Article

On Generalizations of Sampling Theorem and Stability Theorem in Shift-Invariant Subspaces of Lebesgue and Wiener Amalgam Spaces with Mixed-Norms

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Abstract: In this paper, we establish generalized sampling theorems, generalized stability theorems and new inequalities in the setting of shift-invariant subspaces of Lebesgue and Wiener amalgam spaces with mixed-norms. A convergence theorem of general iteration algorithms for sampling in some shift-invariant subspaces of $L_{\vec{p}}(\mathbb{R}^d)$ are also given.

1. Introduction and Preliminaries

We first introduce and review some spaces that will be used in this paper. In what follows, for any $d \in \mathbb{N}$ (the set of positive integers), we denote $\mathbb{Z}^d$ and $\mathbb{R}^d$ by

$$\mathcal{K}^d = \underbrace{\mathcal{K} \times \mathcal{K} \times \cdots \times \mathcal{K}}_{d \text{ times}} = \{\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) : \alpha_i \in \mathcal{K}, 1 \leq i \leq d\},$$

where $\mathcal{K} = \mathbb{Z}$ (the set of integers) or $\mathbb{R}$ (the set of real numbers). Recall that the discrete space $l_p(\mathbb{Z})$ is defined by

$$l_p(\mathbb{Z}) = \left\{ c = (c_k)_{k \in \mathbb{Z}} : \left(\sum_{k \in \mathbb{Z}} |c_k|^p \right)^{\frac{1}{p}} < \infty \right\} \quad \text{if } 1 \leq p < \infty,$$

and

$$l_p(\mathbb{Z}) = \left\{ c = (c_k)_{k \in \mathbb{Z}} : \sup_{k \in \mathbb{Z}} |c_k| < \infty \right\} \quad \text{if } p = \infty.$$

For $c = (c_k)_{k \in \mathbb{Z}} \in l_p(\mathbb{Z})$, we define $l_p(\mathbb{Z})$-norm of c by

$$\|c\|_{l_p} = \begin{cases} \left(\sum_{k \in \mathbb{Z}} |c_k|^p \right)^{\frac{1}{p}}, & \text{if } 1 \leq p < \infty; \\ \sup_{k \in \mathbb{Z}} |c_k|, & \text{if } p = \infty. \end{cases}$$

Let $p \in [1, \infty]$. The Wiener amalgam spaces $(W\mathfrak{L})_p(\mathbb{R}^d)$ is the set of all measurable functions g on $\mathbb{R}^d$ such that their norms $\|g(x)\|_{(W\mathfrak{L})_p(\mathbb{R}^d)}$ defined by

$$\|g(x)\|_{(W\mathfrak{L})_p(\mathbb{R}^d)} = \begin{cases} \left(\sum_{k \in \mathbb{Z}^d} \text{esssup}_{x \in [0,1]^d} |g(x+k)|^p \right)^{\frac{1}{p}} < \infty, & \text{if } 1 \le p < \infty; \\ \sup_{k \in \mathbb{Z}^d} \left(\text{esssup}_{x \in [0,1]^d} |g(x+k)| \right) < \infty. & \text{if } p = \infty. \end{cases}$$

Throughout this paper, we denote

$$[1, \infty]^d = \{(r_1, \ldots, r_d) : 1 \le r_i \le \infty \text{ for } i = 1, 2, \ldots, d\}.$$

The following mixed-norm spaces will be discussed in this paper.

Definition 1 (see [1,2]). *Let $\vec{p} = (p_1, \ldots, p_d) \in [1, \infty]^d$ be a mixed-norm index. The mixed-norm discrete space $l_{\vec{p}}(\mathbb{Z}^d)$ is defined by*

$$l_{\vec{p}}\left(\mathbb{Z}^d\right) = \left\{ c : \left\| \left(\ldots \|c(k_1, k_2, \ldots, k_d)\|_{l_{p_1(k_1)}} \ldots \right) \right\|_{l_{p_d(k_d)}} < \infty \right\}$$

with its norm

$$\|c\|_{l_{\vec{p}}} = \left\| \left(\ldots \|c(k_1, k_2, \ldots, k_d)\|_{l_{p_1(k_1)}} \ldots \right) \right\|_{l_{p_d(k_d)}} \quad \text{for } c \in l_{\vec{p}}\left(\mathbb{Z}^d\right).$$

Definition 2 (see [1,2]). *Let $\vec{p} = (p_1, \ldots, p_d) \in [1, \infty]^d$ be a mixed-norm index. The mixed-norm Lebesgue space $L_{\vec{p}}(\mathbb{R}^d)$ is the set of all measurable functions g on $\mathbb{R}^d$ such that*

$$\left(\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \left[\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |g(x_1, x_2, \ldots, x_d)|^{p_1} dx_1 \right)^{\frac{p_2}{p_1}} dx_2 \right]^{\frac{p_3}{p_2}} dx_3 \ldots dx_d \right)^{\frac{1}{p_d}} < \infty.$$

For $g \in L_{\vec{p}}(\mathbb{R}^d)$, we define $L_{\vec{p}}(\mathbb{R}^d)$-norm of g by

$$\|g\|_{L_{\vec{p}}(\mathbb{R}^d)} = \left(\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \left[\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |g(x_1, x_2, \ldots, x_d)|^{p_1} dx_1 \right)^{\frac{p_2}{p_1}} dx_2 \right]^{\frac{p_3}{p_2}} dx_3 \ldots dx_d \right)^{\frac{1}{p_d}}$$

$$= \left\| \ldots \left\| \|g(x_1, x_2, \ldots, x_d)\|_{L_{p_1(x_1)}} \right\|_{L_{p_2(x_2)}} \ldots \right\|_{L_{p_d(x_d)}}. \tag{1}$$

If $p_i = \infty$ for $i = 1, \ldots, d$, then the relevant L_{p_i}-norms in (1) are replaced by L_∞-norms. To simplify the notation $\|g\|_{L_{\vec{p}}(\mathbb{R}^d)}$, we also abbreviate this to $\|g\|_{L_{\vec{p}}}$ or $\|g\|_{\vec{p}}$.

