Article

A Dynamic Distance Measure of Picture Fuzzy Sets and Its Application

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Abstract: Picture fuzzy sets, which are the extension of intuitionistic fuzzy sets, can deal with inconsistent information better in practical applications. A distance measure is an important mathematical tool to calculate the difference degree between picture fuzzy sets. Although some distance measures of picture fuzzy sets have been constructed, there are some unreasonable and counterintuitive cases. The main reason is that the existing distance measures do not or seldom consider the refusal degree of picture fuzzy sets. In order to solve these unreasonable and counterintuitive cases, in this paper, we propose a dynamic distance measure of picture fuzzy sets based on a picture fuzzy point operator. Through a numerical comparison and multi-criteria decision-making problems, we show that the proposed distance measure is reasonable and effective.

Keywords: picture fuzzy set; picture fuzzy point operator; distance measure; multi-criteria decision-making

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1. Introduction

The theory of fuzzy sets [1], which can deal with fuzzy information in practical applications, is a very important theory. It has been applied in decision-making [2], image edge detection [3], medical diagnosis [4,5], three-way decision-making [6,7], and so on. The fuzzy set theory cannot handle incomplete information, because the sum of the membership degree and non-membership degree is one. The intuitionistic fuzzy set was introduced in 1986 [8], which is composed of the membership degree and the non-membership degree. The hesitancy degree can be obtained by the membership degree and the non-membership degree. Since then, it has been used to solve pattern recognition problems [9–12], decision-making problems [13–17], medical diagnosis problems [18–20], and so on. However, it cannot express uncertain and incomplete information more accurately in real situations. For example, in the election of a village leader, the voting results can be divided into four categories: vote for, abstain from voting, vote against, and refuse to vote. Abstaining from voting means that the ballot is left blank, rejecting both voting for and voting against the candidate, but still casting the ballot. Refusing to vote means either an invalid ballot or bypassing the vote [21]. In this situation, the intuitionistic fuzzy set fails to attain any satisfactory result. In order to solve these problems, the picture fuzzy set (PFS) was introduced by Cuong in 2013 [22], which is formed by the membership degree, non-membership degree, refusal degree, and neutral degree. Compared to the intuitionistic fuzzy set, the PFS divides the hesitancy degree into two parts, i.e., the refusal degree and the neutral degree, which can easily handle uncertain and incomplete information more accurately in real situations such as human opinions involving four responses of the types: yes, abstain, no, and refusal. Because of this advantage, many have started to study the PFS theory.

A distance measure as an important mathematical tool to distinguish the difference between fuzzy sets has been widely used to solve multi-criteria decision-making problems,

medical diagnosis problems, pattern recognition problems, and so on. Since Cuong [22] proposed PFSs, distance measures of PFSs have been proposed in the literature. Dinh and Thao [23] proposed distance and similarity measures and used them to solve multi-criteria decision-making problems. Since Dinh and Thao's measures [23] failed to consider the refusal degree, Singh et al. [24] developed new distance and similarity measures. Although Singh et al.'s distance measures take into account the refusal degree, there are some counterintuitive situations (see Example 2). In 2017, Wei [25] extended the cosine similarity measures of intuitionistic fuzzy sets to PFSs and gave distance measures based on cosine similarity measures. In 2018, Wei [25] proposed arccos distance measures, which cannot solve the problem of $\frac{0}{0}$. Son [21] proposed the generalized picture distance measure and used it to handle picture fuzzy clustering problems. Palash [26] pointed out that Son's distance measure [21] has counterintuitive examples and proposed a new distance measure. Although some distance measures have been proposed, they are cases in which they are counterintuitive (see Example 2), which causes a great deal of trouble for actual users. The main reason is that the existing distance measures do not or seldom consider the refusal degree of picture fuzzy sets.

The ability of a distance measure to distinguish the difference between fuzzy sets is determined by the form of its expression and the information that is contained in the expression: the more information that the distance measure has, the stronger its ability to distinguish is. Therefore, if we want the distance measure to be more effective, we need to construct a new distance measure from a new perspective, which contains more information. In this paper, we construct a dynamic distance measure of PFSs based on the picture fuzzy point operator, which not only fully considers the rejection degree of PFSs, but also reflects the attitude of decision-makers who refuse to participate. In order to show that the proposed distance measure is more effective and reasonable than other distance measures, the proposed distance measure is compared with the existing distance measures by a numerical example and multi-attribute decision-making problems.

The rest of this paper is organized as follows. In Section 2, we review some definitions and distance measures. In Section 3, we give a picture fuzzy point operator and discuss the properties. In Section 4, we develop new distance measures based on the picture fuzzy point operation. In Section 5, the proposed distance measure is used to solve the problems of multi-criteria decision-making. Finally, some conclusions are given in Section 6.

2. Preliminaries

In this section, we review the basic concepts and some distance measures.

Definition 1. Reference [27]: Let X be a universe. A PFS A on X is an object having the form: $A = \{ \langle \mu_A(x), \eta_A(x), \nu_A(x) \rangle | x \in X \}$, where $\mu_A(x)$ is the positive degree, $\eta_A(x)$ is the neutral degree, and $\nu_A(x)$ is the negative degree. They meet the conditions: $\mu_A(x), \eta_A(x), \nu_A(x) \in [0, 1]$, and $0 \leq \mu_A(x) + \eta_A(x) + \nu_A(x) \leq 1, \forall x \in X$. The refusal degree $\rho_A(x)$ is calculated by $1 - (\mu_A(x) + \eta_A(x) + \nu_A(x))$, $\forall x \in X$. $\langle \mu_A, \eta_A, \nu_A \rangle$ is called a picture fuzzy number (PFN).

If $\eta_A(x) = 0$, then A is an intuitionistic fuzzy set. If $\eta_A(x) = 0$ and $\mu(x) + \nu(x) = 1$, then A is a fuzzy set.

Definition 2. Reference [27]: Let A and $B \in$ PFSs on X , then

$A \subseteq B$ iff $\mu_A(x) \leq \mu_B(x), \eta_A(x) \geq \eta_B(x), \nu_A(x) \geq \nu_B(x)$ for any $x \in X$.

$A = B$ iff $A \subseteq B$ and $B \subseteq A$, i.e., $\mu_A(x) = \mu_B(x), \eta_A(x) = \eta_B(x), \nu_A(x) = \nu_B(x)$ for any $x \in X$.

$A^c = \langle \nu_A(x), \eta_A(x), \mu_A(x) \rangle$ for any $x \in X$.

Definition 3. Reference [24]: Let X be a universe and A, B, C be three PFSs on X , then the distance between A and B is defined as $d(A, B)$, which satisfies the following properties:

(D1) $0 \leq d(A, B) \leq 1$;

(D2) $d(A, B) = 0$ iff $A = B$;

(D3) $d(A, B) = d(B, A)$;

(D4) If $A \subseteq B \subseteq C$, then $d(A, B) \leq d(A, C)$, $d(B, C) \leq d(A, C)$.

We review some existing distances between PFSs. Let $X = \{x_1, x_2, \dots, x_n\}$, $A, B \in \text{PFSs}(X)$. Some existing measures of the distance between A and B are given as follows:

Dinh and Thao's distance measures [23]:

$$d_H(A, B) = \frac{1}{3n} \sum_{i=1}^n [|\mu_A(x_i) - \mu_B(x_i)| + |\eta_A(x_i) - \eta_B(x_i)| + |\nu_A(x_i) - \nu_B(x_i)|], \quad (1)$$

$$d_E(A, B) = \frac{1}{3n} \left\{ \sum_{i=1}^n [|\mu_A(x_i) - \mu_B(x_i)|^2 + |\eta_A(x_i) - \eta_B(x_i)|^2 + |\nu_A(x_i) - \nu_B(x_i)|^2] \right\}^{\frac{1}{2}}, \quad (2)$$

$$d_{MH}(A, B) = \frac{1}{n} \sum_{i=1}^n \max [|\mu_A(x_i) - \mu_B(x_i)|, |\eta_A(x_i) - \eta_B(x_i)|, |\nu_A(x_i) - \nu_B(x_i)|], \quad (3)$$

$$d_{Em}(A, B) = \frac{1}{n} \left\{ \sum_{i=1}^n \max [|\mu_A(x_i) - \mu_B(x_i)|^2, |\eta_A(x_i) - \eta_B(x_i)|^2, |\nu_A(x_i) - \nu_B(x_i)|^2] \right\}^{\frac{1}{2}}. \quad (4)$$

Wei's distance measures [25]:

$$d_{CS1}(A, B) = 1 - \frac{1}{n} \sum_{i=1}^n \cos \left\{ \frac{\pi}{2} [|\mu_A(x_i) - \mu_B(x_i)| \vee |\eta_A(x_i) - \eta_B(x_i)| \vee |\nu_A(x_i) - \nu_B(x_i)|] \right\}, \quad (5)$$

$$d_{CS2}(A, B) = 1 - \frac{1}{n} \sum_{i=1}^n \cos \left\{ \frac{\pi}{4} [|\mu_A(x_i) - \mu_B(x_i)| + |\eta_A(x_i) - \eta_B(x_i)| + |\nu_A(x_i) - \nu_B(x_i)|] \right\}, \quad (6)$$

$$d_{CS3}(A, B) = 1 - \frac{1}{n} \sum_{i=1}^n \cos \left\{ \frac{\pi}{2} [|\mu_A(x_i) - \mu_B(x_i)| \vee |\eta_A(x_i) - \eta_B(x_i)| \vee |\nu_A(x_i) - \nu_B(x_i)| \vee |\rho_A(x_i) - \rho_B(x_i)|] \right\}, \quad (7)$$

$$d_{CS4}(A, B) = 1 - \frac{1}{n} \sum_{i=1}^n \cos \left\{ \frac{\pi}{4} [|\mu_A(x_i) - \mu_B(x_i)| + |\eta_A(x_i) - \eta_B(x_i)| + |\nu_A(x_i) - \nu_B(x_i)| + |\rho_A(x_i) - \rho_B(x_i)|] \right\}, \quad (8)$$

$$d_{CT1}(A, B) = 1 - \frac{1}{n} \sum_{i=1}^n \cot \left[\frac{\pi}{4} + \frac{\pi}{4} \{ |\mu_A(x_i) - \mu_B(x_i)| \vee |\eta_A(x_i) - \eta_B(x_i)| \vee |\nu_A(x_i) - \nu_B(x_i)| \} \right], \quad (9)$$

$$d_{CT2}(A, B) = 1 - \frac{1}{n} \sum_{i=1}^n \cot \left[\frac{\pi}{4} + \frac{\pi}{4} \{ |\mu_A(x_i) - \mu_B(x_i)| \vee |\eta_A(x_i) - \eta_B(x_i)| \vee |\nu_A(x_i) - \nu_B(x_i)| \vee |\rho_A(x_i) - \rho_B(x_i)| \} \right]. \quad (10)$$

