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Monotone Iterative Technique for the Periodic Solutions of High-Order Delayed Differential Equations in Abstract Spaces

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Abstract: This paper deals with the existence of ω-periodic solutions for nth-order ordinary differential equation involving fixed delay in Banach space E. $L_n u(t) = f(t, u(t), u(t-\tau)), \quad t \in \mathbb{R}$, where $L_n u(t) := u^{(n)}(t) + \sum_{i=0}^{n-1} a_i u^{(i)}(t), a_i \in \mathbb{R}, i = 0, 1, \cdots, n-1$, are constants, $f(t, x, y) : \mathbb{R} \times E \times E \longrightarrow E$ is continuous and ω-periodic with respect to t, $\tau > 0$. By applying the approach of upper and lower solutions and the monotone iterative technique, some existence and uniqueness theorems are proved under essential conditions.

Keywords: nth-order ordinary differential equations; delay; ω-periodic solutions; the measure of noncompactness; upper and lower solutions

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1. Introduction

The properties of periodic solutions of differential equations are significant problems in application science. A great number of works have focused on the existence of periodic solutions of differential equations, but they mainly studied the self-adjoint equations. For the case of non-self-adjoint differential equations, the researches are seldom because of their complex spectral structure. Since the nth-order differential equations are typical non-self-adjoint differential equations, it is very important both in theory and practice to prove the existence theorems of periodic solutions for nth-order ordinary differential equations. Recently, there are many beautiful results are obtained, for instance, see Cabada [1–3], Li [4–6], Liu [7] and V. Seda [8] and the references therein. The higher-order differential equation and its application in optimization and control theory were also studied, see [9–11] and the references therein. In some publications, the maximum principle is essential in the proof of main results. In [4], by using the obtained maximum principle, Li extended the results of Cabada in [1–3] and proved some existence results for the nth-order periodic boundary value problem of ordinary differential equations. Later, Li in [5] discussed the existence as well as the uniqueness of solutions for the nth-order periodic boundary value problem under spectral conditions. The maximum principle was also used in [6] to deal with the periodic boundary value problem of nth-order ordinary differential equation

$$\begin{cases} L_n u(t) = f(t, u(t)), & 0 \le t \le \omega, \\ u^{(i)}(0) = u^{(i)}(\omega), & i = 0, 1, 2, \cdots, n-1, \end{cases}$$

where $L_n u(t) := u^{(n)}(t) + \sum_{i=0}^{n-1} a_i u^{(i)}(t)$, $a_i \in \mathbb{R}$, $i = 0, 1, \cdots, n-1$, are constants, $f : [0, \omega] \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous mapping. By using the obtained maximum principle, the author proved some existence and uniqueness theorems. In [7], Liu investigated the existence results of periodic solutions for the two special cases of nth-order delay differential equation by applying the coincidence degree theory, but the above mentioned literatures did not consider the periodic solutions for the general delayed differential equations in abstract spaces.

In the present work, we consider the existence as well as the uniqueness of ω-periodic solutions for nth-order ordinary differential equation involving delay in Banach space E

$$L_n u(t) = f(t, u(t), u(t-\tau)), \quad t \in \mathbb{R}, \tag{1}$$

where $f(t,x,y) : \mathbb{R} \times E \times E \longrightarrow E$ is a continuous mapping and it is ω-periodic with respect to t and $\tau > 0$. Firstly, we establish the maximum principle to the corresponding linear delayed equation

$$L_n u(t) + bu(t-\tau) = h(t), \quad \forall t \in \mathbb{R},$$

where $h : \mathbb{R} \to E$ is an ω-periodic continuous function and $b \geq 0$ is a constant. Then, by applying the obtained maximum principle, some existence and uniqueness theorems are proved by applying the fixed point approach and monotone iterative technique.

The next Table 1 describes several symbols which will be later used within the body of the manuscript.

Table 1. Symbols used in this paper.

E	the ordered and separable Banach space
$\mathbb{R}$	$(-\infty, +\infty)$
$\mathbb{R}^+$	$[0, +\infty)$
$\mathbb{N}$	the set of natural number
$\mathbb{C}$	the complex plane

2. Preliminaries

Let $J := [0, \omega]$ and $C_\omega(\mathbb{R}, \mathbb{R})$ be the set of all continuous and ω-periodic functions. Then $C_\omega(\mathbb{R}, \mathbb{R})$ is a Banach space equipped with norm $\|u\|_C := \max_{t \in [0,\omega]} |u(t)|$ and $C(J, \mathbb{R})$ is also the Banach space. In general, $C^n(J, \mathbb{R})$ is the Banach space of nth-order continuous and differentiable functions.

For all $h \in C(J, \mathbb{R})$, we know that the linear periodic boundary value problem(LPBVP)

$$\begin{cases} L_n u(t) = h(t), \quad t \in J, \\ u^{(i)}(0) = u^{(i)}(\omega) \end{cases} \tag{2}$$

possesses a unique solution $u \in C^n(J, \mathbb{R})$:

$$u(t) = \int_0^\omega G_n(t,s) h(s) ds,$$

where

$$G_n(t,s) = \begin{cases} r_n(t-s), \quad 0 \leq s \leq t \leq \omega, \\ r_n(\omega + t - s), \quad 0 \leq t \leq s \leq \omega, \end{cases}$$

where $r_n(t) \in C^\infty(I, \mathbb{R})$ is the unique solution of the LBVP

$$\begin{cases} L_n v(t) = 0, \quad t \in J, \\ v^{(i)}(0) = v^{(i)}(\omega), \quad i = 0, 1, \ldots\ldots, n-2, \\ v^{(n-1)}(0) - v^{(n-1)}(\omega) = -1. \end{cases} \tag{3}$$

Let $P_n(\lambda)$ be the characteristic polynomial of L_n defined by

$$P_n(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_0. \tag{4}$$

And let $\mathcal{N}(P_n(\lambda))$ be the set of null points of $P_n(\lambda)$ in $\mathbb{C}$. For the LBVP (3), we assume the following hypothesis.