The mixed-norm Lebesgue space $L_{\vec{p}}(\mathbb{R}^d)$ is a generalization of the classical Lebesgue space $L_p(\mathbb{R}^d)$. This class of function spaces focus on independent variables of possibly different meanings. Mixed-norm Lebesgue space studies can be traced back to the years of 1960s in [1,2]. The space such like $L_{\vec{p}}(\mathbb{R}^d)$ has many practical or theoretical applications. For example, functions in partial differential equations defined by spacial and time variables may be in some mixed-norm spaces. In addition, inhomogeneous Besov spaces based on mixed Lebesgue norms are studied recently in [3–5]. Moreover, mixed-norm Triebel–Lizorkin spaces or Hardy spaces are also studied in [6,7]. Besides, sampling theory is also studied based on mixed-norm theories in [8], which only distinguishes variables in time and space. With the help of [9], we will give a generalization of such sampling problem

in sbuspaces of $L_{\vec{p}}(\mathbb{R}^d)$ cases. That is to say, the results in this paper unify and extend the revelent existing results in [8,9] and so on.

Definition 3 (see [8,9]). *Let $\vec{p} = (p_1, \ldots, p_d) \in [1, \infty]^d$ be a mixed-norm index. The mixed-norm Wiener amalgam spaces $(W\mathfrak{L})_{\vec{p}}(\mathbb{R}^d)$ is the set of all measurable functions g on $\mathbb{R}^d$ such that*

$$\left\| \cdots \|g(x_1, x_2, \cdots, x_d)\|_{(W\mathfrak{L})_{p_1}(x_1)(\mathbb{R})} \cdots \right\|_{(W\mathfrak{L})_{p_d}(x_d)(\mathbb{R})} < \infty.$$

For $g \in (W\mathfrak{L})_{\vec{p}}(\mathbb{R}^d)$, we define $(W\mathfrak{L})_{\vec{p}}(\mathbb{R}^d)$-norm of g by

$$\|g\|_{(W\mathfrak{L})_{\vec{p}}(\mathbb{R}^d)} = \left\| \cdots \|g(x_1, x_2, \cdots, x_d)\|_{(W\mathfrak{L})_{p_1}(x_1)(\mathbb{R})} \cdots \right\|_{(W\mathfrak{L})_{p_d}(x_d)(\mathbb{R})},$$

where

$$\|g(x)\|_{(W\mathfrak{L})_p(x)(\mathbb{R})} = \left[\sum_{k \in \mathbb{Z}} \mathrm{esssup}_{x \in [0,1]} |g(x+k)|^p \right]^{\frac{1}{p}} \quad \text{if } 1 \leq p < \infty,$$

and

$$\|g(x)\|_{(W\mathfrak{L})_\infty(x)(\mathbb{R})} = \sup_{k \in \mathbb{Z}} \mathrm{esssup}_{x \in [0,1]} |g(x+k)| \quad \text{if } p = \infty.$$

Remark 1.

(i) *In fact, we have $(W\mathfrak{L})_{\vec{p}}(\mathbb{R}^d) \subset L_{\vec{p}}(\mathbb{R}^d)$, but the converse $L_{\vec{p}}(\mathbb{R}^d) \subset (W\mathfrak{L})_{\vec{p}}(\mathbb{R}^d)$ is not true. Indeed, since*

$$\|g(x)\|_{L_p(\mathbb{R})} = \left(\int_{\mathbb{R}} |g(x)|^p dx \right)^{\frac{1}{p}} = \left(\sum_{k \in \mathbb{Z}} \int_{[0,1]} |g(x+k)|^p dx \right)^{\frac{1}{p}}$$

$$\leq \left(\sum_{k \in \mathbb{Z}} \int_{[0,1]} \mathrm{esssup}_{x \in [0,1]} |g(x+k)|^p dx \right)^{\frac{1}{p}}$$

$$= \left(\int_{[0,1]} \sum_{k \in \mathbb{Z}} \mathrm{esssup}_{x \in [0,1]} |g(x+k)|^p dx \right)^{\frac{1}{p}}$$

$$= \|g(x)\|_{(W\mathfrak{L})_p(\mathbb{R})},$$

by induction on $\vec{p}$, one has

$$\|g(x)\|_{L_{\vec{p}}(\mathbb{R}^d)} \leq \|g(x)\|_{(W\mathfrak{L})_{\vec{p}}(\mathbb{R}^d)}. \tag{2}$$

Hence $(W\mathfrak{L})_{\vec{p}}(\mathbb{R}^d) \subset L_{\vec{p}}(\mathbb{R}^d)$.

(ii) *It is worth mentioning that when $\vec{p} = (p, p, \cdots, p)$, we only get*

$$\|g(x)\|_{(W\mathfrak{L})_{\vec{p}}(\mathbb{R}^d)} \leq \|g(x)\|_{(W\mathfrak{L})_p(\mathbb{R}^d)}$$

and hence

$$(W\mathfrak{L})_p(\mathbb{R}^d) \subset (W\mathfrak{L})_{\vec{p}}(\mathbb{R}^d).$$

For example, when $\vec{p} = (1, 1, \cdots, 1) = \vec{1}$, we have

$$(W\mathfrak{L})_1(\mathbb{R}^d) \subset (W\mathfrak{L})_{\vec{1}}(\mathbb{R}^d).$$

In classical sampling theory, Shannon sampling theorem [10] shows the basic theory of signal analysis and communication system, which mainly used the bandlimited functions. In shift-invariant spaces, it is known that sampling that are not bandlimited is a realistic

model for a large number of scientific applications [11]. Especially, sampling study of subspaces generated by wavelets was fascinating in 1990s–2000s (see [12,13]). Until now, the study of sampling in shift-invariant subspaces of $L_p(\mathbb{R}^d)$ (see [9]) has aroused many mathematicians [14–16]. Therefore, a natural research path is to extend the sampling theory to the mixed-norm Lebesgue space $L_{\vec{p}}(\mathbb{R}^d)$. Of course, $L_{\vec{p}}(\mathbb{R}^d)$ is a kind of Banach space which inherits many outstanding properties of the traditional $L_p(\mathbb{R}^d)$ space. However, due to the non-commutability of the integral order in the definition of $L_{\vec{p}}(\mathbb{R}^d)$, this has brought various challenges to scholars in studying related mathematical problems. For example, due to the non-commutative order of integrals, it is very difficult to characterize the relevant mixed-norm spaces (see, e.g., [6,17]).

In this article, motivated by the above studies, we explore the promotion of the results known in the literature in shift-invariant subspaces of mixed-norm Lebesgue spaces $L_{\vec{p}}(\mathbb{R}^d)$ and establish new results to unify and refine these existing results. We establish generalized sampling theorems, generalized stability theorems and new inequalities in the setting of shift-invariant subspaces of Lebesgue and Wiener amalgam spaces with mixed-norms. A convergence theorem of general iteration algorithms for sampling in some shift-invariant subspaces of $L_{\vec{p}}(\mathbb{R}^d)$ are also given. Our new results promotes the existing conclusions of shift-invariant subspaces, such as [8,9]. We hope that it can be used in the study of frame or random sampling under mixed norms in the future.