Singh et al. distance measures [24]:

$$d_{ph}(A, B) = \frac{1}{4n} \sum_{i=1}^n [|\mu_A(x_i) - \mu_B(x_i)| + |\eta_A(x_i) - \eta_B(x_i)| + |\nu_A(x_i) - \nu_B(x_i)| + |\rho_A(x_i) - \rho_B(x_i)|], \quad (11)$$

$$d_{pe}(A, B) = \sqrt{\frac{1}{4n} \sum_{i=1}^n [|\mu_A(x_i) - \mu_B(x_i)|^2 + |\eta_A(x_i) - \eta_B(x_i)|^2 + |\nu_A(x_i) - \nu_B(x_i)|^2 + |\rho_A(x_i) - \rho_B(x_i)|^2]}, \quad (12)$$

$$d_{gpnhh}(A, B) = \frac{1}{4n} \sum_{i=1}^n \max(|\mu_A(x_i) - \mu_B(x_i)|, |\eta_A(x_i) - \eta_B(x_i)|, |\nu_A(x_i) - \nu_B(x_i)|, |\rho_A(x_i) - \rho_B(x_i)|), \quad (13)$$

$$d_{gpneh}(A, B) = \sqrt{\frac{1}{4n} \sum_{i=1}^n \max((\mu_A(x_i) - \mu_B(x_i))^2, (\eta_A(x_i) - \eta_B(x_i))^2, (\nu_A(x_i) - \nu_B(x_i))^2, (\rho_A(x_i) - \rho_B(x_i))^2)}. \quad (14)$$

Wei's distance measure [28]:

$$d_c(A, B) = \frac{1}{\pi} \arccos \left(\frac{1}{n} \sum_{i=1}^n \frac{\mu_A(x_i)\mu_B(x_i) + \eta_A(x_i)\eta_B(x_i) + \nu_A(x_i)\nu_B(x_i)}{\sqrt{\mu_A^2(x_i) + \eta_A^2(x_i) + \nu_A^2(x_i)} \sqrt{\mu_B^2(x_i) + \eta_B^2(x_i) + \nu_B^2(x_i)}} \right). \quad (15)$$

Son's distance measure [21]:

$$d_G(A, B) = \frac{(\frac{1}{n} \sum_{i=1}^n (\frac{\Delta_{\mu_i}^p + \Delta_{\eta_i}^p + \Delta_{\nu_i}^p}{3} + \max\{\Delta_{T_i}^p, \Delta_{L_i}^p, \Delta_{V_i}^p\}))^{1/p}}{(\frac{1}{n} \sum_{i=1}^n (\frac{\Delta_{\mu_i}^p + \Delta_{\eta_i}^p + \Delta_{\nu_i}^p}{3} + \max\{\Delta_{\mu_i}^p, \Delta_{\eta_i}^p, \Delta_{\nu_i}^p\}))^{1/p} + (\max\{\Phi_i^A, \Phi_i^B\} + \frac{1}{n} \sum_{i=1}^n |\Phi_i^A - \Phi_i^B|^p)^{1/p} + 1}, \quad (16)$$

where $\Delta_{\mu_i} = |\mu_A(x_i) - \mu_B(x_i)|$; $\Delta_{\eta_i} = |\eta_A(x_i) - \eta_B(x_i)|$; $\Delta_{\nu_i} = |\nu_A(x_i) - \nu_B(x_i)|$; $\Phi_i^A = |\mu_A(x_i) + \eta_A(x_i) + \nu_A(x_i)|$; $\Phi_i^B = |\mu_B(x_i) + \eta_B(x_i) + \nu_B(x_i)|$.

Palash's distance measure [26]:

$$d_P(A, B) = \frac{(\frac{1}{n} \sum_{i=1}^n (\frac{\Delta_{\mu_i}^p + \Delta_{\eta_i}^p + \Delta_{v_i}^p}{3} + \max\{\Delta_{\mu_i}^p, \Delta_{\eta_i}^p, \Delta_{v_i}^p\}))^{1/p}}{(\frac{1}{n} \sum_{i=1}^n (\frac{\Delta_{\mu_i}^p + \Delta_{\eta_i}^p + \Delta_{v_i}^p}{3} + \max\{\Delta_{\mu_i}^p, \Delta_{\eta_i}^p, \Delta_{v_i}^p\}))^{1/p} + \frac{1}{n} \sum_{i=1}^n (\max\{\Phi_i^A, \Phi_i^B\} + |\Phi_i^A - \Phi_i^B|)^{1/p} + 1}, \quad (17)$$

where $\Delta_{\mu_i} = |\mu_A(x_i) - \mu_B(x_i)|$; $\Delta_{\eta_i} = |\eta_A(x_i) - \eta_B(x_i)|$; $\Delta_{v_i} = |v_A(x_i) - v_B(x_i)|$; $\Phi_i^A = |\mu_A(x_i) + \eta_A(x_i) + v_A(x_i)|$; $\Phi_i^B = |\mu_B(x_i) + \eta_B(x_i) + v_B(x_i)|$.

3. Picture Fuzzy Point Operator

In the process of decision-making, in order to get more accurate results, decision makers need to obtain more accurate information. Therefore, it is necessary to reduce the uncertain information. In this section, we propose the picture fuzzy point operator and discuss its related properties.

Definition 4. Let X be a universe, $x \in X$, $A = \{\langle \mu_A(x), \eta_A(x), v_A(x) \rangle | x \in X\}$. A picture fuzzy point operator, $P_{\alpha_x, \beta_x, \gamma_x}: PFSs(X) \rightarrow PFSs(X)$, is defined by:

$$P_{\alpha_x, \beta_x, \gamma_x}(A) = \{\langle \mu_A(x) + \alpha_x \rho_A(x), \eta_A(x) + \beta_x \rho_A(x), v_A(x) + \gamma_x \rho_A(x) \rangle | x \in X\}, \quad (18)$$

where $\alpha_x, \beta_x, \gamma_x \in [0, 1]$, and $\alpha_x + \beta_x + \gamma_x \leq 1$.

If PFS A degenerates to the intuitionistic fuzzy set, then $P_{\alpha_x, \beta_x, \gamma_x}(A)$ degenerates to Atanassov's point operator [29].

Remark 1.

(1) During the picture fuzzy point's, $P_{\alpha_x, \beta_x, \gamma_x}(A)$, transformation, the refusal degree $\rho_A(x)$ is divided into four parts: $\alpha_x \rho_A(x)$, $\beta_x \rho_A(x)$, $\gamma_x \rho_A(x)$, and $(1 - \alpha_x - \beta_x - \gamma_x) \rho_A(x)$ for the first time through a given pair $(\alpha_x, \beta_x, \gamma_x)$ ($\alpha_x, \beta_x, \gamma_x \in [0, 1]$ and $\alpha_x + \beta_x + \gamma_x \in [0, 1]$). Then, we can obtain new information about x through the picture fuzzy point operator $P_{\alpha_x, \beta_x, \gamma_x}$. This is very necessary in practical application. For example, in the process of voting, let $x \in X$; the values $\mu_A(x)$, $\eta_A(x)$, $v_A(x)$ and $\rho_A(x)$ can be explained as "vote for", "abstain from voting", "vote against", and "refuse to vote", respectively. For the "refuse to vote", some people are inclined to vote in the affirmative, some may be neutral, some may vote against, and the rest still refuse to vote. The value $\rho_A(x)$ can be divided into four parts: $\alpha_x \rho_A(x)$, $\beta_x \rho_A(x)$, $\gamma_x \rho_A(x)$, and $(1 - \alpha_x - \beta_x - \gamma_x) \rho_A(x)$ by a given pair $(\alpha_x, \beta_x, \gamma_x)$. Therefore, the values of "vote for", "abstain from voting", "vote against" and "refuse to vote" are $\mu_A(x) + \alpha_x \rho_A(x)$, $\eta_A(x) + \beta_x \rho_A(x)$, $v_A(x) + \gamma_x \rho_A(x)$, and $(1 - \alpha_x - \beta_x - \gamma_x) \rho_A(x)$, respectively. Obviously, $(1 - \alpha_x - \beta_x - \gamma_x) \rho_A(x) \leq \rho_A(x)$.

(2) If PFS A degenerates to the fuzzy set, then $P_{\alpha_x, \beta_x, \gamma_x}(A) = A$ is a fuzzy set.

(3) Let $\alpha_x = \frac{\mu_A(x)}{\mu_A(x) + \eta_A(x) + v_A(x)}$, $\beta_x = \frac{\eta_A(x)}{\mu_A(x) + \eta_A(x) + v_A(x)}$, $\gamma_x = \frac{v_A(x)}{\mu_A(x) + \eta_A(x) + v_A(x)}$. If $\mu_A(x) + \eta_A(x) + v_A(x) \neq 0$, then $P_{\alpha_x, \beta_x, \gamma_x}(A) = \langle \alpha_x, \beta_x, \gamma_x \rangle$ is an intuitionistic fuzzy set.

Example 1. In a democratic election, the parliament issues 100 votes for a candidate. The voting results are divided into four groups: "vote for" (45), "abstain from voting" (10), "vote against" (15) and "refuse to vote" (30). This can be represented by a PFN $A = \langle 0.45, 0.10, 0.15 \rangle$. For those who refuse to vote, fifty percent prefer to vote in the affirmative, 20% may abstain, 20% vote against, and the rest still refuse to vote, which can be represented by a pair $(\alpha, \beta, \gamma) = (0.50, 0.20, 0.20)$. The final vote may be: "vote for" (60), "abstain from voting" (16), "vote against" (21) and "refuse to vote" (3), which can be represented by a PFN $B = \langle 0.60, 0.16, 0.21 \rangle$.

Definition 5. Let X be a universe, $x \in X$, $A = \{\langle \mu_A(x), \eta_A(x), v_A(x) \rangle | x \in X\}$, $P_{\alpha_x, \beta_x, \gamma_x}^m(A)$, defined by:

$$P_{\alpha_x, \beta_x, \gamma_x}^m(A) = P_{\alpha_x, \beta_x, \gamma_x}(P_{\alpha_x, \beta_x, \gamma_x}^{m-1}(A)), \quad (19)$$

where $\alpha_x, \beta_x, \gamma_x \in [0, 1]$, $0 < \alpha_x + \beta_x + \gamma_x \leq 1$, m is any natural number, $P_{\alpha_x, \beta_x, \gamma_x}^0(A) = A$.

Remark 2.

