa_1-1/2+i(x+y))\Gamma(\kappa_2+1/2-i(x+y))\Gamma(\varrho_2+1/2+i(x+y))$$
$$\times\Gamma(\varrho_1-1/2-i(x+y))\Gamma(\kappa_1+iy)\Gamma(\varrho_1-iy)$$
$$\times\ {}_4E_{n,k}(ix,iy;\kappa_1,\kappa_2,\varrho_2,\varrho_1)\ {}_4E_{r,s}(-ix,-iy;\varrho_1,\varrho_2,\kappa_2,\kappa_1)dydx$$
$$=\frac{2^{2-(\kappa_1+\varrho_1)}\pi^2(n-k)!k!\Gamma(\kappa_1+\varrho_1-n-k)\Gamma(\kappa_1+\varrho_1-k+1)}{(\kappa_1+\varrho_1-2n-1)(\kappa_1+\varrho_1-2k)\Gamma(\kappa_1+\kappa_2+\varrho_1+\varrho_2-(n-k))}$$
$$\times\frac{\Gamma^2(\kappa_2+\varrho_2+2k+1)\Gamma(\kappa_1+\kappa_2)\Gamma(\varrho_1+\varrho_2)}{\Gamma(\kappa_2+\varrho_2+n+k+1)}\delta_{n,r}\delta_{k,s},$$

for $\kappa_1,\varrho_1>1/2$, $\kappa_2,\varrho_2>-1/2$ *and* $\kappa_1+\varrho_1>2n+1$. *Please note that the weight function is positive for* $\kappa_1=\varrho_1$ *and* $\kappa_2=\varrho_2$.

3.5. Fourier Transform of the Polynomials ${}_5Q_{n,k}^{(p,q)}(x,y)$

Let us define

$$h_{n,k}(x,y;\kappa_1,\kappa_2,\lambda,\mu)=e^{\kappa_2x-\frac{1+e^x}{2e^y}}(1+e^x)^{-\left(\kappa_1+\kappa_2-\frac{1}{2}\right)}\left(\frac{e^y}{1+e^x}\right)^{-\kappa_1}{}_5Q_{n,k}^{(\lambda,\mu)}(e^x,e^y), \tag{51}$$

where $\kappa_1,\kappa_2,\lambda$ and μ are real parameters, and the polynomials ${}_5Q_{n,k}^{(p,q)}(x,y)$ are defined in (20). By using the substitutions $e^x=u$, $e^y=v$ and $\frac{v}{1+u}=t$, respectively, and applying the identities (41) and (46), we get

$$\mathcal{F}(h_{n,k}(x,y;\kappa_1,\kappa_2,\lambda,\mu))$$
$$=(-1)^n\frac{\Gamma(\mu+n-k+1)}{\Gamma(\mu+1)}\left[\sum_{l_1=0}^{n-k}\frac{(-(n-k))_{l_1}(n+k+2-\lambda)_{l_1}}{(\mu+1)_{l_1}(-1)^{l_1}l_1!}\right.$$
$$\left.\times\frac{\Gamma(\kappa_2-i\xi_1+l_1)\Gamma\left(\kappa_1-k+i(\xi_1+\xi_2)-\frac{1}{2}-l_1\right)}{\Gamma\left(\kappa_1+\kappa_2-k+i\xi_2-\frac{1}{2}\right)}\right]$$
$$\times\left[\sum_{l_2=0}^{k}\frac{(-k)_{l_2}(1-\lambda+k)_{l_2}}{(-1)^{l_2}l_2!}2^{\kappa_1+i\xi_2-l_2}\Gamma(\kappa_1+i\xi_2-l_2)\right].$$

Therefore, we can conclude that

$$\mathcal{F}(h_{n,k}(x,y;\kappa_1,\kappa_2,\lambda,\mu))=\frac{(-1)^n\Gamma(\mu+n-k+1)}{\Gamma(\mu+1)}$$
$$\times C_5(k,\kappa_1,\kappa_2,\xi_1,\xi_2)\Theta_5(n,k,\kappa_1,\kappa_2,\lambda,\mu,\xi_1,\xi_2),$$

where

$$C_5(k,\kappa_1,\kappa_2,\xi_1,\xi_2)=\frac{2^{\kappa_1+i\xi_2}\Gamma(\kappa_1+i\xi_2)\Gamma(\kappa_1-k+i(\xi_1+\xi_2)-1/2)\Gamma(\kappa_2-i\xi_1)}{\Gamma(\kappa_1+\kappa_2-k+i\xi_2-1/2)},$$

and

$$\Theta_5(n,k,\kappa_1,\kappa_2,\lambda,\mu,\xi_1,\xi_2)=\ {}_3F_2\left(\begin{matrix}-(n-k),\ n+k+2-\lambda,\ \kappa_2-i\xi_1\\ \mu+1,\ k-\kappa_1+3/2-i(\xi_1+\xi_2)\end{matrix}\ \Big|\ 1\right)$$
$$\times\ {}_2F_1\left(\begin{matrix}-k,\ k+1-\lambda\\ 1-\kappa_1-i\xi_2\end{matrix}\ \Big|\ \frac{1}{2}\right).$$

Hence, from the Parseval identity (39), we obtain

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_{n,k}(x,y;\kappa_1,\kappa_2,\lambda,\mu)h_{r,s}(x,y;\varrho_2,\varrho_1,\alpha,\beta)dydx \tag{52}$$
$$= \int_0^{\infty}\int_0^{\infty} u^{\kappa_2+\varrho_2-1}(1+u)^{-(\kappa_2+\varrho_2-1)}v^{-(\kappa_1+\varrho_1+1)}e^{-\frac{1+u}{v}}$$
$$\times\ {}_5Q_{n,k}^{(\lambda,\mu)}(u,v)\ {}_5Q_{r,s}^{(\alpha,\beta)}(u,v)dvdu$$
$$= \frac{(-1)^{n+r}}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\Gamma(\mu+n-k+1)\Gamma(\beta+r-s+1)}{\Gamma(\mu+1)\Gamma(\beta+1)}C_5(k,\kappa_1,\kappa_2,\xi_1,\xi_2)$$
$$\times\ \overline{C_5(s,\varrho_1,\varrho_2,\xi_1,\xi_2)}\Theta_5(n,k,\kappa_1,\kappa_2,\lambda,\mu,\xi_1,\xi_2)\overline{\Theta_5(r,s,\varrho_1,\varrho_2,\alpha,\beta,\xi_1,\xi_2)}d\xi_1 d\xi_2.$$

Now by taking $\kappa_2+\varrho_2-1=\mu=\beta$ and $\kappa_1+\varrho_1+1=\lambda=\alpha$ in (52) and using the orthogonality relation (21) in the left-hand side of (52), we obtain

$$\frac{4\pi^2(n-k)!k!\Gamma(\kappa_1+\varrho_1-n-k)\Gamma(\kappa_1+\varrho_1-k+1)}{(\kappa_1+\varrho_1-2n-1)(\kappa_1+\varrho_1-2k)\Gamma(\kappa_1+\kappa_2+\varrho_1+\varrho_2-n-k-1)}$$
$$\times\ \frac{\Gamma^2(\kappa_2+\varrho_2)}{\Gamma(\kappa_2+\varrho_2+n-k)}\delta_{n,r}\delta_{k,s}$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\overline{C_5(s,\varrho_1,\varrho_2,\xi_1,\xi_2)\Theta_5(r,s,\varrho_1,\varrho_2,\varrho_1+\kappa_1+1,\varrho_2+\kappa_2-1,\xi_1,\xi_2)}$$
$$\times\ C_5(k,\kappa_1,\kappa_2,\xi_1,\xi_2)\Theta_5(n,k,\kappa_1,\kappa_2,\kappa_1+\varrho_1+1,\kappa_2+\varrho_2-1,\xi_1,\xi_2)d\xi_1 d\xi_2.$$

Theorem 5. *The special function*

$$ {}_5E_{n,k}(x,y;\kappa_1,\kappa_2,\varrho_2,\varrho_1) = \frac{(3/2-\kappa_1-\kappa_2-y)_k}{(3/2-\kappa_1-(x+y))_k}$$
$$\times\Theta_5(n,k,\kappa_1,\kappa_2,\kappa_1+\varrho_1+1,\kappa_2+\varrho_2-1,-ix,-iy),$$

has an orthogonality relation of form

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\Gamma(\kappa_1+i(x+y)-1/2)\Gamma(\varrho_1-i(x+y)-1/2)}{\Gamma(\kappa_1+\kappa_2+iy-1/2)\Gamma(\varrho_1+\varrho_2-iy-1/2)}$$
$$\times\ \Gamma(\kappa_1+iy)\Gamma(\varrho_1-iy)\Gamma(\kappa_2-ix)\Gamma(\varrho_2+ix)$$
$$\times\ {}_5E_{n,k}(ix,iy;\kappa_1,\kappa_2,\varrho_2,\varrho_1)$$
$$\times\ {}_5E_{r,s}(-ix,-iy;\varrho_1,\varrho_2,\kappa_2,\kappa_1)dxdy$$
$$= \frac{2^{2-(\kappa_1+\varrho_1)}\pi^2(n-k)!k!\Gamma(\kappa_1+\varrho_1-n-k)}{(\kappa_1+\varrho_1-2n-1)\Gamma(\kappa_1+\kappa_2+\varrho_1+\varrho_2-n-k-1)}$$
$$\times\ \frac{\Gamma(\kappa_1+\varrho_1-k+1)\Gamma^2(\kappa_2+\varrho_2)}{(\kappa_1+\varrho_1-2k)\Gamma(\kappa_2+\varrho_2+n-k)}\delta_{n,r}\delta_{k,s},$$

for $\kappa_1,\varrho_1>n+1/2$ and $\kappa_2,\varrho_2>0$. Please note that the weight function of the orthogonality relation is positive for $\kappa_1=\varrho_1$ and $\kappa_2=\varrho_2$.