2. New Stability Theorems in $L_{\vec{p}}(\mathbb{R}^d)$ and $(W\mathfrak{L})_{\vec{p}}(\mathbb{R}^d)$

In this section, we import a series of results for supporting stability theorem in the setting of $L_{\vec{p}}(\mathbb{R}^d)$ and $(W\mathfrak{L})_{\vec{p}}(\mathbb{R}^d)$. The following known lemma ([18], Theorem 6.18) is fundamental for our proofs.

Lemma 1. *Let $1 \leq p \leq \infty$, $c \in l_p$ and $h \in l_1$, then $\| \sum_k c(k)h(l-k) \|_{l_p} \leq \|c\|_{l_p} \|h\|_{l_1}$.*

With the help of Lemma 1, we now have a estimation of upper bound for stability theorem in Theorem 3, which calls the help of $(W\mathfrak{L})_1(\mathbb{R}^d)$ (the upper bound positive constant is $\|\eta(x_1, x_2)\|_{(W\mathfrak{L})_1(\mathbb{R}^d)(\mathbb{R}^2)}$).

Theorem 1. *Assume $\vec{p} \in [1, \infty]^d$, $c \in l_{\vec{p}}$ and $\eta(x) \in (W\mathfrak{L})_{\vec{1}}(\mathbb{R}^d)$. Then*

$$\left\| \sum_k c_k \eta(x-k) \right\|_{L_{\vec{p}}(\mathbb{R}^d)} \leq \|c\|_{l_{\vec{p}}} \|\eta\|_{(W\mathfrak{L})_{\vec{1}}(\mathbb{R}^d)}.$$

Proof. For $1 \leq p_1 < \infty$, by Lemma 1, we have

$$\left\| \sum_{k_1} c_{k_1,k_2} \eta(x_1 - k_1, x_2 - k_2) \right\|_{L_{p_1}(x_1)(\mathbb{R})}^{p_1}$$

$$= \int_{\mathbb{R}} |\sum_{k_1} c_{k_1,k_2} \eta(x_1 - k_1, x_2 - k_2)|^{p_1} dx_1$$

$$= \int_{[0,1]} \sum_{l_1 \in \mathbb{Z}} |\sum_{k_1} c_{k_1,k_2} \eta(x_1 - k_1 - l_1, x_2 - k_2)|^{p_1} dx_1$$

$$\leq \int_{[0,1]} \|c_{k_1,k_2}\|_{l_{p_1}(k_1)}^{p_1} \|\eta(x_1 - k_1 - l_1, x_2 - k_2)\|_{l_1(k_1)}^{p_1} dx_1$$

$$= \|c_{k_1,k_2}\|_{l_{p_1}(k_1)}^{p_1} \|\eta(x_1, x_2 - k_2)\|_{(W\mathfrak{L})_1(x_1)(\mathbb{R})}^{p_1},$$

which leads to

$$\left\| \sum_{k_1} c_{k_1,k_2} \eta(x_1 - k_1, x_2 - k_2) \right\|_{L_{p_1}(x_1)(\mathbb{R})} \leq \|c_{k_1,k_2}\|_{l_{p_1}(k_1)} \|\eta(x_1, x_2 - k_2)\|_{(W\mathfrak{L})_1(x_1)(\mathbb{R})}.$$

A similar argument could be made for $p_1 = \infty$. So we can obtain

$$
\left\| \sum_{k_2} \sum_{k_1} c_{k_1,k_2} \eta(x_1 - k_1, x_2 - k_2) \right\|_{L_{(p_1,p_2)}(\mathbb{R}^2)}
$$
$$
\leq \left\| \Sigma_{k_2} \left\| \Sigma_{k_1} c_{k_1,k_2} \eta(x_1 - k_1, x_2 - k_2) \right\|_{L_{p_1}(x_1)(\mathbb{R})} \right\|_{L_{p_2}(x_2)(\mathbb{R})}
$$
$$
\leq \left\| \Sigma_{k_2} \| c_{k_1,k_2} \|_{l_{p_1}(k_1)} \| \eta(x_1, x_2 - k_2) \|_{(W\mathfrak{L})_1(x_1)(\mathbb{R})} \right\|_{L_{p_2}(x_2)(\mathbb{R})}
$$
$$
\leq \left\| \| c_{k_1,k_2} \|_{l_{p_1}(k_1)} \right\|_{l_{p_2}(k_2)} \left\| \| \eta(x_1, x_2) \|_{(W\mathfrak{L})_1(x_1)(\mathbb{R})} \right\|_{(W\mathfrak{L})_1(x_2)(\mathbb{R})}
$$
$$
\leq \| c_{k_1,k_2} \|_{l_{(p_1,p_2)}} \| \eta(x_1, x_2) \|_{(W\mathfrak{L})_{\vec{1}}(\mathbb{R}^2)}.
$$

Therefore the conclusion holds by induction on $\vec{p}$. $\quad\square$

It is easy to see that when $(W\mathfrak{L})_{\vec{1}}$ is replaced by $(W\mathfrak{L})_1$, we have the same results as in Theorem 1 and its proof is similar because of $\|g\|_{(W\mathfrak{L})_{\vec{1}}(\mathbb{R}^d)} \leq \|g\|_{(W\mathfrak{L})_1(\mathbb{R}^d)}$:

Corollary 1. *Assume* $\vec{p} \in [1, \infty]^d$, $c \in l_{\vec{p}}$ *and* $\eta(x) \in (W\mathfrak{L})_1(\mathbb{R}^d)$. *Then*

$$
\left\| \sum_k c_k \eta(x - k) \right\|_{L_{\vec{p}}(\mathbb{R}^d)} \leq \|c\|_{l_{\vec{p}}} \|\eta\|_{(W\mathfrak{L})_1(\mathbb{R}^d)}.
$$

The shift-invariant subspace $\mathrm{Span}(\eta)$ (we usually call this space generated by η) is defined by

$$
\mathrm{Span}(\eta) := \left\{ f = \sum_k c_k \eta(x - k) : c \in l_{\vec{p}}(\mathbb{Z}^d), \eta \in (W\mathfrak{L})_1(\mathbb{R}^d) \right\}.
$$

The following two known results are important for proving our new stability theorem.