(1) On the basis of Definition 5, $P_{\alpha_x, \beta_x, \gamma_x}^m(A)$ and $\rho_{P_{\alpha_x, \beta_x, \gamma_x}^m(A)}(x)$ are calculated, respectively, as:

$$P_{\alpha_x, \beta_x, \gamma_x}^m(A) = \left\{ \mu_A(x) + \alpha_x \rho_A(x) \frac{1 - (1 - \alpha_x - \beta_x - \gamma_x)^m}{\alpha_x + \beta_x + \gamma_x}, \right. \\ \left. A(x) + x A(x) \frac{1 - (1 - \alpha_x - \beta_x - \gamma_x)^m}{\alpha_x + \beta_x + \gamma_x}, \right. \\ \left. \nu_A(x) + \gamma_x A(x) \frac{1 - (1 - \alpha_x - \beta_x - \gamma_x)^m}{\alpha_x + \beta_x + \gamma_x} \right\} \mid x \in X, \quad (20)$$

$$\rho_{P_{\alpha_x, \beta_x, \gamma_x}^m(A)}(x) = (1 - \alpha_x - \beta_x - \gamma_x)^m \rho_A(x). \quad (21)$$

(2) Based on Equation (21), we get $\rho_{P_{\alpha_x, \beta_x, \gamma_x}^m(A)}(x) \leq \rho_{P_{\alpha_x, \beta_x, \gamma_x}^{m-1}(A)}(x)$.

Definition 6. Let X be a universe, $x \in X$, $A \in \text{PFSs}(X)$, $\lim_{m \rightarrow \lambda} P_{\alpha_x, \beta_x, \gamma_x}^m(A)$, defined by

$$\lim_{m \rightarrow \lambda} P_{\alpha_x, \beta_x, \gamma_x}^m(A) = \left\{ \left\langle \lim_{m \rightarrow \lambda} \mu_{P_{\alpha_x, \beta_x, \gamma_x}^m(A)}(x), \lim_{m \rightarrow \lambda} \eta_{P_{\alpha_x, \beta_x, \gamma_x}^m(A)}(x), \lim_{m \rightarrow \lambda} \nu_{P_{\alpha_x, \beta_x, \gamma_x}^m(A)}(x) \right\rangle \mid x \in X \right\},$$

where λ is any positive integer.

Remark 3.

(1) On the basis of Definition 6 and Remark 2 (1), $\lim_{m \rightarrow +\infty} P_{\alpha_x, \beta_x, \gamma_x}^m(A)$ is calculated as:

$$\lim_{m \rightarrow +\infty} P_{\alpha_x, \beta_x, \gamma_x}^m(A) = \left\{ \mu_A(x) + \alpha_x \rho_A(x) \frac{1}{\alpha_x + \beta_x + \gamma_x}, \right. \\ \left. \eta_A(x) + \beta_x \rho_A(x) \frac{1}{\alpha_x + \beta_x + \gamma_x}, \right. \\ \left. \nu_A(x) + \gamma_x \rho_A(x) \frac{1}{\alpha_x + \beta_x + \gamma_x} \right\} \mid x \in X. \quad (22)$$

(2) Based on Equation (21), we can get $\lim_{m \rightarrow +\infty} \rho_{P_{\alpha_x, \beta_x, \gamma_x}^m(A)}(x) = (1 - \alpha_x - \beta_x - \gamma_x)^m \rho_A(x) = 0$.

4. Distance Measure Based on the Picture Fuzzy Point Operator**4.1. Distance between Picture Fuzzy Sets**

Theorem 1. Let $X = \{x_1, x_2, x_3, \dots, x_n\}$ be a universe. Suppose that A, B are any two PFSs on X , $\alpha_{x_i}, \beta_{x_i}, \gamma_{x_i} \in [0, 1]$, and $0 < \alpha_{x_i} + \beta_{x_i} + \gamma_{x_i} \leq 1$, m being any natural number, $p \in [1, +\infty)$. A function $d_m: \text{PFSs}(X) \times \text{PFSs}(X) \rightarrow [0, 1]$ is defined by:

$$d_m(A, B) = \sqrt[p]{\frac{1}{3n} \sum_{i=1}^n (|\varphi_A(x_i) - \varphi_B(x_i)|^p + |\psi_A(x_i) - \psi_B(x_i)|^p + |\phi_A(x_i) - \phi_B(x_i)|^p)}, \quad (23)$$

$$\text{where } \varphi_A(x_i) = \mu_A(x_i) + \alpha_{x_i} \rho_A(x_i) \frac{1 - (1 - \alpha_{x_i} - \beta_{x_i} - \gamma_{x_i})^m}{\alpha_{x_i} + \beta_{x_i} + \gamma_{x_i}}, \psi_A(x_i) = \eta_A(x_i) + \beta_{x_i} \rho_A(x_i) \frac{1 - (1 - \alpha_{x_i} - \beta_{x_i} - \gamma_{x_i})^m}{\alpha_{x_i} + \beta_{x_i} + \gamma_{x_i}}, \phi_A(x_i) = \nu_A(x_i) + \gamma_{x_i} \rho_A(x_i) \frac{1 - (1 - \alpha_{x_i} - \beta_{x_i} - \gamma_{x_i})^m}{\alpha_{x_i} + \beta_{x_i} + \gamma_{x_i}}, \\ \varphi_B(x_i) = \mu_B(x_i) + \alpha_{x_i} \rho_B(x_i) \frac{1 - (1 - \alpha_{x_i} - \beta_{x_i} - \gamma_{x_i})^m}{\alpha_{x_i} + \beta_{x_i} + \gamma_{x_i}}, \psi_B(x_i) = \eta_B(x_i) + \beta_{x_i} \rho_B(x_i) \frac{1 - (1 - \alpha_{x_i} - \beta_{x_i} - \gamma_{x_i})^m}{\alpha_{x_i} + \beta_{x_i} + \gamma_{x_i}}, \phi_B(x_i) = \nu_B(x_i) + \gamma_{x_i} \rho_B(x_i) \frac{1 - (1 - \alpha_{x_i} - \beta_{x_i} - \gamma_{x_i})^m}{\alpha_{x_i} + \beta_{x_i} + \gamma_{x_i}}.$$

Then, d_m is a distance between A and B .

Proof. Let $X = \{x_1, x_2, x_3, \dots, x_n\}$ be a universe. Suppose that A, B, C are any three PFSs on X .

$$(D1) \varphi_A(x_i) - \varphi_B(x_i) = \mu_A(x_i) - \mu_B(x_i) + \alpha_{x_i} \frac{1 - (1 - \alpha_{x_i} - \beta_{x_i} - \gamma_{x_i})^m}{\alpha_{x_i} + \beta_{x_i} + \gamma_{x_i}} (\rho_A(x_i) - \rho_B(x_i)).$$

$$\text{Let } m_{x_i} = \alpha_{x_i} \frac{1-(1-\alpha_{x_i}-\beta_{x_i}-\gamma_{x_i})^m}{\alpha_{x_i}+\beta_{x_i}+\gamma_{x_i}}, \quad n_{x_i} = \beta_{x_i} \frac{1-(1-\alpha_{x_i}-\beta_{x_i}-\gamma_{x_i})^m}{\alpha_{x_i}+\beta_{x_i}+\gamma_{x_i}}, \quad t_{x_i} = \gamma_{x_i} \frac{1-(1-\alpha_{x_i}-\beta_{x_i}-\gamma_{x_i})^m}{\alpha_{x_i}+\beta_{x_i}+\gamma_{x_i}}. \quad \square$$

Since $0 < \alpha_{x_i} + \beta_{x_i} + \gamma_{x_i} \leq 1$, then $1 - (1 - \alpha_{x_i} - \beta_{x_i} - \gamma_{x_i})^m \in (0, 1]$.

Therefore, $0 < m_{x_i} \leq 1, 0 < n_{x_i} \leq 1, 0 < t_{x_i} \leq 1$.

By $\rho_A(x_i) = 1 - (\mu_A(x_i) + \eta_A(x_i) + \nu_A(x_i))$ and $\rho_B(x_i) = 1 - (\mu_B(x_i) + \eta_B(x_i) + \nu_B(x_i))$, we have:

$$\begin{aligned} \varphi_A(x_i) - \varphi_B(x_i) &= \mu_A(x_i) - \mu_B(x_i) + \alpha_{x_i} \frac{1 - (1 - \alpha_{x_i} - \beta_{x_i} - \gamma_{x_i})^m}{\alpha_{x_i} + \beta_{x_i} + \gamma_{x_i}} (\rho_A(x_i) - \rho_B(x_i)) \\ &= \mu_A(x_i) - \mu_B(x_i) + m_{x_i} (\rho_A(x_i) - \rho_B(x_i)) \\ &= (1 - m_{x_i})(\mu_A(x_i) - \mu_B(x_i)) + m_{x_i}(\eta_B(x_i) + \nu_B(x_i) - \eta_A(x_i) - \nu_A(x_i)). \end{aligned} \quad (24)$$

$$\psi_A(x_i) - \psi_B(x_i) = (1 - n_{x_i})(\eta_A(x_i) - \eta_B(x_i)) + n_{x_i}(\mu_B(x_i) + \nu_B(x_i) - \mu_A(x_i) - \nu_A(x_i)). \quad (25)$$

$$\phi_A(x_i) - \phi_B(x_i) = (1 - t_{x_i})(\nu_A(x_i) - \nu_B(x_i)) + t_{x_i}(\eta_B(x_i) + \mu_B(x_i) - \eta_A(x_i) - \mu_A(x_i)). \quad (26)$$

Since $\mu_A(x_i), \mu_B(x_i) \in [0, 1]$, $\eta_B(x_i) + \nu_B(x_i) \in [0, 1]$ and $\eta_A(x_i) + \nu_A(x_i) \in [0, 1]$, we get:

$$\begin{aligned} -1 + m_{x_i} &\leq (1 - m_{x_i})(\mu_A(x_i) - \mu_B(x_i)) \leq 1 - m_{x_i}, \\ -m_{x_i} &\leq m_{x_i}(\eta_B(x_i) + \nu_B(x_i) - \eta_A(x_i) - \nu_A(x_i)) \leq m_{x_i}, \end{aligned}$$

Similarly,

$$\begin{aligned} -1 + n_{x_i} &\leq (1 - n_{x_i})(\eta_A(x_i) - \eta_B(x_i)) \leq 1 - n_{x_i}, \\ -n_{x_i} &\leq n_{x_i}(\mu_B(x_i) + \nu_B(x_i) - \mu_A(x_i) - \nu_A(x_i)) \leq n_{x_i}, \\ -1 + t_{x_i} &\leq (1 - t_{x_i})(\nu_A(x_i) - \nu_B(x_i)) \leq 1 - t_{x_i}, \\ -t_{x_i} &\leq t_{x_i}(\eta_B(x_i) + \mu_B(x_i) - \eta_A(x_i) - \mu_A(x_i)) \leq t_{x_i}; \end{aligned}$$

hence,

$$\begin{aligned} -1 &\leq (1 - m_{x_i})(\mu_A(x_i) - \mu_B(x_i)) + m_{x_i}(\eta_B(x_i) + \nu_B(x_i) - \eta_A(x_i) - \nu_A(x_i)) \leq 1, \\ -1 &\leq (1 - n_{x_i})(\eta_A(x_i) - \eta_B(x_i)) + n_{x_i}(\mu_B(x_i) + \nu_B(x_i) - \mu_A(x_i) - \nu_A(x_i)) \leq 1, \\ -1 &\leq (1 - t_{x_i})(\nu_A(x_i) - \nu_B(x_i)) + t_{x_i}(\eta_B(x_i) + \mu_B(x_i) - \eta_A(x_i) - \mu_A(x_i)) \leq 1. \end{aligned}$$

i.e.,

$$\begin{aligned} |\varphi_A(x_i) - \varphi_B(x_i)| &\leq 1, \\ |\psi_A(x_i) - \psi_B(x_i)| &\leq 1, \\ |\phi_A(x_i) - \phi_B(x_i)| &\leq 1. \end{aligned}$$

Therefore,

$$0 \leq \sqrt[p]{\frac{1}{3n} \sum_{i=1}^n (|\varphi_A(x_i) - \varphi_B(x_i)|^p + |\psi_A(x_i) - \psi_B(x_i)|^p + |\phi_A(x_i) - \phi_B(x_i)|^p)} \leq 1,$$

i.e., $d_m(A, B) \in [0, 1]$.

