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Some Results on Ricci Almost Solitons

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Abstract: We find three necessary and sufficient conditions for an n-dimensional compact Ricci almost soliton $(M, g, \mathbf{w}, \sigma)$ to be a trivial Ricci soliton under the assumption that the soliton vector field $\mathbf{w}$ is a geodesic vector field (a vector field with integral curves geodesics). The first result uses condition $r^2 \leq n\sigma r$ on a nonzero scalar curvature r; the second result uses the condition that the soliton vector field $\mathbf{w}$ is an eigen vector of the Ricci operator with constant eigenvalue λ satisfying $n^2\lambda^2 \geq r^2$; the third result uses a suitable lower bound on the Ricci curvature $S(\mathbf{w}, \mathbf{w})$. Finally, we show that an n-dimensional connected Ricci almost soliton $(M, g, \mathbf{w}, \sigma)$ with soliton vector field $\mathbf{w}$ is a geodesic vector field with a trivial Ricci soliton, if and only if, $n\sigma - r$ is a constant along integral curves of $\mathbf{w}$ and the Ricci curvature $S(\mathbf{w}, \mathbf{w})$ has a suitable lower bound.

Keywords: Ricci soliton; Ricci almost soliton; Einstein manifolds; trivial Ricci soliton

MSC: 2000; 83F05; 53C25

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1. Introduction

Given a Riemannian manifold (M, g), the sectional curvature distributions divide the manifold into three portions, one where sectional curvatures are positive, another where sectional curvatures are negative and the third, where sectional curvatures are zero. Hamilton, in his quest to solve Poincare conjecture, realized the role of a heat equation that evenly distributes temperature on the region, and considered a heat equation for the evolving metric known as Ricci flow, for an excellent description on this topic, we refer to (cf. [1,2]). A Ricci flow on a Riemannian manifold (M, g) is the following PDE for the evolving metric g_t

$$\frac{\partial g_t}{\partial t} = -2S(g_i), \quad t \in [0, T],$$

where $S(g_i)$ is the Ricci tensor of the metric g_t. A stable solution of the above Ricci flow of the form $g_t = f(t)\psi_t^*(g)$ with initial condition $g_0 = g$ is called a Ricci soliton, where ψ_t are diffeomorphisms of M and $f(t)$ is the scaling function, $\psi_0 = id$, $f(0) = 1$. In [3], the authors considered the stable solution of the Ricci flow of the form $g_t = f(t, x)\psi_t^*(g)$ (that is, allowing the scaling function to be a function of both time t and the local coordinates on M) and called the solution Ricci almost soliton. In [2], the authors introduced the notion of Riemann flow on a Riemannian manifold (M, g) as the following PDE for the evolving metric g_t

$$\frac{\partial G_t}{\partial t} = -2R(g_i), \quad t \in [0, T].$$

where G_t is $(0, 4)$-tensor field defined by

$$G_t(X, Y; Z, W) = g(Y, Z)g(X, W) - g(X, Z)g(Y, W)$$

and $R(g_i)$ is the Riemann curvature tensor field of the metric g_t. The stable solution of the Riemann flow with initial condition $g_0 = g$ is called the Riemann soliton, which is an interesting generalization of the Ricci soliton and is a current topic of research, for details see [2].

A Ricci almost soliton is a Riemannian manifold (M,g) that admits a smooth vector field $\mathbf{w}$, satisfying

$$\frac{1}{2}\pounds_{\mathbf{w}}g + S = \sigma g, \tag{1}$$

where $\pounds_{\mathbf{w}}g$ is the Lie differentiation of the metric g with respect to $\mathbf{w}$, S is the Ricci tensor and σ is a smooth function on M. We denote a Ricci almost soliton by $(M,g,\mathbf{w},\sigma)$. The notion of Ricci almost soliton is introduced in (cf. [3]) in an attempt to generalize Ricci solitons, by replacing the soliton constant with the smooth function σ. Geometry of Ricci solitons and Ricci almost solitons has been subject of immense interest due to their elegant geometry as well as applications (cf. [1,3–15]). Given a Ricci almost soliton $(M,g,\mathbf{w},\sigma)$, we call $\mathbf{w}$ the *soliton vector field* and the smooth function σ the *potential function*. A Ricci almost soliton $(M,g,\mathbf{w},\sigma)$ is said to be trivial if it is a Ricci soliton, that is, the potential function σ is a constant and a Ricci soliton is trivial if the soliton vector field is Killing. For examples of compact and non-compact non-trivial Ricci almost solitons, we refer to [3,4].