3.6. Fourier Transform of the Polynomials ${}_6Q_{n,k}^{(p,q)}(x,y)$

Let us define

$$h_{n,k}(x,y;\kappa_1,\kappa_2,\lambda,\mu) = \left(1+e^{y-x}\right)^{-(\kappa_1+\kappa_2)} \exp\left[\kappa_2 y - \left(\kappa_1+\kappa_2-\frac{1}{2}\right)x - \frac{e^{-x}}{2}\right] \times {}_6Q_{n,k}^{(\lambda,\mu)}(e^x,e^y), \tag{53}$$

where $\kappa_1, \kappa_2, \lambda$ and μ are real parameters, and the polynomials ${}_6Q_{n,k}^{(p,q)}(x,y)$ are defined in (22). If we apply the Fourier transform for the function $h_{n,k}(x,y;\kappa_1,\kappa_2,\lambda,\mu)$ given by (53), we get

$$\begin{aligned}
\mathcal{F}(h_{n,k}(x,y;\kappa_1,\kappa_2,\lambda,\mu)) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(\xi_1 x+\xi_2 y)} e^{\kappa_2 y-\left(\kappa_1+\kappa_2-\frac{1}{2}\right)x-\frac{e^{-x}}{2}} \\
&\quad \times \left(1+e^{y-x}\right)^{-(\kappa_1+\kappa_2)} {}_6Q_{n,k}^{(\lambda,\mu)}(e^x,e^y)dxdy \\
&= \left(\int_0^{\infty} u^{-\left(\kappa_1-k+i(\xi_1+\xi_2)+\frac{1}{2}\right)} e^{-\frac{1}{2u}} N_{n-k}^{(\lambda-2k-1)}(u)du\right) \\
&\quad \times \left(\int_0^{\infty} t^{\kappa_2-i\xi_2-1}(1+t)^{-(\kappa_1+\kappa_2)} M_k^{(\lambda,\mu)}(t)dt\right) \\
&= \frac{(-1)^n\Gamma(\mu+k+1)}{\Gamma(\mu+1)}\left[\sum_{l_1=0}^{n-k}\frac{(-(n-k))_{l_1}(n+k+2-\lambda)_{l_1}}{(-1)^{l_1}l_1!}\right. \\
&\quad \left.\times\left(\int_0^{\infty} u^{-\left(\kappa_1-k+i(\xi_1+\xi_2)+\frac{1}{2}-l_1\right)} e^{-\frac{1}{2u}}du\right)\right] \\
&\quad \times\left[\sum_{l_2=0}^{k}\frac{(-k)_{l_2}(k+1-\lambda)_{l_2}}{(\mu+1)_{l_2}(-1)^{l_2}l_2!}\left(\int_0^{\infty} t^{\kappa_2-i\xi_2-1+l_2}(1+t)^{-(\kappa_1+\kappa_2)}dt\right)\right],
\end{aligned}$$

by using the relations (41) and (46), we can conclude that

$$\begin{aligned}
\mathcal{F}(h_{n,k}(x,y;\kappa_1,\kappa_2,\lambda,\mu)) &= \frac{(-1)^n\Gamma(k+\mu+1)}{\Gamma(\mu+1)\Gamma(\kappa_1+\kappa_2)} \\
&\quad \times C_6(k,\kappa_1,\kappa_2,\xi_1,\xi_2)\Theta_6(n,k,\kappa_1,\kappa_2,\lambda,\mu,\xi_1,\xi_2),
\end{aligned}$$

where

$$\begin{aligned}
C_6(k,\kappa_1,\kappa_2,\xi_1,\xi_2) &= 2^{\kappa_1-k+i(\xi_1+\xi_2)-\frac{1}{2}}\Gamma(\kappa_1-k-1/2+i(\xi_1+\xi_2)) \\
&\quad \times\Gamma(\kappa_1+i\xi_2)\Gamma(\kappa_2-i\xi_2),
\end{aligned}$$

and

$$\begin{aligned}
\Theta_6(n,k,\kappa_1,\kappa_2,\lambda,\mu,\xi_1,\xi_2) &= {}_2F_1\left(\begin{matrix}-(n-k),\ n+k+2-\lambda \\ k-\kappa_1+3/2-i(\xi_1+\xi_2)\end{matrix}\ \Big|\ \frac{1}{2}\right) \\
&\quad \times {}_3F_2\left(\begin{matrix}-k,\ k+1-\lambda,\ \kappa_2-i\xi_2 \\ \mu+1,\ 1-\kappa_1-i\xi_2\end{matrix}\ \Big|\ 1\right).
\end{aligned}$$

Hence, from the Parseval identity (39), we obtain

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_{n,k}(x,y;\kappa_1,\kappa_2,\lambda,\mu)h_{r,s}(x,y;\varrho_1,\varrho_2,\alpha,\beta)dydx \\
&=\int_{0}^{\infty}\int_{0}^{\infty} v^{\kappa_2+\varrho_2-1}(u+v)^{-(\kappa_1+\kappa_2+\varrho_1+\varrho_2)}e^{-1/u}\,{}_6Q_{n,k}^{(\lambda,\mu)}(u,v)\,{}_6Q_{r,s}^{(\alpha,\beta)}(u,v)dvdu \\
&=\frac{(-1)^{n+r}}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\Gamma(k+\mu+1)\Gamma(s+\beta+1)}{\Gamma(\mu+1)\Gamma(\beta+1)\Gamma(\kappa_1+\kappa_2)\Gamma(\varrho_1+\varrho_2)} \\
&\times C_6(k,\kappa_1,\kappa_2,\xi_1,\xi_2)\overline{C_6(s,\varrho_1,\varrho_2,\xi_1,\xi_2)} \\
&\times \Theta_6(n,k,\kappa_1,\kappa_2,\lambda,\mu,\xi_1,\xi_2)\overline{\Theta_6(r,s,\varrho_1,\varrho_2,\alpha,\beta,\xi_1,\xi_2)}d\xi_1 d\xi_2.
\end{aligned} \tag{54}$$

Now by taking $\kappa_2+\varrho_2-1=\mu=\beta$ and $\kappa_1+\varrho_1+1=\lambda=\alpha$ in (54) and then use the orthogonality relation (23) in the left-hand side of (54), we obtain

$$\begin{aligned}
&\frac{4\pi^2(n-k)!k!\Gamma(\kappa_1+\varrho_1-n-k)\Gamma(\kappa_1+\varrho_1-k+1)}{(\kappa_1+\varrho_1-2n-1)(\kappa_1+\varrho_1-2k)} \\
&\times\frac{\Gamma^2(\kappa_2+\varrho_2)\Gamma(\kappa_1+\kappa_2)\Gamma(\varrho_2+\varrho_1)}{\Gamma(\kappa_1+\kappa_2+\varrho_1+\varrho_2-k)\Gamma(\kappa_2+\varrho_2+k)}\delta_{n,r}\delta_{k,s} \\
&=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\overline{C_6(s,\varrho_1,\varrho_2,\xi_1,\xi_2)}C_6(k,\kappa_1,\kappa_2,\xi_1,\xi_2) \\
&\times\overline{\Theta_6(r,s,\varrho_1,\varrho_2,\varrho_1+\kappa_1+1,\varrho_2+\kappa_2-1,\xi_1,\xi_2)} \\
&\times\Theta_6(n,k,\kappa_1,\kappa_2,\kappa_1+\varrho_1+1,\kappa_2+\varrho_2-1,\xi_1,\xi_2)d\xi_1 d\xi_2.
\end{aligned}$$