Hypothesis 1 (H1). $\mathcal{N}(P_n(\lambda)) \cap \{\frac{2k\pi}{\omega}i : k = 0, \pm 1, \pm 2, \dots\} = \emptyset.$

Lemma 1. *If the Hypothesis 1 (H1) holds, then the LBVP (3) possesses a unique solution $r_n(t) \in C^\infty(I, \mathbb{R})$.*

Proof of Lemma 1. Denote by $U(t) := (u(t), u^{'}(t), \dots u^{(n-1)}(t))^T$ and $B := (0, 0, \dots 0, 1)^T$. Then the LBVP (3) equivalents to the linear system

$$\begin{cases} U^{'}(t) = AU(t), \quad t \in I, \\ U(0) - U(\omega) = B, \end{cases} \tag{5}$$

where A is defined by

$$A = \begin{pmatrix} 0 & 1 & 0 & . & . & . & 0 \\ 0 & 0 & 1 & . & . & . & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & 1 \\ -a_0 & -a_1 & -a_2 & . & . & . & -a_{n-1} \end{pmatrix}.$$

If we take $U(0)$ as the initial value, the first equation of (5) has a unique solution expressed by

$$U(t) = e^{tA}U(0).$$

Then $U(t)$ satisfies $U(0) - U(\omega) = B$ if and only if

$$(I - e^{\omega A})U(0) = B.$$

By the Hypothesis 1 (H1), we know that $(I - e^{\omega A})^{-1}$ exists and

$$U(0) = (I - e^{\omega A})^{-1}B.$$

This implies that the linear system (5) has a unique solution

$$V_0(t) = e^{tA}((I - e^{\omega A})^{-1}B). \tag{6}$$

The first component of $V_0(t)$ is denoted by $r_n(t)$, then it follows from (6) that $r_n(t) \in C^\infty(I, \mathbb{R})$ and it is a unique solution of the LBVP (3). □

Remark 1. *Clearly, for each $h \in C(J, \mathbb{R})$, the LPBVP (2) possesses a unique solution if and only if the LBVP(3) has a unique solution.*

Lemma 2. *If the Hypothesis 1 (H1) holds, then for each $h \in C_\omega(\mathbb{R}, \mathbb{R})$ and $t \in \mathbb{R}$, the linear equation*

$$L_n u(t) = h(t) \tag{7}$$

possesses a unique solution $u := T_n h \in C^n_\omega(\mathbb{R}, \mathbb{R})$, and $T_n : C_\omega(\mathbb{R}, \mathbb{R}) \longrightarrow C_\omega(\mathbb{R}, \mathbb{R})$ is a bounded linear operator satisfying $\|T_n\| = \frac{1}{|a_0|}$ when $a_0 \neq 0$.

Proof of Lemma 2. If $h \in C_\omega(\mathbb{R},\mathbb{R})$, since the ω-periodic solution of (7) is equivalent to the solution of the LPBVP (2), by Lemma 1, the linear Equation (7) possesses a unique ω-periodic solution

$$u(t) = \int_{t-\omega}^{t} G_n(t,s)h(s)ds := (T_nh)(t), \quad t \in \mathbb{R}.$$

Clearly, $T_n : C_\omega(\mathbb{R},\mathbb{R}) \longrightarrow C^n_\omega(\mathbb{R},\mathbb{R}) \hookrightarrow C_\omega(\mathbb{R},\mathbb{R})$, and

$$\begin{aligned} \|T_nh(t)\| &\leq \|\int_0^\omega G_n(t,s)ds\|\|h\|_C \\ &= \|\int_0^\omega r_n(s)ds\|\|h\|_C \\ &= \frac{1}{|a_0|}\|h\|_C, \end{aligned}$$

that is, $\|T_n\| \leq \frac{1}{|a_0|}$.

On the other hand, let $h_0(t) \equiv 1$. Then $h_0 \in C_\omega(\mathbb{R},\mathbb{R})$ and $\|h_0\|_C = 1$. Thus,

$$\begin{aligned} \|T_nh_0(t)\| &= \|\int_{t-\omega}^{t} G_n(t,s)h_0(s)ds\| \\ &= \|\int_0^\omega G_n(t,s)ds\| \\ &= \frac{1}{|a_0|}, \end{aligned}$$

that is, $\|T_n\| \geq \frac{1}{|a_0|}$. Consequently, we obtain $\|T_n\| = \frac{1}{|a_0|}$. □

Let $(E, \|\cdot\|, \leq)$ be an ordered and separable Banach space, $K := \{x \in E : x \geq \theta\}$ be a positive cone of E, where θ denotes the zero element of E. Then K is a normal cone with the constant N. Denote by $C_\omega(\mathbb{R}, E)$ the set of E-valued continuous and ω-periodic functions. Then $C_\omega(\mathbb{R}, E)$ is a Banach space whose norm is defined by $\|u\|_C := \max_{t\in[0,\omega]} \|u(t)\|$ for every $u \in C_\omega(\mathbb{R}, E)$. Let $K_C := C_\omega(\mathbb{R}, K) = \{u \in C_\omega(\mathbb{R}, E) : u(t) \in K, \forall t \in \mathbb{R}\}$. Then K_C is also a normal cone with the same constant of cone K, and $C_\omega(\mathbb{R}, E)$ is an ordered Banach space. Generally, $C^n_\omega(\mathbb{R}, E)$ is the Banach space of all ω-periodic and nth-order continuous differentiable functions for $n \in \mathbb{N}$.

Now, for any $h \in C_\omega(\mathbb{R}, E)$, we consider the linear delayed differential equation(LDDE)

$$L_nu(t) + bu(t-\tau) = h(t), \quad t \in \mathbb{R}, \tag{8}$$

where $b \geq 0$ and $\tau > 0$ are constants.

