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Spatial Moduli of Non-Differentiability for Time-Fractional SPIDEs and Their Gradient

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Abstract: High order and fractional PDEs have become prominent in theory and in modeling many phenomena. In this paper, we study spatial moduli of non-differentiability for the fourth order time fractional stochastic partial integro-differential equations (SPIDEs) and their gradient, driven by space-time white noise. We use the underlying explicit kernels and spectral/harmonic analysis, yielding spatial moduli of non-differentiability for time fractional SPIDEs and their gradient. On one hand, this work builds on the recent works on delicate analysis of regularities of general Gaussian processes and stochastic heat equation driven by space-time white noise. On the other hand, it builds on and complements Allouba and Xiao’s earlier works on spatial uniform and local moduli of continuity of time fractional SPIDEs and their gradient.

Keywords: time fractional SPIDEs; space-time white noise; modulus of non-differentiability; Hölder regularity



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1. Introduction

In recent years, many authors have applied fractional and higher order evolution equations as (stochastic) models in mathematical physics, fluids dynamics, turbulence and mathematical finance [1–19]. The time fractional stochastic partial integro-differential equations (SPIDEs) are related to slow diffusion or diffusion in material with memory (see [12,13,20–24] for connected deterministic PDEs, and see [25–29] for connected stochastic PDEs, and see [30–32] for the associated stochastic integral equations (SIEs)). Among others, the fourth order time fractional SPIDEs are related to Brownian-time processes (BTPs); they form a unifying class for some different exciting processes like the iterated Brownian motion (IBM) of Burdzy [3,7,32,33] and the Brownian-snake of Le Gall [32,34].

The fundamental kernel associated with the deterministic version of the time-fractional SPIDE is built on the BTP [20,31,35] and extensions thereof. In this article, we give exact, dimension-dependent, spatial moduli of non-differentiability for the important class of stochastic equation:

$$\begin{cases} C_{\partial_t^\beta} U_\beta = \frac{1}{2} \Delta U_\beta + I_t^{1-\beta} \left(\frac{\partial^{d+1} W}{\partial t \partial x} \right), & (t, x) \in \mathring{\mathbb{R}}_+ \times \mathbb{R}^d; \\ U_\beta(0, x) = u_0(x), & x \in \mathbb{R}^d, \end{cases} \quad (1)$$

where $\mathring{\mathbb{R}}_+ = (0, \infty)$, the noise term $\partial^{d+1} W / \partial t \partial x$ is the space-time white noise corresponding to the real-valued Brownian sheet W on $\mathring{\mathbb{R}}_+ \times \mathbb{R}^d$, $d = 1, 2, 3$; the time fractional derivative of order β , $C_{\partial_t^\beta}$, is the Caputo fractional operator

$$C_{\partial_t^\beta} f(t) := \begin{cases} \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{f'(\tau)}{(t-\tau)^\beta} d\tau, & \text{if } 0 < \beta < 1; \\ \frac{d}{dt} f(t), & \text{if } \beta = 1, \end{cases}$$

and the time fractional integral of order α , I_t^α , is the Riemann-Liouville fractional integral of order α :

$$I_t^\alpha f := \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau, \quad \text{for } t > 0 \text{ and } \alpha > 0,$$

and $I_t^0 = I$, the identity operator. Here $\Gamma(s) = \int_0^\infty x^{s-1} e^{-x} dx$, $s > 0$, is the Gamma function. The initial data u_0 here is assumed Borel measurable, deterministic, and that there is a constant $0 < \gamma \leq 1$ such that

$$u_0 \in C_b^{2^{k+1}-2, \gamma}(\mathbb{R}^d; \mathbb{R}), \quad \text{for } 2^{k-1} < \beta^{-1} \leq 2^k, k \in \mathbb{N}, \quad (2)$$

where $C_b^{\alpha, \gamma}(\mathbb{R}^d; \mathbb{R})$ denotes the set of α -continuously differentiable functions on \mathbb{R}^d whose α -derivative is locally Hölder continuous with exponent γ .

Of course, Equation (1) is the formal (and nonrigorous) equation. Its rigorous formulation, which we work with in this paper, is given in mild form as kernel stochastic integral equation (SIE). This SIE was first introduced and treated by [21,30,31,35–38]. We give them below in Section 2, along with some relevant details.

The existence/uniqueness as well as sharp dimension-dependent L^p and Hölder regularity of the linear and nonlinear noise version of Equation (1) were investigated in [30,36–38]. These results naturally lead to the following list of motivating questions:

- Are the solutions to Equation (1) spatially continuously differentiable?
- What are the exact moduli of continuity?
- What are the exact moduli of non-differentiability?

It was studied in [39] that the exact uniform and local moduli of continuity for the time fractional SPIDE in the time variable t and space variable x , separately. In fact, it was established in [39] that the exact, dimension-dependent, spatio-temporal, uniform and local moduli of continuity for the fourth order time fractional SPIDEs and their gradient. These results give the answers to spatially continuity and exact moduli of continuity of the solutions to Equation (1), and give partial answers to above questions. In this paper, we are concerned with spatially differentiability of the solutions to Equation (1). We delve into the exact moduli of non-differentiability of the process U_β and its gradient in space. It builds on and complements works in [39], and together answers all of the above questions.

The rest of the paper is organized as follows. In Section 2, we discuss the rigorous time-fractional SPIDE kernel SIE (mild) formulation, spatial spectral density and spatial zero-one laws for time fractional SPIDEs and their gradient by using the time-fractional SPIDE kernel SIE formulation and spectral/harmonic analysis. In Section 3, we investigate the exact spatial moduli of non-differentiability for time fractional SPIDEs and their gradient by making use of the theory on limsup random fractals in [40]. In order to apply their results, the Gaussian correlation inequality in [41] will also play an important role. In the final section, the results are summarized and discussed.