Theorem 6. *The special function*

$$\begin{aligned}
{}_6E_{n,k}(x,y;\kappa_1,\kappa_2,\varrho_2,\varrho_1) &= \frac{1}{(3/2-\kappa_1-(x+y))_k} \\
&\times\Theta_6(n,k,\kappa_1,\kappa_2,\kappa_1+\varrho_1+1,\kappa_2+\varrho_2-1,-ix,-iy),
\end{aligned}$$

has an orthogonality relation in form

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Gamma\left(\kappa_1+i(x+y)-\frac{1}{2}\right)\Gamma\left(\varrho_1-i(x+y)-\frac{1}{2}\right) \\
&\times\Gamma(\kappa_1+iy)\Gamma(\varrho_1-iy)\Gamma(\kappa_2-iy)\Gamma(\varrho_2+iy) \\
&\times\,{}_6E_{n,k}(ix,iy;\kappa_1,\kappa_2,\varrho_2,\varrho_1)\,{}_6E_{r,s}(-ix,-iy;\varrho_1,\varrho_2,\kappa_2,\kappa_1)dydx \\
&=\frac{2^{2k+3-(\kappa_1+\varrho_1)}\pi^2(n-k)!k!\Gamma(\kappa_1+\varrho_1-n-k)\Gamma(\kappa_1+\varrho_1-k+1)}{(\kappa_1+\varrho_1-2n-1)(\kappa_1+\varrho_1-2k)} \\
&\times\frac{\Gamma^2(\kappa_2+\varrho_2)\Gamma(\kappa_1+\kappa_2)\Gamma(\varrho_1+\varrho_2)}{\Gamma(\kappa_1+\kappa_2+\varrho_1+\varrho_2-k)\Gamma(\kappa_2+\varrho_2+k)}\delta_{n,r}\delta_{k,s},
\end{aligned}$$

for $\kappa_1,\varrho_1>1/2,\ \kappa_2,\varrho_2>0$ *and* $\kappa_1+\varrho_1>2n+1$. *Please note that the weight function is positive for* $\kappa_1=\varrho_1$ *and* $\kappa_2=\varrho_2$.

3.7. Fourier Transform of the Polynomials ${}_7Q_{n,k}^{(p,q,u,v)}(x,y)$

Let us define

$$h_{n,k}(x,y;\kappa_1,\kappa_2,\kappa_3,\kappa_4,\lambda,\mu,\eta,\tau)=(1+e^x)^{-(\kappa_1+\kappa_2)}(1+e^y)^{-(\kappa_3+\kappa_4)} \times \exp(\kappa_2 x+\kappa_4 y)\,{}_7Q_{n,k}^{(\lambda,\mu,\eta,\tau)}(e^x,e^y), \tag{55}$$

where $\kappa_1,\ \kappa_2,\ \kappa_3,\ \kappa_4,\ \lambda,\ \mu,\ \eta,\ \tau$ are real parameters. Similar to the results in the paper [21], we get

$$\begin{aligned}
&\mathcal{F}(h_{n,k}(x,y;\kappa_1,\kappa_2,\kappa_3,\kappa_4,\lambda,\mu,\eta,\tau))\\
&=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(\xi_1 x+\xi_2 y)}e^{\kappa_2 x+\kappa_4 y}(1+e^x)^{-(\kappa_1+\kappa_2)}(1+e^y)^{-(\kappa_3+\kappa_4)}\,{}_7Q_{n,k}^{(\lambda,\mu,\eta,\tau)}(e^x,e^y)dxdy\\
&=\frac{(-1)^n\Gamma(\mu+n-k+1)\Gamma(\tau+k+1)}{\Gamma(\mu+1)\Gamma(\tau+1)}C_7(\kappa_1,\kappa_2,\kappa_3,\kappa_4,\xi_1,\xi_2)\\
&\times\Theta_7(n,k,\kappa_1,\kappa_2,\kappa_3,\kappa_4,\lambda,\mu,\eta,\tau,\xi_1,\xi_2),
\end{aligned}$$

where

$$C_7(\kappa_1,\kappa_2,\kappa_3,\kappa_4,\xi_1,\xi_2)=\frac{\Gamma(\kappa_1+i\xi_1)\Gamma(\kappa_2-i\xi_1)\Gamma(\kappa_3+i\xi_2)\Gamma(\kappa_4-i\xi_2)}{\Gamma(\kappa_1+\kappa_2)\Gamma(\kappa_3+\kappa_4)},$$

and

$$\begin{aligned}
\Theta_7(n,k,\kappa_1,\kappa_2,\kappa_3,\kappa_4,\lambda,\mu,\eta,\tau,\xi_1,\xi_2)=&\ {}_3F_2\left(\begin{matrix}-k,\ k+1-\eta,\ \kappa_4-i\xi_2\\ \tau+1,\ 1-\kappa_3-i\xi_2\end{matrix}\ \middle|\ 1\right)\\
&\times {}_3F_2\left(\begin{matrix}-(n-k),\ n-k+1-\lambda,\ \kappa_2-i\xi_1\\ \mu+1,\ 1-\kappa_1-i\xi_1\end{matrix}\ \middle|\ 1\right).
\end{aligned}$$

Hence, from the Parseval identity (39) and then using the orthogonality relation of ${}_7Q_{n,k}^{(p,q,u,v)}(x,y)$ given by (24), after the necessary arrangements, we can give the next theorem:

Theorem 7. *The special function*

$$\begin{aligned}
{}_7E_{n,k}(x,y;\kappa_1,\kappa_2,\kappa_3,\kappa_4,\varrho_4,\varrho_3,\varrho_2,\varrho_1)=&\frac{\Gamma(\kappa_2+\varrho_2+n-k)\Gamma(\kappa_4+\varrho_4+k)}{\Gamma(\kappa_2+\varrho_2)\Gamma(\kappa_4+\varrho_4)}\\
\times\Theta_7(n,k,\kappa_1,\kappa_2,\kappa_3,\kappa_4,\kappa_1+\varrho_1+1,\kappa_2+\varrho_2-1,&\kappa_3+\varrho_3+1,\kappa_4+\varrho_4-1,-ix,-iy),
\end{aligned}$$

has an orthogonality relation of form

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Gamma(\kappa_1+ix)\Gamma(\kappa_2-ix)\Gamma(\kappa_3+iy)\Gamma(\kappa_4-iy)\Gamma(\varrho_1-ix)\Gamma(\varrho_2+ix)\\
&\times\Gamma(\varrho_3-iy)\Gamma(\varrho_4+iy)\,{}_7E_{n,k}(ix,iy;\kappa_1,\kappa_2,\kappa_3,\kappa_4,\varrho_4,\varrho_3,\varrho_2,\varrho_1)\\
&\times\ {}_7E_{r,s}(-ix,-iy;\varrho_1,\varrho_2,\varrho_3,\varrho_4,\kappa_4,\kappa_3,\kappa_2,\kappa_1)dxdy\\
&=\frac{4\pi^2(n-k)!k!\Gamma(\kappa_1+\varrho_1+1-(n-k))\Gamma(\kappa_3+\varrho_3+1-k)\Gamma(n-k+\kappa_2+\varrho_2)}{(\kappa_1+\varrho_1-2(n-k))(\kappa_3+\varrho_3-2k)\Gamma(\kappa_1+\kappa_2+\varrho_1+\varrho_2-(n-k))}\\
&\times\frac{\Gamma(k+\kappa_4+\varrho_4)\Gamma(\kappa_1+\kappa_2)\Gamma(\kappa_3+\kappa_4)\Gamma(\varrho_1+\varrho_2)\Gamma(\varrho_3+\varrho_4)}{\Gamma(\kappa_3+\kappa_4+\varrho_3+\varrho_4-k)}\delta_{n,r}\delta_{k,s},
\end{aligned}$$

where $\kappa_1,\kappa_2,\kappa_3,\kappa_4,\varrho_1,\varrho_2,\varrho_3,\varrho_4>0,\ \kappa_3+\varrho_3>2n$ *and* $\kappa_1+\varrho_1>2n$. *Please note that the weight function of this orthogonality relation is positive for* $\kappa_1=\varrho_1,\ \kappa_2=\varrho_2,\ \kappa_3=\varrho_3,\ \kappa_4=\varrho_4$ *or* $\kappa_1=\kappa_2,\ \varrho_1=\varrho_2,\ \kappa_3=\kappa_4,\ \varrho_3=\varrho_4$.