Recall that a Ricci soliton is a generalization of an Einstein manifold and a Ricci almost soliton is a generalization of a Ricci soliton. Note that if the soliton vector field $\mathbf{w}$ of a Ricci almost soliton is Killing and $\dim M > 2$, then a Ricci almost soliton is a trivial Ricci soliton. In the geometry of a Ricci almost soliton, there are two important questions, the first one is to find conditions under which it is a trivial, that is, it is a Ricci soliton and the other is to find conditions under which it is a trivial Ricci soliton, that is, the soliton vector field $\mathbf{w}$ is Killing. A Ricci almost soliton could be an Einstein manifold without being trivial, as suggested by the example $(\mathbf{S}^n(c),g,\mathbf{w},\sigma)$, where $\mathbf{S}^n(c)$ is the sphere of constant curvature c and $\mathbf{w} = \operatorname{grad} h$ for some smooth function on the sphere (cf. [4,11]). In [11], the author has proved a necessary and sufficient condition for a Ricci almost soliton $(M,g,\mathbf{w},\sigma)$ to be a Ricci soliton, is that the soliton vector field $\mathbf{w}$ is an infinitesimal harmonic transformation. In [12], the author has proved that a Ricci almost soliton $(M,g,\mathbf{w},\sigma)$ is a Ricci soliton if and only if, the soliton vector field $\mathbf{w}$ satisfies $\square\mathbf{w} = 0$, where $\square$ is the de-Rham Laplace operator. Similarly in [10], several results are proved in finding conditions under which a compact Ricci almost soliton is a trivial Ricci soliton.

Recall that the integral curves of a Killing vector field of constant length are geodesics. However, a vector field that has all its integral curves geodesics (a geodesic vector field) need not be Killing; for instance, the Reeb vector field on a trans-Sasakian manifold (cf. [16]) or the Reeb vector field on a Kenmotsu manifold (cf. [17]). For properties of geodesic vector fields, we refer to [18]. In this article, we impose the condition on the soliton vector field $\mathbf{w}$ of a Ricci almost soliton $(M,g,\mathbf{w},\sigma)$ to be a geodesic vector field and analyze the situations under which it is either a Ricci soliton or a trivial Ricci soliton. It should be mentioned that in [12], geodesic vector fields are used in a different context.

Let $(M,g,\mathbf{w},\sigma)$ be an n-dimensional Ricci almost soliton, we denote by S the Ricci tensor and by r the scalar curvature of $(M,g,\mathbf{w},\sigma)$. In this paper, we show that for a compact $(M,g,\mathbf{w},\sigma)$ with $\mathbf{w}$ a geodesic vector field and nonzero scalar curvature r satisfying $r^2 \leq n\sigma r$ is necessary and sufficient to be trivial (cf. Theorem 1). Similarly, we show that if a compact $(M,g,\mathbf{w},\sigma)$ with $\mathbf{w}$ a geodesic vector field satisfies $Q(\mathbf{w}) = \lambda\mathbf{w}$ for a constant λ with $r^2 \leq n^2\lambda^2$, if and only if, it is a trivial Ricci soliton (cf. Theorem 2). We also, show that if a compact $(M,g,\mathbf{w},\sigma)$ with $\mathbf{w}$ a geodesic vector field has an appropriate lower bound for the Ricci curvature in the direction of $\mathbf{w}$, if and only if, it is trivial (cf. Theorem 3). Finally, we show that for a connected $(M,g,\mathbf{w},\sigma)$ with $\mathbf{w}$ a geodesic vector field and Ricci curvature $S(\mathbf{w},\mathbf{w})$ has certain lower bound and the function $n\sigma - r$ is a constant on integral curves of $\mathbf{w}$ if and only if $(M,g,\mathbf{w},\sigma)$ is a trivial Ricci soliton (cf. Theorem 4).