2. Methodology

2.1. Rigorous Kernel Stochastic Integral Equations Formulations

For the time fractional SPIDE, as in [39], we use the density of an inverse stable Lévy time Brownian motion to define their rigorous mild SIE formulation. This density, as shown in [30,31,36], is the solution to the time-fractional PDE:

$$\begin{cases} C_{\partial_t^\beta} U_\beta = \frac{1}{2} \Delta U_\beta, & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d; \\ U_\beta(0, x) = \delta(x), & x \in \mathbb{R}^d, \end{cases} \quad (3)$$

where $\delta(x)$ is the usual Dirac delta function. This solution is the transition density of a d -dimensional β -inverse-stable-Lévy-time Brownian motion (β -ISLTBM), starting from $x \in \mathbb{R}^d$, $\mathbb{B}_{A_\beta}^x := \{B^x(A_\beta(t)), t \geq 0\}$, in which the inverse stable Lévy motion A_β of index

$\beta \in (0, 1/2]$ acts as the time clock for an independent d -dimensional Brownian motion B^x (see [14,30,42,43]), given by:

$$\mathbb{K}_{t;x,y}^{(\beta,d)} = \int_0^\infty K_{s;x,y}^{BM^d} K_{t,0,s}^{A_\beta} ds, \tag{4}$$

where $K_{s;x,y}^{BM^d} = \frac{-|x-y|^2/2s}{(2\pi s)^{d/2}}$ and $K_{t,0,s}^{A_\beta} = t\beta^{-1}s^{-1-1/\beta}g_\beta(ts^{-1/\beta})$. Here, $g_\beta(u)$ is the density of a stable subordinator and its Laplace transform is e^{-s^β} . In the case $\beta = 1/2$, the kernel $\mathbb{K}_{t;x}^{(\beta,d)}$ is the density of the Brownian-time Brownian motion (BTBM) as in [20,21,32]; and when $\beta \in \{1/2^k; k \in \mathbb{N}\}$, the kernel $\mathbb{K}_{t;x,y}^{(\beta,d)}$ is the density of k -iterated BTBM as detailed in [30,31].

Let $b : \mathbb{R} \rightarrow \mathbb{R}$ be Borel measurable. The nonlinear drift-diffusion time-fractional SPIDE is

$$\begin{cases} C_{\partial_t^\beta U_\beta} = \frac{1}{2}\Delta U_\beta + I_t^{1-\beta} \left[b(U_\beta) + a(U_\beta) \frac{\partial^{d+1}W}{\partial t \partial x} \right], & (t, x) \in \mathring{\mathbb{R}}_+ \times \mathbb{R}^d; \\ U_\beta(0, x) = u_0(x), & x \in \mathbb{R}^d. \end{cases} \tag{5}$$

Then, the rigorous time-fractional SPIDE kernel SIE (mild) formulation is the stochastic integral equation

$$U_\beta(t, x) = \int_{\mathbb{R}^d} \mathbb{K}_{t;x,y}^{(\beta,d)} u_0(y) dy + \int_{\mathbb{R}^d} \int_0^t \mathbb{K}_{t-s;x,y}^{(\beta,d)} [b(U_\beta(s, y)) ds dy + a(U_\beta(s, y)) W(ds \times dy)] \tag{6}$$

(see p. 530 in [32], and Definition 1.1 and Equation (1.11) in [36]). Of course, the mild formulation of (1.1) is then obtained by setting $a \equiv 1$ and $b \equiv 0$ in Equation (5).

Notation 1. Positive and finite constants (independent of x) in Section i are numbered as $c_{i,1}, c_{i,2}, \dots$

We conclude this section by citing the following spatial Fourier transform of the β -time-fractional (including the $\beta = 1/2$ BTBM case) kernels from Lemma 2.1 in [39].

Lemma 1. (Spatial Fourier transforms) Let $\mathbb{K}_{t;x,y}^{(\beta,d)}$ be the β -time-fractional kernel, and let $0 < \beta < 1$. The spatial Fourier transform of the β -time-fractional kernel is given by

$$\hat{\mathbb{K}}_{t;x,\xi}^{(\beta,d)} = (2\pi)^{-\frac{d}{2}} e^{-i\langle x,\xi \rangle} E_\beta \left(-\frac{|\xi|^2}{2} t^\beta \right), \tag{7}$$

where

$$E_\beta(u) = \sum_{k=0}^\infty \frac{u^k}{\Gamma(1 + \beta k)} \tag{8}$$

is the well known Mittag-Leffler function. Here, the following symmetric form of the spatial Fourier transform has been used: $\hat{f}(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(u) e^{-i\xi \cdot u} du$.

2.2. Spatial Spectral Density for Time Fractional SPIDEs and Their Gradient

Our spatial results are crucially depend on the following Lemma. In this lemma, (a) is Lemma 4.1 in [39], and (b) follows from (4.27) in [39].

Lemma 2. (Spatial spectral density). Let $t \in \mathring{\mathbb{R}}_+$ be fixed and $0 < \beta \leq 1/2$, and assume that $u_0 = 0$ in Equation (1).

(a) Let $d = 1, 2, 3$. The centered Gaussian random field $\{U_\beta(t, x), x \in \mathbb{R}^d\}$ is stationary with spectral density

$$S_\beta(t, \xi) = (2\pi)^{-d} \int_0^t E_\beta^2 \left(-\frac{|\xi|^2}{2} (t-r)^\beta \right) dr, \quad \forall \xi \in \mathbb{R}^d. \quad (9)$$

(b) Let $d = 1$. The centered Gaussian random field $\{\partial_x U_\beta(t, x), x \in \mathbb{R}\}$ is stationary with spectral density

$$\tilde{S}_\beta(t, \xi) = \xi^2 S_\beta(t, \xi), \quad \forall \xi \in \mathbb{R}. \quad (10)$$

2.3. Spatial Zero-One Laws for Time Fractional SPIDEs and Their Gradient

We establish spatial zero-one laws for time fractional SPIDEs and their gradient to have moduli of non-differentiability, which may be of independent interest. Fix $t \in \mathbb{R}_+$. For compact rectangle $I_{\text{space}} \subset \mathbb{R}^d$, $x \in \mathbb{R}^d$ and $h \in \mathbb{R}_+$, we consider $M_\beta(x, h) = \sup_{y \in I_{\text{space}}: |y| \leq 1} |U_\beta(t, x + hy) - U_\beta(t, x)|$ and $V_\beta(x, h) = \sup_{y \in I_{\text{space}}: |y| \leq 1} |\partial_{x+hy} U_\beta(t, x + hy) - \partial_x U_\beta(t, x)|$.

Proposition 1. Let $t \in \mathbb{R}_+$ be fixed and $\beta \in (0, 1/2]$, and assume that $u_0 = 0$ in Equation (1).