3.8. Fourier Transform of the Polynomials ${}_8Q_{n,k}^{(p,q)}(x,y)$

Let us define

$$h_{n,k}(x,y;\kappa_1,\kappa_2,\lambda,\mu)=\exp\left(-\kappa_1 x-\kappa_2 y-\frac{e^{-x}+e^{-y}}{2}\right){}_8Q_{n,k}^{(\lambda,\mu)}(e^x,e^y), \tag{56}$$

where $\kappa_1,\kappa_2,\lambda$ and μ are real parameters, and the polynomials ${}_8Q_{n,k}^{(p,q)}(x,y)$ are defined in (26). As a result of the calculations in the paper [21], we get

$$\begin{aligned}\mathcal{F}(h_{n,k}(x,y;\kappa_1,\kappa_2,\lambda,\mu)) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(\xi_1 x+\xi_2 y)}e^{-\kappa_1 x-\kappa_2 y-\frac{e^{-x}+e^{-y}}{2}}\,{}_8Q_{n,k}^{(\lambda,\mu)}(e^x,e^y)dxdy\\ &=(-1)^n C_8(\kappa_1,\kappa_2,\xi_1,\xi_2)\Theta_8(n,k,\kappa_1,\kappa_2,\lambda,\mu,\xi_1,\xi_2),\end{aligned}$$

where

$$C_8(\kappa_1,\kappa_2,\xi_1,\xi_2)=2^{\kappa_1+\kappa_2+i(\xi_1+\xi_2)}\Gamma(\kappa_1+i\xi_1)\Gamma(\kappa_2+i\xi_2),$$

and

$$\begin{aligned}\Theta_8(n,k,\kappa_1,\kappa_2,\lambda,\mu,\xi_1,\xi_2)=&\ {}_2F_1\left(\begin{matrix}-(n-k),\ n-k+1-\lambda\\ 1-\kappa_1-i\xi_1\end{matrix}\ \Big|\ \frac{1}{2}\right)\\ &\times\ {}_2F_1\left(\begin{matrix}-k,\ k+1-\mu\\ 1-\kappa_2-i\xi_2\end{matrix}\ \Big|\ \frac{1}{2}\right).\end{aligned}$$

By using the Parseval's identity (39) and the orthogonality relation (27) for the polynomials ${}_8Q_{n,k}^{(\lambda,\mu)}(x,y)$, we can give next result.

Theorem 8. *The special function*

$${}_8E_{n,k}(x,y;\kappa_1,\kappa_2,\varrho_2,\varrho_1)=\Theta_8(n,k,\kappa_1,\kappa_2,\kappa_1+\varrho_1+1,\kappa_2+\varrho_2+1,-ix,-iy),$$

has an orthogonality relation as follow

$$\begin{aligned}&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Gamma(\kappa_1+ix)\Gamma(\kappa_2+iy)\Gamma(\varrho_1-ix)\Gamma(\varrho_2-iy)\\ &\times\ {}_8E_{n,k}(ix,iy;\kappa_1,\kappa_2,\varrho_2,\varrho_1)\,{}_8E_{r,s}(-ix,-iy;\varrho_1,\varrho_2,\kappa_2,\kappa_1)dxdy\\ &=\frac{\pi^2(n-k)!k!\Gamma(\kappa_1+\varrho_1-n+k+1)\Gamma(\kappa_2+\varrho_2+1-k)}{2^{\kappa_1+\kappa_2+\varrho_1+\varrho_2-2}(\kappa_1+\varrho_1-2(n-k))(\kappa_2+\varrho_2-2k)}\delta_{n,r}\delta_{k,s},\end{aligned}$$

where $\kappa_1,\kappa_2,\varrho_1,\varrho_2>0$, $\kappa_1+\varrho_1>2n$ and $\kappa_2+\varrho_2>2n$. Please note that the weight function of this orthogonality relation is positive for $\kappa_1=\varrho_1$ and $\kappa_2=\varrho_2$.

3.9. Fourier Transform of the Polynomials ${}_9Q_{n,k}^{(p,q,u)}(x,y)$

Let us define

$$h_{n,k,}(x,y;\kappa_1,\kappa_2,\kappa_3,\lambda,\mu,\eta)=e^{\kappa_2 x-\kappa_3 y-\frac{e^{-y}}{2}}(1+e^x)^{-(\kappa_1+\kappa_2)}\,{}_9Q_{n,k}^{(\lambda,\mu,\eta)}(e^x,e^y), \tag{57}$$

where $\kappa_1,\kappa_2,\kappa_3,\lambda,\mu$ and η are real parameters, the polynomials ${}_9Q_{n,k}^{(p,q,u)}(x,y)$ are defined by (28). In view of the calculations in the paper [21], we get

$$\begin{aligned}
&\mathcal{F}(h_{n,k,}(x,y;\kappa_1,\kappa_2,\kappa_3,\lambda,\mu,\eta))\\
&=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(\xi_1 x+\xi_2 y)}e^{\kappa_2 x-\kappa_3 y-\frac{e^{-y}}{2}}(1+e^x)^{-(\kappa_1+\kappa_2)}\,{}_9Q_{n,k}^{(\lambda,\mu,\eta)}(e^x,e^y)dxdy\\
&=\frac{(-1)^n\Gamma(n-k+\mu+1)}{\Gamma(\mu+1)}C_9(\kappa_1,\kappa_2,\kappa_3,\xi_1,\xi_2)\Theta_9(n,k,\kappa_1,\kappa_2,\kappa_3,\lambda,\mu,\eta,\xi_1,\xi_2),
\end{aligned}$$

where

$$C_9(\kappa_1,\kappa_2,\kappa_3,\xi_1,\xi_2)=\frac{2^{\kappa_3+i\xi_2}}{\Gamma(\kappa_1+\kappa_2)}\Gamma(\kappa_1+i\xi_1)\Gamma(\kappa_2-i\xi_1)\Gamma(\kappa_3+i\xi_2),$$

and

$$\begin{aligned}
\Theta_9(n,k,\kappa_1,\kappa_2,\kappa_3,\lambda,\mu,\eta,\xi_1,\xi_2)=&\ {}_2F_1\left(\begin{matrix}-k,\ k+1-\eta\\ 1-\kappa_3-i\xi_2\end{matrix}\ \Big|\ \frac{1}{2}\right)\\
&\times {}_3F_2\left(\begin{matrix}-(n-k),\ n-k+1-\lambda,\ \kappa_2-i\xi_1\\ \mu+1,\ 1-\kappa_1-i\xi_1\end{matrix}\ \Big|\ 1\right).
\end{aligned}$$

Hence, from the Parseval identity (39) and the orthogonality relation (29),

Theorem 9. *The special function*

$$\begin{aligned}
&{}_9E_{n,k}(x,y;\kappa_1,\kappa_2,\kappa_3,\varrho_3,\varrho_2,\varrho_1)\\
&=\Theta_9(n,k,\kappa_1,\kappa_2,\kappa_3,\kappa_1+\varrho_1+1,\kappa_2+\varrho_2-1,\kappa_3+\varrho_3+1,-ix,-iy),
\end{aligned}$$

has an orthogonality relation of form

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Gamma(\kappa_1+ix)\Gamma(\kappa_2-ix)\Gamma(\kappa_3+iy)\Gamma(\varrho_1-ix)\Gamma(\varrho_2+ix)\Gamma(\varrho_3-iy)\\
&\times\ {}_9E_{n,k}(ix,iy;\kappa_1,\kappa_2,\kappa_3,\varrho_3,\varrho_2,\varrho_1)\,{}_9E_{r,s}(-ix,-iy;\varrho_1,\varrho_2,\varrho_3,\kappa_3,\kappa_2,\kappa_1)dxdy\\
&=\frac{2^{2-(\kappa_3+\varrho_3)}\pi^2(n-k)!k!\Gamma(\kappa_1+\varrho_1+1-(n-k))\Gamma(\kappa_3+\varrho_3+1-k)}{(\kappa_1+\varrho_1-2(n-k))(\kappa_3+\varrho_3-2k)}\\
&\quad\times\frac{\Gamma^2(\kappa_2+\varrho_2)\Gamma(\kappa_1+\kappa_2)\Gamma(\varrho_1+\varrho_2)}{\Gamma(\kappa_1+\varrho_1+\kappa_2+\varrho_2-(n-k))\Gamma(\kappa_2+\varrho_2+n-k)}\delta_{n,r}\delta_{k,s},
\end{aligned}$$

for $\kappa_1,\kappa_2,\kappa_3,\varrho_1,\varrho_2,\varrho_3>0$, $\kappa_3+\varrho_3>2n$ and $\kappa_1+\varrho_1>2n$. Please note that the weight function of this orthogonality relation is positive for $\kappa_1=\varrho_1$, $\kappa_2=\varrho_2$, $\kappa_3=\varrho_3$ or $\kappa_1=\kappa_2$, $\varrho_1=\varrho_2$, $\kappa_3=\varrho_3$.