Lemma 2 ([19]). *(Mixed-norm Hölder inequality) Let* $1 \leq \vec{p}, \vec{q} \leq \infty$ *with* $\frac{1}{p_i} + \frac{1}{q_i} = 1$ *for* $i = 1, 2, \cdots, d$. *Let* $g_1(x) \in L_{\vec{p}}(\mathbb{R}^d)$ *and* $g_2(x) \in L_{\vec{q}}(\mathbb{R}^d)$. *Then*

$$
\|g_1(x)g_2(x)\|_{L_1(\mathbb{R}^d)} \leq \|g_1(x)\|_{L_{\vec{p}}(\mathbb{R}^d)} \|g_2(x)\|_{L_{\vec{q}}(\mathbb{R}^d)}.
$$

Theorem 2 ([9]). *Let* $\eta \in (W\mathfrak{L})_1(\mathbb{R}^d)$ *and* C_1 *and* C_2 *be positive constants. Then*

$$
C_1 \|c\|_{l_2} \leq \| \sum_k c_k \eta(x - k) \|_{L_2(\mathbb{R}^d)} \leq C_2 \|c\|_{l_2}
$$

holds if and only if one of the following conditions holds:

(i)   *There exists a function* $h \in \mathrm{Span}(\eta)$ *such that*

$$
\langle \eta(x - l), h(x) \rangle = \delta_{0,l}, \; l \in \mathbb{Z}^d,
$$

     and $\delta_{0,l}$ *is defined by* $\delta = 1$ *for* $l = 0$, $\delta = 0$ *for* $l \neq 0$;

(ii) $\sum_l |\hat{\eta}(\xi + 2\pi l)|^2 > 0$ *for every* $\xi \in \mathbb{R}^d$.

Now, we establish the following new stability theorem.

Theorem 3. *Let $\vec{p} \in [1, \infty]^d$, $c \in l_{\vec{p}}(\mathbb{Z}^d)$ and $\eta \in (W\mathfrak{L})_1(\mathbb{R}^d)$. Suppose that $\sum_l |\hat{\eta}(\xi + 2\pi l)|^2 > 0$ for every $\xi \in \mathbb{R}^d$. Then*

$$C_1 \|c\|_{l_{\vec{p}}} \leq \| \sum_k c_k \eta(x-k) \|_{L_{\vec{p}}(\mathbb{R}^d)} \leq C_2 \|c\|_{l_{\vec{p}}} \tag{3}$$

and

$$C_1 \|c\|_{l_{\vec{p}}} \leq \| \sum_k c_k \eta(x-k) \|_{(W\mathfrak{L})_{\vec{p}}(\mathbb{R}^d)} \leq C_2 \|c\|_{l_{\vec{p}}} \tag{4}$$

hold for some positive constants C_1 and C_2.

Proof. We first verify (3). On one hand, by Theorem 1, it leads to

$$\left\| \sum_k c_k \eta(x-k) \right\|_{L_{\vec{p}}(\mathbb{R}^d)} \leq \|c\|_{l_{\vec{p}}} \|\eta\|_{(W\mathfrak{L})_{\vec{1}}(\mathbb{R}^d)}.$$

This means $C_2 = \|\eta\|_{(W\mathfrak{L})_{\vec{1}}(\mathbb{R}^d)} \leq \|\eta\|_{(W\mathfrak{L})_1(\mathbb{R}^d)}$ and the upper bound of (3) is obtained.

For lower bound on the other hand, let $h_\eta(x) = \sum_k c_k \eta(x-k)$. By applying Theorem 2, we can find a function $h \in \text{Span}(\eta)$ such that

$$\langle \eta(x-l), h(x) \rangle = \delta_{0,l}, \ l \in \mathbb{Z}^d.$$

Then

$$\int_{\mathbb{R}^d} h_\eta(x) \overline{h(x-k)} dx$$

$$= \int_{\mathbb{R}^d} (\sum_m c_m \eta(x-m)) \overline{h(x-k)} dx$$

$$= \sum_m c_m \int_{\mathbb{R}^d} \eta(x-m+k) \overline{h(x)} dx = c_k.$$

Let $\{(\tilde{c}_k)_k\} \in l_{\vec{q}}$ with $\frac{1}{p_i} + \frac{1}{q_i} = 1$ for $i \in \{1, 2, \cdots, d\}$. Thus

$$|\langle c, \tilde{c} \rangle| = \left| \sum_k c_k \overline{\tilde{c}_k} \right|$$

$$= \left| \sum_k \overline{\tilde{c}_k} \int_{\mathbb{R}^d} h_\eta(x) \overline{h(x-k)} dx \right|$$

$$= \left| \int_{\mathbb{R}^d} h_\eta(x) \sum_k \overline{\tilde{c}_k} \overline{h(x-k)} dx \right|.$$

With the help of Lemma 2 and Theorem 1, one has

$$|\langle c, \tilde{c} \rangle| \leq \|h_\eta(x)\|_{L_{\vec{p}}(\mathbb{R}^d)} \| \sum_k \overline{\tilde{c}_k} \overline{h(x-k)} \|_{L_{\vec{q}}(\mathbb{R}^d)}$$

$$\leq \|h_\eta(x)\|_{L_{\vec{p}}(\mathbb{R}^d)} \|\tilde{c}\|_{l_{\vec{q}}} \|h(x)\|_{(W\mathfrak{L})_{\vec{1}}(\mathbb{R}^d)}.$$

It follows that

$$\|c\|_{l_{\vec{p}}} \leq \|h_\eta(x)\|_{L_{\vec{p}}(\mathbb{R}^d)} \|h(x)\|_{(W\mathfrak{L})_{\vec{1}}(\mathbb{R}^d)} \leq \|h_\eta(x)\|_{L_{\vec{p}}(\mathbb{R}^d)} \|h(x)\|_{(W\mathfrak{L})_1(\mathbb{R}^d)}.$$

Therefore, we get

$$\frac{1}{\|h(x)\|_{(W\mathfrak{L})_1(\mathbb{R}^d)}} \|c\|_{l_{\vec{p}}} \leq \|h_\eta(x)\|_{L_{\vec{p}}(\mathbb{R}^d)}.$$

Take $C_1 = \frac{1}{\|h(x)\|_{(W\mathfrak{L})_1(\mathbb{R}^d)}}$ and hence the lower bound is arrived.