(D2) If $d_m(A, B) = 0$, then $|\varphi_A(x_i) - \varphi_B(x_i)| = 0, |\psi_A(x_i) - \psi_B(x_i)| = 0, |\phi_A(x_i) - \phi_B(x_i)| = 0$, i.e.,

$$\begin{aligned}\mu_A(x_i) + \alpha_{x_i}\rho_A(x_i) \frac{1 - (1 - \alpha_{x_i} - \beta_{x_i} - \gamma_{x_i})^m}{\alpha_{x_i} + \beta_{x_i} + \gamma_{x_i}} &= \mu_B(x_i) + \alpha_{x_i}\rho_B(x_i) \frac{1 - (1 - \alpha_{x_i} - \beta_{x_i} - \gamma_{x_i})^m}{\alpha_{x_i} + \beta_{x_i} + \gamma_{x_i}}, \\ \eta_A(x_i) + \beta_{x_i}\rho_A(x_i) \frac{1 - (1 - \alpha_{x_i} - \beta_{x_i} - \gamma_{x_i})^m}{\alpha_{x_i} + \beta_{x_i} + \gamma_{x_i}} &= \eta_B(x_i) + \beta_{x_i}\rho_B(x_i) \frac{1 - (1 - \alpha_{x_i} - \beta_{x_i} - \gamma_{x_i})^m}{\alpha_{x_i} + \beta_{x_i} + \gamma_{x_i}}, \\ \nu_A(x_i) + \gamma_{x_i}\rho_A(x_i) \frac{1 - (1 - \alpha_{x_i} - \beta_{x_i} - \gamma_{x_i})^m}{\alpha_{x_i} + \beta_{x_i} + \gamma_{x_i}} &= \nu_B(x_i) + \gamma_{x_i}\rho_B(x_i) \frac{1 - (1 - \alpha_{x_i} - \beta_{x_i} - \gamma_{x_i})^m}{\alpha_{x_i} + \beta_{x_i} + \gamma_{x_i}}.\end{aligned}$$

Since $m_{x_i} = \alpha_{x_i} \frac{1 - (1 - \alpha_{x_i} - \beta_{x_i} - \gamma_{x_i})^m}{\alpha_{x_i} + \beta_{x_i} + \gamma_{x_i}}$, $n_{x_i} = \beta_{x_i} \frac{1 - (1 - \alpha_{x_i} - \beta_{x_i} - \gamma_{x_i})^m}{\alpha_{x_i} + \beta_{x_i} + \gamma_{x_i}}$, $t_{x_i} = \gamma_{x_i} \frac{1 - (1 - \alpha_{x_i} - \beta_{x_i} - \gamma_{x_i})^m}{\alpha_{x_i} + \beta_{x_i} + \gamma_{x_i}}$, then we have:

$$\begin{aligned}(1 - m_{x_i})(\mu_A(x_i) - \mu_B(x_i)) - m_{x_i}(\eta_A(x_i) - \eta_B(x_i)) - m_{x_i}(\nu_A(x_i) - \nu_B(x_i)) &= 0, \\ -n_{x_i}(\mu_A(x_i) - \mu_B(x_i)) + (1 - n_{x_i})(\eta_A(x_i) - \eta_B(x_i)) - n_{x_i}(\nu_A(x_i) - \nu_B(x_i)) &= 0, \\ -t_{x_i}(\mu_A(x_i) - \mu_B(x_i)) - t_{x_i}(\eta_A(x_i) - \eta_B(x_i)) + (1 - t_{x_i})(\nu_A(x_i) - \nu_B(x_i)) &= 0.\end{aligned}$$

i.e.,

$$\begin{pmatrix} 1 - m_{x_i} & -m_{x_i} & -m_{x_i} \\ -n_{x_i} & 1 - n_{x_i} & -n_{x_i} \\ -t_{x_i} & -t_{x_i} & 1 - t_{x_i} \end{pmatrix} \begin{pmatrix} \mu_A(x_i) - \mu_B(x_i) \\ \eta_A(x_i) - \eta_B(x_i) \\ \nu_A(x_i) - \nu_B(x_i) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

By the definition of the absolute value, we have:

$$\begin{vmatrix} 1 - m_{x_i} & -m_{x_i} & -m_{x_i} \\ -n_{x_i} & 1 - n_{x_i} & -n_{x_i} \\ -t_{x_i} & -t_{x_i} & 1 - t_{x_i} \end{vmatrix} = 1 - m_{x_i} - n_{x_i} - t_{x_i}.$$

Since $1 - m_{x_i} - n_{x_i} - t_{x_i} = (1 - \alpha_{x_i} - \beta_{x_i} - \gamma_{x_i})^m > 0$, therefore,

$$\begin{pmatrix} \mu_A(x_i) - \mu_B(x_i) \\ \eta_A(x_i) - \eta_B(x_i) \\ \nu_A(x_i) - \nu_B(x_i) \end{pmatrix} = \begin{pmatrix} 1 - m_{x_i} & -m_{x_i} & -m_{x_i} \\ -n_{x_i} & 1 - n_{x_i} & -n_{x_i} \\ -t_{x_i} & -t_{x_i} & 1 - t_{x_i} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

i.e., $\mu_A(x_i) = \mu_B(x_i)$, $\eta_A(x_i) = \eta_B(x_i)$, $\nu_A(x_i) = \nu_B(x_i)$.

Therefore, $A = B$.

If $A = B$, then we have $\mu_A(x_i) = \mu_B(x_i)$, $\eta_A(x_i) = \eta_B(x_i)$ and $\nu_A(x_i) = \nu_B(x_i)$.

Through Equations (24)–(26), we have:

$$\varphi_A(x_i) - \varphi_B(x_i) = 0,$$

$$\psi_A(x_i) - \psi_B(x_i) = 0,$$

$$\phi_A(x_i) - \phi_B(x_i) = 0.$$

Therefore, $d_m(A, B) = 0$.

$$\begin{aligned}(\text{D3}) \text{ Since } |\varphi_A(x_i) - \varphi_B(x_i)| &= |\mu_A(x_i) - \mu_B(x_i) + m_{x_i}(\rho_A(x_i) - \rho_B(x_i))| \\ &= |-(\mu_A(x_i) - \mu_B(x_i) + m_{x_i}(\rho_A(x_i) - \rho_B(x_i)))| \\ &= |\mu_B(x_i) - \mu_A(x_i) + m_{x_i}(\rho_B(x_i) - \rho_A(x_i))| \\ &= |\varphi_B(x_i) - \varphi_A(x_i)|, \\ |\psi_A(x_i) - \psi_B(x_i)| &= |\eta_A(x_i) - \eta_B(x_i) + n_{x_i}(\rho_A(x_i) - \rho_B(x_i))| \\ &= |-(\eta_A(x_i) - \eta_B(x_i) + n_{x_i}(\rho_A(x_i) - \rho_B(x_i)))| \\ &= |\eta_B(x_i) - \eta_A(x_i) + n_{x_i}(\rho_B(x_i) - \rho_A(x_i))| \\ &= |\psi_B(x_i) - \psi_A(x_i)|, \\ |\phi_A(x_i) - \phi_B(x_i)| &= |\nu_A(x_i) - \nu_B(x_i) + t_{x_i}(\rho_A(x_i) - \rho_B(x_i))| \\ &= |-(\nu_A(x_i) - \nu_B(x_i) + t_{x_i}(\rho_A(x_i) - \rho_B(x_i)))| \\ &= |\nu_B(x_i) - \nu_A(x_i) + t_{x_i}(\rho_B(x_i) - \rho_A(x_i))|\end{aligned}$$

$$= |\phi_B(x_i) - \phi_A(x_i)|,$$

hence

$$\begin{aligned} & \sqrt[p]{\frac{1}{3n} \sum_{i=1}^n (|\varphi_A(x_i) - \varphi_B(x_i)|^p + |\psi_A(x_i) - \psi_B(x_i)|^p + |\phi_A(x_i) - \phi_B(x_i)|^p)} \\ &= \sqrt[p]{\frac{1}{3n} \sum_{i=1}^n (|\varphi_B(x_i) - \varphi_A(x_i)|^p + |\psi_B(x_i) - \psi_A(x_i)|^p + |\phi_B(x_i) - \phi_A(x_i)|^p)}, \end{aligned}$$

i.e., $d_m(A, B) = d_m(B, A)$.

(D4) When $A \subseteq B \subseteq C$, i.e., $\mu_A(x_i) \leq \mu_B(x_i) \leq \mu_C(x_i)$, $\eta_A(x_i) \geq \eta_B(x_i) \geq \eta_C(x_i)$ and $\nu_A(x_i) \geq \nu_B(x_i) \geq \nu_C(x_i)$, then we can obtain:

$$\mu_C(x_i) - \mu_A(x_i) \geq \mu_B(x_i) - \mu_A(x_i) \geq 0,$$

$$\mu_C(x_i) - \mu_A(x_i) \geq \mu_C(x_i) - \mu_B(x_i) \geq 0,$$

$$\eta_A(x_i) - \eta_C(x_i) \geq \eta_A(x_i) - \eta_B(x_i) \geq 0,$$

$$\eta_A(x_i) - \eta_C(x_i) \geq \eta_B(x_i) - \eta_C(x_i) \geq 0,$$

$$\nu_A(x_i) - \nu_C(x_i) \geq \nu_A(x_i) - \nu_B(x_i) \geq 0,$$

$$\nu_A(x_i) - \nu_C(x_i) \geq \nu_B(x_i) - \nu_C(x_i) \geq 0.$$

Since,

$$|\varphi_A(x_i) - \varphi_B(x_i)| = (1 - m_{x_i})(\mu_B(x_i) - \mu_A(x_i)) + m_{x_i}(\eta_A(x_i) - \eta_B(x_i)) + m_{x_i}(\nu_A(x_i) - \nu_B(x_i)),$$

$$|\varphi_A(x_i) - \varphi_C(x_i)| = (1 - m_{x_i})(\mu_C(x_i) - \mu_A(x_i)) + m_{x_i}(\eta_A(x_i) - \eta_C(x_i)) + m_{x_i}(\nu_A(x_i) - \nu_C(x_i)),$$

$$|\varphi_B(x_i) - \varphi_C(x_i)| = (1 - m_{x_i})(\mu_C(x_i) - \mu_B(x_i)) + m_{x_i}(\eta_B(x_i) - \eta_C(x_i)) + m_{x_i}(\nu_B(x_i) - \nu_C(x_i)),$$

therefore, $|\varphi_A(x_i) - \varphi_C(x_i)| \geq |\varphi_A(x_i) - \varphi_B(x_i)|$, $|\varphi_A(x_i) - \varphi_C(x_i)| \geq |\varphi_B(x_i) - \varphi_C(x_i)|$.