2. Preliminaries

On an n-dimensional Ricci almost soliton $(M, g, \mathbf{w}, \sigma)$, we denote by $\mathfrak{X}(M)$ the Lie algebra of smooth vector fields on M and by ∇_X, $X \in \mathfrak{X}(M)$ the covariant derivative with respect to X. The curvature tensor field of Ricci almost soliton $(M, g, \mathbf{w}, \sigma)$ is given by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \quad X, Y, Z \in \mathfrak{X}(M)$$

and the Ricci tensor S of $(M, g, \mathbf{w}, \sigma)$ is given by

$$S(X,Y) = \sum_{i=1}^{n} g(R(e_i, X)Y, e_i),$$

where $\{e_1, \dots, e_n\}$ is a local orthonormal frame. Note that the Ricci tensor S and the Ricci operator Q defined by $S(X,Y) = g(QX,Y)$, are both symmetric. The scalar curvature r of the Ricci almost soliton $(M, g, \mathbf{w}, \sigma)$ is given by $r = TrQ$ and its gradient grad$r$ satisfies

$$\frac{1}{2}\text{grad } r = \sum_{i=1}^{n} (\nabla Q)(e_i, e_i), \tag{2}$$

where $\{e_1, \dots, e_n\}$ is a local orthonormal frame and $(\nabla Q)(X,Y) = \nabla_X QY - Q(\nabla_X Y)$.

Let β be the 1-form dual to soliton vector field $\mathbf{w}$. Then, we define a skew symmetric operator F by

$$\frac{1}{2} d\beta(X,Y) = g(FX, Y). \tag{3}$$

We call the operator F the associated operator of the Ricci almost soliton $(M, g, \mathbf{w}, \sigma)$.Then, using Equations (1) and (3) and Koszul's formula (cf. [8]), we have

$$\nabla_X \mathbf{w} = \sigma X - Q(X) + FX, \quad X \in \mathfrak{X}(M). \tag{4}$$

Using Equation (4), for the Ricci almost soliton $(M, g, \mathbf{w}, \sigma)$, we have

$$\begin{aligned} R(X,Y)\mathbf{w} = {} & X(\sigma)Y - Y(\sigma)X - (\nabla Q)(X,Y) + (\nabla Q)(Y,X) \\ & + (\nabla F)(X,Y) - (\nabla F)(Y,X). \end{aligned} \tag{5}$$

On using Equation (2) and symmetry of Q and skew-symmetry of F in above equation, we conclude

$$S(Y, \mathbf{w}) = -(n-1)Y(\sigma) + \frac{1}{2}Y(r) - \sum_{i=1}^{n} g(Y, (\nabla F)(e_i, e_i)). \tag{6}$$

Thus, we have

$$Q(\mathbf{w}) = -(n-1)\nabla\sigma + \frac{1}{2}\text{grad } r - \sum_{i=1}^{n} (\nabla F)(e_i, e_i), \tag{7}$$

Using Equation (4), we compute

$$\text{div } \mathbf{w} = (n\sigma - r). \tag{8}$$

The divergence of the vector field $F\mathbf{w}$ is given by

$$\text{div } F\,\mathbf{w} = -\|F\|^2 - \sum_{i=1}^{n} g(\mathbf{w}, (\nabla F)(e_i, e_i)), \tag{9}$$

where the squared norm $\|F\|^2$ is

$$\|F\|^2 = \sum_{i=1}^{n} g(Fe_i, Fe_i)$$

and we have used the symmetry and skew-symmetry of the operators Q and F to conclude

$$\sum_{i=1}^{n} g(Fe_i, Qe_i) = 0. \tag{10}$$

Using Equation (9), we get the following.

Lemma 1. *Let $(M, g, \mathbf{w}, \sigma)$ be an n-dimensional compact Ricci almost soliton with associated operator F. Then, for a local orthonormal frame $\{e_1, \ldots, e_n\}$ on M*

$$\int_M \left\{ \|F\|^2 + \sum_{i=1}^{n} g(\mathbf{w}, (\nabla F)(e_i, e_i)) \right\} = 0.$$

Using Equation (1), for an n-dimensional Ricci almost soliton $(M, g, \mathbf{w}, \sigma)$, we have

$$\frac{1}{2}£_{\mathbf{w}} g = \sigma g - S$$

and for a local orthonormal frame $\{e_1, \ldots, e_n\}$, we have

$$\frac{1}{4}|£_{\mathbf{w}} g|^2 = \sum_{i,j=1}^{n} \left((£_{\mathbf{w}} g)(e_i, e_j) \right)^2 = n\sigma^2 + \|Q\|^2 - 2\sigma r.$$

Thus, we have the following.