Next, we prove (4). Using Lemma 1, we obtain

$$\|h_\eta(x)\|_{(W\mathfrak{L})_{(p_1,p_2)}(\mathbb{R}^2)}$$

$$= \{\textstyle\sum_{k_2} \text{esssup}_{x_2\in[0,1]}[\sum_{k_1}\text{esssup}_{x_1\in[0,1]}|h_\eta(x_1-k_1,x_2-k_2)|^{p_1}]^{\frac{p_2}{p_1}}\}^{\frac{1}{p_2}}$$

$$= \{\textstyle\sum_{k_2} \text{esssup}_{x_2\in[0,1]}[\sum_{k_1}\text{esssup}_{x_1\in[0,1]}(\sum_{l_1,l_2}|c_{l_1,l_2}||\eta(x_1-k_1-l_1,x_2-k_2-l_2)|)^{p_1}]^{\frac{p_2}{p_1}}\}^{\frac{1}{p_2}}$$

$$\leq \{\textstyle\sum_{k_2}[\sum_{k_1}(\sum_{l_1}\sum_{l_2}|c_{l_1,l_2}|\text{esssup}_{x_1,x_2\in[0,1]}|\eta(x_1-k_1-l_1,x_2-k_2-l_2)|)^{p_1}]^{\frac{p_2}{p_1}}\}^{\frac{1}{p_2}}$$

$$\leq \|c_{l_1,l_2}\|_{l_{(p_1,p_2)}}\textstyle\sum_{k_2}[\sum_{k_1}(\text{esssup}_{x_1,x_2\in[0,1]}|\eta(x_1-k_1,x_2-k_2)|)]$$

$$= \|c_{l_1,l_2}\|_{l_{(p_1,p_2)}}\|\eta(x_1,x_2)\|_{(W\mathfrak{L})_1(\mathbb{R}^2)},$$

So the upper bound of (4) is arrived. Moreover, by (2),

$$\|h_\eta(x)\|_{L_{(p_1,p_2)}(\mathbb{R}^2)} \leq C\|h_\eta(x)\|_{(W\mathfrak{L})_{(p_1,p_2)}(\mathbb{R}^2)}.$$

Putting this with the lower bound of (3), we have the lower bound of (4).

A similar argument could be extended to the case of $p = (p_1, p_2, \cdots, p_d)$ with $d \geq 3$. The proof is completed. □

Remark 2. *Here, from the proof of upper bound in (4), we cannot have the same result by $\eta \in (W\mathfrak{L})_{\vec{1}}(\mathbb{R}^d)$ because we only have $\|g(x)\|_{(W\mathfrak{L})_{\vec{1}}(\mathbb{R}^d)} \leq \|g(x)\|_{(W\mathfrak{L})_1(\mathbb{R}^d)}$. In fact, we have another proof of (3) in [19]. In fact, in [19], we use the condition of $\eta \in \mathfrak{L}_{\vec{\infty}}(\mathbb{R}^d)$, not $\eta \in (W\mathfrak{L})_1(\mathbb{R}^d)$. Note that*

$$\|g\|_{\mathfrak{L}_{\vec{\infty}}(\mathbb{R}^d)} = esssup_{x_d\in[0,1]}\left(\sum_{k_d\in\mathbb{Z}}\left\{\cdots\left[esssup_{x_1\in[0,1]}\left(\sum_{k_1\in\mathbb{Z}}|g(x_1+k_1,\ldots,x_d+k_d)|\right)\right]\cdots\right\}\right).$$

Then $\|g(x)\|_{\mathfrak{L}_{\vec{\infty}}(\mathbb{R}^d)} \leq \|g(x)\|_{(W\mathfrak{L})_{\vec{1}}(\mathbb{R}^d)}$ by their definitions.

3. Sampling Theorems in Span(η) with Mixed-Norms

In order to conduct sampling studies, we usually assume that the objective function or signal $g(x)$ is continuous. Furthermore, we also need the help of 1-order modulus of continuity of $g(x)$ to establish the convergence of the summation of sampling series.

Definition 4 (see [8,9])**.** *Let $t > 0$ and $g(x)$ be a continuous function on $\mathbb{R}^d$. Then 1-order modulus of continuity of $g(x)$ is defined by*

$$m(g)_t(x) = \sup_{|h|\leq t}|g(x+h) - g(x)|.$$

The following result will show that Definition 4 is well-defined.

Proposition 1. *Let $g \in (W\mathfrak{L})_1(\mathbb{R}^d)$. If g is continuous, then $m(g)_t(x) \in (W\mathfrak{L})_{\vec{1}}(\mathbb{R}^d)$ and $m(g)_t(x) \in (W\mathfrak{L})_1(\mathbb{R}^d)$.*

Proof. Assume $0 < t < 1$. Then

$$m(g)_t(x_1 + l_1, x_2 + l_2, \cdots, x_d + l_d)$$

$$= \sup_{|h| \leq t} |g(x_1 + l_1 + h_1, x_2 + l_2 + h_2, \cdots, x_d + l_d + h_d) - g(x_1 + l_1, x_2 + l_2, \cdots, x_d + l_d)|$$

$$\leq \sup_{|h| \leq t} |g(x_1 + l_1 + h_1, x_2 + l_2 + h_2, \cdots, x_d + l_d + h_d)|$$

$$+ \sup_{|h| \leq t} |g(x_1 + l_1, x_2 + l_2, \cdots, x_d + l_d)|$$

It is easy to see that

$$\sum_{l_d} \sup_{x_d \in [0,1]} \cdots \sum_{l_1} \sup_{x_1 \in [0,1]} \sup_{|h| \leq t} |g(x_1 + l_1, x_2 + l_2, \cdots, x_d + l_d)|$$
$$= \sum_{l_d} \sup_{x_d \in [0,1]} \cdots \sum_{l_1} \sup_{x_1 \in [0,1]} |g(x_1 + l_1, x_2 + l_2, \cdots, x_d + l_d)|$$

$$= \|g\|_{(W\mathfrak{L})_{\bar{1}}(\mathbb{R}^d)}.$$

On the other hand, since

$$\sup_{x_1 \in [0,1]} |g(x_1 + l_1 + h_1, x_2 + l_2 + h_2, \cdots, x_d + l_d + h_d)|$$
$$\leq \sup_{x_1 \in [-1,2]} |g(x_1 + l_1, x_2 + l_2 + h_2, \cdots, x_d + l_d + h_d)|,$$

which leads to

$$\sum_{l_1 \in \mathbb{Z}} \sup_{x_1 \in [0,1]} |g(x_1 + l_1 + h_1, x_2 + l_2 + h_2, \cdots, x_d + l_d + h_d)|$$
$$\leq \sum_{l_1 \in \mathbb{Z}} \sup_{x_1 \in [-1,2]} |g(x_1 + l_1, x_2 + l_2 + h_2, \cdots, x_d + l_d + h_d)|$$
$$\leq \sum_{l_1 \in \mathbb{Z}} \sum_{k_1 \in [-1,2] \cap \mathbb{Z}} \sup_{x_1 \in [0,1]} |g(x_1 + l_1 + k_1, x_2 + l_2 + h_2, \cdots, x_d + l_d + h_d)|$$
$$= \sum_{k_1 \in [-1,2] \cap \mathbb{Z}} \sum_{l_1 \in \mathbb{Z}} \sup_{x_1 \in [0,1]} |g(x_1 + l_1 + k_1, x_2 + l_2 + h_2, \cdots, x_d + l_d + h_d)|$$
$$\leq 4 \sum_{l_1 \in \mathbb{Z}} \sup_{x_1 \in [0,1]} |g(x_1 + l_1, x_2 + l_2 + h_2, \cdots, x_d + l_d + h_d)|.$$