(a) Suppose $d = 1, 2, 3$. For any compact rectangle $I_{\text{space}} \subset \mathbb{R}^d$, there exists a constant $0 \leq C_{2,1} \leq \infty$ such that

$$\liminf_{h \rightarrow 0^+} \gamma_1(h) \inf_{x \in I_{\text{space}}} M_\beta(x, h) = C_{2,1} \quad \text{a.s.} \quad (11)$$

where

$$\gamma_1(h) = \begin{cases} (h |\log(h)|^{-1/3})^{-1/2}; & \text{if } 0 < \beta < 1/2, \\ (h |\log(h)|^{2/3})^{-1/2}; & \text{if } \beta = 1/2. \end{cases} \quad (12)$$

(b) Suppose $d = 1$. For any compact rectangle $I_{\text{space}} \subset \mathbb{R}$, there exists a constant $0 \leq C_{2,2} \leq \infty$ such that

$$\liminf_{h \rightarrow 0^+} \gamma_2(h) \inf_{x \in I_{\text{space}}} V_\beta(x, h) = C_{2,2} \quad \text{a.s.} \quad (13)$$

where

$$\gamma_2(h) = \begin{cases} (h |\log(h)|^{-1})^{-1/2}; & \text{if } 0 < \beta < 1/2, \\ h^{-1/2}; & \text{if } \beta = 1/2. \end{cases} \quad (14)$$

Remark 1. Equation (11) establishes zero-one law for the minimum oscillation $\inf_{x \in I_{\text{space}}} M_\beta(x, h)$ of the sample function $x \mapsto U_\beta(t, x)$ over the compact rectangle I_{space} . Equation (13) establishes zero-one law for the minimum oscillation $\inf_{x \in I_{\text{space}}} V_\beta(x, h)$ of the sample function $x \mapsto \partial_x U_\beta(t, x)$ over the compact rectangle I_{space} .

Remark 2. From the proof below, Equations (11) and (13) hold for any $\gamma_1(h)$ and $\gamma_2(h)$ whenever $\gamma_i(h)h |\log(h)|^{1/2} \rightarrow 0$, $i = 1, 2$, respectively. Here, Equations (12) and (14) come from Theorem 1 below for convenience.

Proof of Proposition 1. Since the proof of Equation (13) is similar to Equation (11), we only prove Equation (11). Let $\Omega_1 := O(0, 1) \subset \mathbb{R}^d$ and for $n \geq 2$, $\Omega_n := O(0, n) \setminus O(0, n-1) \subset \mathbb{R}^d$ such that $\Omega_1, \Omega_2, \dots$, are mutually disjoint, where the following notation is used: $O(\lambda, r) = \{z \in \mathbb{R}^d : \|z - \lambda\| \leq r\}$. For $n \geq 1$ and $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$, let

$$\zeta_n(t, x) = \zeta_n(\beta; t, x) := \int_{\Omega_n} \int_0^t \mathbb{K}_{t-r, x, z}^{(\beta, d)} W(dr \times dz),$$

Then $\zeta_n = \{\zeta_n(t, x), (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d\}$, $n = 1, 2, \dots$, are independent Gaussian fields. By Equation (6), we express

$$U_\beta(t, x) = \sum_{n=1}^{\infty} \zeta_n(t, x), \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d.$$

Equip $I_{\text{space}} = [0, 1]^d$ with the canonical metric

$$d_{\xi_n}(x, y) = (E[\xi_n(t, x) - \xi_n(t, y)]^2)^{1/2}, \quad x, y \in I_{\text{space}}, \tag{15}$$

and denote by $N(d_{\xi_n}, I_{\text{space}}, \delta)$ the smallest number of d_{ξ_n} -balls of radius $\delta > 0$ needed to cover I_{space} . Since $\{U_\beta(t, x), x \in \mathbb{R}^d\}$ is a stationary Gaussian random field with spectral density $S_\beta(t, \xi)$, by the definition of spectral density, one has

$$\begin{aligned} d_{\xi_n}(x, y) &= \left(2 \int_{\Omega_n} (1 - \cos(\langle (x - y), \xi \rangle)) S_\beta(t, \xi) d\xi\right)^{1/2} \\ &\leq |x - y| \left(\int_{\Omega_n} |\xi|^2 S_\beta(t, \xi) d\xi\right)^{1/2} \\ &=: |x - y| K_n, \quad x, y \in \mathbb{R}^d. \end{aligned} \tag{16}$$

To obtain the last inequality, in the integral we bound $1 - \cos(\langle u, v \rangle)$ by $|u|^2|v|^2/2$ for $u, v \in \mathbb{R}^d$. Then, by Theorem 4.1 in Meerschaert et al. [44], one has

$$\sup_{x, y \in I_{\text{space}}: |y| \leq h} \frac{|\xi_n(t, x + y) - \xi_n(t, x)|}{\tau(h)} \leq c_{2,3} \text{ a.s.}, \tag{17}$$

where $\tau(h) = h\sqrt{|\log(h)|}$. Set

$$X_M(t, x) = \sum_{n=1}^M \xi_n(t, x), \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d.$$

Then, by Equation (17), one has

$$\lim_{h \rightarrow 0^+} \sup_{x, y \in I_{\text{space}}: |y| \leq h} \gamma_1(h) |X_M(t, x + y) - X_M(t, x)| = 0 \text{ a.s.} \tag{18}$$

Therefore, the random variable

$$\liminf_{h \rightarrow 0^+} \gamma_1(h) \inf_{x \in I_{\text{space}}} M_\beta(x, h) \tag{19}$$

is measurable with respect to the tail field of $\{\xi_n\}_{n=1}^\infty$ and hence is constant almost surely. This implies Equation (11). \square

3. Results

3.1. Extremes for Time Fractional SPIDEs and Their Gradient

Without loss of generality, we again assume that $u_0 = 0$, and the random field solution U_β is given by Equation (1). Fix an arbitrary $t > 0$ throughout this subsection. Our spatial results are depend on the following the following small ball probability estimates for time fractional SPIDEs and their gradient.

Lemma 3. Let $t \in \mathring{\mathbb{R}}_+$ be fixed and $0 < \beta \leq 1/2$, and assume that $u_0 = 0$ in Equation (1).