For $G_n(t,s)$, if $r_n(t) > 0$, $G_n(t,s) > 0$. Hence, by Lemma 2, when $r_n(t) > 0$ and the Hypothesis 1 (H1) holds, the operator $T_n : C_\omega(\mathbb{R}, E) \to C_\omega(\mathbb{R}, E)$ is a positive operator. Let $m_n := \min_{t\in I} r_n(t)$ and $M_n := \max_{t\in I} r_n(t)$. It is clear that $0 < m_n \leq r_n(t) \leq M_n$. By Lemma 2, we obtain the following lemma.

Lemma 3. *Let the Hypothesis 1 (H1) hold, $a_0 > 0, 0 \leq b < \frac{m_n a_0}{M_n}$. Then for any $h \in C_\omega(\mathbb{R}, E)$, the LDDE (8) possesses a unique ω-periodic solution $u := Th \in C_\omega(\mathbb{R}, E)$ satisfying $\|T\| \leq \frac{1}{a_0-b}$. Furthermore, if $r_n(t) > 0$, $T : C_\omega(\mathbb{R}, E) \to C_\omega(\mathbb{R}, E)$ is a linear bounded and positive operator.*

Proof of Lemma 3. By Lemma 2, it is easy to see that the LDDE (8) possesses a solution

$$u(t) = \int_{t-\omega}^{t} G_n(t,s)[h(s) - bu(s-\tau)]ds. \tag{9}$$

Define $B_1 : C_\omega(\mathbb{R}, E) \longrightarrow C_\omega(\mathbb{R}, E)$ by

$$B_1 u(t) = bu(t-\tau). \tag{10}$$

Obviously, $B_1 : C_\omega(\mathbb{R}, E) \longrightarrow C_\omega(\mathbb{R}, E)$ is a linear operator and $\|B_1\| \leq b$. Then (9) and (10) yield

$$\begin{aligned} u(t) &= \int_{t-\omega}^{t} G_n(t,s)h(s)ds - \int_{t-\omega}^{t} G_n(t,s)B_1 u(s)ds \\ &= (T_n h)(t) - (T_n B_1 u)(t). \end{aligned}$$

This implies that

$$(I + T_n B_1)u(t) = (T_n h)(t). \tag{11}$$

Since $\|T_n B_1\| \leq \|T_n\| \cdot \|B_1\| \leq \frac{b}{a_0} < 1$, the perturbation theorem yields that $(I + T_n B_1)^{-1}$ exists and

$$(I + T_n B_1)^{-1} = \sum_{i=0}^{\infty} (-1)^i (T_n B_1)^i = \sum_{i=0}^{\infty} (T_n B_1)^{2i} (I - T_n B_1), \tag{12}$$

which implies

$$\|(I + T_n B_1)^{-1}\| \leq \frac{1}{1 - \|T_n B_1\|} \leq \frac{a_0}{a_0 - b}.$$

Hence, by (11), we conclude that

$$u(t) = (I + T_n B_1)^{-1}(T_n h)(t) := (Th)(t), \qquad t \in \mathbb{R}. \tag{13}$$

Consequently, $u(t)$ is an ω-periodic solution of the LDDE (8). It follows from (13) that

$$\begin{aligned} \|Th(t)\| &= \|(I + T_n B_1)^{-1} T_n h(t)\| \\ &\leq \|(I + T_n B_1)^{-1}\| \|T_n\| \|h\|_C \\ &\leq \frac{a_0}{a_0 - b} \cdot \frac{1}{a_0} \cdot \|h\|_C \\ &= \frac{1}{a_0 - b} \|h\|_C. \end{aligned}$$

Hence

$$\|T\| \leq \frac{1}{a_0 - b}.$$

Next, we prove that $T : C_\omega(\mathbb{R}, E) \to C_\omega(\mathbb{R}, E)$ is a positive operator when $r_n(t) > 0$. By (12), for any $h \in C_\omega(\mathbb{R}, K)$, we have

$$Th(t) = \sum_{i=0}^{\infty} (T_n B_1)^{2i} (I - T_n B_1)(T_n h)(t), \quad t \in \mathbb{R}.$$

Form the above equality, it remains to prove the positivity of $(I - T_n B_1)T_n$. Since

$$(T_n h)(t) = \int_{t-\omega}^{t} G_n(t,s)h(s)ds \geq m_n \int_0^{\omega} h(s)ds$$

and

$$(T_n h)(t) \leq M_n \int_0^{\omega} h(s)ds,$$

it follows that

$$\begin{aligned}(I - T_n B_1)(T_n h)(t) &= T_n h(t) - (T_n B_1)(T_n h)(t)\\ &\geq (m_n - \frac{bM_n}{a_0}) \int_0^{\omega} h(s)ds.\end{aligned}$$

The condition $h \in C_\omega(\mathbb{R}, K)$ implies $h(t) \not\equiv 0$ for $t \in \mathbb{R}$. Then there exist a small interval $[c, d] \subset [0, \omega]$ and a constant $\varepsilon > 0$ such that

$$h(t) > \varepsilon, \quad t \in [c, d].$$

Hence, $\int_0^{\omega} h(s)ds \geq \int_c^d h(s)ds > \varepsilon(d - c) > 0$ and

$$(I - T_n B_1)(T_n h)(t) \geq (m_n - \frac{bM_n}{a_0})\varepsilon(d - c) > 0.$$

Consequently, the operator $T : C_\omega(\mathbb{R}, E) \longrightarrow C_\omega(\mathbb{R}, E)$ is positive. □

In Lemma 3, the condition $r_n(t) > 0$ is essential. We now introduce a condition to guarantee $r_n(t) > 0$ for all $t \in \mathbb{R}$:

Hypothesis 2 (H2). $\mathcal{N}(P_n(\lambda)) \subset \{\xi \in \mathbb{C} : |Im\xi| < \frac{\pi}{\omega}\}$. *See Theorem 1.5 of [6] for more detail.*

Lemma 4. *Assume that $a_0 > 0$ and $P_n(\lambda)$, defined by (4), satisfies the Hypothesis 2 (H2). Then the Hypothesis 1 (H1) holds and the LBVP (3) possesses a unique solution $r_n(t) > 0$ for all $t \in \mathbb{R}$.*

Hence, from Lemmas 3 and 4, the following lemma is easy to obtain.