So, we have

$$\sum_{l_d} \sup_{x_d \in [0,1]} \cdots \sum_{l_1} \sup_{x_1 \in [0,1]} \sup_{|h| \leq t} |g(x_1 + l_1 + h_1, x_2 + l_2 + h_2, \cdots, x_d + l_d + h_d)|$$
$$\leq 4^d \sum_{l_d} \sup_{x_d \in [0,1]} \cdots \sum_{l_1} \sup_{x_1 \in [0,1]} |g(x_1 + l_1, x_2 + l_2, \cdots, x_d + l_d)|$$

$$\leq C \|g\|_{(W\mathfrak{L})_{\bar{1}}(\mathbb{R}^d)},$$

where $C = 4^d$ for $0 < t < 1$.

Put all the above together, we get

$$\|m(g)_t(x)\|_{(W\mathfrak{L})_{\bar{1}}(\mathbb{R}^d)} \leq C \|g(x)\|_{(W\mathfrak{L})_{\bar{1}}(\mathbb{R}^d)}.$$

Similarly, since

$$\sum_{l=(l_1,\cdots,l_d)\in\mathbb{Z}^d} \sup_{x\in[0,1]^d} \sup_{|h|\leq t} |m(g)(x_1+l_1+h_1,x_2+l_2+h_2,\cdots,x_d+l_d+h_d)|$$

$$\leq \sum_{l\in\mathbb{Z}^d} \sup_{x\in[0,1]^d} \sup_{|h|\leq t} |g(x_1+l_1+h_1,x_2+l_2+h_2,\cdots,x_d+l_d+h_d)|$$

$$+\sum_{l\in\mathbb{Z}^d} \sup_{x\in[0,1]^d} \sup_{|h|\leq t} |g(x_1+l_1,x_2+l_2,\cdots,x_d+l_d)|$$

$$\leq 2\sum_{l\in\mathbb{Z}^d} \sup_{x\in[-1,2]^d} |g(x_1+l_1,x_2+l_2,\cdots,x_d+l_d)|$$

$$= 2\sum_{l\in\mathbb{Z}^d} \sum_{k\in[-1,2]^d\cap\mathbb{Z}^d} \sup_{x\in[0,1]^d} |g(x_1+l_1+k_1,x_2+l_2+k_2,\cdots,x_d+l_d+k_d)|$$

$$= 2\sum_{k\in[-1,2]^d\cap\mathbb{Z}^d} \sum_{l\in\mathbb{Z}^d} \sup_{x\in[0,1]^d} |g(x_1+l_1+k_1,x_2+l_2+k_2,\cdots,x_d+l_d+k_d)|$$

$$= 2\cdot 4^d \sum_{l\in\mathbb{Z}^d} \sup_{x\in[0,1]^d} |g(x_1+l_1+k_1,x_2+l_2+k_2,\cdots,x_d+l_d+k_d)|$$

$$= C\|g\|_{(W\mathfrak{L})_1(\mathbb{R}^d)},$$

we obtain

$$\|m(g)_t(x)\|_{(W\mathfrak{L})_1(\mathbb{R}^d)} \leq C\|g(x)\|_{(W\mathfrak{L})_1(\mathbb{R}^d)}.$$

The proof is completed. □

Proposition 2. *Let $g \in (W\mathfrak{L})_1(\mathbb{R}^d)$. If g is continuous, then*

$$\lim_{t\to 0^+} \|m(g)_t(x)\|_{(W\mathfrak{L})_1(\mathbb{R}^d)} = 0.$$

Proof. By Proposition 1, for every fixed $\varepsilon > 0$, there exists $M \in \mathbb{N}$, such that

$$\sum_{|l|\geq M} \sup_{x\in[0,1]^d} |m(g)(x+l)| \leq \varepsilon.$$

Since g is continuous, there exists some $t_M > 0$ satisfying the following estimation:

$$\sup_{x\in[0,1]^d} \sup_{|h|<t_M} |g(x+l+h) - g(x+l)| \leq \frac{\varepsilon}{M^d}$$

with $|l| < M$ and $t < t_M$. Thus,

$$\sum_{|l|<M} \sup_{x\in[0,1]^d} \sup_{|h|<t_M} |g(x+l+h) - g(x+l)| \leq \sum_{|l|<M} \frac{\varepsilon}{M^d} \leq 2\varepsilon.$$

Observe that for any $0 < t < t_M$,

$$\|m(g)_t(x)\|_{(W\mathfrak{L})_1(\mathbb{R}^d)} = \sum_{|l|\geq M} \sup_{x\in[0,1]^d} |m(g)_t(x+l)| + \sum_{|l|<M} \sup_{x\in[0,1]^d} \sup_{|h|<t_M} |g(x+l+h) - g(x+l)| \leq 3\varepsilon,$$

this shows $\lim_{t\to 0^+} \|m(g)_t(x)\|_{(W\mathfrak{L})_1(\mathbb{R}^d)} = 0$. The proof is completed. □

Theorem 4. *Let $\eta \in (W\mathfrak{L})_1(\mathbb{R}^d)$ and $g \in Span(\eta)$. If η is continuous, then the following hold:*

(i) $m(g)_t(x) \in L_{\vec{p}}$;

(ii) *for any $\varepsilon > 0$, there exists some $T > 0$ such that*

$$\|m(g)_t(x)\|_{L_{\vec{p}}(\mathbb{R}^d)} \leq \varepsilon\|g\|_{L_{\vec{p}}(\mathbb{R}^d)}$$

for $0 < t < T$.