3.10. Fourier Transform of the Polynomials ${}_{13}Q_{n,k}^{(p,q)}(x,y)$

Let us define

$$h_{n,k}(x,y;\kappa_1,\kappa_2,\lambda,\mu)=\left(1+x^2\right)^{-(\kappa_1-1/4)}\left(1+y^2\right)^{-(\kappa_2-1/4)}\,{}_{13}Q_{n,k}^{(\lambda,\mu)}(x,y), \tag{58}$$

where $\kappa_1, \kappa_2, \lambda$ and μ are real parameters, and the polynomials ${}_{13}Q_{n,k}^{(p,q)}(x,y)$ is defined in (33). Applying the Fourier transform for the function given in (58), we get

$$\begin{aligned}\mathcal{F}(h_{n,k}(x,y;\kappa_1,\kappa_2,\lambda,\mu)) &= \frac{2^n\Gamma(\lambda)\Gamma(\mu)}{\Gamma(\lambda-(n-k))\Gamma(\mu-k)}\\ \times&\left[\sum_{l_1=0}^{[\frac{n-k}{2}]}\frac{(-1)^{l_1}\left(-\frac{n-k}{2}\right)_{l_1}\left(-\frac{n-k-1}{2}\right)_{l_1}}{(\lambda-(n-k))_{l_1}l_1!}\int_{-\infty}^{\infty}e^{-i\xi_1 x}\left(1+x^2\right)^{-(\kappa_1-1/4)}x^{n-k-2l_1}dx\right]\\ \times&\left[\sum_{l_2=0}^{[\frac{k}{2}]}\frac{(-1)^{l_2}\left(-\frac{k}{2}\right)_{l_2}\left(-\frac{k-1}{2}\right)_{l_2}}{(\mu-k)_{l_2}l_2!}\int_{-\infty}^{\infty}e^{-i\xi_2 y}\left(1+y^2\right)^{-(\kappa_2-1/4)}y^{k-2l_2}dy\right].\end{aligned} \tag{59}$$

In here, if we take as

$$\begin{aligned}S_{n-k,l_1}(\kappa_1,\xi_1) &= \int_{-\infty}^{\infty}e^{-i\xi_1 x}\left(1+x^2\right)^{-(\kappa_1-1/4)}x^{n-k-2l_1}dx\\ &= \int_{-\infty}^{\infty}\left(\sum_{j=0}^{\infty}\frac{(-i\xi_1 x)^j}{j!}\right)\left(1+x^2\right)^{-(\kappa_1-1/4)}x^{n-k-2l_1}dx\\ &= \sum_{j=0}^{\infty}\frac{(-i\xi_1)^j}{j!}\int_{-\infty}^{\infty}\left(1+x^2\right)^{-(\kappa_1-1/4)}x^{n-k+j-2l_1}dx,\end{aligned}$$

then, for $j=0,2,4,\ldots$,

$$\begin{aligned}S_{2m,l_1}(\kappa_1,\xi_1) &= \sum_{r=0}^{\infty}\frac{(-i\xi_1)^{2r}}{(2r)!}\int_{-\infty}^{\infty}\left(1+x^2\right)^{-(\kappa_1-1/4)}x^{2m+2r-2l_1}dx\\ &= \sum_{r=0}^{\infty}\frac{(-1)^r\left(\frac{\xi_1^2}{4}\right)^r}{r!\left(\frac{1}{2}\right)_r}\int_0^{\infty}(1+t)^{-(\kappa_1-1/4)}t^{m+r-l_1-1/2}dt\\ &= \sum_{r=0}^{\infty}\frac{(-1)^r\left(\frac{\xi_1^2}{4}\right)^r}{r!\left(\frac{1}{2}\right)_r}\frac{\Gamma(m+r-l_1+1/2)\Gamma(\kappa_1-m-r+l_1-3/4)}{\Gamma(\kappa_1-1/4)}\\ &= B(m+1/2-l_1,\ \kappa_1-m-3/4+l_1)\sum_{r=0}^{\infty}\frac{(m-l_1+1/2)_r\left(\frac{\xi_1^2}{4}\right)^r}{(7/4+m-\kappa_1-l_1)_r\left(\frac{1}{2}\right)_r r!},\end{aligned} \tag{60}$$

where

$$(2r)! = 2^{2r}r!\left(\frac{1}{2}\right)_r \tag{61}$$

and the relations (41) are used. Besides that, $S_{2m,l_1}(\kappa_1,\xi_1)=0$ for $j=1,3,5,\ldots$.

On the other hand, for $j = 1, 3, 5, \ldots,$

$$
\begin{aligned}
S_{2m+1,l_1}(\kappa_1,\xi_1) &= \sum_{r=0}^{\infty} \frac{(-i\xi_1)^{2r+1}}{(2r+1)!} \int_{-\infty}^{\infty} \left(1+x^2\right)^{-(\kappa_1-1/4)} x^{2m+2r+2-2l_1} dx \\
&= (-i\xi_1) \sum_{r=0}^{\infty} \frac{(-1)^r \left(\xi_1^2\right)^r}{\Gamma(2)(2)_{2r}} \int_0^{\infty} (1+t)^{-(\kappa_1-1/4)} t^{m+r+1/2-l_1} dt \\
&= (-i\xi_1) \sum_{r=0}^{\infty} \frac{(-1)^r \left(\xi_1^2\right)^r}{2^{2r} \prod\limits_{s=1}^{2} \left(\frac{2+s-1}{2}\right)_r} \frac{\Gamma(m+r+3/2-l_1)\Gamma(\kappa_1-m-r-7/4+l_1)}{\Gamma(\kappa_1-1/4)} \\
&= (-i\xi_1) B(m+3/2-l_1,\ \kappa_1-m-7/4+l_1) \sum_{r=0}^{\infty} \frac{(m+3/2-l_1)_r \left(\frac{\xi_1^2}{4}\right)^r}{(11/4+m-\kappa_1-l_1)_r \left(\frac{3}{2}\right)_r r!},
\end{aligned}
\tag{62}
$$

where the relations (41) and

$$
(-k)_{2l} = 2^{2l} \prod_{s=1}^{2} \left(\frac{-k+s-1}{2}\right)_l, \quad 0 \le l \le k/2 \tag{63}
$$

are used. Also, $S_{2m+1,l_1}(\kappa_1,\xi_1) = 0$ for $j = 0, 2, 4, \ldots$. Thus, from the relations (60) and (62),

$$
\begin{aligned}
S_{n-k,l_1}(\kappa_1,\xi_1) &= (-i\xi_1)^{\frac{1-(-1)^{n-k}}{2}} \frac{\Gamma\left(\left[\frac{n-k+1}{2}\right]+1/2-l_1\right)\Gamma\left(\kappa_1-\left[\frac{n-k+1}{2}\right]-3/4+l_1\right)}{\Gamma(\kappa_1-1/4)} \\
&\times \sum_{r=0}^{\infty} \frac{\left(\left[\frac{n-k+1}{2}\right]+1/2-l_1\right)_r \left(\frac{\xi_1^2}{4}\right)^r}{\left(7/4+\left[\frac{n-k+1}{2}\right]-\kappa_1-l_1\right)_r \left(1-\frac{(-1)^{n-k}}{2}\right)_r r!},
\end{aligned}
$$

and similarly the integral $S_{k,l_2}(\kappa_2,\xi_2)$ is calculated as

$$
\begin{aligned}
S_{k,l_2}(\kappa_2,\xi_2) &= \int_{-\infty}^{\infty} e^{-i\xi_2 y} \left(1+y^2\right)^{-(\kappa_2-1/4)} y^{k-2l_2} dy \\
&= (-i\xi_2)^{\frac{1-(-1)^k}{2}} \frac{\Gamma\left(\left[\frac{k+1}{2}\right]+1/2-l_2\right)\Gamma\left(\kappa_2-\left[\frac{k+1}{2}\right]-3/4+l_2\right)}{\Gamma(\kappa_2-1/4)} \\
&\times \sum_{r=0}^{\infty} \frac{\left(\left[\frac{k+1}{2}\right]+1/2-l_2\right)_r \left(\frac{\xi_2^2}{4}\right)^r}{\left(7/4+\left[\frac{k+1}{2}\right]-\kappa_2-l_2\right)_r \left(1-\frac{(-1)^k}{2}\right)_r r!}.
\end{aligned}
$$

Now, substituting the calculated integrals $S_{n-k,l_1}(\kappa_1,\xi_1)$ and $S_{k,l_2}(\kappa_2,\xi_2)$ in (59) we get

$$
\begin{aligned}
\mathcal{F}(h_{n,k}(x,y;\kappa_1,\kappa_2,\lambda,\mu)) &= \frac{2^n \Gamma(\lambda)\Gamma(\mu) C_{13}(n,k,\kappa_1,\kappa_2)}{\Gamma(\lambda-(n-k))\Gamma(\mu-k)} \\
&\times \Theta_{13}(n,k,\kappa_1,\kappa_2,\lambda,\mu,\xi_1,\xi_2),
\end{aligned}
$$

where

$$C_{13}(n,k,\kappa_1,\kappa_2) = \Gamma\left(\left[\frac{n-k+1}{2}\right]+1/2\right)\Gamma\left(\left[\frac{k+1}{2}\right]+1/2\right) \times \frac{\Gamma\left(\kappa_1-\left[\frac{n-k+1}{2}\right]-3/4\right)\Gamma\left(\kappa_2-\left[\frac{k+1}{2}\right]-3/4\right)}{\Gamma(\kappa_1-1/4)\Gamma(\kappa_2-1/4)},$$