In the same way, we get: $|\psi_A(x_i) - \psi_C(x_i)| \geq |\psi_A(x_i) - \psi_B(x_i)|$, $|\psi_A(x_i) - \psi_C(x_i)| \geq |\psi_B(x_i) - \psi_C(x_i)|$, $|\phi_A(x_i) - \phi_C(x_i)| \geq |\phi_A(x_i) - \phi_B(x_i)|$, $|\phi_A(x_i) - \phi_C(x_i)| \geq |\phi_B(x_i) - \phi_C(x_i)|$.

Hence, $d_m(A, C) \geq d_m(A, B)$, $d_m(A, C) \geq d_m(B, C)$.

Through the above analysis, we prove that $d_m(A, B)$ is the distance between A and B on X .

Remark 4.

(1) From Theorem 1, according to the picture fuzzy point operator, $d_m(A, B)$ can be obtained by the positive degree, the neutral degree, and the negative degree of the element x_i in the set A and B and a part of picture fuzzy index (i.e., a part of the rejected information) obtained by the m times redistribution to the picture fuzzy index using $(\alpha_{x_i}, \beta_{x_i}, \gamma_{x_i})$; the larger the positive integer m is, the more information that the distance measure has, and the stronger the distinguishing ability is.

(2) According to Theorem 1, for any two PFSSs A, B on $X = \{x_1, x_2, x_3, \dots, x_n\}$, if $\rho_A(x_i) = \rho_B(x_i)$ and $p = 1$, then $d_m(A, B) = \frac{1}{3n} \sum_{i=1}^n [|\mu_A(x_i) - \mu_B(x_i)| + |\eta_A(x_i) - \eta_B(x_i)| + |\nu_A(x_i) - \nu_B(x_i)|]$, which is the same as the distance measure $d_H(A, B)$ in [23].

(3) According to Theorem 1, for any two PFSSs A, B on $X = \{x_1, x_2, x_3, \dots, x_n\}$, if $m = \infty$, then Equation (23) can be re-expressed as follows:

$$d_\infty(A, B) = \sqrt[p]{\frac{1}{3n} \sum_{i=1}^n (|\varphi'_A(x_i) - \varphi'_B(x_i)|^p + |\psi'_A(x_i) - \psi'_B(x_i)|^p + |\phi'_A(x_i) - \phi'_B(x_i)|^p)} \quad (p \in [1, +\infty)), \quad (27)$$

where $\varphi'_A(x_i) = \mu_A(x_i) + \alpha_{x_i}\rho_A(x_i)\frac{1}{\alpha_{x_i}+\beta_{x_i}+\gamma_{x_i}}$, $\psi'_A(x_i) = \eta_A(x_i) + \beta_{x_i}\rho_A(x_i)\frac{1}{\alpha_{x_i}+\beta_{x_i}+\gamma_{x_i}}$, $\varphi'_A(x_i) = \nu_A(x_i) + \gamma_{x_i}\rho_A(x_i)\frac{1}{\alpha_{x_i}+\beta_{x_i}+\gamma_{x_i}}$, $\varphi'_B(x_i) = \mu_B(x_i) + \alpha_{x_i}\rho_B(x_i)\frac{1}{\alpha_{x_i}+\beta_{x_i}+\gamma_{x_i}}$, $\psi'_B(x_i) = \eta_B(x_i) + \beta_{x_i}\rho_B(x_i)\frac{1}{\alpha_{x_i}+\beta_{x_i}+\gamma_{x_i}}$, $\varphi'_B(x_i) = \nu_B(x_i) + \gamma_{x_i}\rho_B(x_i)\frac{1}{\alpha_{x_i}+\beta_{x_i}+\gamma_{x_i}}$.

Theorem 2. Let $X = \{x_1, x_2, x_3, \dots, x_n\}$ be a universe. Suppose that A, B are any two PFSSs on X , $\omega_i \in [0, 1]$, $\sum_{i=1}^n \omega_i = 1$, $\alpha_{x_i}, \beta_{x_i}, \gamma_{x_i} \in [0, 1]$, and $0 < \alpha_{x_i} + \beta_{x_i} + \gamma_{x_i} \leq 1$, m being any natural number, $p \in [1, +\infty)$. A function $d_{m,\omega} : \text{PFSSs}(X) \times \text{PFSSs}(X) \rightarrow [0, 1]$ is defined by:

$$d_{m,\omega}(A, B) = \sqrt[p]{\frac{1}{3} \sum_{i=1}^n \omega_i (|\varphi_A(x_i) - \varphi_B(x_i)|^p + |\psi_A(x_i) - \psi_B(x_i)|^p + |\phi_A(x_i) - \phi_B(x_i)|^p)}, \quad (28)$$

where $\varphi_A(x_i) = \mu_A(x_i) + \alpha_{x_i}\rho_A(x_i)\frac{1-(1-\alpha_{x_i}-\beta_{x_i}-\gamma_{x_i})^m}{\alpha_{x_i}+\beta_{x_i}+\gamma_{x_i}}$, $\psi_A(x_i) = \eta_A(x_i) + \beta_{x_i}\rho_A(x_i)\frac{1-(1-\alpha_{x_i}-\beta_{x_i}-\gamma_{x_i})^m}{\alpha_{x_i}+\beta_{x_i}+\gamma_{x_i}}$, $\phi_A(x_i) = \nu_A(x_i) + \gamma_{x_i}\rho_A(x_i)\frac{1-(1-\alpha_{x_i}-\beta_{x_i}-\gamma_{x_i})^m}{\alpha_{x_i}+\beta_{x_i}+\gamma_{x_i}}$, $\varphi_B(x_i) = \mu_B(x_i) + \alpha_{x_i}\rho_B(x_i)\frac{1-(1-\alpha_{x_i}-\beta_{x_i}-\gamma_{x_i})^m}{\alpha_{x_i}+\beta_{x_i}+\gamma_{x_i}}$, $\psi_B(x_i) = \eta_B(x_i) + \beta_{x_i}\rho_B(x_i)\frac{1-(1-\alpha_{x_i}-\beta_{x_i}-\gamma_{x_i})^m}{\alpha_{x_i}+\beta_{x_i}+\gamma_{x_i}}$, $\phi_B(x_i) = \nu_B(x_i) + \gamma_{x_i}\rho_B(x_i)\frac{1-(1-\alpha_{x_i}-\beta_{x_i}-\gamma_{x_i})^m}{\alpha_{x_i}+\beta_{x_i}+\gamma_{x_i}}$.

Then, $d_{m,\omega}$ is a weighted distance between A and B .

Proof. The proof is similar to Theorem 1. $\square$

Remark 5. According to Theorem 2, for any two PFSSs A, B on $X = \{x_1, x_2, x_3, \dots, x_n\}$, if $m = \infty$, then Equation (28) can be re-expressed as follows:

$$d_{\infty,\omega}(A, B) = \sqrt[p]{\frac{1}{3} \sum_{i=1}^n \omega_i (|\varphi'_A(x_i) - \varphi'_B(x_i)|^p + |\psi'_A(x_i) - \psi'_B(x_i)|^p + |\phi'_A(x_i) - \phi'_B(x_i)|^p)} (p \in [1, +\infty)). \quad (29)$$

where $\varphi'_A(x_i) = \mu_A(x_i) + \alpha_{x_i}\rho_A(x_i)\frac{1}{\alpha_{x_i}+\beta_{x_i}+\gamma_{x_i}}$, $\psi'_A(x_i) = \eta_A(x_i) + \beta_{x_i}\rho_A(x_i)\frac{1}{\alpha_{x_i}+\beta_{x_i}+\gamma_{x_i}}$, $\varphi'_A(x_i) = \nu_A(x_i) + \gamma_{x_i}\rho_A(x_i)\frac{1}{\alpha_{x_i}+\beta_{x_i}+\gamma_{x_i}}$, $\varphi'_B(x_i) = \mu_B(x_i) + \alpha_{x_i}\rho_B(x_i)\frac{1}{\alpha_{x_i}+\beta_{x_i}+\gamma_{x_i}}$, $\psi'_B(x_i) = \eta_B(x_i) + \beta_{x_i}\rho_B(x_i)\frac{1}{\alpha_{x_i}+\beta_{x_i}+\gamma_{x_i}}$, $\varphi'_B(x_i) = \nu_B(x_i) + \gamma_{x_i}\rho_B(x_i)\frac{1}{\alpha_{x_i}+\beta_{x_i}+\gamma_{x_i}}$.

5. Experiments and Analysis

In order to illustrate the rationality and superiority of the proposed distance measure, we give a comprehensive analysis and comparison between the proposed distance measure and the existing distance measures through a numerical example and multi-attribute decision-making problems.

5.1. Numerical Comparisons of Some Distance Measures

Example 2. Let $A_i, B_i \in \text{PFSSs}$ ($i = \{1, 2, 3, 4, 5, 6\}$). The distances between A_i and B_i based on different distance measures can be found in Table 1.

By analyzing Table 1, we can easily find that:

- (i) The distance measures $d_H, d_E, d_{MH}, d_{Em}, d_{CS2}, d_{CT1}, d_{pe}$ cannot recognize the difference between A_1, B_1 and A_2, B_2 in the first column and the second column, since the distance between A_1 and B_1 is equal the distance between A_2 and B_2 under $A_1 = A_2, B_1 \neq B_2$.
- (ii) The distance measures $d_{MH}, d_{Em}, d_{CS1}, d_{CS3}, d_{CS4}, d_{CT1}, d_{CT2}, d_{ph}, d_{gpnhh}, d_{gpnheh}$ have counterintuitive situations by analyzing the third column and the fourth column. Although the distance measures $d_G, p = 2$ and $d_P, p = 2$ can recognize the difference between A_3, B_3 and A_4, B_4 , the results are not unreasonable because $B_3 < B_4 < A_3 = A_4$.