Proof. Since $g \in \text{Span}(\eta)$, $g = \sum_k c_k \eta(x-k)$ for $c \in l_{\vec{p}}$. So

$$
\begin{aligned}
m(g)_t(x) &= \sup_{|h| \leq t} |g(x+h) - g(x)| \\
&\leq \sup_{|h| \leq t} \sum_k |c_k| |\eta(x-k+h) - \eta(x-k)| \\
&\leq \sum_k |c_k| \sup_{|h| \leq t} |\eta(x-k+h) - \eta(x-k)| \\
&= \sum_k |c_k| m(\eta)_t(x-k).
\end{aligned}
$$

By Theorem 3, one has

$$
\|m(g)_t(x)\|_{L_{\vec{p}}(\mathbb{R}^d)} \leq \|c\|_{l_{\vec{p}}} \|m(\eta)_t(x)\|_{(W\mathfrak{L})_1(\mathbb{R}^d)}.
$$

Combining this with (3) yield

$$
\|m(g)_t(x)\|_{L_{\vec{p}}(\mathbb{R}^d)} \leq C\|g\|_{L_{\vec{p}}(\mathbb{R}^d)} \|m(\eta)_t(x)\|_{(W\mathfrak{L})_1(\mathbb{R}^d)},
$$

which means $m(g)_t(x) \in L_{\vec{p}}(\mathbb{R}^d)$. Hence (i) is proved. To see (ii), let $\varepsilon > 0$ be given. Then, by Proposition 2, there exists $T > 0$ such that

$$
\|m(g)_t(x)\|_{L_{\vec{p}}(\mathbb{R}^d)} \leq \varepsilon \|g\|_{L_{\vec{p}}(\mathbb{R}^d)}
$$

for $0 < t < T$. The proof is completed. □

Definition 5 (see [8])**.** *Let $\Delta_1, \Delta_2, \cdots, \Delta_d$ be countable index sets. The sampling point set*

$$
S = \{x_k = (x_{k_1}, x_{k_2}, \cdots, x_{k_d}) : k = (k_1, k_2, \cdots, k_d) \in (\Delta_1, \Delta_2, \cdots, \Delta_d)\}
$$

is said to be strongly separated if $\inf_{k_i \neq k_j} |x_{k_i} - x_{k_j}| \geq \epsilon > 0$, where ϵ is fixed.

Before presenting the sampling theorem in this section, we will show some strongly separated properties of the sampling points meticulously.

Theorem 5. *Let $\Delta_1, \Delta_2, \cdots, \Delta_d$ be countable index sets. Suppose that the sampling point set $S = \{x_k = (x_{k_1}, x_{k_2}, \cdots, x_{k_d}) : k = (k_1, k_2, \cdots, k_d) \in (\Delta_1, \Delta_2, \cdots, \Delta_d)\}$ is strongly separated. Then*

$$
\|g(x_{k_1}, x_{k_2}, \cdots, x_{k_d})\|_{l_{\vec{p}}} \leq C_{\epsilon, \vec{p}} \|g\|_{(W\mathfrak{L})_{\vec{p}}(\mathbb{R}^d)}
$$

for a continuous function $f \in (W\mathfrak{L})_{\vec{p}}(\mathbb{R}^d)$ and some positive constant $C_{\epsilon, \vec{p}}$.

Proof. Note that $\|g(x_{k_1}, x_{k_2}, \cdots, x_{k_d})\|_{l_{\vec{p}}} := \left\| \cdots \|g(x_{k_1}, x_{k_2}, \cdots, x_{k_d})\|_{l_{p_1}(k_1)} \cdots \right\|_{l_{p_d}(x_d)}$. So, we have

$$\|g(x_{k_1}, x_{k_2}, \cdots, x_{k_d})\|_{l_{p_1}(k_1)}$$

$$= [\textstyle\sum_{k_1 \in \Delta_1} |g(x_{k_1}, x_{k_2}, \cdots, x_{k_d})|^{p_1}]^{\frac{1}{p_1}}$$

$$= [\textstyle\sum_{l_1 \in \mathbb{Z}} \sum_{x_{k_1} \in [l_1, l_1+1]} |g(x_{k_1}, x_{k_2}, \cdots, x_{k_d})|^{p_1}]^{\frac{1}{p_1}}$$

$$\leq [\textstyle\sum_{l_1 \in \mathbb{Z}} \sum_{x_{k_1} \in [l_1, l_1+1]} \sup_{x_1 \in [l_1, l_1+1]} |g(x_1, x_{k_2}, \cdots, x_{k_d})|^{p_1}]^{\frac{1}{p_1}}$$

$$\leq [\textstyle\sum_{l_1 \in \mathbb{Z}} (\lfloor \tfrac{1}{\epsilon} \rfloor + 1) \sup_{x_1 \in [l_1, l_1+1]} |g(x_1, x_{k_2}, \cdots, x_{k_d})|^{p_1}]^{\frac{1}{p_1}}$$

$$\leq [\textstyle\sum_{l_1 \in \mathbb{Z}} (\lfloor \tfrac{1}{\epsilon} \rfloor + 1) \sup_{x_1 \in [0,1]} |g(x_1 + l_1, x_{k_2}, \cdots, x_{k_d})|^{p_1}]^{\frac{1}{p_1}}$$

$$= (\lfloor \tfrac{1}{\epsilon} \rfloor + 1)^{\frac{1}{p_1}} [\textstyle\sum_{l_1 \in \mathbb{Z}} \sup_{x_1 \in [0,1]} |g(x_1 + l_1, x_{k_2}, \cdots, x_{k_d})|^{p_1}]^{\frac{1}{p_1}},$$

with $\lfloor x \rfloor$ is the biggest integer less than or equals to x. By induction, we get

$$\left\| \cdots \|g(x_{k_1}, x_{k_2}, \cdots, x_{k_d})\|_{l_{p_1}(k_1)} \cdots \right\|_{l_{p_d}(x_d)}$$

$$\leq (\lfloor \tfrac{1}{\epsilon} \rfloor + 1)^{\frac{1}{p_1}} \cdots (\lfloor \tfrac{1}{\epsilon} \rfloor + 1)^{\frac{1}{p_d}}$$

$$\left\{ \textstyle\sum_{l_d \in \mathbb{Z}} \left[\cdots [\textstyle\sum_{l_1 \in \mathbb{Z}} \sup_{x_1 \in [0,1]} |g(x_1 + l_1, \cdots, x_d + l_d)|^{p_1}]^{\frac{1}{p_1}} \cdots \right]^{\frac{p_d}{p_{d-1}}} \right\}^{\frac{1}{p_d}}$$

$$= C_{\epsilon, \vec{p}} \|g\|_{(W\mathfrak{L})_{\vec{p}}(\mathbb{R}^d)}.$$

The proof is completed. □

Definition 6 (see [8]). *The sampling points S is called r-dense if the balls $B_r(x)$ satisfy*

$$\bigcup_{x \in S} B_R(x) = \mathbb{R}^d \quad \text{for any } R > r.$$

Let $\Delta_1, \Delta_2, \cdots, \Delta_d$ be countable index sets. Recall that the collection $\{u_k(x)\}_{k \in (\Delta_1, \Delta_2, \cdots, \Delta_d)}$ is a *partition of unity* if it satisfies the following three conditions:

(1) $0 \leq u_k(x) \leq 1$ for $k = (k_1, k_2, \cdots, k_d) \in (\Delta_1, \Delta_2, \cdots, \Delta_d)$;
(2) $\text{Supp} u_k(x) \subseteq B_r(x)$, where $B_r(x)$ is the open ball centered at $x = (x_1, x_2, \cdots, x_d)$ with radius $r > 0$;
(3) $\sum_k u_k(x) = 1$.