Lemma 5. *Let the Hypothesis 2 (H2) hold and $a_0 > 0, 0 \leq b < \frac{m_n a_0}{M_n}$. If $u \in C_\omega(\mathbb{R}, E)$ satisfies*

$$L_n u(t) + bu(t - \tau) \geq \theta, \quad t \in \mathbb{R},$$

then, for any $t \in \mathbb{R}$, $u(t) \geq \theta$.

Proof of Lemma 5. Let $h(t) := L_n u(t) + bu(t - \tau) \geq \theta$. Then $h \in C_\omega(\mathbb{R}, K)$. So, for any $t \in \mathbb{R}$, Lemma 3 yields $u(t) \geq \theta$. □

Let $\beta_E(\cdot)$ and $\beta_C(\cdot)$ denote the Kuratowski's measure of non-compactness(MNC) of bounded subsets in E and $C_\omega(\mathbb{R}, E)$, respectively. For every bounded subset $D \subset C_\omega(\mathbb{R}, E)$, $\beta_E(D(t)) \leq \beta_C(D)$ for all $t \in \mathbb{R}$, where $D(t) := \{u(t) : u \in D\}$. For more detail of the MNC, we refer to [12,13] and the references therein. The following lemmas can be found in [12,14], which are more useful in our arguments.

Lemma 6. *Let D be a equicontinuous and bounded subset of $C(J, E)$. Then $\beta_E(D(t))$ is continuous and*

$$\beta_C(D) = \max_{t \in J} \beta_E(D(t)).$$

Lemma 7. *Let D be bounded in E. Then there is a countable subset D_0 in D such that*

$$\beta_E(D) \leq 2\beta_E(D_0).$$

Lemma 8. *Let E be a separable Banach space and $D = \{u_n\}$ be a countable and bounded subset of $C(J, E)$. Then $\beta_E(D(t))$ is Lebesgue integrable on J and*

$$\beta_E(\{\int_J u_n(t)dt\}) \leq 2 \int_J \beta_E(D(t))dt.$$

By Lemma 3, we present the definition of ω-periodic solution of Equation (1) as follows.

Definition 1. *A function $u \in C_\omega(\mathbb{R}, E)$ is called an ω-periodic solution of Equation (1) if it satisfies the integral equation*

$$u(t) = (I + T_n B_1)^{-1} \int_{t-\omega}^{t} G_n(t,s)[f(s,u(s),u(s-\tau)) + bu(s-\tau)]ds, \quad t \in \mathbb{R},$$

where $b \geq 0$ is a constant and $B_1 : C_\omega(\mathbb{R}, E) \longrightarrow C_\omega(\mathbb{R}, E)$ is defined as in (10).

To end this section, we introduce the definitions of lower and upper ω-periodic solutions of Equation (1).

Definition 2. *If $v_0 \in C^n_\omega(\mathbb{R}, E)$ satisfies*

$$L_n v_0(t) \leq f(t, v_0(t), v_0(t-\tau)), \qquad \forall t \in \mathbb{R}, \tag{14}$$

then it is called the lower ω-periodic solution of Equation (1). If we inverse the inequality in (14), then it is called the upper ω-periodic solution of Equation (1).

3. The Method of Upper and Lower Solutions and the Monotone Iterative Technique

In this section, by utilizing the Sadovskii's fixed point theorem, we first consider the existence of ω-periodic solutions of Equation (1) between the lower and upper ω-periodic solutions. Then the monotone iterative technique is applied to study the existence as well as the uniqueness of ω-periodic solutions of Equation (1). At last, A sufficient condition is established for the existence of lower and upper ω-periodic solutions of the Equation (1).

At first, we make the following assumptions:

Hypothesis 3 (H3). *There is $b \in [0, \frac{m_n a_0}{M_n})$ such that*

$$f(t,x,y) - f(t, v_0(t), v_0(t-\tau)) \geq b(v_0(t-\tau) - y)$$

and

$$f(t,x,y) - f(t, w_0(t), w_0(t-\tau)) \leq b(w_0(t-\tau) - y)$$

for all $t \in \mathbb{R}$, $v_0(t) \leq x \leq w_0(t)$ and $v_0(t-\tau) \leq y \leq w_0(t-\tau)$.

Hypothesis 4 (H4). *There is $L_1 \in (0, \frac{a_0 - b}{8})$ such that*

$$\beta_E(f(t, D_1, D_2) + bD_2) \leq L_1(\beta_E(D_1) + \beta_E(D_2)), \qquad t \in \mathbb{R},$$

for any countable subsets $D_i \subset E, i = 1, 2$.

Theorem 1. *Let the Hypothesis 2 (H2) hold and $a_0 > 0$. If Equation (1) possesses lower and upper ω-periodic solutions v_0 and w_0 satisfying $v_0 \leq w_0$, and the Hypothesis 3 (H3) and Hypothesis 4 (H4) are satisfied, then Equation (1) possesses at least one ω-periodic solution on $\mathbb{R}$.*

Proof of Theorem 1. Since Equation (1) can be rewritten as

$$L_n u(t) + bu(t-\tau) = f(t, u(t), u(t-\tau)) + bu(t-\tau), \quad t \in \mathbb{R},$$

by Lemma 3 and Definition 1, we define $Q : C_\omega(\mathbb{R}, E) \longrightarrow C_\omega(\mathbb{R}, E)$ by

$$(Qu)(t) = (I + T_n B_1)^{-1} \int_{t-\omega}^{t} G_n(t,s)[f(s,u(s),u(s-\tau)) + bu(s-\tau)]ds, t \in \mathbb{R}. \tag{15}$$

Let $D = [v_0, w_0]$. It is obvious that $D \subset C_\omega(\mathbb{R}, E)$ is nonempty bounded, convex and closed. We will apply the approach of fixed point to discuss the existence of fixed points of Q in D. These fixed points are the ω-periodic solutions of Equation (1) between v_0 and w_0 due to Lemma 3 and Definition 1.