Lemma 2. *Let $(M, g, \mathbf{w}, \sigma)$ be an n-dimensional Ricci almost soliton. Then*

$$\frac{1}{4}|£_{\mathbf{w}} g|^2 = \left(\|Q\|^2 - \frac{r^2}{n} \right) + \frac{1}{n}(n\sigma - r)^2.$$

3. Ricci Almost Solitons with Geodesic Soliton Vector Field

Let $(M, g, \mathbf{w}, \sigma)$ be an n-dimensional Ricci almost soliton. We use the notion of geodesic vector field used in [18] to find necessary and sufficient conditions for a Ricci almost soliton to be a trivial Ricci soliton. Recall that if the potential function σ is a constant, then a Ricci almost soliton becomes a Ricci soliton and in addition, if the soliton vector field is a Killing vector field and $n > 2$, then the Ricci almost solitons is a trivial Ricci soliton, that is, an Einstein manifold. Note that with $\mathbf{w}$ being a geodesic vector field, that is, integral curves of $\mathbf{w}$ are geodesics, equivalently $\nabla_{\mathbf{w}}\mathbf{w} = 0$ is too far from $\mathbf{w}$ being a Killing vector field. On one hand, if a Killing vector field ξ on a Riemannian manifold (M, g) is not of constant length, then $\nabla_\xi \xi \neq 0$. On the other hand, there are examples of unit vector fields which are geodesic vector fields and are not Killing vector fields. For instance, the Reeb vector field ξ of a Trans-Sasakian manifold or a Kenmotsu manifold (also of a nearly Sasakian manifold) is a geodesic vector field that is not Killing (cf. [11,16]). In this section, we set the condition on the soliton vector field $\mathbf{w}$ of the Ricci almost soliton $(M, g, \mathbf{w}, \sigma)$ to be a geodesic vector field and find additional conditions so that $(M, g, \mathbf{w}, \sigma)$ becomes a trivial Ricci soliton.

Let the soliton vector field $\mathbf{w}$ of Ricci almost soliton $(M, g, \mathbf{w}, \sigma)$ be a geodesic vector field. Then, we have

$$\nabla_{\mathbf{w}}\mathbf{w} = 0 \tag{11}$$

and the Equation (4), gives
$$Q(\mathbf{w}) = \sigma\mathbf{w} + Fw. \tag{12}$$

Theorem 1. *Let $(M,g,\mathbf{w},\sigma)$ be an n-dimensional compact Ricci almost soliton, $n > 2$, with nonzero scalar curvature r and soliton vector field $\mathbf{w}$ a geodesic vector field. Then, $(M,g,\mathbf{w},\sigma)$ is a trivial Ricci soliton of nonzero scalar curvature, if and only if, $r^2 \leq n\sigma r$.*

Proof. Assume that $\mathbf{w}$ is a geodesic vector field. Then, taking covariant derivative in Equation (12) with respect to $X \in \mathfrak{X}(M)$ and using Equation (4), we obtain

$$\begin{aligned}(\nabla Q)(X,\mathbf{w}) + Q(\sigma X - Q(X) + FX) &= \sigma(\sigma X - Q(X) + FX) + (\nabla F)(X,\mathbf{w}) \\ &\quad + X(\sigma)\mathbf{w} + F(\sigma X - Q(X) + FX).\end{aligned}$$

Using a local orthonormal frame $\{e_1,\dots,e_n\}$ and taking $X = e_i$ in the above equation and then taking the inner product with e_i and summing, we conclude

$$\frac{1}{2}\mathbf{w}(r) + \sigma r - \|Q\|^2 = \mathbf{w}(\sigma) + n\sigma^2 - \sigma r - \sum_{i=1}^{n} g(\mathbf{w},(\nabla F)(e_i,e_i) - \|F\|^2,$$

where we have used Equations (2) and (10) and symmetry, skew-symmetry of the operators Q and F respectively. Integrating the above equation and using Lemma 1, we obtain

$$\int_M \left(\mathbf{w}\left(\frac{1}{2}r - \sigma\right) + 2\sigma r - n\sigma^2 - \|Q\|^2\right) = 0. \tag{13}$$