(a) Suppose $d = 3$. Then there exist positive and finite constants $c_{3,1}$ and $c_{3,2}$ depending only on β such that for all $x_0 \in [0, 1]^3$, $r > 0$ and $u \in (0, 1)$,

$$\exp\left(-\frac{c_{3,1} r^3}{(\phi_\beta^{-1}(u^2))^3}\right) \leq \mathbb{P}(M_\beta(x_0, r) \leq u) \leq \exp\left(-\frac{c_{3,2} r^3}{(\phi_\beta^{-1}(u^2))^3}\right). \tag{20}$$

(b) Suppose $d = 1$. Then there exist positive and finite constants $c_{3,3}$ and $c_{3,4}$ depending only on β such that for all $x_0 \in [0, 1]$, $r > 0$ and $u \in (0, 1)$,

$$\exp\left(-\frac{c_{3,3}r}{\phi_\beta^{-1}(u^2)}\right) \leq \mathbb{P}(V_\beta(x_0, r) \leq u) \leq \exp\left(-\frac{c_{3,4}r}{\phi_\beta^{-1}(u^2)}\right). \tag{21}$$

In the above, $\phi_\beta^{-1}(u) = \inf\{v : \phi_\beta(v) > u\}$ is the right-continuous inverse function of ϕ_β , which is defined on $(0, \infty)$ by

$$\phi_\beta(u) = \begin{cases} u; & \text{if } 0 < \beta < 1/2, \\ u|\log(u)|; & \text{if } \beta = 1/2. \end{cases} \tag{22}$$

Proof. It follows from Lemma 4.4 in [39] that for every fixed $t > 0$, the time-fractional SIPDE solution $\{U_\beta(t, x); x \in \mathbb{R}^3\}$ is spatially strongly locally nondeterministic. Namely, for every $M > 0$, there exists a finite constant $K > 0$ (depending on t and M) such that for every $n \geq 1$ and for every $x, y_1, \dots, y_n \in [-M, M]^3$,

$$\text{Var}[U_\beta(t, x)|U_\beta(t, y_1), \dots, U_\beta(t, y_n)] \geq K \min_{1 \leq j \leq n} \phi_\beta(|x - y_j|), \tag{23}$$

where $y_0 = 0$ and the function ϕ_β is defined in Equation (22). Also, it follows from Lemma 4.4 in [39] that

$$\mathbb{E}[(U_\beta(t, x) - U_\beta(t, y))^2] \asymp \phi_\beta(|x - y|); \quad \forall x, y \in [-M, M]^3. \tag{24}$$

Thus, by Theorem 3.1 in [45] or Lemma 2.2 in [46], Equations (23) and (24) yield Equation (20).

Similarly, since, by (4.31) and (4.33) in [39], Equations (23) and (24) also hold for $\partial_x U_\beta$ instead of U_β , one has Equation (21) holds. This completes the proof. \square

We also need the following lemma, which is Theorem 1.1 in [41].

Lemma 4. Let $G' = (G'_1, G'_2)$ be an \mathbb{R}^n -valued normal random vector with mean vector 0, where $G_1 = (X_1, \dots, X_k)'$, $G_2 = (X_{k+1}, \dots, X_n)'$ and $1 \leq k < n$. Then $\forall x > 0$,

$$\mathbb{P}(\|G\|_\infty \leq x) \leq \rho \mathbb{P}(\|G_1\|_\infty \leq x) \mathbb{P}(\|G_2\|_\infty \leq x), \tag{25}$$

where $\|\mathbf{x}\|_\infty$ denotes the maximum norm of a vector \mathbf{x} and

$$\rho = \left(\frac{\det(\mathbb{E}[G_1 G_1']) \det(\mathbb{E}[G_2 G_2'])}{\det(\mathbb{E}[G G'])}\right)^{1/2}. \tag{26}$$

We also need the following lemma, which is Lemma 2.4 in [47].

Lemma 5. Let $A = (a_{ij}, 1 \leq i, j \leq 2p)$ be a positive semidefinite matrix given by

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where A_{11} and A_{22} are two $p \times p$ matrices. Put $S_i = \sum_{j=p+1}^{2p} |a_{ij}|$ for $1 \leq i \leq p$ and $= \sum_{j=1}^p |a_{ij}|$ for $p + 1 \leq i \leq 2p$. Assume the following conditions are satisfied:

(i) There is a constant δ such that for all $1 \leq i \leq 2p$,

$$S_i < \delta. \tag{27}$$

(ii) There exists a finite constant $B > 0$ such that for all $1 \leq i \leq 2p$,

$$\frac{\det(A^{(i)})}{\det(A)} \leq B, \tag{28}$$

where $A^{(i)}$ is the submatrix of A obtained by deleting the i th row and i th column.

Then

$$\det(A) \geq e^{-2\delta Bp} \det(A_{11}) \det(A_{22}). \tag{29}$$

3.2. Spatial Moduli of Non-Differentiability for Time Fractional SPIDEs and Their Gradient

We investigate the exact spatial moduli of non-differentiability for time fractional SPIDE $U_\beta(t, x)$ and the gradient process $\partial_x U_\beta(t, x)$.

Theorem 1. (Spatial moduli of non-differentiability) Let $t \in \mathring{\mathbb{R}}_+$ be fixed and $0 < \beta \leq 1/2$, and assume that $u_0 = 0$ in Equation (1).

(a) Suppose $d = 3$. For any compact rectangle $I_{\text{space}} \subset \mathbb{R}^3$,

$$\liminf_{h \rightarrow 0^+} \gamma_1(h) \inf_{x \in I_{\text{space}}} M_\beta(x, h) = c_{3,5} \quad \text{a.s.} \tag{30}$$

Consequently, the sample paths of $U_\beta(t, x)$ are almost surely nowhere differentiable in all directions of x .

(b) Suppose $d = 1$. For any compact rectangle $I_{\text{space}} \subset \mathbb{R}$,

$$\liminf_{h \rightarrow 0^+} \gamma_2(h) \inf_{x \in I_{\text{space}}} V_\beta(x, h) = c_{3,6} \quad \text{a.s.} \tag{31}$$

Consequently, the sample paths of $\partial_x U_\beta(t, x)$ are almost surely nowhere differentiable in x .

Remark 3. The following are some remarks on Theorem 1.