First of all, we prove $Q(D) \subset D$. Let $u \in D$. Then

$$v_0(t) \leq u(t) \leq w_0(t)$$

and

$$v_0(t-\tau) \leq u(t-\tau) \leq w_0(t-\tau)$$

for all $t \in \mathbb{R}$. By the Hypothesis 3 (H3), we have

$$f(t, u(t), u(t-\tau)) + bu(t-\tau) \geq f(t, v_0(t), v_0(t-\tau)) + bv_0(t-\tau), \quad t \in \mathbb{R}$$

and

$$f(t, u(t), u(t-\tau)) + bu(t-\tau) \leq f(t, w_0(t), w_0(t-\tau)) + bw_0(t-\tau), \quad t \in \mathbb{R}.$$

For any $u \in D$, let $v = Qu$, then

$$\begin{aligned}
& L_n(v - v_0)(t) + b(v - v_0)(t-\tau) \\
= \; & L_n v(t) + bv(t-\tau) - (L_n v_0(t) + bv_0(t-\tau)) \\
\geq \; & f(t, u(t), u(t-\tau)) + bu(t-\tau) - f(t, v_0(t), v_0(t-\tau)) - bv_0(t-\tau) \\
\geq \; & 0
\end{aligned}$$

and

$$\begin{aligned}
& L_n(w_0 - v)(t) + b(w_0 - v)(t-\tau) \\
= \; & L_n w_0(t) + bw_n(t-\tau) - (L_n v(t) + bv(t-\tau)) \\
\geq \; & f(t, w_0(t), w_0(t-\tau)) + bw_0(t-\tau) - f(t, u(t), u(t-\tau)) - bu(t-\tau) \\
\geq \; & 0.
\end{aligned}$$

By Lemma 5, it follows that $v - v_0 \geq \theta$ and $w_0 - v \geq \theta$. Therefore, $v = Qu \in [v_0, w_0]$, that is, $Q(D) \subset D$ is continuous.

Secondly, we prove the equi-continuity of $Q(D)$. For any $u \in D$ and $0 \leq t_1 \leq t_2 \leq \omega$, since $r_n \in C^\infty(I, \mathbb{R})$, by the definition of $G_n(t, s)$, we have

$$|G_n(t_2, s) - G_n(t_1, s)| \to 0, \quad (t_2 - t_1 \to 0).$$

Together this fact with the definition of Q, we obtain that

$$\|(Qu)(t_2) - (Qu)(t_1)\| \to 0$$

as $t_2 - t_1 \to 0$ independent of $u \in D$. Therefore, the set $Q(D)$ is equi-continuous.

It remains to prove that $Q : D \longrightarrow D$ is a condensing mapping. By Lemma 7, since $Q(D)$ is bounded, there is a countable subset $D_0 \subset D$ such that

$$\beta_E(Q(D)(t)) \leq 2\beta_E(Q(D_0)(t)).$$

Hence, Lemma 8 and the Hypothesis 4 (H4) yield

$$\begin{aligned}
\beta_E(Q(D_0)(t)) &= \beta_E\Big((I+T_nB_1)^{-1}\int_{t-\omega}^{t} G_n(t,s)[f(s,D_0(s),D_0(s-\tau))+bD_0(s-\tau)]ds\Big)\\
&\le \|(I+T_nB_1)^{-1}\|\beta_E\Big(\int_{t-\omega}^{t} G_n(t,s)[f(s,D_0(s),D_0(s-\tau))+bD_0(s-\tau)]ds\Big)\\
&\le \frac{2a_0}{a_0-b}\int_{t-\omega}^{t} G_n(t,s)\beta_E\Big(f(s,D_0(s),D_0(s-\tau))+bD_0(s-\tau)\Big)ds\\
&\le \frac{2a_0}{a_0-b}\int_{t-\omega}^{t} G_n(t,s)[L_1(\beta_E(D_0(s))+\beta_E(D_0(s-\tau)))]ds\\
&\le \frac{4a_0L_1}{a_0-b}\int_{0}^{\omega} G_n(t,s)\beta_C(D_0)ds\\
&\le \frac{4L_1}{a_0-b}\beta_C(D).
\end{aligned}$$

By the equi-continuity and boundedness of $Q(D)$, we have

$$\beta_C(Q(D)) = \max_{t\in I}\beta_E(Q(D)(t)) \le 2\max_{t\in I}\beta_E(Q(D_0)(t)) \le \frac{8L_1}{a_0-b}\beta_C(D).$$

Since $\frac{8L_1}{a_0-b} < 1$, it follows that $Q : D \longrightarrow D$ is a condensing operator. Therefore, the Sadovskii's fixed point theorem guarantees that there is at least one fixed point of Q in D. So, the Equation (1) possesses at least one ω-periodic solution in D. □

If we replace the conditions Hypothesis 3 (H3) and Hypothesis 4 (H4) in Theorem 1 by

Hypothesis 5 (H5). *There is a constant $b \in [0, \frac{m_n a_0}{M_n})$ such that*

$$f(t,x_2,y_2) - f(t,x_1,y_1) \ge b(y_1 - y_2), \quad \forall t \in \mathbb{R}$$

for any $v_0(t) \le x_1 \le x_2 \le w_0(t)$ and $v_0(t-\tau) \le y_1 \le y_2 \le w_0(t-\tau)$.