Now, using Equation (8) and $\operatorname{div}(fX) = X(f) + f\operatorname{div}(X)$, we have

$$\operatorname{div}\left((\frac{1}{2}r - \sigma)\mathbf{w}\right) = \mathbf{w}\left(\frac{1}{2}r - \sigma\right) + \left(\frac{1}{2}r - \sigma\right)(n\sigma - r),$$

inserting the above equation into Equation (13), we arrive at

$$\int_M \left(-\left(\frac{1}{2}r - \sigma\right)(n\sigma - r) + 2\sigma r - n\sigma^2 - \|Q\|^2\right) = 0.$$

Thus, we have

$$\int_M \left(\frac{1}{n}r^2 - \|Q\|^2\right) = \int_M \left(\left(\frac{1}{n} - \frac{1}{2}\right)r^2 + \left(\frac{n}{2} - 1\right)\sigma r\right),$$

that is,
$$\int_M \left(\frac{1}{n}r^2 - \|Q\|^2\right) = \frac{n-2}{2n}\int_M \left(n\sigma r - r^2\right). \tag{14}$$

Now, using the Schwarz's inequality $\|Q\|^2 \geq \frac{r^2}{n}$ and the condition in the hypothesis $r^2 \leq n\sigma r$ in the above equation, we conclude, $\|Q\|^2 = \frac{r^2}{n}$ and this equality holds, if and only if,
$$Q = \frac{r}{n}I. \tag{15}$$

Thus, the Equation (14) implies

$$\frac{n-2}{2n}\int_M \left(n\sigma r - r^2\right) = 0$$

and as $n > 2$ and $r^2 \leq n\sigma r$, the above equation implies $r^2 = n\sigma r$. Moreover, as $n > 2$, Equation (15) implies that the nonzero scalar curvature r is constant. This proves $r = n\sigma$ and joining this conclusion together with Equation (15) and Lemma 2, we obtain

$$\pounds_{\mathbf{w}} g = 0$$

with $\sigma = \frac{r}{n}$ a constant. Hence, $(M, g, \mathbf{w}, \sigma)$ is a trivial Ricci soliton. The converse is trivial, as we could choose $\mathbf{w} = 0$ on an Einstein manifold with nonzero scalar curvature. □

Theorem 2. *Let $(M, g, \mathbf{w}, \sigma)$ be an n-dimensional compact and connected Ricci almost soliton , $n > 2$, with soliton vector field $\mathbf{w}$ a geodesic vector field. Then, $Q(\mathbf{w}) = \lambda \mathbf{w}$ for a constant λ satisfying $n^2\lambda^2 \geq r^2$, if and only if, $(M, g, \mathbf{w}, \sigma)$ is a trivial Ricci soliton.*

Proof. Suppose that $\mathbf{w}$ is a geodesic vector field with $Q(\mathbf{w}) = \lambda \mathbf{w}$, for a constant λ satisfying $n^2\lambda^2 \geq r^2$. Using Equation (12), we get $(\lambda - \sigma)\mathbf{w} = F\mathbf{w}$ and taking the inner product with $\mathbf{w}$, we have

$$(\lambda - \sigma)\|\mathbf{w}\|^2 = 0.$$

Then, on connected M, we have either $\mathbf{w} = 0$ or $\sigma = \lambda$. If $\mathbf{w} = 0$, as $n > 2$, we get that $(M, g, \mathbf{w}, \sigma)$ is a trivial Ricci soliton. Thus, we shall concentrate on the case $\sigma = \lambda$, which makes $(M, g, \mathbf{w}, \sigma)$ a Ricci soliton. Now, since a compact Ricci soliton is a gradient Ricci soliton (cf. [1]), that is, the soliton vector field $\mathbf{w}$ is the gradient of a smooth function and as such it is closed, which implies $F = 0$. Then, Equation (7) takes the form $Q(\mathbf{w}) = \frac{1}{2}\mathrm{grad}\, r$ and we have

$$\lambda \mathbf{w} = \frac{1}{2}\mathrm{grad}\, r. \tag{16}$$

We use Equations (4) and (16) to compute the Hessian operator H_r given by

$$H_r(X) = \nabla_X \,\mathrm{grad}\, r = 2\lambda(\lambda X - Q(X)), \quad X \in \mathfrak{X}(M). \tag{17}$$