- Equations (30) and (31) describe the uniform minimal oscillations of the sample functions of $U_\beta(t, x)$ and $\partial_x U_\beta(t, x)$, respectively. To be precise, Equation (30) implies the size of the minimum oscillation $\inf_{x \in I_{\text{space}}} M_\beta(x, h)$ of the sample function $x \mapsto U_\beta(t, x)$ over the compact rectangle I_{space} is $\gamma_1(h)$ (up to a constant factor). Equation (31) implies the size of the minimum oscillation $\inf_{x \in I_{\text{space}}} V_\beta(x, h)$ of the sample function $x \mapsto \partial_x U_\beta(t, x)$ over the compact rectangle I_{space} is $\gamma_2(h)$ (up to a constant factor).
- Together with the Khinchin-type law of the iterated logarithm and the uniform modulus of continuity in [39], they provide complete information on the regularity properties of $U_\beta(t, x)$ and $\partial_x U_\beta(t, x)$.

Proof of Theorem 1. Since the proof of Equation (31) is similar to Equation (30), we only prove Equation (30). To prove Equation (30), we claim first the following two inequalities:

$$\liminf_{h \rightarrow 0^+} \gamma_1(h) \inf_{x \in I_{\text{space}}} M_\beta(x, h) \geq c_{3,7} \quad \text{a.s.} \tag{32}$$

and

$$\liminf_{h \rightarrow 0^+} \gamma_1(h) \inf_{x \in I_{\text{space}}} M_\beta(x, h) \leq c_{3,8} \quad \text{a.s.} \tag{33}$$

where $c_{3,7} < c_{3,8}$ and $c_{3,8} > (c_{3,1}/2)^{1/6}$. It follows from Equations (32), (33) and (11) that Equation (30) holds and thereby we complete the proof.

It remains to prove Equations (32) and (33). We prove Equation (32) first. Without loss of generality, we assume $I_{\text{space}} = [0, 1]^3$.

Fix an arbitrary $\theta > 1$. For $n \in \mathbb{Z}_+$, let $h_n = \theta^{-n}$ and $\rho_n = \theta^{4n}$. For $\mathbf{i} = (i_1, i_2, i_3) \in \mathbb{Z}_+^3$ and $n \geq 1$, we define two sets A_n and $A_{n,\mathbf{i}}$ as follows:

$$\begin{aligned} A_n &= \{h \in (0, 1) : h_{n+1} < h \leq h_n\}, \\ A_{n,\mathbf{i}} &= \{x = (x_1, x_2, x_3)' \in I : \mathbf{i}\rho_n^{-1} < x \leq (\mathbf{i} + \mathbf{1})\rho_n^{-1}\}, \end{aligned}$$

where $\mathbf{1}$ is a vector with elements 1. Observe that for all $h \in (0, 1)$, there exists a set A_n such that $h \in A_n$, and for all $x \in I$, there exists a set $A_{n,\mathbf{i}}$ such that $x \in A_{n,\mathbf{i}}$. Let $x_{\mathbf{i},n} := \mathbf{i}\rho_n^{-1}$ be a point in $A_{n,\mathbf{i}}$, $\mathbf{i} \in [0, \rho_n]^3 \cap \mathbb{Z}_+^3$. Note that

$$\phi_\beta^{-1}(u) \sim \begin{cases} u; & \text{if } 0 < \beta < 1/2, \\ u(\log(1/u))^{-1}; & \text{if } \beta = 1/2 \end{cases} \tag{34}$$

as $u \rightarrow 0$. Thus, by Equations (20) and (34), one has

$$\begin{aligned} &\mathbb{P}\left(\gamma_1(h_n) \min_{\mathbf{i} \in [0, \rho_n]^3 \cap \mathbb{Z}_+^3} M_\beta(x_{\mathbf{i},n}, h_n) \leq c_{3,7}\right) \\ &\leq \sum_{\mathbf{i} \in [0, \rho_n]^3 \cap \mathbb{Z}_+^3} \mathbb{P}(M_\beta(x_{\mathbf{i},n}, h_n) \leq c_{3,7}\gamma_1(h_n)^{-1}) \\ &\leq \theta^{-(c_{3,2}/c_{3,7}^6)^n}. \end{aligned}$$

Hence, the sum of above probabilities with respect to n is finite and hence, by Borel-Cantelli lemma, one has

$$\liminf_{n \rightarrow \infty} \gamma_1(h_n) \min_{\mathbf{i} \in [0, \rho_n]^3 \cap \mathbb{Z}_+^3} M_\beta(x_{\mathbf{i},n}, h_n) \geq c_{3,7} \quad \text{a.s.} \tag{35}$$

It follows from Theorem 4.1 in Meerschaert et al. [44] that

$$\limsup_{n \rightarrow \infty} \gamma_1(h_n) \sup_{x \in [0,2]^3} M_\beta(x, h_n) = 0 \quad \text{a.s.} \tag{36}$$

Since the function $x \mapsto \gamma_1(x)$ is decreasing for $x \in (0, 1)$, one has

$$\begin{aligned} &\liminf_{h \rightarrow 0^+} \gamma_1(h) \inf_{x \in [0,1]^3} M_\beta(x, h) \\ &\geq \liminf_{n \rightarrow \infty} \inf_{h \in A_n} \min_{\mathbf{i} \in [0, \rho_n]^3 \cap \mathbb{Z}_+^3} \inf_{x \in A_{n,\mathbf{i}}} \gamma_1(h) M_\beta(x, h) \\ &\geq \liminf_{n \rightarrow \infty} \min_{\mathbf{i} \in [0, \rho_n]^3 \cap \mathbb{Z}_+^3} \inf_{x \in A_{n,\mathbf{i}}} (\gamma_1(h_{n+1}) / \gamma_1(h_n)) \gamma_1(h_n) M_\beta(x, h_n) \\ &\geq \liminf_{n \rightarrow \infty} \min_{\mathbf{i} \in [0, \rho_n]^3 \cap \mathbb{Z}_+^3} (\gamma_1(h_{n+1}) / \gamma_1(h_n)) \gamma_1(h_n) M_\beta(x_{\mathbf{i},n}, h_n) \\ &\quad - \limsup_{n \rightarrow \infty} \max_{\mathbf{i} \in [0, \rho_n]^3 \cap \mathbb{Z}_+^3} \sup_{x \in A_{n,\mathbf{i}}} (\gamma_1(h_{n+1}) / \gamma_1(h_n)) \gamma_1(h_n) |M_\beta(x, h_n) - M_\beta(x_{\mathbf{i},n}, h_n)| \\ &\geq \liminf_{n \rightarrow \infty} \min_{\mathbf{i} \in [0, \rho_n]^3 \cap \mathbb{Z}_+^3} \gamma_1(h_n) M_\beta(x_{\mathbf{i},n}, h_n) \\ &\quad - 2 \limsup_{n \rightarrow \infty} \sup_{x \in [0,2]^3} \gamma_1(h_n) M_\beta(x, h_n). \end{aligned} \tag{37}$$

It follows from Equations (35)–(37) that Equation (32) holds.