- (iii) By analyzing the fifth column and the sixth column, we find that the distance measures $d_H, d_E, d_{MH}, d_{Em}, d_{CS1}, d_{CS2}, d_{CS3}, d_{CS4}, d_{CT1}, d_{CT2}, d_{ph}, d_{gpnhh}, d_{gpneh}, d_G, p = 1, d_G, p = 2, d_P, p = 1, d_P, p = 2$ have counterintuitive situations.
- (iv) By looking at the second column, we find that the distance measure d_C produces an undefined situation.
- (v) Nevertheless, when we compare the distance measure results of the first and second columns or the third and fourth columns or the fifth and sixth columns, the proposed distance measure is more effective than some other distance measures at distinguishing the differences between PFSs.

Table 1. Distance measures between picture fuzzy sets A and B ($\alpha = 0.5, \beta = 0.2, \gamma = 0.1$ in $d_1, d_2, d_3, d_4, d_\infty$).

	1	2	3	4	5	6
A_i	$\langle 0.5, 0.5, 0.0 \rangle$	$\langle 0.5, 0.5, 0.0 \rangle$	$\langle 0.1, 0.4, 0.0 \rangle$	$\langle 0.1, 0.4, 0.0 \rangle$	$\langle 0.4, 0.3, 0.2 \rangle$	$\langle 0.4, 0.3, 0.2 \rangle$
B_i	$\langle 1.0, 0.0, 0.0 \rangle$	$\langle 0.0, 0.0, 0.0 \rangle$	$\langle 0.1, 0.0, 0.3 \rangle$	$\langle 0.1, 0.0, 0.0 \rangle$	$\langle 0.4, 0.4, 0.2 \rangle$	$\langle 0.4, 0.3, 0.3 \rangle$
d_H [23]	0.3333	0.3333	0.2333	0.1333	0.3333	0.3333
d_E [23]	0.2357	0.2357	0.1667	0.1333	0.3333	0.3333
d_{MH} [23]	0.5000	0.5000	0.4000	0.4000	0.1000	0.1000
d_{Em} [23]	0.2500	0.2500	0.1600	0.1600	0.0100	0.0100
d_{CS1} [25]	0.2927	0.2929	0.1910	0.1910	0.0123	0.0123
d_{CS2} [25]	0.3727	0.3727	0.2352	0.0789	0.0050	0.0050
d_{CS3} [25]	0.2929	1.0000	0.1910	0.1910	0.0123	0.0123
d_{CS4} [25]	0.0192	0.2929	0.0079	0.0079	0.0001	0.0001
d_{CT1} [25]	0.5858	0.5858	0.4905	0.4905	0.1459	0.1459
d_{CT2} [25]	0.5858	1.0000	0.4905	0.4905	0.1459	0.1459
d_{ph} [24]	0.2500	0.5000	0.2000	0.2000	0.0500	0.0500
d_{pe} [24]	0.3536	0.3536	0.2540	0.2000	0.0500	0.0502
d_{gpnhh} [24]	0.1250	0.2500	0.1000	0.1000	0.0250	0.0250
d_{gpneh} [24]	0.2500	0.5000	0.2000	0.2000	0.0500	0.0500
d_c [28]	0.2500	NaN	0.4756	0.4220	0.0441	0.0510
$d_G, p = 1$ [21]	0.2174	0.2941	0.2235	0.2192	0.0412	0.0412
$d_G, p = 2$ [21]	0.3134	0.3923	0.3314	0.3625	0.0712	0.0712
$d_P, p = 1$ [26]	0.2174	0.2941	0.2235	0.2192	0.0412	0.0412
$d_P, p = 2$ [26]	0.2110	0.2440	0.2096	0.2197	0.0444	0.0444
$d_1, p = 2$	0.4082	0.1826	0.2846	0.2191	0.0548	0.0606
$d_2, p = 2$	0.4082	0.1751	0.2842	0.2253	0.0563	0.0630
$d_3, p = 2$	0.4082	0.1763	0.2841	0.2269	0.0567	0.0636
$d_\infty, p = 2$	0.4082	0.1768	0.2841	0.2273	0.0568	0.0637

5.2. Algorithm and Applications

Algorithm for Multi-Attribute Decision-Making

Let $C = \{c_1, c_2, \dots, c_m\}$ be a set of attributes. There are n alternatives $A_i = \{A_{ij}\} = \{\langle \mu_{A_i}(c_j), \eta_{A_i}(c_j), \nu_{A_i}(c_j) \rangle \mid c_j \in C\}$, where $\mu_{A_i}(c_j), \eta_{A_i}(c_j), \nu_{A_i}(c_j) \in [0, 1], 0 \leq \mu_{A_i}(c_j) + \eta_{A_i}(c_j) + \nu_{A_i}(c_j) \leq 1$, $\mu_{A_i}(c_j)$ is a positive membership, which is used to describe the degree to which alternative A_i satisfies the criteria c_j ($i = \{1, 2, \dots, n\}, j = \{1, 2, \dots, m\}$).

$\eta_{A_i}(c_j)$ is a neutral degree, which is used to describe the degree to which alternative A_i does not satisfy the criteria c_j . $\nu_{A_i}(c_j)$ indicates the degree to which alternative A_i does not fulfill the criteria c_j . Which is the best alternative? The decision-making steps are as follows:

Step 1. Give the decision matrix.

Step 2. Normalize the decision matrix.

In MADC problems, attributes can often be divided into two categories: cost type and benefit type. The cost attribute needs to be changed into the benefit attribute by the following formula in the decision-making process. If all attributes are beneficial, there is no need for the transformation.

$$\bar{A}_{ij} = \begin{cases} A_{ij}, & \text{for benefit attribute } C_j, \\ A_{ij}^c, & \text{for cost attribute } C_j, \end{cases} \quad (30)$$

where $A_{ij}^c = \langle \nu_{A_i}(x_j), \eta_{A_i}(x_j), \mu_{A_i}(x_j) \rangle$, $i = \{1, 2, \dots, n\}$, $j = \{1, 2, \dots, m\}$.

Step 3. Determine the ideal solution.

Let $\bar{A}^+ = (c_1^+, c_2^+, \dots, c_m^+)$ be the positive ideal solution, which can be obtained by:

$$C_j^+ = \langle \max_i \mu_{\bar{A}_i}(c_j), \min_i \eta_{\bar{A}_i}(c_j), \min_i \nu_{\bar{A}_i}(c_j) \rangle. \quad (31)$$

Step 4. Calculate the distance between alternatives $\bar{A}_i$ and $\bar{A}^+$.

Calculate the distance between each alternative $\bar{A}_i$ and $\bar{A}^+$ by using Theorem 2.

Step 5. Rank all alternatives.

We sort the distances in Step 4. Based on the principle of minimum distance, we give the order of all alternatives.

Step 6. Calculate the degree of confidence.

Calculate the degree of confidence Doc , $Doc^{(i_0)} = \sum_{i=1, i \neq i_0}^n |d(\bar{A}_{i_0}, \bar{A}^+) - d(\bar{A}_i, \bar{A}^+)|$ [30],

where $d(\bar{A}_{i_0}, \bar{A}^+) = \min_{1 \leq i \leq n} \{d(\bar{A}_i, \bar{A}^+)\}$. If the $Doc^{(i_0)}$ is larger, the result of the distance measure is more credible.

5.3. Applications to Multi-Attribute Decision-Making

Example 3. There are four alternatives A_1, A_2, A_3, A_4 with three different attributes c_1, c_2, c_3 , which are described by PFNs, as shown in Table 2. We want to choose the best alternative from A_1, A_2, A_3, A_4 .

The decision results based on different distances are given in Table 3. Since $d_2(A_4, A^+) < d_2(A_3, A^+) < d_2(A_2, A^+) < d_2(A_1, A^+)$, then we have that the best choice is A_4 , which coincides with the decision result of distance measure d_{CS2} . The decision result based on the proposed distance measure is more credible because of the high credibility of the measure. $d_H, d_E, d_{MH}, d_{Em}, d_{CS1}, d_{CS3}, d_{CS4}, d_{CT1}, d_{CT2}, d_{ph}, d_{pe}, d_{gpnhh}, d_{gpnch}, d_c, d_G$, and d_P cannot provide the decision results. Therefore, the proposed distance measure can effectively solve the decision-making problem that cannot be solved by the existing distance measures. This shows that the proposed distance measure is reasonable and effective.

Table 2. Four alternatives with three different attributes in Example 3.

	c_1	c_2	c_3
A_1	$\langle 0.2, 0.1, 0.2 \rangle$	$\langle 0.4, 0.3, 0.2 \rangle$	$\langle 0.4, 0.1, 0.3 \rangle$
A_2	$\langle 0.2, 0.2, 0.3 \rangle$	$\langle 0.4, 0.2, 0.1 \rangle$	$\langle 0.4, 0.1, 0.3 \rangle$
A_3	$\langle 0.2, 0.2, 0.3 \rangle$	$\langle 0.5, 0.3, 0.1 \rangle$	$\langle 0.5, 0.1, 0.3 \rangle$
A_4	$\langle 0.2, 0.1, 0.3 \rangle$	$\langle 0.4, 0.3, 0.2 \rangle$	$\langle 0.4, 0.1, 0.3 \rangle$

Table 3. The results of the distance measure and decision results in Example 3.

S	Results of Distance Measures				Decision-Making		
	$d(A_1, A^+)$	$d(A_2, A^+)$	$d(A_3, A^+)$	$d(A_4, A^+)$	Ranking	Result	$Doc^{(4)}$
d_H [23]	0.0444	0.0444	0.0333	0.0333	$A_1 = A_2 < A_3 = A_4$	No	No
d_E [23]	0.0703	0.0703	0.0609	0.0609	$A_1 = A_2 < A_3 = A_4$	No	No
d_{MH} [23]	0.1333	0.1333	0.1000	0.1000	$A_1 = A_2 < A_3 = A_4$	No	No
d_{Em} [23]	0.2108	0.2108	0.1826	0.1826	$A_1 = A_2 < A_3 = A_4$	No	No
d_{CS1} [25]	0.1000	0.1000	0.1000	0.1000	$A_1 = A_2 = A_3 = A_4$	No	No
d_{CS2} [25]	0.0062	0.0102	0.0051	0.0031	$A_2 < A_1 < A_3 < A_4$	A_4	0.0122
d_{CS3} [25]	0.0245	0.0245	0.0245	0.0245	$A_1 = A_2 = A_3 = A_4$	No	No
d_{CS4} [25]	0.0143	0.0143	0.0143	0.0143	$A_1 = A_2 = A_3 = A_4$	No	No
d_{CT1} [25]	0.1459	0.0973	0.0973	0.1459	$A_1 = A_4 < A_2 = A_3$	No	No
d_{CT2} [25]	0.1884	0.1884	0.1884	0.1884	$A_1 = A_2 = A_3 = A_4$	No	No
d_{ph} [24]	0.0500	0.0500	0.0500	0.0500	$A_1 = A_2 = A_3 = A_4$	No	No
d_{pe} [24]	0.0816	0.0816	0.0816	0.0816	$A_1 = A_2 = A_3 = A_4$	No	No
d_{gpnhh} [24]	0.0333	0.0333	0.0333	0.0333	$A_1 = A_2 = A_3 = A_4$	No	No
d_{gpneh} [24]	0.0050	0.0041	0.0041	0.0041	$A_1 < A_2 = A_3 = A_4$	No	No
d_c [28]	0.4200	0.4200	0.4200	0.4200	$A_1 = A_2 = A_3 = A_4$	No	No
$d_G, p = 1$ [21]	0.0792	0.0792	0.0597	0.0597	$A_1 = A_2 < A_3 = A_4$	No	No
$d_P, p = 1$ [26]	0.0792	0.0792	0.0597	0.0597	$A_1 = A_2 < A_3 = A_4$	No	No
$d_2, p = 2$	0.3598	0.3575	0.3516	0.3396	$A_1 < A_2 < A_3 < A_4$	A_4	0.0501

Example 4. Reference [31]: There are five alternatives A_1, A_2, A_3, A_4, A_5 with four different attributes c_1, c_2, c_3, c_4 , which are described by PFNs, as shown in Table 4. We want to choose the best alternative from A_1, A_2, A_3, A_4, A_5 .