and

$$\begin{aligned}\Theta_{13}(n,k,\kappa_1,\kappa_2,\lambda,\mu,\xi_1,\xi_2) &= (-i\xi_1)^{\frac{1-(-1)^{n-k}}{2}}(-i\xi_2)^{\frac{1-(-1)^k}{2}} \\ &\times\left[\sum_{l_1=0}^{\left[\frac{n-k}{2}\right]}\frac{\left(-\frac{n-k}{2}\right)_{l_1}\left(-\frac{n-k-1}{2}\right)_{l_1}\left(\kappa_1-\left[\frac{n-k+1}{2}\right]-3/4\right)_{l_1}}{(\lambda-(n-k))_{l_1}\left(1/2-\left[\frac{n-k+1}{2}\right]\right)_{l_1}l_1!}\right. \\ &\times\left.{}_1F_2\left(\begin{matrix}1/2+\left[\frac{n-k+1}{2}\right]-l_1\\ 7/4+\left[\frac{n-k+1}{2}\right]-\kappa_1-l_1,\ 1-\frac{(-1)^{n-k}}{2}\end{matrix}\ \middle|\ \frac{\xi_1^2}{4}\right)\right] \\ &\times\left[\sum_{l_2=0}^{\left[\frac{k}{2}\right]}\frac{\left(-\frac{k}{2}\right)_{l_2}\left(-\frac{k-1}{2}\right)_{l_2}\left(\kappa_2-\left[\frac{k+1}{2}\right]-3/4\right)_{l_2}}{(\mu-k)_{l_2}\left(1/2-\left[\frac{k+1}{2}\right]\right)_{l_2}l_2!}\right. \\ &\times\left.{}_1F_2\left(\begin{matrix}1/2+\left[\frac{k+1}{2}\right]-l_2\\ 7/4+\left[\frac{k+1}{2}\right]-\kappa_2-l_2,\ 1-\frac{(-1)^{k}}{2}\end{matrix}\ \middle|\ \frac{\xi_2^2}{4}\right)\right],\end{aligned}$$

where ${}_1F_2$ is the special case of the hypergeometric function. If the results are replaced in the Parseval identity (39), we obtain

$$\begin{aligned}&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_{n,k}(x,y;\kappa_1,\kappa_2,\lambda,\mu)h_{r,s}(x,y;\varrho_1,\varrho_2,\alpha,\beta) \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(1+x^2\right)^{-(\kappa_1+\varrho_1-1/2)}\left(1+y^2\right)^{-(\kappa_2+\varrho_2-1/2)}\ {}_{13}Q_{n,k}^{(\lambda,\mu)}(x,y)\ {}_{13}Q_{r,s}^{(\alpha,\beta)}(x,y)dydx \\ &= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{2^{n+r}\Gamma(\lambda)\Gamma(\mu)\Gamma(\alpha)\Gamma(\beta)C_{13}(n,k,\kappa_1,\kappa_2)\overline{C_{13}(r,s,\varrho_1,\varrho_2)}}{\Gamma(\lambda-(n-k))\Gamma(\mu-k)\Gamma(\alpha-(r-s))\Gamma(\beta-s)} \\ &\times\Theta_{13}(n,k,\kappa_1,\kappa_2,\lambda,\mu,\xi_1,\xi_2)\overline{\Theta_{13}(r,s,\varrho_1,\varrho_2,\alpha,\beta,\xi_1,\xi_2)}d\xi_1 d\xi_2.\end{aligned}$$

Now by taking $\lambda=\alpha=\kappa_1+\varrho_1$ and $\mu=\beta=\kappa_2+\varrho_2$ in the left-hand side of the above equality and applying the orthogonality relation (34), it can be written that

$$\begin{aligned}&\frac{2^{2(\kappa_1+\kappa_2+\varrho_1+\varrho_2-n-1)}\pi^2(n-k)!k!\Gamma^2(\kappa_1+\varrho_1-(n-k))\Gamma^2(\kappa_2+\varrho_2-k)\delta_{n,r}\delta_{k,s}}{(\kappa_1+\varrho_1-(n-k)-1)(\kappa_2+\varrho_2-k-1)\Gamma(2(\kappa_1+\varrho_1)-(n-k)-1)\Gamma(2(\kappa_2+\varrho_2)-k-1)} \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\overline{C_{13}(r,s,\varrho_1,\varrho_2)\Theta_{13}(r,s,\varrho_1,\varrho_2,\varrho_1+\kappa_1,\varrho_2+\kappa_2,\xi_1,\xi_2)} \\ &\quad\times C_{13}(n,k,\kappa_1,\kappa_2)\Theta_{13}(n,k,\kappa_1,\kappa_2,\kappa_1+\varrho_1,\kappa_2+\varrho_2,\xi_1,\xi_2)d\xi_1 d\xi_2,\end{aligned}$$

in view of the relation

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z)\Gamma(z+1/2). \qquad (64)$$

Theorem 10. *The special function*

$$ {}_{13}E_{n,k}(x,y;\kappa_1,\kappa_2,\varrho_2,\varrho_1) = \Theta_{13}(n,k,\kappa_1,\kappa_2,\kappa_1+\varrho_1,\kappa_2+\varrho_2,-ix,-iy), $$

has an orthogonality relation of form

$$ \begin{aligned} &\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} {}_{13}E_{n,k}(ix,iy;\kappa_1,\kappa_2,\varrho_2,\varrho_1)\,{}_{13}E_{r,s}(-ix,-iy;\varrho_1,\varrho_2,\kappa_2,\kappa_1)dxdy \\ &= \frac{2^{2(\kappa_1+\kappa_2+\varrho_1+\varrho_2-n-1)}\pi^2(n-k)!k!\Gamma^2(\kappa_1+\varrho_1-(n-k))\Gamma^2(\kappa_2+\varrho_2-k)}{(\kappa_1+\varrho_1-(n-k)-1)\Gamma(2(\kappa_1+\varrho_1)-(n-k)-1)\Gamma^2\left(\left[\frac{n-k+1}{2}\right]+1/2\right)} \\ &\times \frac{\Gamma(\kappa_1-1/4)\Gamma(\kappa_2-1/4)}{\Gamma(2(\kappa_2+\varrho_2)-k-1)\Gamma\left(\kappa_1-\left[\frac{n-k+1}{2}\right]-3/4\right)\Gamma\left(\varrho_1-\left[\frac{n-k+1}{2}\right]-3/4\right)} \\ &\times \frac{\Gamma(\varrho_2-1/4)\Gamma(\varrho_1-1/4)\delta_{n,r}\delta_{k,s}}{(\kappa_2+\varrho_2-k-1)\Gamma\left(\kappa_2-\left[\frac{k+1}{2}\right]-3/4\right)\Gamma\left(\varrho_2-\left[\frac{k+1}{2}\right]-3/4\right)\Gamma^2\left(\left[\frac{k+1}{2}\right]+1/2\right)}, \end{aligned} $$

for $\kappa_1,\varrho_1 > \left[\frac{n+1}{2}\right]+3/4$ *and* $\kappa_2,\varrho_2 > \left[\frac{n+1}{2}\right]+3/4$.

3.11. Fourier Transform of the Polynomials ${}_{14}Q_{n,k}^{(p,q,u)}(x,y)$

Let us define

$$ h_{n,k}(x,y;\kappa_1,\kappa_2,\kappa_3,\lambda,\mu,\eta) = e^{\kappa_2 x}(1+e^x)^{-(\kappa_1+\kappa_2)}\left(1+y^2\right)^{-(\kappa_3-1/4)}\,{}_{14}Q_{n,k}^{(\lambda,\mu,\eta)}(e^x,y), \quad (65) $$

where $\kappa_1,\kappa_2,\kappa_3,\lambda,\mu$ and η are real parameters, and the polynomials ${}_{14}Q_{n,k}^{(p,q,u)}(x,y)$ is defined in (35). By applying similar calculations in the paper [21] and in the previous subsection, we get

$$ \begin{aligned} \mathcal{F}(h_{n,k}(x,y;\kappa_1,\kappa_2,\kappa_3,\lambda,\mu,\eta)) = &\frac{(-1)^{n-k}2^k\Gamma(\mu+n-k+1)\Gamma(\eta)}{\Gamma(\mu+1)\Gamma(\eta-k)} \\ &\times C_{14}(k,\kappa_1,\kappa_2,\kappa_3,\xi_1)\Theta_{14}(n,k,\kappa_1,\kappa_2,\kappa_3,\lambda,\mu,\eta,\xi_1,\xi_2), \end{aligned} $$

where

$$ \begin{aligned} C_{14}(k,\kappa_1,\kappa_2,\kappa_3,\xi_1) = &\frac{\Gamma\left(\left[\frac{k+1}{2}\right]+1/2\right)\Gamma\left(\kappa_3-\left[\frac{k+1}{2}\right]-3/4\right)}{\Gamma(\kappa_1+\kappa_2)\Gamma(\kappa_3-1/4)} \\ &\times\Gamma(\kappa_1+i\xi_1)\Gamma(\kappa_2-i\xi_1), \end{aligned} $$

and

$$ \begin{aligned} \Theta_{14}(n,k,\kappa_1,\kappa_2,\kappa_3,\lambda,\mu,\eta,\xi_1,\xi_2) = &(-i\xi_2)^{\frac{1-(-1)^k}{2}} \\ &\times {}_3F_2\left(\begin{matrix}-(n-k),\ n-k+1-\lambda,\ \kappa_2-i\xi_1\\ \mu+1,\ 1-\kappa_1-i\xi_1\end{matrix}\ \middle|\ 1\right) \\ &\times\sum_{l_2=0}^{\left[\frac{k}{2}\right]}\frac{\left(-\frac{k}{2}\right)_{l_2}\left(-\frac{k-1}{2}\right)_{l_2}\left(\kappa_3-\left[\frac{k+1}{2}\right]-3/4\right)_{l_2}}{(\eta-k)_{l_2}\left(1/2-\left[\frac{k+1}{2}\right]\right)_{l_2}l_2!} \\ &\times {}_1F_2\left(\begin{matrix}\left[\frac{k+1}{2}\right]+1/2-l_2\\ 7/4+\left[\frac{k+1}{2}\right]-\kappa_3-l_2,\ 1-\frac{(-1)^k}{2}\end{matrix}\ \middle|\ \frac{\xi_2^2}{4}\right). \end{aligned} $$