Next we prove Equation (33). For convenience, a typical parameter (“space point”) $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ is sometimes also written as $\langle x_i \rangle$, or $\langle c \rangle$, if $x_1 = x_2 = x_3 = c$. Denote by $\lfloor u \rfloor \leq u < \lfloor u \rfloor + 1$ the integer part of $u \in \mathbb{R}_+$. For each $n \geq 1$, we denote $h_n = 2^{-n}$, $\Theta = \Theta_n = h_n^2$, $\vartheta = \vartheta_n = |\log(h_n)|^{-1}$, $D_n = \lfloor \vartheta^{-1} h_n^{-2} \rfloor$, $\mathcal{S}_n = \{1, \dots, D_n\}$, and $\mathcal{T}_n = \{\mathbf{i} = (i_1, i_2, i_3)' \in \mathbb{Z}_+^3 : i_k \in \{1, \dots, \lfloor \vartheta^{-1} \rfloor\}, k = 1, 2, 3\}$. For each $i \in \mathcal{S}_n$, we define

$$x_{i,n} = i\vartheta \langle \Theta \rangle.$$

We define $Y_{i,n}$, for $i \in S_n$, to be 1 or 0 according as the random variable

$$\max_{\langle k_\ell \rangle \in \mathcal{T}_n} \gamma_1(h_n) |U_\beta(t, x_{i,n} + \vartheta \langle k_\ell \Theta \rangle) - U_\beta(t, x_{i,n})| \leq c_{3,8}$$

is or not. Define $S_n := \sum_{i \in S_n} Y_{i,n}$. For each $n \geq 1$, the mean $p_n := \mathbb{E}[Y_{i,n}]$ is the same for all $i \in S_n$, and that, by Equation (20), one has uniformly over $i \in S_n$, as $n \rightarrow \infty$,

$$\begin{aligned} p_n = \mathbb{P}(Y_{i,n} = 1) &= \mathbb{P}(Y_{1,n} = 1) \\ &\geq \mathbb{P}\left(\gamma_1(h_n) M_\beta(x_{i,n}, \Theta) \leq c_{3,8}\right) \\ &\geq \exp((c_{3,1}/c_{3,8}^6) \log(h_n)). \end{aligned} \tag{38}$$

We need only show that $S_n > 0$ for infinitely many n . We want to estimate

$$\text{Var}(S_n) = \sum_{i,j \in S_n} \text{Cov}(Y_{i,n}, Y_{j,n}). \tag{39}$$

Put $\Lambda = \Lambda_n = \vartheta^{-12}$. We make the following claim: $\forall \tau > 0$, whenever $|i - j| \geq \Lambda$,

$$\mathbb{P}(Y_{i,n} = 1, Y_{j,n} = 1) \leq (1 + \tau)(\mathbb{P}(Y_{1,n} = 1))^2. \tag{40}$$

Before we prove Equation (40), we complete the proof of Equation (33) and thereby of Theorem 1.

It follows from Equation (39) that for all $\tau > 0$, whenever $i, j \in S_n$ satisfy $|j - i| \geq \Lambda$, then $\text{Cov}(Y_{i,n}, Y_{j,n}) \leq \tau \mathbb{E}[Y_{i,n}] \mathbb{E}[Y_{i,n}]$. Thus, by Equation (38),

$$\text{Var}(S_n) \leq \tau D_n^2 p_n^2 + \sum_{i,j \in S_n: |i-j| \leq \Lambda} \text{Cov}(Y_{i,n}, Y_{j,n}).$$

For the remaining covariance, use the fact that all $Y_{i,n}$'s are either 0 or 1. In particular, $\text{Cov}(Y_{i,n}, Y_{j,n}) \leq \mathbb{E}[Y_{i,n}] = p_n$. Thus

$$\text{Var}(S_n) \leq \tau D_n^2 p_n^2 + D_n p_n \Lambda.$$

Combining this with the Chebyshev's inequality, we obtain:

$$\mathbb{P}(S_n = 0) \leq \frac{\text{Var}(S_n)}{(\mathbb{E}[S_n])^2} \leq \tau + \frac{\Lambda}{D_n p_n} \tag{41}$$

since $\mathbb{E}[S_n] = D_n p_n$. Noting $c_{3,8} > (c_{3,1}/2)^{1/6}$, one has $c_{3,1}/c_{3,8}^6 < 2$. Thus, by Equation (38), $\Lambda/(D_n p_n) \leq h_n^{2-c_{3,1}/c_{3,8}^6} |\log(h_n)|^{13} \rightarrow 0$. Hence, by Equation (41) and the arbitrariness of τ , we see that $\mathbb{P}(S_n = 0) \rightarrow 0$ as $n \rightarrow \infty$. Finally

$$\mathbb{P}(S_n > 0 \text{ i.o.}) > \limsup_{n \rightarrow \infty} \mathbb{P}(S_n > 0) = 1.$$

This yields

$$\liminf_{n \rightarrow \infty} \min_{i \in S_n} \max_{\langle k_\ell \rangle \in \mathcal{T}_n} \gamma_1(h_n) |U_\beta(t, x_{i,n} + \vartheta \langle k_\ell \Theta \rangle) - U_\beta(t, x_{i,n})| \leq c_{3,8} \text{ a.s.} \tag{42}$$

Thus,

$$\liminf_{n \rightarrow \infty} \inf_{x \in [0,1]^3} \max_{\langle k_\ell \rangle \in \mathcal{T}_n} \gamma_1(h_n) |U_\beta(t, x + \vartheta \langle k_\ell \Theta \rangle) - U_\beta(t, x)| \leq c_{3,8} \text{ a.s.} \tag{43}$$