Let $\{u_k(x)\}_{k \in (\Delta_1, \Delta_2, \cdots, \Delta_d)}$ is a *partition of unity*. Then we can define the sampling operator by

$$P_S g(x) = \sum_{k \in (\Delta_1, \Delta_2, \cdots, \Delta_d)} g(x_k) u_k(x)$$

with the sampling sequence $g(x_k) = g(x_{k_1}, x_{k_2}, \cdots, x_{k_d})$. Usually, we take the sampling point set S as strongly separated.

Finally, applying Theorem 4, we establish the following new sampling theorem.

Theorem 6. *Let $\eta \in (W\mathfrak{L})_1(\mathbb{R}^d)$, $g \in Span(\eta)$ and $p \in [1, \infty]^d$. If η is continuous and S is R_0-dense and strongly separated, then for each $\varepsilon > 0$,*

$$\|g(x) - P_S g(x)\|_{L_{\vec{p}}(\mathbb{R}^d)} \leq \varepsilon \|g\|_{L_{\vec{p}}(\mathbb{R}^d)}$$

for any $0 < r < R_0$.

Proof. Since

$$g(x) - P_S g(x) = \sum_{k \in (\Delta_1, \cdots, \Delta_d)} \left(g(x_1, \cdots, x_d) - g(x_{k_1}, \cdots, x_{k_d}) \right) u_k(x)$$

$$= \sum_{k \in (\Delta_1, \cdots, \Delta_d)} \left(g(x_1, \cdots, x_d) - g(x_1 + h_{x_{k_1}}, \cdots, x_d + h_{x_{k_d}}) \right) u_k(x),$$

by Theorem 4, we have

$$\|g - P_S g(x)\|_{L_{\vec{p}}}$$

$$\leq \left\| \sum_{k \in (\Delta_1, \cdots, \Delta_d)} \left| \left(g(x_1, \cdots, x_d) - g(x_1 + h_{x_{k_1}}, \cdots, x_d + h_{x_{k_d}}) \right) \right| u_k(x) \right\|_{L_{\vec{p}}(\mathbb{R}^d)}$$

$$\leq \left\| \sum_{k \in (\Delta_1, \cdots, \Delta_d)} \sup_{|h| \leq r} \left| \left(g(x_1, \cdots, x_d) - g(x_1 + h_{x_{k_1}}, \cdots, x_d + h_{x_{k_d}}) \right) \right| u_k(x) \right\|_{L_{\vec{p}}(\mathbb{R}^d)}$$

$$\leq \left\| m(g)_r(x) \sum_{k \in (\Delta_1, \cdots, \Delta_d)} u_k(x) \right\|_{L_{\vec{p}}(\mathbb{R}^d)}$$

$$= \|m(g)_r(x)\|_{L_{\vec{p}}}$$

$$\leq \varepsilon \|g\|_{L_{\vec{p}}(\mathbb{R}^d)},$$

where $h = (h_{x_{k_1}}, \cdots, h_{x_{k_d}})$. The proof is completed. □

Remark 3. *An new iterative algorithm can be established as follows:*
  Define

$$g_1(x) := P_S g$$

and

$$g_{n+1}(x) := P_S(g - g_n)(x) + g_n(x) \quad \text{for } n \in \mathbb{N},$$

where S is a fixed sampling point set with $0 < \varepsilon < 1$ in Theorem 6. It is easy to see $g - g_1 \in Span(\eta)$. Hence, by induction, we get $g - g_n \in Span(\eta)$. Applying Theorem 6, we obtain

$$\|g - g_{n+1}\|_{L_{\vec{p}}(\mathbb{R}^d)} = \|g - g_n - P_S(g - g_n)\|_{L_{\vec{p}}(\mathbb{R}^d)}$$

$$\leq \varepsilon \|g - g_n\|_{L_{\vec{p}}(\mathbb{R}^d)}$$

$$\leq \varepsilon^{n-1} \|g - g_1\|_{L_{\vec{p}}(\mathbb{R}^d)}$$

$$\leq \varepsilon^n \|g\|_{L_{\vec{p}}(\mathbb{R}^d)},$$

which imply

$$\lim_{n \to +\infty} \|g - g_n\|_{L_{\vec{p}}(\mathbb{R}^d)} = 0.$$

This proves the strong convergence of our iterative algorithm.

Remark 4. *As an example, let*

$$\eta(x_1, x_2, \cdots, x_d) = \varphi_1(x_1) \otimes \varphi_2(x_2) \otimes \cdots \otimes \varphi_d(x_d),$$

where $\otimes$ means the traditional tensor product. In wavelet theory, we can take the functions φ_i, $1 \leq i \leq d$, as the orthonormal or biorthogonal scaling functions with compact support. It is not hard to verify that these scaling functions satisfy $\|\eta\|_{(W\mathfrak{L})_{\vec{1}}(\mathbb{R}^d)} < \infty$ or $\|\eta\|_{(W\mathfrak{L})_1(\mathbb{R}^d)} < \infty$ and $\sum_l |\hat{\eta}(\xi + 2\pi l)|^2 > 0$ for every $\xi \in \mathbb{R}^d$.

4. Conclusions

It is well-known that sampling theory and stability theory are fascinating theories that has a wide range of applications in different branches of mathematics. In this article, inspired by previous research, we explore the promotion of the results known in the literature in shift-invariant subspaces of mixed-norm Lebesgue spaces $L_{\vec{p}}(\mathbb{R}^d)$ and establish new results to unify and refine these existing results. We establish generalized stability theorems, generalized sampling theorems and new inequalities in shift-invariant subspaces of Lebesgue and Wiener amalgam spaces with mixed-norms. A convergence theorem of general iteration algorithms for sampling in some shift-invariant subspaces of $L_{\vec{p}}(\mathbb{R}^d)$ is also given. Our new results promote the existing conclusions in shift-invariant subspaces. We hope that these new results can be used in the investigation of frame or random sampling under mixed norms in the future. Furthermore, the study of mathematical models and numerical experiment results is also an important goal of our future research.

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