Hence, from the Parseval identity (39), we obtain

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_{n,k}(x,y;\kappa_1,\kappa_2,\kappa_3,\lambda,\mu,\eta)h_{r,s}(x,y;\varrho_1,\varrho_2,\varrho_3,\alpha,\beta,\gamma)dxdy \\
&= \int_0^{\infty}\int_{-\infty}^{\infty} t^{\kappa_2+\varrho_2-1}(1+t)^{-(\kappa_1+\varrho_1+\kappa_2+\varrho_2)}\left(1+y^2\right)^{-(\kappa_3+\varrho_3-1/2)} {}_{14}Q_{n,k}^{(\lambda,\mu,\eta)}(t,y) \\
&\qquad\qquad \times {}_{14}Q_{r,s}^{(\alpha,\beta,\gamma)}(t,y)dydt \\
&= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{2^{k+s}\Gamma(\mu+n-k+1)\Gamma(\beta+r-s+1)\Gamma(\eta)\Gamma(\gamma)}{\Gamma(\mu+1)\Gamma(\beta+1)\Gamma(\eta-k)\Gamma(\gamma-s)} \\
&\times (-1)^{n-k}(-1)^{r-s}\overline{C_{14}(s,\varrho_1,\varrho_2,\varrho_3,\xi_1)\Theta_{14}(r,s,\varrho_1,\varrho_2,\varrho_3,\alpha,\beta,\gamma,\xi_1,\xi_2)} \\
&\times C_{14}(k,\kappa_1,\kappa_2,\kappa_3,\xi_1)\Theta_{14}(n,k,\kappa_1,\kappa_2,\kappa_3,\lambda,\mu,\eta,\xi_1,\xi_2)d\xi_1 d\xi_2.
\end{aligned} \tag{66}$$

Now by taking $\kappa_1+\varrho_1+1=\lambda=\alpha$, $\kappa_2+\varrho_2-1=\mu=\beta$ and $\kappa_3+\varrho_3=\eta=\gamma$ in the left-hand side of (66) and then using the orthogonality relation (36), in view of the relation (64), we obtain

$$\begin{aligned}
&\frac{2^{2(\kappa_3+\varrho_3-k)}\pi^2(n-k)!k!\Gamma(\kappa_1+\varrho_1+1-(n-k))}{(\kappa_1+\varrho_1-2(n-k))(\kappa_3+\varrho_3-k-1)\Gamma(\kappa_1+\kappa_2+\varrho_1+\varrho_2-(n-k))} \\
&\qquad \times \frac{\Gamma^2(\kappa_2+\varrho_2)\Gamma^2(\kappa_3+\varrho_3-k)}{\Gamma(\kappa_2+\varrho_2+n-k)\Gamma(2(\kappa_3+\varrho_3)-k-1)}\delta_{n,r}\delta_{k,s} \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\overline{C_{14}(s,\varrho_1,\varrho_2,\varrho_3,\xi_1)}C_{14}(k,\kappa_1,\kappa_2,\kappa_3,\xi_1) \\
&\times \overline{\Theta_{14}(r,s,\varrho_1,\varrho_2,\varrho_3,\varrho_1+\kappa_1+1,\varrho_2+\kappa_2-1,\varrho_3+\kappa_3,\xi_1,\xi_2)} \\
&\times \Theta_{14}(n,k,\kappa_1,\kappa_2,\kappa_3,\kappa_1+\varrho_1+1,\kappa_2+\varrho_2-1,\kappa_3+\varrho_3,\xi_1,\xi_2)d\xi_1 d\xi_2.
\end{aligned}$$

Theorem 11. *The special function*

$$\begin{aligned}
&{}_{14}E_{n,k}(x,y;\kappa_1,\kappa_2,\kappa_3,\varrho_3,\varrho_2,\varrho_1) \\
&= \Theta_{14}(n,k,\kappa_1,\kappa_2,\kappa_3,\kappa_1+\varrho_1+1,\kappa_2+\varrho_2-1,\kappa_3+\varrho_3,-ix,-iy),
\end{aligned}$$

has an orthogonality relation in form

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Gamma(\kappa_1+ix)\Gamma(\kappa_2-ix)\,{}_{14}E_{n,k}(ix,iy;\kappa_1,\kappa_2,\kappa_3,\varrho_3,\varrho_2,\varrho_1) \\
&\times \Gamma(\varrho_1-ix)\Gamma(\varrho_2+ix)\,{}_{14}E_{r,s}(-ix,-iy;\varrho_1,\varrho_2,\varrho_3,\kappa_3,\kappa_2,\kappa_1)dxdy \\
&= \frac{2^{2(\kappa_3+\varrho_3-k)}\pi^2(n-k)!k!\Gamma(\kappa_1+\varrho_1+1-(n-k))}{(\kappa_1+\varrho_1-2(n-k))(\kappa_3+\varrho_3-k-1)\Gamma(\kappa_1+\kappa_2+\varrho_1+\varrho_2-(n-k))} \\
&\times \frac{\Gamma^2(\kappa_2+\varrho_2)\Gamma^2(\kappa_3+\varrho_3-k)}{\Gamma(\kappa_2+\varrho_2+n-k)\Gamma(2(\kappa_3+\varrho_3)-k-1)} \\
&\times \frac{\Gamma(\kappa_1+\kappa_2)\Gamma(\varrho_1+\varrho_2)\Gamma(\kappa_3-1/4)\Gamma(\varrho_3-1/4)\delta_{n,r}\delta_{k,s}}{\Gamma^2\left(\left[\frac{k+1}{2}\right]+1/2\right)\Gamma\left(\kappa_3-\left[\frac{k+1}{2}\right]-3/4\right)\Gamma\left(\varrho_3-\left[\frac{k+1}{2}\right]-3/4\right)},
\end{aligned}$$

for $\kappa_1,\kappa_2,\varrho_1,\varrho_2>0$, $\kappa_1+\varrho_1>2n$ and $\kappa_3,\varrho_3>\left[\frac{n+1}{2}\right]+3/4$. Please note that the weight function is positive for $\kappa_1=\kappa_2$ and $\varrho_1=\varrho_2$ or $\kappa_1=\varrho_1$ and $\kappa_2=\varrho_2$.

3.12. Fourier Transform of the Polynomials ${}_{15}Q_{n,k}^{(p,q)}(x,y)$

Let us define

$$h_{n,k}(x,y;\kappa_1,\kappa_2,\lambda,\mu) = e^{-\kappa_1 x}e^{-\frac{e^{-x}}{2}}\left(1+y^2\right)^{-(\kappa_2-1/4)}\,{}_{15}Q_{n,k}^{(\lambda,\mu)}(e^x,y), \tag{67}$$

where $\kappa_1,\kappa_2,\lambda$ and μ are real parameters, and the polynomials ${}_{15}Q_{n,k}^{(p,q)}(x,y)$ is defined by (37). In view of the calculations in the paper [21] and in similar calculations given for ${}_{13}Q_{n,k}^{(p,q)}(x,y)$, we get

$$\begin{aligned}
&\mathcal{F}(h_{n,k}(x,y;\kappa_1,\kappa_2,\lambda,\mu))\\
&= (-1)^{n-k}2^{k+\kappa_1+i\xi_1}\Gamma(\kappa_1+i\xi_1)\frac{\Gamma(\mu)}{\Gamma(\mu-k)}\\
&\times\, {}_2F_1\left(\begin{matrix}-(n-k),\ n-k+1-\lambda\\ 1-\kappa_1-i\xi_1\end{matrix}\ \Big|\ \frac{1}{2}\right)\\
&\times\sum_{l_2=0}^{[k/2]}\frac{(-1)^{l_2}\left(-\frac{k}{2}\right)_{l_2}\left(-\frac{k-1}{2}\right)_{l_2}}{(\mu-k)_{l_2}l_2!}\int_{-\infty}^{\infty}e^{-i\xi_2 y}\left(1+y^2\right)^{-(\kappa_2-1/4)}y^{k-2l_2}dy\\
&= \frac{(-1)^{n-k}\Gamma(\mu)}{\Gamma(\mu-k)}C_{15}(k,\kappa_1,\kappa_2,\xi_1,\xi_2)\Theta_{15}(n,k,\kappa_1,\kappa_2,\lambda,\mu,\xi_1,\xi_2),
\end{aligned}$$