Note that

$$\begin{aligned}
 & \liminf_{n \rightarrow \infty} \inf_{x \in [0,1]^3} \gamma_1(h_n) M_\beta(x, h_n) \\
 & \leq \liminf_{n \rightarrow \infty} \inf_{x \in [0,1]^3} \max_{\langle k_\ell \rangle \in \mathcal{T}_n} \sup_{\langle k_{\ell-1} \rangle \leq \langle u_\ell \rangle \leq \langle k_\ell \rangle} \gamma_1(h_n) |U_\beta(t, x + \vartheta \langle u_\ell \Theta \rangle) - U_\beta(t, x)| \\
 & \leq \liminf_{n \rightarrow \infty} \inf_{x \in [0,1]^3} \max_{\langle k_\ell \rangle \in \mathcal{T}_n} \gamma_1(h_n) |U_\beta(t, x + \vartheta \langle k_\ell \Theta \rangle) - U_\beta(t, x)| \\
 & \quad + \limsup_{n \rightarrow \infty} \sup_{x \in [0,1]^3} \max_{\langle k_\ell \rangle \in \mathcal{T}_n} \sup_{\langle k_{\ell-1} \rangle \leq \langle u_\ell \rangle \leq \langle k_\ell \rangle} \gamma_1(h_n) |U_\beta(t, x + \vartheta \langle u_\ell \Theta \rangle) - U_\beta(t, x + \vartheta \langle k_\ell \Theta \rangle)| \\
 & \leq \liminf_{n \rightarrow \infty} \inf_{x \in [0,1]^3} \max_{\langle k_\ell \rangle \in \mathcal{T}_n} \gamma_1(h_n) |U_\beta(t, x + \vartheta \langle k_\ell \Theta \rangle) - U_\beta(t, x)| \\
 & \quad + \limsup_{n \rightarrow \infty} \sup_{x \in [0,2]^3} \sup_{\langle |y_\ell| \rangle \leq \vartheta \langle \Theta \rangle} \gamma_1(h_n) |U_\beta(t, x + y) - U_\beta(t, x)|.
 \end{aligned} \tag{44}$$

It follows from Theorem 4.1 in Meerschaert et al. [44] that

$$\limsup_{n \rightarrow \infty} \sup_{x \in [0,2]^3} \sup_{\langle |y_\ell| \rangle \leq \vartheta \langle \Theta \rangle} \gamma_1(h_n) |U_\beta(t, x + y) - U_\beta(t, x)| = 0 \quad \text{a.s.} \tag{45}$$

Hence, by Equations (43)–(45), Equation (33) holds.

Therefore, it remains to prove Equation (40). Since, it follows from Lemma 2 that $\{U_\beta(t, x), x \in \mathbb{R}^3\}$ is stationary, one can define $\sigma(y) = \mathbb{E}[(U_\beta(t, x + y) - U_\beta(t, x))^2]$. With $F_{i,n}(x) = U_\beta(t, x_{i,n} + x) - U_\beta(t, x_{i,n}), x \in [0, 1]^3$, one has for $i, j \in \mathcal{S}_n$ and $x, y \in [0, 1]^3$,

$$\begin{aligned}
 & \mathbb{E}[F_{i,n}(x)F_{j,n}(y)] \\
 & = \frac{1}{2}(-\sigma((x_{j,n} - x_{i,n}) + (y - x)) + \sigma((x_{j,n} - x_{i,n}) - x) + \sigma((x_{j,n} - x_{i,n}) + y) - \sigma(x_{j,n} - x_{i,n})).
 \end{aligned} \tag{46}$$

Put $\nabla \sigma = (\partial \sigma / \partial h_1, \partial \sigma / \partial h_2, \partial \sigma / \partial h_3)$, $a_{lm} = (\partial^2 \sigma / (\partial h_l \partial h_m))((x_{j,n} - x_{i,n}) + \eta_3(y - \eta_1 x + \eta_2 x))$ ($1 \leq l, m \leq 3$) and the matrix $\Omega = (a_{lm})_{3 \times 3}$, where $\eta_1, \eta_2, \eta_3 \in [0, 1]$. We use Taylor expansion to see that

$$\begin{aligned}
 & \mathbb{E}[F_{i,n}(x)F_{j,n}(y)] \\
 & = (\nabla \sigma((x_{j,n} - x_{i,n}) + (y - \eta_1 x)) - \nabla \sigma((x_{j,n} - x_{i,n}) - \eta_2 x))x' \\
 & = (y - \eta_1 x + \eta_2 x) \Omega x'.
 \end{aligned} \tag{47}$$

In the sequel, for ease of exposition, we arrange all points in \mathcal{T}_n according to the following rule: for two points $\langle k_\ell \rangle = (k_1, k_2, k_3), \langle m_\ell \rangle = (m_1, m_2, m_3) \in \mathcal{T}_n$, we define $\langle k_\ell \rangle \prec \langle m_\ell \rangle$ if there exists $1 \leq \ell \leq 3$ such that $k_1 = m_1, \dots, k_{\ell-1} = m_{\ell-1}, k_\ell < m_\ell$ with convention $k_0 = m_0 = 0$. Fix $i, j \in \mathcal{S}_n$. Then, we consider Gaussian random vectors $G_1 := (F_{i,n}(\vartheta \langle k_\ell \Theta \rangle), \langle k_\ell \rangle \in \mathcal{T}_n)'$ and $G_2 := (F_{j,n}(\vartheta \langle m_\ell \Theta \rangle), \langle m_\ell \rangle \in \mathcal{T}_n)'$ and $G = (G_1', G_2')'$. With Σ the covariance matrix of G , we have

$$\Sigma = \begin{pmatrix} \Sigma_1 & \Sigma_2 \\ \Sigma_2' & \Sigma_1 \end{pmatrix},$$

where $\Sigma_1 = \mathbb{E}[G_1 G_1']$ and $\Sigma_2 = \mathbb{E}[G_2 G_1']$. Put $g(h_n) = c_{3,8} \gamma_1(h_n)^{-1}$. By Lemma 4, we have

$$\begin{aligned}
 & \mathbb{P}\left(\max_{\langle k_\ell \rangle \in \mathcal{T}_n} |F_{i,n}(\vartheta \langle k_\ell \Theta \rangle)| \leq g(h_n), \max_{\langle m_\ell \rangle \in \mathcal{T}_n} |F_{j,n}(\vartheta \langle m_\ell \Theta \rangle)| \leq g(h_n)\right) \\
 & \leq \rho \mathbb{P}\left(\max_{\langle k_\ell \rangle \in \mathcal{T}_n} |F_{i,n}(\vartheta \langle k_\ell \Theta \rangle)| \leq g(h_n)\right) \mathbb{P}\left(\max_{\langle m_\ell \rangle \in \mathcal{T}_n} |F_{j,n}(\vartheta \langle m_\ell \Theta \rangle)| \leq g(h_n)\right),
 \end{aligned} \tag{48}$$

where

$$\rho = \left(\frac{\det(\Sigma_1) \det(\Sigma_1)}{\det(\Sigma)}\right)^{1/2}.$$

We will make use of Lemma 5 (with $p = \vartheta^{-3}$) to consider the determinant of $(2\vartheta^{-3}) \times (2\vartheta^{-3})$ matrix Σ . We first verify that the positive semidefinite matrix Σ satisfies Conditions (i)–(ii) of Lemma 5.