The Laplacian of r is computed using Equation (8) and we obtain

$$\Delta r = \mathrm{div}(\mathrm{grad}\, r) = 2\lambda(n\sigma - r) = 2\lambda(n\lambda - r). \tag{18}$$

Equation (17) implies

$$\|H_r\|^2 = 4\lambda^2\left(n\lambda^2 + \|Q\|^2 - 2\lambda r\right). \tag{19}$$

Using the Bochner's formula

$$\int_M \left(S(\mathrm{grad}\, r, \mathrm{grad}\, r) + \|H_r\|^2 - (\Delta r)^2\right) = 0$$

and Equations (16), (18) and (19), we have

$$\int_M \left(S(\mathbf{w}, \mathbf{w}) + n\lambda^2 + \|Q\|^2 - 2\lambda r - (n\lambda - r)^2\right) = 0. \tag{20}$$

Note that $Q(\mathbf{w}) = \lambda \mathbf{w}$, gives $S(\mathbf{w}, \mathbf{w}) = \lambda\|\mathbf{w}\|^2$, which in view of (16) implies

$$S(\mathbf{w}, \mathbf{w}) = \frac{1}{4\lambda}\|\mathrm{grad}\, r\|^2.$$

Using the above equation in Equation (20), we conclude

$$\int_M \left(\frac{1}{4\lambda}\|\mathrm{grad}\, r\|^2 + n\lambda^2 + \|Q\|^2 - 2\lambda r - (n\lambda - r)^2\right) = 0. \tag{21}$$

Now, using Equation (18), we have

$$\int_M \|\text{grad } r\|^2 = 2\lambda \int_M r(n\lambda - r)$$

and inserting the above equation in Equation (21), we obtain

$$\int_M \left(\frac{1}{2}\left(n\lambda r - r^2\right) + n\lambda^2 + \|Q\|^2 - 2\lambda r - (n\lambda - r)^2 \right) = 0. \tag{22}$$

Using Equation (8), we have

$$\int_M (n\lambda - r) = 0 \tag{23}$$

and using Equation (23) in Equation (22), we arrive at

$$\int_M \left(\frac{1}{2}\left(n^2\lambda^2 - r^2\right) + n\lambda^2 + \|Q\|^2 - 2n\lambda^2 - n^2\lambda^2 + 2n^2\lambda^2 - r^2 \right) = 0,$$

that is,

$$\int_M \left(\frac{r^2}{n} - \|Q\|^2 \right) = \frac{3n-2}{2n} \int_M \left(n^2\lambda^2 - r^2 \right). \tag{24}$$

Now, using the Schwarz's inequality and $n^2\lambda^2 \geq r^2$, we conclude $\|Q\|^2 = \frac{r^2}{n}$ and this inequality holds, if and only if,

$$Q = \frac{r}{n} I. \tag{25}$$

Since, $n > 2$, we conclude that r is a constant and, therefore, Equation (23) implies $r = n\lambda$. Thus, Lemma 2 and Equation (25) yield

$$\pounds_{\mathbf{w}} g = 0.$$

Hence, $(M, g, \mathbf{w}, \sigma)$ is a trivial Ricci soliton. The converse is trivial. □

Theorem 3. *Let $(M, g, \mathbf{w}, \sigma)$ be an n-dimensional compact and connected Ricci almost soliton, $n > 2$, with the potential vector field $\mathbf{w}$ a geodesic vector field. Then, the Ricci curvature $S(\mathbf{w}, \mathbf{w})$, the associated operator F and the scalar curvature r satisfy*

$$S(\mathbf{w}, \mathbf{w}) \geq \|F\|^2 + \frac{n-1}{n}(n\sigma - r)^2,$$

if and only if, $(M, g, \mathbf{w}, \sigma)$ is a trivial Ricci soliton.