The decision results based on different distance measures are given in Table 5. Since $d_2(A_1, A^+) < d_2(A_3, A^+) < d_2(A_5, A^+) < d_2(A_4, A^+) < d_2(A_2, A^+)$, then we have that the best choice is A_1 , which coincides with the decision result of distance measures $d_E, d_{MH}, d_{Em}, d_{CS1}, d_{CS3}, d_{CS4}, d_{CT1}, d_{CT2}, d_{ph}, d_{pe}, d_{gpnhh}, d_{gpneh}, d_G$, and d_P . Although the credibility measure of the proposed distance measure is not as high as that of some existing distance measures, the existing distance measures can produce unreasonable situations. Therefore, the proposed distance measure is reasonable and effective.

Table 4. The decision matrix in Example 4.

	c_1	c_2	c_3	c_4
A_1	$\langle 0.56, 0.34, 0.10 \rangle$	$\langle 0.90, 0.07, 0.03 \rangle$	$\langle 0.40, 0.33, 0.19 \rangle$	$\langle 0.09, 0.79, 0.03 \rangle$
A_2	$\langle 0.70, 0.10, 0.09 \rangle$	$\langle 0.10, 0.66, 0.20 \rangle$	$\langle 0.06, 0.81, 0.12 \rangle$	$\langle 0.72, 0.14, 0.09 \rangle$
A_3	$\langle 0.88, 0.09, 0.03 \rangle$	$\langle 0.08, 0.10, 0.06 \rangle$	$\langle 0.05, 0.83, 0.09 \rangle$	$\langle 0.65, 0.25, 0.07 \rangle$
A_4	$\langle 0.80, 0.07, 0.04 \rangle$	$\langle 0.70, 0.15, 0.11 \rangle$	$\langle 0.03, 0.88, 0.05 \rangle$	$\langle 0.07, 0.82, 0.05 \rangle$
A_5	$\langle 0.85, 0.06, 0.03 \rangle$	$\langle 0.64, 0.07, 0.22 \rangle$	$\langle 0.06, 0.88, 0.05 \rangle$	$\langle 0.13, 0.77, 0.09 \rangle$

Table 5. The results of the distance measures and decision results in Example 4.

S	Results of Distance Measures					Decision-Making		
	$d(A_1, A^+)$	$d(A_2, A^+)$	$d(A_3, A^+)$	$d(A_4, A^+)$	$d(A_5, A^+)$	Ranking	Results	$Doc^{(1)}$
d_H [23]	0.1742	0.2325	0.1683	0.2275	0.2208	$A_2 < A_4 < A_5 < A_1 < A_3$	A_3	No
d_E [23]	0.0844	0.0989	0.0861	0.0980	0.0940	$A_2 < A_4 < A_5 < A_3 < A_1$	A_1	0.0396
d_{MH} [23]	0.2775	0.3800	0.3650	0.3775	0.3675	$A_2 < A_4 < A_5 < A_3 < A_1$	A_1	0.3800
d_{Em} [23]	0.1845	0.2380	0.2418	0.2252	0.2191	$A_3 < A_2 < A_4 < A_5 < A_1$	A_1	0.1861
d_{CS1} [25]	0.1600	0.2500	0.2600	0.2300	0.2200	$A_3 < A_2 < A_4 < A_5 < A_1$	A_1	0.3200
d_{CS2} [25]	0.1514	0.2302	0.1198	0.2010	0.1902	$A_2 < A_4 < A_5 < A_1 < A_3$	A_3	No
d_{CS3} [25]	0.1563	0.2515	0.2575	0.2314	0.2212	$A_3 < A_2 < A_4 < A_5 < A_1$	A_1	0.3364
d_{CS4} [25]	0.1622	0.2714	0.2721	0.2383	0.2415	$A_3 < A_2 < A_5 < A_4 < A_1$	A_1	0.3745
d_{CT1} [25]	0.3314	0.4370	0.4122	0.4415	0.4303	$A_4 < A_2 < A_5 < A_3 < A_1$	A_1	0.3954
d_{CT2} [25]	0.3314	0.4370	0.4122	0.4415	0.4303	$A_4 < A_2 < A_5 < A_3 < A_1$	A_1	0.4064
d_{ph} [24]	0.1425	0.1988	0.1925	0.1913	0.1750	$A_2 < A_3 < A_4 < A_5 < A_1$	A_1	0.1876
d_{pe} [24]	0.2556	0.3026	0.3249	0.2983	0.2828	$A_3 < A_2 < A_4 < A_5 < A_1$	A_1	0.1859
d_{gpnhh} [24]	0.0694	0.0950	0.0913	0.0944	0.0938	$A_2 < A_4 < A_5 < A_3 < A_1$	A_1	0.0969
d_{gpneh} [24]	0.1088	0.1712	0.2000	0.1393	0.1269	$A_3 < A_2 < A_4 < A_5 < A_1$	A_1	0.2022
d_c [28]	0.4400	0.4400	0.4300	0.4400	0.4300	$A_1 = A_2 = A_4 < A_3 = A_5$	No	No
$d_G, p = 1$ [21]	0.1072	0.1343	0.1121	0.1221	0.1192	$A_2 < A_4 < A_5 < A_3 < A_1$	A_1	0.0589
$d_P, p = 1$ [26]	0.1072	0.1343	0.1121	0.1221	0.1192	$A_2 < A_4 < A_5 < A_3 < A_1$	A_1	0.0589
$d_2, p = 2$	0.3608	0.5303	0.4076	0.4325	0.4260	$A_2 < A_4 < A_5 < A_3 < A_1$	A_1	0.3532

Example 5. There is a committee that wants to choose a viable emerging technology company. Five companies are taken into consideration, which are denoted by A_1, A_2, A_3, A_4, A_5 . The decision-maker makes a decision matrix with four different attributes, including technical advancement (c_1), potential market (c_2), human resources (c_3), the employment formation, and the process of science and technology (c_4). The corresponding attribute weights are 0.2, 0.1, 0.3, and 0.4, which are given by the decision-makers. The decision matrix is given in Table 6.

Table 6. The decision matrix in Example 5.

	C_1	C_2	C_3	C_4
A_1	$\langle 0.45, 0.10, 0.09 \rangle$	$\langle 0.79, 0.08, 0.03 \rangle$	$\langle 0.42, 0.30, 0.10 \rangle$	$\langle 0.08, 0.59, 0.02 \rangle$
A_2	$\langle 0.63, 0.02, 0.08 \rangle$	$\langle 0.13, 0.44, 0.11 \rangle$	$\langle 0.03, 0.52, 0.10 \rangle$	$\langle 0.63, 0.05, 0.08 \rangle$
A_3	$\langle 0.51, 0.03, 0.02 \rangle$	$\langle 0.07, 0.09, 0.05 \rangle$	$\langle 0.04, 0.45, 0.10 \rangle$	$\langle 0.48, 0.06, 0.10 \rangle$
A_4	$\langle 0.55, 0.09, 0.05 \rangle$	$\langle 0.34, 0.16, 0.10 \rangle$	$\langle 0.02, 0.59, 0.05 \rangle$	$\langle 0.08, 0.54, 0.06 \rangle$
A_5	$\langle 0.58, 0.05, 0.02 \rangle$	$\langle 0.48, 0.08, 0.11 \rangle$	$\langle 0.05, 0.47, 0.06 \rangle$	$\langle 0.13, 0.45, 0.09 \rangle$

The decision-making process is as follows:

Step 1. Give the decision matrix.

The decision matrix can be found in Table 6.

Step 2. Normalize the decision matrix.

Since these four attributes belong to benefit attributes, the decision matrix does not need to be normalized.

Step 3. Determine the ideal solution.

By Table 2 and Equation (31), we can obtain the positive ideal solution $\bar{A}^+$, which is presented in Table 7.

Table 7. The positive ideal solution and negative ideal solution.

	C_1	C_2	C_3	C_4
$\bar{A}^+$	$\langle 0.63, 0.02, 0.02 \rangle$	$\langle 0.79, 0.08, 0.03 \rangle$	$\langle 0.42, 0.30, 0.05 \rangle$	$\langle 0.63, 0.05, 0.02 \rangle$

Step 4. Calculate the distance between alternatives $\bar{A}_i$ and $\bar{A}^+$.

By Theorem 2, we can get $d(A_1, \bar{A}^+) = 0.3552$, $d(A_2, \bar{A}^+) = 0.3804$, $d(A_3, \bar{A}^+) = 0.4506$, $d(A_4, \bar{A}^+) = 0.4511$, $d(A_5, \bar{A}^+) = 0.4282$.

Step 5. Rank all alternatives.

Obviously, $d(A_1, \bar{A}^+) < d(A_2, \bar{A}^+) < d(A_5, \bar{A}^+) < d(A_3, \bar{A}^+) < d(A_4, \bar{A}^+)$. Then, $A_1 > A_2 > A_5 > A_3 > A_4$. Therefore, company A_1 is the best choice for the committee.

In order to discuss the inferences of different parameters m and p on the final ranking, we use different values of m, p to rank the five companies. The ranking results are given in Table 8.

By Table 8, we can easily find that different parameters m and p in Theorem 2 can produce different distances between alternatives and the positive ideal solution. When the values of p and m are changed, although we get two different sorting results $A_1 > A_2 > A_5 > A_4 > A_3$ and $A_1 > A_2 > A_5 > A_3 > A_4$, the sorting of A_1, A_2, A_5 is always $A_1 > A_2 > A_5$, which does not affect the final decision.