For $i, j \in \mathcal{S}_n$, $\langle k_\ell \rangle, \langle m_\ell \rangle \in \mathcal{T}_n$ and $1 \leq u, v \leq 3$, we put $\lambda_u = \vartheta \Theta((j-i) + \eta_3(m_u - \eta_1 k_u + \eta_2 k_u))$, $\tilde{a}_{uv} = \partial^2 \sigma(\langle \lambda_\ell \rangle) / \partial \lambda_u \partial \lambda_v$, $f_u = \vartheta \Theta(m_u - \eta_1 k_u + \eta_2 k_u)$, and $g_u = \vartheta \Theta k_u$. Noting $|\tilde{a}_{uv}| \leq c_{3,9} |\lambda_u|^{-1/2} |\lambda_v|^{-1/2}$ if $u \neq v$, and $|\tilde{a}_{uu}| \leq c_{3,10} |\lambda_u|^{-1}$ if $u = v$, it is easy to verify that for all $1 \leq u, v \leq 3$ and $i, j \in \mathcal{S}_n$ with $|j-i| \geq \Lambda$,

$$|f_u \tilde{a}_{uv} g_v| \leq c_{3,11} \vartheta^{11} h_n^2. \quad (49)$$

It follows from Equations (47) (with $x = \vartheta \langle k_\ell \rangle$, $y = \vartheta \langle m_\ell \rangle$) and $h = \langle h_\ell \rangle$) and (49) that for $\langle k_\ell \rangle, \langle m_\ell \rangle \in \mathcal{T}_n$ and $i, j \in \mathcal{S}_n$ with $|j-i| \geq \Lambda$,

$$\begin{aligned} a_{\langle k_\ell \rangle, \langle m_\ell \rangle} &= |\mathbb{E}[F_{i,n}(\vartheta \langle k_\ell \rangle) F_{j,n}(\vartheta \langle m_\ell \rangle)]| \\ &= \vartheta^2 |(\langle m_\ell \rangle - \eta_1 \langle k_\ell \rangle + \eta_2 \langle k_\ell \rangle) \Omega \langle k_\ell \rangle'| \\ &\leq \sum_{u=1}^3 \sum_{v=1}^3 |f_u \tilde{a}_{uv} g_v| \\ &\leq c_{3,12} \vartheta^{11} h_n^2. \end{aligned} \quad (50)$$

Thus, by Equation (50),

$$\sum_{\langle k_\ell \rangle \in \mathcal{T}_n} \sum_{\langle m_\ell \rangle \in \mathcal{T}_n} a_{\langle k_\ell \rangle, \langle m_\ell \rangle} \leq c_{3,12} \vartheta^5 h_n^2. \quad (51)$$

This verifies Condition (i) in Lemma 5 with $\delta = c_{3,12} \vartheta^5 h_n^2$.

In order to verify Condition (ii) in Lemma 5, we make use of the following fact on the conditional variance

$$\text{Var}\left(F_{i,n}(\vartheta \langle k_\ell \rangle) | F_{u,n}(\vartheta \langle m_\ell \rangle), \langle m_\ell \rangle \neq \langle k_\ell \rangle, \langle m_\ell \rangle \in \mathcal{T}_n, u \in \{i, j\}\right) = \frac{\det(\Sigma)}{\det(\Sigma^{(l)})}, \quad (52)$$

where $F_{i,n}(\vartheta \langle k_\ell \rangle)$ is the l th point in U_β according to the above mentioned rule. Thus, by Equation (23), one has

$$\frac{\det(\Sigma)}{\det(\Sigma^{(l)})} \geq \text{Var}\left(U_\beta(t, \vartheta \langle k_\ell \rangle) | U_\beta(t, \vartheta \langle m_\ell \rangle), \langle m_\ell \rangle \neq \langle k_\ell \rangle, \langle m_\ell \rangle \in \mathcal{T}_n\right) \geq c_{3,13} \vartheta h_n^2. \quad (53)$$

This verifies Condition (ii) with $B = (c_{3,13} \vartheta h_n^2)^{-1}$.

Applying Lemma 5 with $p = \vartheta^{-3}$, $\delta = c_{3,12} \vartheta^5 h_n^2$ and $B = (c_{3,13} \vartheta h_n^2)^{-1}$, we obtain

$$\det(\Sigma) \geq e^{-2\delta B p} (\det(\Sigma_1))^2. \quad (54)$$

This, together with Equation (54), yields that

$$\rho \leq e^{\delta B p}. \quad (55)$$

Notice that $\delta B p \rightarrow 0$ as $n \rightarrow \infty$. This, together with Equations (55) and (48), yields that Equation (40) holds. The proof of Theorem 1 is completed. \square

4. Conclusions

In this article, we have presented that the solutions to the fourth order time fractional SPIDEs and their gradient, driven by space-time white noise, are almost surely nowhere differentiable in all directions of space variable x . We have established the exact spatial moduli of non-differentiability, and been concerned with the small fluctuation behavior, with delicate analysis of regularities, for the above class of equations and their gradient. They complement Allouba's earlier works on the spatio-temporal Hölder regularity of time fractional SPIDEs and their gradient. Together with the Khinchin-type law of the iterated logarithm and the uniform modulus of continuity, they provide complete information on the regularity properties of time fractional SPIDEs and their gradient in space.

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Abbreviations

The following abbreviations are used in this manuscript:

SPDE	Stochastic partial differential equation
SPIDEs	Stochastic partial integro-differential equations
SIE	Stochastic integral equation
BTP	Brownian-time process
IBM	Iterated Brownian motion

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