Proof. Suppose $(M, g, \mathbf{w}, \sigma)$ is an n-dimensional compact Ricci almost soliton and $\mathbf{w}$, a geodesic vector field satisfying

$$S(\mathbf{w}, \mathbf{w}) \geq \|F\|^2 + \frac{n-1}{n}(n\sigma - r)^2. \tag{26}$$

Then, for a local orthonormal frame $\{e_1, \ldots, e_n\}$, using Equation (4), we have

$$\|\nabla \mathbf{w}\|^2 = \sum_{i=1}^{n} g(\nabla_{e_i} \mathbf{w}, \nabla_{e_i} \mathbf{w}) = n\sigma^2 + \|Q\|^2 + \|F\|^2 - 2\sigma r. \tag{27}$$

Now, using the following integral formula (cf. [19])

$$\int_M \left(S(\mathbf{w}, \mathbf{w}) + \frac{1}{2}|\pounds_{\mathbf{w}} g| - \|\nabla \mathbf{w}\|^2 - (\text{div } \mathbf{w})^2 \right) = 0$$

and Lemma 2, Equations (8) and (27), we obtain

$$\int_M \left(S(\mathbf{w},\mathbf{w}) + 2\left(\|Q\|^2 - \frac{r^2}{n} \right) + \frac{2}{n}(n\sigma - r)^2 - n\sigma^2 \right.$$
$$\left. - \|Q\|^2 - \|F\|^2 + 2\sigma r - (n\sigma - r)^2 \right) = 0.$$

The above equation can be arranged as

$$\int_M \left(\frac{r^2}{n} - \|Q\|^2 \right) = \int_M \left(S(\mathbf{w},\mathbf{w}) - \|F\|^2 - \left(\frac{n-1}{n} \right)(n\sigma - r)^2 \right),$$

and in view of inequality (26), we conclude

$$\|Q\|^2 = \frac{1}{n} r^2.$$

The above equality holds, if and only if,

$$Q = \frac{r}{n} I \tag{28}$$

and as $n > 2$ the Equation (28) implies r is a constant. Next, as $\mathbf{w}$ is a geodesic vector field, using the above equation together with Equation (12), we conclude

$$\left(\frac{r}{n} - \sigma \right) \mathbf{w} = F\mathbf{w}. \tag{29}$$

Taking the inner product with $\mathbf{w}$ in Equation (29), we get

$$\left(\frac{r}{n} - \sigma \right) \|\mathbf{w}\|^2 = 0$$

and on connected M, we get either $n\sigma = r$ or $\mathbf{w} = 0$. If $\mathbf{w} = 0$, then as $n > 2$, we get $(M, g, \mathbf{w}, \sigma)$ is a trivial Ricci soliton. Moreover, in another case with $Q = \frac{r}{n} I$, Lemma 2 and r a constant implies $(M, g, \mathbf{w}, \sigma)$ is a trivial Ricci soliton. The converse is trivial. □

Theorem 4. *Let $(M, g, \mathbf{w}, \sigma)$ be an n-dimensional connected Ricci almost soliton, $n > 2$, with the soliton vector field $\mathbf{w}$ a geodesic vector field. Then, $(M, g, \mathbf{w}, \sigma)$ is a trivial Ricci soliton, if and only if, the Ricci curvature $S(\mathbf{w},\mathbf{w})$, the associated operator F satisfies $S(\mathbf{w},\mathbf{w}) \geq \|F\|^2$ and the function $n\sigma - r$ is constant on the integral curves of $\mathbf{w}$.*

Proof. Let $(M, g, \mathbf{w}, \sigma)$ be a connected Ricci almost soliton with $\mathbf{w}$, a geodesic vector field. Suppose that the function $n\sigma - r$ is constant on the integral curves of $\mathbf{w}$ and

$$S(\mathbf{w},\mathbf{w}) \geq \|F\|^2. \tag{30}$$

Then, using Equation (5), we have

$$\begin{aligned} R(X,\mathbf{w})\mathbf{w} = {} & X(\sigma)\mathbf{w} - \mathbf{w}(\sigma)X - (\nabla Q)(X,\mathbf{w}) + (\nabla Q)(\mathbf{w},X) \\ & + (\nabla F)(X,\mathbf{w}) - (\nabla F)(\mathbf{w},X). \end{aligned} \tag{31}$$

Taking the covariant derivative in Equation (12) and using Equation (4), we get

$$\begin{aligned} (\nabla Q)(X,\mathbf{w}) + Q(\sigma X - Q(X) + FX) \quad = \quad & \sigma(\sigma X - Q(X) + FX) + (\nabla F)(X,\mathbf{w}) \\ & + X(\sigma)\mathbf{w} + F(\sigma X - Q(X) + FX), \end{aligned}$$

that is,

$$(\nabla Q)(X,\mathbf{w}) - (\nabla F)(X,\mathbf{w}) = X(\sigma)\mathbf{w} + \sigma^2 X - 2\sigma Q(X) + Q^2(X) - Q(FX) + 2\sigma FX - FQ(X) + F^2 X.$$