Hypothesis 6 (H6). *There is a constant $L_2 \in (0, \frac{a_0-b}{4})$ such that*

$$\beta_E(\{f(t,u_n(t),u_n(t-\tau)) + bu_n(t-\tau)\}) \le L_2(\beta_E(\{u_n(t)\}) + \beta_E(\{u_n(t-\tau)\}), \quad \forall t \in \mathbb{R}$$

for every monotonous sequence $\{u_n\} \subset [v_0, w_0]$. Then we can obtain the following theorem by utilizing the monotone iterative technique.

Theorem 2. *Let the Hypothesis 2 (H2) hold and $a_0 > 0$. If Equation (1) possesses lower and upper ω-periodic solutions v_0 and w_0 satisfying $v_0 \le w_0$, and the conditions Hypothesis 5 (H5) and Hypothesis 6 (H6) are satisfied, then there exist minimal and maximal ω-periodic solutions $\underline{u}, \overline{u}$ of Equation (1) between v_0 and w_0. Moreover, $\underline{u}$ and $\overline{u}$ can be derived by iterative sequences starting from v_0 and w_0, respectively.*

Proof of Theorem 2. We first prove that Q has properties:

(i) $v_0 \le Qv_0, \quad Qw_0 \le w_0,$
(ii) $Qu_1 \le Qu_2$ for all $u_1, u_2 \in [v_0, w_0]$ satisfying $u_1 \le u_2$,

where the operator Q is defined as in (15).

Let $v_1 := Qv_0$. Then

$$L_n v_1(t) + bv_1(t-\tau) = f(t, v_0(t), v_0(t-\tau)) + bv_0(t-\tau), \quad t \in \mathbb{R}.$$

Hence Definition 2 yields

$$\begin{aligned}
& L_n(v_1 - v_0)(t) + b(v_1 - v_0)(t-\tau) \\
= \ & L_n v_1(t) + bv_1(t-\tau) - (L_n v_0(t) + bv_0(t-\tau)) \\
\geq \ & f(t, v_0(t), v_0(t-\tau)) + bv_0(t-\tau) - f(t, v_0(t), v_0(t-\tau)) + bv_0(t-\tau) \\
= \ & 0.
\end{aligned}$$

It follows from Lemma 5 that $v_0(t) \leq v_1(t) = (Qv_0)(t)$ for each $t \in \mathbb{R}$. On the other hand, let $w_1 := Qw_0$. Then

$$L_n w_1(t) + bw_1(t-\tau) = f(t, w_0(t), w_0(t-\tau)) + bw_0(t-\tau), \quad t \in \mathbb{R}.$$

Hence, we have

$$\begin{aligned}
& L_n(w_0 - w_1)(t) + b(w_0 - w_1)(t-\tau) \\
\geq \ & f(t, w_0(t), w_0(t-\tau)) + bw_0(t-\tau) - f(t, w_0(t), w_0(t-\tau)) - bw_0(t-\tau) \\
= \ & 0.
\end{aligned}$$

Lemma 5 implies $(Qw_0)(t) = w_1(t) \leq w_0(t)$ for all $t \in \mathbb{R}$. Hence, (i) is satisfied.
For any $u_1, u_2 \in [v_0, w_0]$ with $u_1 \leq u_2$, owing to Hypothesis 5 (H5), we have

$$f(t, u_1(t), u_1(t-\tau)) + bu_1(t-\tau) \leq f(t, u_2(t), u_2(t-\tau)) + bu_2(t-\tau), t \in \mathbb{R}.$$

By (15), $Qu_1 \leq Qu_2$. Hence, (ii) holds.
Secondly, let

$$v_n = Qv_{n-1}, \quad w_n = Qw_{n-1}, \qquad n = 1, 2, \cdots. \tag{16}$$

Then, we deduce from (i) and (ii) that

$$v_0 \leq v_1 \leq \cdots \leq v_n \leq \cdots \leq w_n \leq \cdots \leq w_1 \leq w_0. \tag{17}$$

By the countability and boundedness of $\{v_n\}$, we conclude from Lemma 8 and Hypothesis 6 (H6) that

$$\begin{aligned}
& \beta_E\Big(\{v_n(t)\}\Big) = \beta_E\Big(\{(Qv_{n-1})(t)\}\Big) \\
= \ & \beta_E\Big(\{(I + T_n B_1)^{-1} \int_{t-\omega}^{t} G_n(t,s)[f(s, v_{n-1}(s), v_{n-1}(s-\tau)) + bv_{n-1}(s-\tau)]ds\}\Big) \\
\leq \ & \frac{2a_0}{a_0 - b} \int_{t-\omega}^{t} G_n(t,s) \beta_E\Big(\{f(s, v_{n-1}(s), v_{n-1}(t-\tau) + bv_{n-1}(t-\tau)\}\Big)ds \\
\leq \ & \frac{2a_0 L_2}{a_0 - b} \int_{t-\omega}^{t} G_n(t,s) \Big[\beta_E\Big(\{v_{n-1}(s)\}\Big) + \beta_E\Big(\{v_{n-1}(t-\tau)\}\Big)\Big]ds \\
\leq \ & \frac{4L_2}{a_0 - b} \beta_C(\{v_n\}).
\end{aligned}$$

Furthermore, $\{v_n\}$ is equi-continuous, by Lemma 6, we get

$$0 \leq \beta_C(\{v_n\}) = \max_{t \in I} \beta_E(\{v_n(t)\}) \leq \frac{4L_2}{a_0 - b} \beta_C(\{v_n\}).$$

Hence $\beta(\{v_n\}) = 0$ due to $\frac{4L_2}{a_0 - b} < 1$. Similarly, we obtain $\beta_C(\{w_n\}) = 0$. Hence, the sets $\{v_n\}$ and $\{w_n\}$ have convergent subsequences due to their relative compactness in

$C_\omega(\mathbb{R}, E)$. Since the cone K_C is normal and $\{v_n\}$, $\{w_n\}$ are monotone, we assume that $\{v_n\}$ and $\{w_n\}$ are convergent. That is, there exist $\underline{u}$ and $\overline{u}$ belong to $C_\omega(\mathbb{R}, E)$ such that

$$\underline{u} = \lim_{n\to\infty} v_n, \quad \overline{u} = \lim_{n\to\infty} w_n.$$

Putting $n \to \infty$ in (16), we get

$$\underline{u} = Q\underline{u}, \quad \overline{u} = Q\overline{u}.$$

This means that $\underline{u}$ and $\overline{u}$ are all the fixed points of Q. Consequently, $\underline{u}$ and $\overline{u}$ are ω-periodic solutions of Equation (1).