In order to further verify the effectiveness of the proposed distance measure, we compare the sorting results with those of other methods based on the cross-entropy [32], the distance measures [23,33], and the aggregation operators [31,34,35]. Table 9 provides the comparative analysis results.

Firstly, the ranking result by the method based on Dombi aggregation operators [31] with $p = 3$ or $p = 4$ is the same as that by the proposed dynamic distance measure. Secondly, although the ranking result by the proposed dynamic distance measure is different from that by the method based on the operator [35] and the distance measure of [33], the best alternative is always A_1 . Thirdly, the final ranking result by the method based on the Einstein operator [34] and the distance measure of [23] is not the same as that by other methods, and the best alternative is A_2 . The main reason may be that the refusal degree is not considered. Fourth, when using the method based on the cross-entropy, the best alternative is A_3 , which is not consistent with the result of other methods. The main reason is that the neutral degree and refusal degree are not considered. Fifth, although the proposed distance measure and the method based on Dombi aggregation operators [31] are more flexible in the decision-making process, the result based on the proposed dynamic distance measure is more stable. Because of the change of parameters, it does not affect the best choice. Thus, the proposed distance measure is effective.

Table 8. Ranking order for different parameters of the distance measure in Example 5.

		$d(A_1, \bar{A}^+)$	$d(A_2, \bar{A}^+)$	$d(A_3, \bar{A}^+)$	$d(A_4, \bar{A}^+)$	$d(A_5, \bar{A}^+)$	Ranking Order
$p = 1$	m = 0	0.1308	0.1462	0.2062	0.2059	0.1860	$A_1 > A_2 > A_5 > A_4 > A_3$
	m = 1	0.1271	0.1481	0.2049	0.2047	0.1890	$A_1 > A_2 > A_5 > A_4 > A_3$
	m = 2	0.1240	0.1484	0.2022	0.2040	0.1874	$A_1 > A_2 > A_5 > A_3 > A_4$
	m = 3	0.1216	0.1485	0.1989	0.2036	0.1855	$A_1 > A_2 > A_5 > A_3 > A_4$
	m = 4	0.1198	0.1480	0.1960	0.2035	0.1831	$A_1 > A_2 > A_5 > A_3 > A_4$
	m = 5	0.1183	0.1471	0.1936	0.2035	0.1805	$A_1 > A_2 > A_5 > A_3 > A_4$
	m = 6	0.1170	0.1464	0.1916	0.2036	0.1784	$A_1 > A_2 > A_5 > A_3 > A_4$
	m = 10	0.1138	0.1441	0.1860	0.2020	0.1728	$A_1 > A_2 > A_5 > A_3 > A_4$
	m = 20	0.1110	0.1417	0.1808	0.1969	0.1682	$A_1 > A_2 > A_5 > A_3 > A_4$
	m = ∞	0.1102	0.1408	0.1790	0.1954	0.1668	$A_1 > A_2 > A_5 > A_3 > A_4$
$p = 2$	m = 0	0.3466	0.3711	0.4497	0.4405	0.4205	$A_1 > A_2 > A_5 > A_4 > A_3$
	m = 1	0.3520	0.3762	0.4506	0.4469	0.4254	$A_1 > A_2 > A_5 > A_4 > A_3$
	m = 2	0.3552	0.3804	0.4506	0.4511	0.4282	$A_1 > A_2 > A_5 > A_3 > A_4$
	m = 3	0.3564	0.3833	0.4497	0.4535	0.4293	$A_1 > A_2 > A_5 > A_3 > A_4$
	m = 4	0.3564	0.3852	0.4482	0.4546	0.4292	$A_1 > A_2 > A_5 > A_3 > A_4$
	m = 5	0.3556	0.3863	0.4464	0.4548	0.4284	$A_1 > A_2 > A_5 > A_3 > A_4$
	m = 6	0.3543	0.3868	0.4443	0.4544	0.4273	$A_1 > A_2 > A_5 > A_3 > A_4$
	m = 10	0.3486	0.3857	0.4369	0.4513	0.4224	$A_1 > A_2 > A_5 > A_3 > A_4$
	m = 20	0.3415	0.3816	0.4277	0.4463	0.4161	$A_1 > A_2 > A_5 > A_3 > A_4$
	m = ∞	0.3394	0.3794	0.4244	0.4444	0.4139	$A_1 > A_2 > A_5 > A_3 > A_4$
$p = 3$	m = 0	0.4877	0.5130	0.5870	0.5742	0.5572	$A_1 > A_2 > A_5 > A_4 > A_3$
	m = 1	0.4944	0.5179	0.5875	0.5809	0.5619	$A_1 > A_2 > A_5 > A_4 > A_3$
	m = 2	0.4997	0.5225	0.5880	0.5859	0.5657	$A_1 > A_2 > A_5 > A_4 > A_3$
	m = 3	0.5033	0.5265	0.5882	0.5892	0.5682	$A_1 > A_2 > A_5 > A_3 > A_4$
	m = 4	0.5053	0.5296	0.5878	0.5908	0.5695	$A_1 > A_2 > A_5 > A_3 > A_4$
	m = 5	0.5059	0.5318	0.5869	0.5914	0.5696	$A_1 > A_2 > A_5 > A_3 > A_4$
	m = 6	0.5054	0.5330	0.5856	0.5913	0.5691	$A_1 > A_2 > A_5 > A_3 > A_4$
	m = 10	0.5000	0.5329	0.5789	0.5889	0.5648	$A_1 > A_2 > A_5 > A_3 > A_4$
	m = 20	0.4914	0.5281	0.5691	0.5847	0.5584	$A_1 > A_2 > A_5 > A_3 > A_4$
	m = ∞	0.4889	0.5256	0.5656	0.5831	0.5561	$A_1 > A_2 > A_5 > A_3 > A_4$
$p = 4$	m = 0	0.5813	0.6051	0.6710	0.6575	0.6432	$A_1 > A_2 > A_5 > A_4 > A_3$
	m = 1	0.5876	0.6093	0.6713	0.6635	0.6475	$A_1 > A_2 > A_5 > A_4 > A_3$
	m = 2	0.5929	0.6135	0.6717	0.6683	0.6511	$A_1 > A_2 > A_5 > A_4 > A_3$
	m = 3	0.5971	0.6173	0.6721	0.6715	0.6540	$A_1 > A_2 > A_5 > A_4 > A_3$
	m = 4	0.5999	0.6174	0.6669	0.6740	0.6593	$A_1 > A_2 > A_5 > A_3 > A_4$
	m = 5	0.6014	0.6233	0.6719	0.6739	0.6564	$A_1 > A_2 > A_5 > A_3 > A_4$
	m = 6	0.6017	0.6250	0.6711	0.6741	0.6562	$A_1 > A_2 > A_5 > A_3 > A_4$
	m = 10	0.5970	0.6254	0.6654	0.6724	0.6521	$A_1 > A_2 > A_5 > A_3 > A_4$
	m = 20	0.5883	0.6203	0.6561	0.6688	0.6458	$A_1 > A_2 > A_5 > A_3 > A_4$
	m = ∞	0.5857	0.6179	0.6528	0.6674	0.6436	$A_1 > A_2 > A_5 > A_3 > A_4$
$p = 5$	m = 0	0.6467	0.6687	0.7271	0.7138	0.7018	$A_1 > A_2 > A_5 > A_4 > A_3$
	m = 1	0.6524	0.6724	0.7273	0.7192	0.7056	$A_1 > A_2 > A_5 > A_4 > A_3$
	m = 2	0.6574	0.6760	0.7276	0.7236	0.7089	$A_1 > A_2 > A_5 > A_4 > A_3$
	m = 3	0.6614	0.6795	0.7279	0.7264	0.7117	$A_1 > A_2 > A_5 > A_4 > A_3$
	m = 4	0.6645	0.6827	0.7282	0.7280	0.7136	$A_1 > A_2 > A_5 > A_4 > A_3$
	m = 5	0.6664	0.6853	0.7282	0.7288	0.7145	$A_1 > A_2 > A_5 > A_3 > A_4$
	m = 6	0.6671	0.6872	0.7277	0.7291	0.7144	$A_1 > A_2 > A_5 > A_3 > A_4$
	m = 10	0.66633	0.6880	0.7230	0.7281	0.7106	$A_1 > A_2 > A_5 > A_3 > A_4$
	m = 20	0.6549	0.6829	0.7143	0.7249	0.7045	$A_1 > A_2 > A_5 > A_3 > A_4$
	m = ∞	0.6525	0.6807	0.7113	0.7236	0.7024	$A_1 > A_2 > A_5 > A_3 > A_4$

Table 9. Ranking order based on different methods.

Reference	Methods	Parameter Value	Ranking Order
Jana et al. [31]	Dombi aggregation operators	$p = 1$	$A_2 > A_1 > A_3 > A_5 > A_4$
		$p = 2$	$A_2 > A_1 > A_5 > A_3 > A_4$
		$p = 3$	$A_1 > A_2 > A_5 > A_3 > A_4$
		$p = 4$	$A_1 > A_2 > A_5 > A_3 > A_4$
		$p = 5$	$A_1 > A_2 > A_5 > A_4 > A_3$
Wei [32]	Cross-entropy	\	$A_3 > A_2 > A_1 > A_5 > A_4$
Wei [35]	Weighted geometric operator	\	$A_1 > A_2 > A_3 > A_5 > A_4$
Khan et al. [34]	Einstein weighted averaging operator	\	$A_2 > A_3 > A_1 > A_5 > A_4$
Liu et al. [33]	Hybrid weighted distance measure	\	$A_1 > A_2 > A_3 > A_5 > A_4$
Dinh et al. [23]	Distance measure	\	$A_2 > A_3 > A_1 > A_5 > A_4$
this paper	dynamic distance	$p = 2, m = 3$	$A_1 > A_2 > A_5 > A_3 > A_4$

6. Conclusions

In this paper, we develop a dynamic distance measure based on a picture fuzzy point operator and the Minkowski distance and use it to solve the problems of multi-attribute decision-making. First, we give the concept of the picture fuzzy point operator and discuss its related properties. In addition, a dynamic distance measure for the distance between picture fuzzy sets is constructed, which can overcome the defects of the existing distance measures for the distance between picture fuzzy sets. Moreover, we use the proposed distance measure to solve the problems of multi-attribute decision-making. Through a numerical comparison and the examples of multi-attribute decision-making, we prove that the proposed distance measure is the most reasonable among the existing distance measures for the distance between picture fuzzy sets for multi-attribute decision-making because of the correct decision results. It can better deal with decision-making problems, which the existing distance measures cannot solve.

Versakis and Morabito [3] proved that a particular fuzzy divergence formulation is also a distance measure. Inspired by the result of [3], in the future, we will study the relationship among the distance measure, the fuzzy divergence degree, the fuzzy entropy, and the similarity measure and find more methods to construct distance measures in the picture fuzzy set environment.

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