Inserting this equation, in Equation (31), we obtain

$$R(X,\mathbf{w})\mathbf{w} = (\nabla Q)(\mathbf{w},X) - (\nabla F)(\mathbf{w},X) - \mathbf{w}(\sigma)X - (Q-\sigma I)^2 X + Q(FX) + FQ(X) - 2\sigma FX - F^2 X.$$

Now, for a local orthonormal frame $\{e_1,\dots,e_n\}$, choosing $X = e_i$ in the above equation and taking the inner product with e_i and taking sum of the resulting equation, we conclude

$$S(\mathbf{w},\mathbf{w}) = \mathbf{w}(r) - n\mathbf{w}(\sigma) - \|Q-\sigma I\|^2 + \|F\|^2,$$

that is,

$$S(\mathbf{w},\mathbf{w}) - \|F\|^2 + \mathbf{w}(n\sigma - r) + \|Q-\sigma I\|^2 = 0. \tag{32}$$

Using the condition that the function $n\sigma - r$ is constant on the integral curves of $\mathbf{w}$ and the inequality (30) in the above equation, we conclude

$$Q = \sigma I \tag{33}$$

and as $n > 2$, the above equation implies σ is a constant. Moreover, using $S = \sigma g$ in Equation (1), we conclude $\pounds_{\mathbf{w}} g = 0$, and this proves that $(M,g,\mathbf{w},\sigma)$ is a trivial Ricci soliton. The converse is trivial. □

As a consequence of the proof of the previous Theorem, for a compact Ricci almost soliton, we have the following.

Corollary 1. *Let $(M,g,\mathbf{w},\sigma)$ be an n-dimensional compact Ricci almost soliton, $n > 2$, with soliton vector field $\mathbf{w}$ being a geodesic vector field. Then, $(M,g,\mathbf{w},\sigma)$ is a trivial Ricci soliton, if and only if, the Ricci curvature $S(\mathbf{w},\mathbf{w})$, the associated operator F and the scalar curvature r satisfy*

$$S(\mathbf{w},\mathbf{w}) \geq \|F\|^2 + (n\sigma - r)^2.$$

Proof. Let (M,g,w,σ) be a compact Ricci almost soliton with $\mathbf{w}$ a geodesic vector field and

$$S(\mathbf{w},\mathbf{w}) \geq \|F\|^2 + (n\sigma - r)^2. \tag{34}$$

Note that using Equation (8), we have $\operatorname{div}((n\sigma - r)\mathbf{w}) = \mathbf{w}(n\sigma - r) + (n\sigma - r)^2$, inserting this equation in Equation (32) and integrating the resulting equation, we obtain

$$\int_M \left(S(\mathbf{w},\mathbf{w}) - \|F\|^2 - (n\sigma - r)^2 + \|Q-\sigma I\|^2 \right) = 0,$$

that is,

$$\int_M \|Q-\sigma I\|^2 = \int_M \left(\|F\|^2 + (n\sigma - r)^2 - S(\mathbf{w},\mathbf{w}) \right).$$

Using inequality (34) in the above equation, we obtain $Q = \sigma I$ and the rest of proof follows as in the proof of Theorem 4. □

4. Conclusions

It is known that on the sphere $\mathbf{S}^n(c)$ with canonical metric g, there is a vector field ξ and a non-constant function f, such that $(\mathbf{S}^n(c), g, \xi, f)$ is a Ricci almost soliton (cf. [4–6]). In the geometry of Ricci almost solitons, there are two important questions, the first one is to find conditions under which a Ricci almost soliton is a trivial Ricci soliton and the other is to find conditions under which a compact Ricci almost soliton is isometric to $\mathbf{S}^n(c)$. Geodesic vector fields (vector fields having integral curves geodesics) (cf. [18]) are linked to Killing vector fields and therefore, in this paper, are employed on Ricci almost solitons in reducing them to trivial Ricci solitons. There are other types of vector fields, for instance generalized geodesic vector fields, which are closely related to conformal vector fields (cf. [20]), it will be interesting to investigate the role of generalized geodesic vector fields on compact Ricci almost solitons in making them isometric to the sphere $\mathbf{S}^n(c)$.

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