where

$$\begin{aligned}
C_{15}(k,\kappa_1,\kappa_2,\xi_1,\xi_2) &= 2^{k+\kappa_1+i\xi_1}\Gamma(\kappa_1+i\xi_1)\\
&\times B\left(\left[\frac{k+1}{2}\right]+1/2,\kappa_2-\left[\frac{k+1}{2}\right]-3/4\right),
\end{aligned}$$

and

$$\begin{aligned}
&\Theta_{15}(n,k,\kappa_1,\kappa_2,\lambda,\mu,\xi_1,\xi_2)\\
&= (-i\xi_2)^{\frac{1-(-1)^k}{2}}\,{}_2F_1\left(\begin{matrix}-(n-k),\ n-k+1-\lambda\\ 1-\kappa_1-i\xi_1\end{matrix}\ \Big|\ \frac{1}{2}\right)\\
&\times\sum_{l_2=0}^{[k/2]}\frac{\left(-\frac{k}{2}\right)_{l_2}\left(-\frac{k-1}{2}\right)_{l_2}\left(\kappa_2-\left[\frac{k+1}{2}\right]-3/4\right)_{l_2}}{(\mu-k)_{l_2}\left(1/2-\left[\frac{k+1}{2}\right]\right)_{l_2}l_2!}\\
&\times\,{}_1F_2\left(\begin{matrix}\left[\frac{k+1}{2}\right]+1/2-l_2\\ 7/4+\left[\frac{k+1}{2}\right]-\kappa_2-l_2,\ 1-\frac{(-1)^k}{2}\end{matrix}\ \Big|\ \frac{\xi_2^2}{4}\right).
\end{aligned}$$

Hence, from the Parseval identity (39), we obtain

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}h_{n,k}(x,y;\kappa_1,\kappa_2,\lambda,\mu)h_{r,s}(x,y;\varrho_1,\varrho_2,\alpha,\beta)dxdy\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}t^{-(\kappa_1+\varrho_1+1)}e^{-\frac{1}{t}}\left(1+y^2\right)^{-(\kappa_2+\varrho_2-1/2)}\,{}_{15}Q_{n,k}^{(\lambda,\mu)}(t,y)\,{}_{15}Q_{r,s}^{(\alpha,\beta)}(t,y)dydt\\
&= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{(-1)^{n-k}(-1)^{r-s}\Gamma(\mu)\Gamma(\beta)}{\Gamma(\mu-k)\Gamma(\beta-s)}C_{15}(k,\kappa_1,\kappa_2,\xi_1,\xi_2)\overline{C_{15}(s,\varrho_1,\varrho_2,\xi_1,\xi_2)}\\
&\times\Theta_{15}(n,k,\kappa_1,\kappa_2,\lambda,\mu,\xi_1,\xi_2)\overline{\Theta_{15}(r,s,\varrho_1,\varrho_2,\alpha,\beta,\xi_1,\xi_2)}d\xi_1 d\xi_2.
\end{aligned} \tag{68}$$

Now by taking $\lambda = \alpha = \kappa_1 + \varrho_1 + 1$ and $\mu = \beta = \kappa_2 + \varrho_2$ in the left-hand side of (68), then using the orthogonality relation (38) we have

$$\frac{2^{2(\kappa_2+\varrho_2)}\pi^2(n-k)!k!\Gamma(\kappa_1+\varrho_1+1-(n-k))\Gamma^2(\kappa_2+\varrho_2-k)}{(\kappa_1+\varrho_1-2(n-k))(\kappa_2+\varrho_2-k-1)\Gamma(2(\kappa_2+\varrho_2)-k-1)}\delta_{n,r}\delta_{k,s}$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{C_{15}(s,\varrho_1,\varrho_2,\xi_1,\xi_2)\Theta_{15}(r,s,\varrho_1,\varrho_2,\varrho_1+\kappa_1+1,\varrho_2+\kappa_2,\xi_1,\xi_2)}$$
$$\times\, C_{15}(k,\kappa_1,\kappa_2,\xi_1,\xi_2)\Theta_{15}(n,k,\kappa_1,\kappa_2,\kappa_1+\varrho_1+1,\kappa_2+\varrho_2,\xi_1,\xi_2)d\xi_1 d\xi_2,$$

in view of the relation (64).

Theorem 12. *The special function*

$${}_{15}E_{n,k}(x,y;\kappa_1,\kappa_2,\varrho_2,\varrho_1) = \Theta_{15}(n,k,\kappa_1,\kappa_2,\kappa_1+\varrho_1+1,\kappa_2+\varrho_2,-ix,-iy),$$

has an orthogonality relation of form

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Gamma(\kappa_1+ix)\Gamma(\varrho_1-ix)\;{}_{15}E_{n,k}(ix,iy;\kappa_1,\kappa_2,\varrho_2,\varrho_1)$$
$$\times\;{}_{15}E_{r,s}(-ix,-iy;\varrho_1,\varrho_2,\kappa_2,\kappa_1)dxdy$$
$$= \frac{2^{2(\kappa_2+\varrho_2-k)-(\kappa_1+\varrho_1)}\pi^2(n-k)!k!\Gamma(\kappa_1+\varrho_1+1-(n-k))}{(\kappa_1+\varrho_1-2(n-k))(\kappa_2+\varrho_2-k-1)\Gamma(2(\kappa_2+\varrho_2)-k-1)}$$
$$\times\frac{\Gamma^2(\kappa_2+\varrho_2-k)\Gamma(\kappa_2-1/4)\Gamma(\varrho_2-1/4)\delta_{n,r}\delta_{k,s}}{\Gamma^2\left(\left[\frac{k+1}{2}\right]+1/2\right)\Gamma\left(\kappa_2-\left[\frac{k+1}{2}\right]-3/4\right)\Gamma\left(\varrho_2-\left[\frac{k+1}{2}\right]-3/4\right)},$$

for $\kappa_1,\varrho_1>0,\ \kappa_1+\varrho_1>2n$ and $\kappa_2,\varrho_2>\left[\frac{n+1}{2}\right]+3/4$. Please note that the weight function of the orthogonality relation is positive for $\kappa_1=\varrho_1$.

4. Conclusions

The integral transforms have many applications in mathematics, physics, engineering, and in other scientific disciplines. The most important use of the Fourier transformation is to solve differential equations, integral equations and partial differential equations of the mathematical physics such as Laplace, Heat, Wave equations. Some other applications of the Fourier transform include vibration analysis, sound engineering, communication, data analysis, etc. [10–14]. The Fourier transform is also an important image processing tool, especially in transformation, representation, and encoding, smoothing, and sharpening images [5]. Another use of the Fourier transform is that it allows us to derive new families of functions by means of Parseval identity. During the last several years, there were many papers on the study of orthogonal polynomials and their transformations. The families of orthogonal polynomials which are mapped onto each other can be introduced by using the well-known Fourier transform or other integral transforms. Up to now, Fourier transforms of Jacobi, generalized Ultraspherical, generalized Hermite, Routh-Romanovski polynomials, finite classical orthogonal polynomials, etc. and relations with other polynomials have been studied by many authors. By the motivation of Fourier transforms of orthogonal polynomials mentioned above, a similar method in those papers has been developed for two-dimensional Fourier transforms. This paper first deals with two-dimensional Fourier transforms of some specific functions in terms of finite bivariate orthogonal polynomials. Then, via Fourier transforms of finite bivariate orthogonal polynomials and Parseval identity, the new families of bivariate orthogonal functions have been derived. This study will be a material for researchers who study on orthogonal polynomials and special functions. The method applied for bivariate case in this paper can be applied for some other orthogonal polynomials in multivariate case. Furthermore, it is possible to investigate some partial differential operators whose eigenfunctions are the obtained new orthogonal functions and some their characteristic properties in further research.

Author Contributions: Investigation E.G.L., R.A. and M.M.-J.; writing-original draft E.G.L. and R.A.; writing-review and editing E.G.L., R.A. and M.M.-J. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not Applicable.

Informed Consent Statement: Not Applicable.

Data Availability Statement: Not Applicable.

Acknowledgments: The research of the first and second authors has been supported by Ankara University Scientific Research Project Unit (BAP) Project No:20L0430007. The work of the third author has been supported by the Alexander von Humboldt Foundation under the Grant number: Ref 3.4-IRN-1128637-GF-E.

Conflicts of Interest: The authors declare no conflict of interest.

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