Let $\tilde{u}$ be any fixed point of Q between v_0 and w_0. Then $v_0(t) \le \tilde{u}(t) \le w_0(t)$ for each $t \in \mathbb{R}$. By (17), we deduce that

$$v_1(t) = (Qv_0)(t) \le (Q\tilde{u})(t) \le (Qw_0)(t) = w_1(t), \quad t \in \mathbb{R}.$$

So, $v_1(t) \le \tilde{u}(t) \le w_1(t)$ for each $t \in \mathbb{R}$. Generally, for every $t \in \mathbb{R}$, we conclude that

$$v_n(t) \le \tilde{u}(t) \le w_n(t). \tag{18}$$

Taking $n \to \infty$ in (18), we get

$$\underline{u} \le \tilde{u} \le \overline{u}.$$

Therefore, $\underline{u}$ and $\overline{u}$ are minimal and maximal ω-periodic solutions of Equation (1). □

The MNC conditions are necessary in Theorems 1 and 2, but they are not easy to verify in application. The next theorem establishes sufficient conditions to guarantee the existence as well as the uniqueness of ω-periodic solution of Equation (1), where the nonlinearity f is not asked to satisfy the MNC condition.

Theorem 3. *Let the Hypothesis 2 (H2) hold and $a_0 > 0$. If Equation (1) possesses lower and upper ω-periodic solutions v_0 and w_0 sarisfying $v_0 \le w_0$, and the nonlinearity f satisfies the Hypothesis 5 (H5) and Hypothesis 7 (H7).*

Hypothesis 7 (H7). *there is a constant L_3 satisfying $\max\{2b - a_0, 0\} < L_3 < b$ such that*

$$f(t, x_2, y_2) - f(t, x_1, y_1) \le -L_3(y_2 - y_1), \quad \forall t \in \mathbb{R},$$

where $v_0(t) \le x_1 \le x_2 \le w_0(t)$ and $v_0(t-\tau) \le y_1 \le y_2 \le w_0(t-\tau)$, then there is a unique ω-periodic solution of Equation (1) between v_0 and w_0.

Proof of Theorem 3. Define a mapping Φ by

$$\Phi(u)(\cdot) = f(\cdot, u(\cdot), u(\cdot - \tau)) + bu(\cdot - \tau),$$

then $\Phi : C_\omega(\mathbb{R}, E) \to C_\omega(\mathbb{R}, E)$ is a continuous mapping. By Lemma 3, for any $h \in C_\omega(\mathbb{R}, E)$, the linear equation

$$L_n u(t) + bu(t - \tau) = \Phi(h)(t), \quad t \in \mathbb{R} \tag{19}$$

has a unique ω-periodic solution, which is given by

$$u(t) = T(\Phi(h)(t)) := (Qh)(t), \quad t \in \mathbb{R}.$$

Then $Q : C_\omega(\mathbb{R}, E) \to C_\omega(\mathbb{R}, E)$ is a continuous operator. It follows from (19) that the fixed point of operator Q is the ω-periodic solution of Equation (1).

From the proof of Theorem 2, the operator Q satisfies the properties:

(i) $v_0 \le Qv_0, \quad Qw_0 \le w_0;$

(ii) $Qu_1 \leq Qu_2$ for every $u_1, u_2 \in [v_0, w_0]$ satisfying $u_1 \leq u_2$.

Let $\{v_n\}$ and $\{w_n\}$ be two sequences defined by (16). The properties (i) and (ii) yield that (17) holds. Then for $t \in \mathbb{R}$, we have

$$\begin{aligned}
\theta &\leq w_n(t) - v_n(t) = (Qw_{n-1})(t) - (Qv_{n-1})(t) \\
&= T\Big(\mathcal{F}(w_{n-1})(t) - \mathcal{F}(v_{n-1})(t)\Big) \\
&= T\Big(f(t, w_{n-1}(t), w_{n-1}(t-\tau)) - f(t, v_{n-1}(t), v_{n-1}(t-\tau)) + bw_{n-1}(t) - bv_{n-1}(t)\Big) \\
&\leq (b - L_3)T\Big(w_{n-1}(t) - v_{n-1}(t)\Big) \\
&\leq \cdots \\
&\leq (b - L_3)^n T^n \Big(w_0(t) - v_0(t)\Big).
\end{aligned}$$

The normality of cone K yields

$$\|w_n(t) - v_n(t)\| \leq (b - L_3)^n \|T\|^n N \|w_0(t) - v_0(t)\|, \quad t \in \mathbb{R},$$

which implies that

$$\|w_n - v_n\|_C \leq (b - L_3)^n \|T\|^n N \|w_0 - v_0\|_C.$$

Since $\max\{2b - a_0, 0\} < L_3 < b$, it follows that $0 < \frac{b-L_3}{a_0-b} < 1$. Thus,

$$(b - L_3)^n \|T\|^n \leq (\frac{b - L_3}{a_0 - b})^n \to 0 \quad (n \to \infty).$$

Therefore, there exists a unique function $\widetilde{u}$ belongs to $\cap_{n=1}^{\infty}[v_n, w_n]$ such that

$$v_n \to \widetilde{u}, \quad w_n \to \widetilde{u}$$

as $n \to \infty$. Since $Qv_{n-1} = v_n \leq \widetilde{u} \leq w_n \leq Qw_n$, taking $n \to \infty$ we get $\widetilde{u} = Q\widetilde{u}$. This implies that Equation (1) possesses unique ω-periodic solution. □

In Theorems 1–3, we always suppose that Equation (1) possesses lower and upper ω-periodic solutions v_0 and w_0 satisfying $v_0 \leq w_0$, but it is still a problem whether Equation (1) possesses lower and upper ω-periodic solutions. Next, we will prove that Equation (1) possesses a pair of lower and upper ω-periodic solutions.

Theorem 4. *Let the Hypothesis 2 (H2) hold and $a_0 > 0$. If f satisfies the condition Hypothesis 8 (H8) then Equation (1) possesses lower and upper ω-periodic solutions v_0 and w_0 satisfying $v_0 \leq w_0$.*

Hypothesis 8 (H8). *There exist $L^* \in [0, \frac{m_n a_0}{M_n})$ and $h \in C_\omega(\mathbb{R}, K)$ such that*

$$f(t, u, v) \leq L^* v + h(t), \quad u \geq \theta$$

and

$$f(t, u, v) \geq L^* v - h(t), \quad u \leq \theta$$

for any $u, v \in E$ and $t \in \mathbb{R}$.

Proof of Theorem 4. By Lemma 4, if the condition Hypothesis 2 (H2) holds and $a_0 > 0$, the LBVP (3) possesses a unique solution $r_n(t) > 0$ for $t \in \mathbb{R}$. By the definition of $G_n(t, s)$

and Lemma 2, we know that $T_n : C_\omega(\mathbb{R}, E) \to C_\omega(\mathbb{R}, E)$ is a positive linear bounded operator with $\|T_n\| = \frac{1}{a_0}$. If the condition Hypothesis 8 (H8) holds, we consider the linear differential equation

$$L_n u(t) = L^* u(t-\tau) + h(t), \quad t \in \mathbb{R}. \tag{20}$$

Let

$$B_2 u(t) = L^* u(t-\tau), \quad t \in \mathbb{R}.$$

Then $B_2 : C_\omega(\mathbb{R}, E) \to C_\omega(\mathbb{R}, E)$ is positive and linear bounded, and $\|B_2\| \le L^*$. Lemma 2 yields that Equation (20) possesses a unique ω-periodic solution

$$\widehat{u}(t) = (T_n B_2)\widehat{u}(t) + T_n h(t), \quad t \in \mathbb{R}. \tag{21}$$

Since $\|T_n B_2\| \le \frac{L^*}{a_0} < \frac{m_n}{M_n} \le 1$, we get that $(I - T_n B_2)^{-1}$ exists and $(I - T_n B_2)^{-1} = \sum_{i=0}^{\infty} (T_n B_2)^i$ is a positive linear operator. Hence, from (21), $\widehat{u}(t)$ is given by

$$\widehat{u}(t) = (I - T_n B_2)^{-1} T_n h(t) \quad t \in \mathbb{R}$$

and $\widehat{u}(t) \ge 0$ for any $t \in \mathbb{R}$ owing to $h(t) \in K$. Let $v_0 = -\widehat{u}$ and $w_0 = \widehat{u}$, by the Hypothesis 8 (H8), we have

$$\begin{aligned} L_n v_0(t) &= L_n(-\widehat{u}(t)) = -L_n \widehat{u}(t) = L^*(-\widehat{u}(t-\tau)) - h(t) \\ &\le f(t, -\widehat{u}(t), -\widehat{u}(t-\tau)) \\ &= f(t, v_0(t), v_0(t-\tau)) \end{aligned}$$

and

$$\begin{aligned} L_n w_0(t) &= L_n(\widehat{u}(t)) = L^*(\widehat{u}(t-\tau)) + h(t) \\ &\ge f(t, \widehat{u}(t), \widehat{u}(t-\tau)) \\ &= f(t, w_0(t), w_0(t-\tau)). \end{aligned}$$

Hence, the Equation (1) possesses lower and upper ω-periodic solutions v_0 and w_0 satisfying $v_0 \le w_0$. □

Example 1. *Consider the following fourth-order ordinary differential equation in Banach space E*

$$u^{(4)}(t) + u'''(t) + u''(t) + u'(t) + u(t) = F(t, u(t), u(t-\tau)), \quad t \in \mathbb{R}, \tag{22}$$

where $u \in C_\omega(\mathbb{R}, E)$ and $F(t, u, v) : \mathbb{R} \times E \times E \to E$ is continuous and ω-periodic with respect to t. We suppose that the following conditions hold.

Hypothesis 9 (P1). $\{\lambda \in \mathbb{C} : \sum_{i=0}^{4} \lambda^i = 0\} \subset \{\xi \in \mathbb{C} : |Im\xi| < \frac{\pi}{\omega}\}$.

Hypothesis 10 (P2). *There exist $L^* \in [0, \frac{m_n}{M_n})$ and $h \in C_\omega(\mathbb{R}, K)$ such that*

$$F(t, u, v) \le L^* v + h(t), \quad u \ge \theta$$

and

$$F(t, u, v) \ge L^* v - h(t), \quad u \le \theta$$

for any $u, v \in E$ and $t \in \mathbb{R}$.

Then the condition Hypothesis 9 (P1) implies Hypothesis 2 (H2). if we choose $f(t, u(t), u(t-\tau)) = F(t, u(t), u(t-\tau))$, the Hypothesis 10 (P2) yields Hypothesis 8 (H8).

Thus, by Theorem 4, the fourth-order ordinary differential Equation (22) possesses lower and upper ω-periodic solutions v_0 and w_0 satisfying $v_0 \leq w_0$.

4. Conclusions

In this work, the maximum principle of the linear problem involving delay term is first established. Then the approach of upper and lower solutions and the monotone iterative technique are applied to consider the existence as well as the uniqueness of ω-periodic solutions for the nth-order ordinary differential Equation (1) by using the obtained maximum principle. The existence of lower and upper ω-periodic solutions of Equation (1) is also discussed in this paper. The results extend and improve some existing works.

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